

On possibilistic modal logics over Gödel logic

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1 Introduction

Possibilistic logic (see e.g. [4]) is a well known formal system to reasoning graded beliefs by means of necessity and possibility measures. From a logical point of view, possibilistic logic can be seen as a graded extension of the non-nested fragment of the well-known modal logic of belief KD45. When we go beyond the classical framework of Boolean algebras of events to many-valued frameworks, one has to come up with appropriate extensions of the notion of necessity and possibility measures for many-valued events [3]. In this abstract, we consider the problem of defining a proper Gödel modal logic capturing a suitable possibilistic semantics for the possibility and necessity operators and its relation to the generalized KD45 Gödel modal logic recently defined by Caicedo and Rodríguez [1].

After this short introduction we first summarize the main results by Caicedo-Rodríguez about a complete many-valued Gödel modal logic $KD45(G)$ with respect to a given Kripke style semantics and then we consider our many-valued possibilistic Kripke semantics, and pose an open problem: are the two semantics equivalent? This problem is addressed and solved in the final section for the particular case over finite-valued Gödel logic with Δ and truth constants.

2 Caicedo-Rodríguez's Gödel modal logic

In their paper [1], Caicedo and Rodríguez consider a modal logic over Gödel logic. The language is defined from a set of propositional variables Var , connectives \wedge, \rightarrow , truth-constant $\bar{0}$, and modalities \Box and \Diamond . The semantics is given by $[0, 1]$ -valued Kripke models $M = (W, e, R)$, where W is a set of worlds, $R = W \times W \rightarrow [0, 1]$ is a many-valued accessibility relation and $e : W \times Var \rightarrow [0, 1]$ is such that, for every $w \in W$, $e(w, \cdot)$ is a $[0, 1]$ -valued

truth evaluation of propositional variables. The truth degree of formulas relative to a world $w \in W$ is defined recursively as follows:

- $\|p\|_{w,M} = e(w, p)$ for each propositional variable p , and $\|\bar{0}\|_{w,M} = 0$
- using Gödel truth-functions for propositional combinations, i.e.

$$\|\varphi \wedge \psi\|_{w,M} = \min(\|\varphi\|_{w,M}, \|\psi\|_{w,M})$$

$$\|\varphi \rightarrow \psi\|_{w,M} = \|\varphi\|_{w,M} \Rightarrow_G \|\psi\|_{w,M}$$
- for modal formulas

$$\|\Box\varphi\|_{w,M} = \inf_{w' \in W} \{R(w, w') \Rightarrow_G \|\varphi\|_{w',M}\}$$

$$\|\Diamond\varphi\|_{w,M} = \sup_{w' \in W} \{\min(R(w, w'), \|\varphi\|_{w',M})\}$$

where \Rightarrow_G denotes Gödel truth-function for implication.

Let \mathcal{K} be the class of all many-valued Kripke models. In [1] they show that the set of valid formulas in \mathcal{K} , $Val(\mathcal{K}) = \{\varphi \mid \|\varphi\|_{w,M} = 1 \text{ for all } M = (W, R, e) \text{ and } w \in W\}$, is axiomatized by adding the following additional axioms and rule to those of Gödel logic G:

$$\begin{array}{ll} (K_{\Box}) & \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi) \\ (K_{\Diamond}) & \Diamond(\varphi \vee \psi) \rightarrow (\Diamond\varphi \vee \Diamond\psi) \\ (F_{\Diamond}) & \neg\Diamond\perp \\ (P) & \Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi) \\ (FS2) & (\Diamond\varphi \rightarrow \Box\psi) \rightarrow \Box(\varphi \rightarrow \psi) \\ (Nec) & \text{from } \varphi \text{ infer } \Box\varphi \end{array}$$

They further consider the subclass $\mathcal{KD45} \subseteq \mathcal{K}$ of many-valued Kripke models $M = (W, e, R)$ such that the accessibility relation R satisfies the following properties:

- Serial*: for every $w \in W$ there exists $w' \in W$ such that $R(w, w') = 1$
- Transitive*: for every $w, w', w'' \in W$, $\min(R(w, w'), R(w', w'')) \leq R(w, w'')$
- Euclidean*: for every $w, w', w'' \in W$, $\min(R(w, w'), R(w, w'')) \leq R(w', w'')$

And they show that the set $Val(\mathcal{KD45})$ of valid formulas in the class $\mathcal{KD45}$ is axiomatized by adding the following additional axioms:

$$\begin{array}{ll} (D) & \Diamond\top \\ (4_{\Box}) & \Box\varphi \rightarrow \Box\Box\varphi \quad (4_{\Diamond}) \quad \Diamond\Diamond\varphi \rightarrow \Diamond\varphi \\ (5_{\Box}) & \Diamond\Box\varphi \rightarrow \Box\varphi \quad (4_{\Diamond}) \quad \Diamond\varphi \rightarrow \Box\Diamond\varphi \end{array}$$

We will call $KD45(G)$ the logic defined by all the above axioms and rules.

3 A possibilistic semantics for Gödel modal logic and an open problem

Now we extend the possibilistic semantics for the fragment of non-nested modal formulas of the above logics defined in [3] to the full modal language in the

obvious way.

The possibilistic semantics is given by Kripke structures $M = (W, e, \pi)$ where W is a set of worlds, $e : W \times Var \rightarrow [0, 1]$ is an evaluation of propositional variables in each world, and $\pi : W \rightarrow [0, 1]$ is now a normalized possibility distribution on the set of worlds, i.e. a mapping such that $\sup_{w \in W} \pi(w) = 1$. Then, the truth-value in each world w of formulas is recursively defined as above with the exception of the following rules for \Box and \Diamond :

$$\begin{aligned} \|\Box\varphi\|_{w,M} &= \inf_{w \in W} \{\pi(w) \Rightarrow_G \|\varphi\|_{w,M}\} \\ \|\Diamond\varphi\|_{w,M} &= \sup_{w \in W} \{\min(\pi(w), \|\varphi\|_{w,M})\} \end{aligned}$$

Note that if Φ is a propositional combination of modal formulas, then its truth-value does not depend on the particular world. Let us call \mathcal{POS} the class of all possibilistic Kripke structures as defined above and let us denote by $Val(\mathcal{POS})$ the set of valid formulas in the class \mathcal{POS} .

Any possibilistic Kripke structure $M = (W, e, \pi)$ can be considered as a $\mathcal{KD45}$ structure by equivalently expressing M as (W, e, R_π) where $R_\pi(w, w') = \pi(w')$ and observing that R_π is serial, transitive and euclidean in the above sense. Therefore $\mathcal{POS} \subset \mathcal{KD45}$ and hence $Val(\mathcal{KD45}) \subseteq Val(\mathcal{POS})$.

Note that in the classical case (where truth-evaluations, accessibility relations and possibility distributions are $\{0, 1\}$ -valued) the other inclusion also holds. Indeed, it is well known that the semantics provided by the class of Kripke frames with serial, transitive and euclidean accessibility relations is equivalent to the class of Kripke frames with semi-universal accessibility relations, that is, relations of the form $R = W \times E$, where $\emptyset \neq E \subseteq W$, but the latter models are nothing else than $\{0, 1\}$ -valued possibilistic models.

However, over Gödel logic we have been unable so far to neither prove nor disprove whether $Val(\mathcal{POS}) \subseteq Val(\mathcal{KD45})$ holds, and thus we formulate the following open problem:

Open problem: $Val(\mathcal{POS}) \subseteq Val(\mathcal{KD45})$?

In the next section, we solve this problem in a very particular case, the one when the underlying logic is a finitely-valued Gödel logic expanded with truth-constants and the the Monteiro-Baaz Δ connective.

4 The case of finitely-valued Gödel logic with truth-constants and Δ

Let us consider the logic $G_{\Delta,k}^c$ defined as the $(k+1)$ -valued Gödel logic expanded with truth-constants \bar{r} for $r \in S_k = \{0, 1/k, \dots, 1\}$ and with the Monteiro-Baaz Δ connective (see e.g. [5]). We further expand the language with modal operators \Box and \Diamond .

Over the modal language, let us consider the possibilistic Kripke-style semantics as defined in the previous section but over the set of truth-values

S_k instead of $[0, 1]$, i.e. structures $M = (W, e, \pi)$, where now, for each $w \in W$, $e(w, \cdot)$ is a valuation of propositional variables on S_k , $\pi : W \rightarrow S_k$ is a normalized possibility distribution and the truth-value of a formula φ in a world $e(w, \varphi)$ is defined according to the truth functions of $G_{\Delta, k}^c$ and to the rules:

$$\begin{aligned} e(w, \diamond\varphi) &= \sup_{w' \in W} \min(\pi(w'), e(w', \varphi)) \\ e(w, \square\varphi) &= \inf_{w' \in W} \pi(w) \Rightarrow_G e(w', \varphi) \end{aligned}$$

Since the evaluation of the modal formulas does not depend on the world w , we can simply write $e(\diamond\varphi)$ and $e(\square\varphi)$. Let us denote by $(r)\varphi$ the (Boolean) formula $\Delta(\bar{r} \leftrightarrow \varphi)$. As observed in [7], notice that we equivalently have $e(\diamond\varphi) = \sup_{r \in S_k} \min(e(\diamond(r)\varphi), r)$ and $e(\square\varphi) = \inf_{r \in S_k} e(\diamond(r)\varphi) \Rightarrow_G r$.

Inspired in [6], we define the logic $\text{Poss-G}_{\Delta, k}^c$ as the extension of $G_{\Delta, k}^c$ with the following axioms: ¹

- (N1) $\square(\varphi \rightarrow \psi) \rightarrow (\square\varphi \rightarrow \square\psi)$
- (N2) $\square\nu \leftrightarrow \nu$,
- (N3) $\square(\nu \rightarrow \varphi) \leftrightarrow (\nu \rightarrow \square\varphi)$
- (II1) $\diamond(\varphi \vee \psi) \leftrightarrow (\diamond\varphi \vee \diamond\psi)$
- (II2) $\diamond\nu \leftrightarrow \nu$,
- (II3) $\diamond(\varphi \wedge \nu) \leftrightarrow (\diamond\varphi \wedge \nu)$
- (NII1) $\square\varphi \rightarrow \diamond\varphi$
- (NII2) $\square(\varphi \rightarrow \nu) \leftrightarrow (\diamond\varphi \rightarrow \nu)$

where ν is any propositional combination (including truth-constants and the Δ connective) of modal formulas, together with the necessitation inference rule for \square and the following definitional axiom:

$$(\square\text{-def}) \quad \square\varphi \leftrightarrow \bigwedge_{r \in S_k} (\diamond(r)\varphi \rightarrow \bar{r})$$

Let us denote by \vdash_\diamond the notion of proof in $\text{Poss-G}_{\Delta, k}^c$ and by \vdash the notion of proof in $G_{\Delta, k}^c$.

Lemma 1 *Let Λ be the set of instances of the modal axioms closed by necessitation. Then: $\Gamma \vdash_\diamond \varphi$ iff $\Gamma \cup \Lambda \vdash \varphi$.*

Theorem 2 *The logic $\text{Poss-G}_{\Delta, k}^c$ is sound and complete with respect to the class of possibilistic Kripke models.*

Proof: Assume $\not\vdash_\diamond \psi$. Then by strong completeness of $G_{\Delta, k}^c$, there exist a $G_{\Delta, k}^c$ -evaluation v_0 model of Γ such that $v_0(\psi) < 1$. Actually we can restrict ourselves to formulas built from the propositional variables appearing in φ (together with connectives, truth-constants and modal operators).

Let Ω be the set of all $G_{\Delta, k}^c$ -evaluations on the expanded language of $\text{KD45-G}_{\Delta, k}^c$ taking all formulas $\diamond\varphi$ as new propositional variables, and define an equivalence on Ω as follows: $v \approx v'$ iff, for all φ , $v(\diamond\varphi) = v'(\diamond\varphi)$.

Define the possibilistic Kripke model $M^* = (W, e, \pi)$ as follows:

¹Actually, restricted to the language of $\text{KD45}(G)$, this system is equivalent to the one presented for $\text{KD45}(G)$.

- $W = \{v \in \Omega \mid v \approx v_0, v \text{ model of } \Lambda, \text{ and } v(p) = v_0(p) \text{ for all propositional variable } p \text{ not appearing in } \psi\}$. Note that W is finite since, in fact, in W there are at most as many elements as different G_k -evaluations of the propositional variables appearing in ψ .
- For each φ , define $[\varphi] : W \rightarrow S_k$ by putting $[\varphi](v) = v(\varphi)$. Notice that $\{[\varphi] \mid \varphi\} = (S_k)^W$.
- Define a mapping $\Pi : (S_k)^W \rightarrow S_k$ by putting $\Pi([\varphi]) = v_0(\diamond\varphi)$. Thanks to the fact that v_0 is a model of Λ , it turns out that Π is a possibility measure on $(S_k)^W$ satisfying all necessary conditions considered in [3] to guarantee the existence of a normalized $\pi : W \rightarrow S_k$ such that, for all φ , $\Pi([\varphi]) = \max_{w \in W} \min(\pi(w), [\varphi](w)) = \max_{w \in W} \min(\pi(w), w(\varphi))$.
- Finally, for each $w \in W$, define $e(w, \varphi) = w(\varphi)$ for each $w \in W$. By construction, we clearly have $e(w, \diamond\varphi) = w(\diamond\varphi) = v_0(\diamond\varphi) = \max_{w \in W} \min(\pi(w), w(\varphi)) = \max_{w \in W} \min(\pi(w), e(w, \varphi))$.

Therefore, so defined, $M^* = (W, e, \pi)$ is a possibilistic model such that $e(v_0, \psi) = v_0(\psi) < 1$. ■

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