Three-valued Fuzzy Logics

Francesc Esteva, Tommaso Flaminio, Lluís Godo

Artificial Intelligence Research Institute (IIIA)
Spanish National Research Council (CSIC)
Campus UAB, 08193 Bellaterra, Spain
{esteva,godo,tommaso}@iiia.csic.es

Abstract

The aim of this chapter is to present three-valued fuzzy logics from two different perspectives. First, we will consider the systems of three-valued logic arising from the formal realm of Mathematical Fuzzy Logic, when moving from the unit real interval $[0, 1]$ to the set $\{0, 1/2, 1\}$ as intended set of truth-values. Second, we will consider logics arising as a natural generalization of classical logic to deal with concepts admitting borderline cases. Connections between both approaches are also highlighted.

1 Introduction

It is common to fix the birth of many-valued logics around 1920, when the first three-valued logics of Łukasiewicz, Kleene, Bochvar were initially introduced. Remarkably, these logics are distinguishable by the intended semantics given to the third-value, that is the new intermediate value between 0 and 1. These three-valued logics and their further generalizations to larger truth-value sets share the property of truth-functionality, that is, the truth-value of a compound formula is determined by the values of its subformulas. Truth-functionality for many-valued logics marks a clear difference from other non-classical graded logics which aim at formalizing uncertain reasoning and modeling intentional notions, such as probability or possibilistic logics, which are well-known to be not truth-functional. Among all the many-valued logics, the family of so-called *fuzzy* logics usually refer to those whose truth-value set is linearly ordered [5]. In most cases the whole real unit interval $[0, 1]$ or a finite subset of it is taken as set of truth-values. This allows to model a notion of *comparative truth* underlying the interpretation of gradual properties.

In this chapter we will hence focus on three-valued fuzzy logics arising from the setting introduced by Hájek in his monograph [39] that subsequently led to the development of the framework that is nowadays known as Mathematical Fuzzy Logic. More precisely, in the next Section 2 we will give a necessary general and historical perspective on main systems of fuzzy logic, while in Section 3 we will present two main ways to deal with three-valuedness in that setting. In particular, we will present two paths leading to three-valued fuzzy logics: the first one consists in restricting the usual truth-value set from the real unit interval $[0, 1]$ to $\{0, 1/2, 1\}$ and will be discussed in Subsection 3.1; the second one is a generalization of the classical two-valued setting to a three-valued one in order to accommodate concepts for which there exist borderline cases, and this will be the topic of Subsection 3.2. Then, in Section 4 we will focus on the two proper three-valued fuzzy logics that arise in our setting, namely the three-valued Łukasiewicz logic (Subsection 4.1) and the three-valued Gödel logic (Subsection 4.2). Furthermore, we will compare these systems in Subsection 4.3. In Section 5, we will present a different approach to fuzziness that considers degree-preserving and matrix logic companions of three-valued fuzzy logics. Finally, in Section 6 we provide a three-valued logical formalization for reasoning with fuzzy concepts based on the intuitive semantics presented in Subsection 3.2. We conclude with Section 7 where we draw some conclusions.

Parts of the contents of Section 2 come from the survey papers [34] and [26].

2 Preliminaries: systems of mathematical fuzzy logic

Consider a non-empty set X of objects. A *fuzzy set* A in X was described by Zadeh as a function $\mu_A : X \to [0,1]$, called the *membership function of* A, which associates each element x of X with a real number $\mu_A(a)$ in $[0,1]$ and which represents the *grade* (or *degree*) *of membership* of x to A. Fuzzy sets hence generalise (classical) sets: if μ_A only takes values $\{0,1\}$ then, upon identifying A with its characteristic function, μ_A describes a (classical) subset of X.

Besides defining the notion of fuzzy set, in [65] Zadeh also discussed on how to extend the usual operations of intersection, union and complementation to this setting. The natural choice he made was the following: the intersection of two fuzzy sets μ_A and μ_B is modelled by the minimum function, i.e. for each $x \in X$, $\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$, their union by the maximum function, i.e. $\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x))$, and the complement of μ_A was computed as $\mu_{\overline{A}}(x) = 1 - \mu_A(x)$.

Actually, this fuzzy set-theoretic setup was considered as a basis where to develop early logical fuzzy logic frameworks, see [51, 32, 57]. However, since the early 1980s it became common in the community of fuzzy set theorists to consider a wider approach based on t-norms as suitable candidates for connectives generalising intersection operations for fuzzy sets, see e.g. [3, 22, 59, 49, 64]. These t-norms, a shorthand for "triangular norms", first became important in discussions of the triangle inequality within probabilistic metric spaces, see the monographs [62] and later [50, 2]. These are binary operations on the real unit interval that endow it with the structure of an ordered commutative monoid with 1 as unit element. The three most prominent and well-known t-norms are the so-called Łukasiewicz, Product and Gödel t-norms, denoted respectively T_L, T_P and T_G, and respectively defined as $T_L(x,y) = \max\{x+y-1, 0\}$, $T_P(x,y) = x \cdot y$, and $T_G(x,y) = \min\{x,y\}$, for every $x, y \in [0,1]$.

The general understanding in the context of fuzzy connectives is that t-norms form a suitable class of generalized conjunction operators.

In an analogous way, the so-called t-conorms appeared as suitable operations for the union operations of fuzzy sets as dual counterparts of t-norms. Indeed, as t-norms, t-conorms are binary operations on the real unit interval providing $[0,1]$ with the structure of ordered commutative monoid, but now replacing 1 by 0 as unit element. More precisely, if $n : [0,1] \to [0,1]$ is a bijective, involutive order-reversing mapping such that $n(0) = 1$ (and hence $n(1) = 0$), then given a t-norm T, the operation S defined as $S(x,y) = n(T(n(x), n(y))$ is a t-conorm, and vice versa, T can be defined from S and n in an analogous way. The following are the corresponding t-conorms of T_L, T_P and T_G t-norms for $n(x) = 1 - x$: $S_L(x,y) = \min\{1, x+y\}$, $S_P(x,y) = x + y - x \cdot y$, and $S_G(x,y) = \max\{x,y\}$, for every $x, y \in [0,1]$.

Although nowadays it seems completely natural to relate fuzzy set theory and formal systems of many-valued logic, that was not the case when fuzzy sets were introduced by Zadeh in the '60s of the last century [65]. In fact, Zadeh himself, although presenting fuzzy sets as a tool to model vague notions —and by doing so he surely recognized the many-valuedness of his own approach— he did not relate fuzzy sets to many-valued logics at the beginning. Not surprisingly, also the overwhelming majority of papers on fuzzy set theory that followed [65] treated fuzzy sets in the standard mathematical context, i.e. with an implicit reference to a naive understanding of classical logic as argumentation structure.

Goguen was the first among Zadeh's immediate followers who emphasized an intimate relationship between fuzzy sets and many-valued logics. In his 1969 paper [36], he considers membership degrees as generalized truth values, i.e. as truth degrees. Additionally he sketches a "solution" of the sorites paradox, i.e. the heap paradox, using – but only implicitly – the ordinary product in $[0,1]$ as a generalized conjunction operation. Based on these ideas, he proposes, as suitable structures for the membership degrees of fuzzy sets, completely distributive lattice-ordered monoids $(A, \leq, *, 0, 1)$ enriched, whenever definable, with an *implication*-like operation

$\Rightarrow$ which is the (right) residuum of the monoidal operation $*$, and hence characterized by the well known adjointness condition

$$a * b \leq c \quad \text{iff} \quad b \leq a \Rightarrow c\,, \tag{1}$$

and with the "implies falsum"-negation $\neg$, i.e. defined as $\neg a = a \Rightarrow 0$. In other words, these structures were completely distributive commutative, bounded and integral residuated lattices.

Gottwald in [37] noticed that a monoid over the real unit interval $[0,1]$ defined by a left continuous t-norm $*$ always has a residuum $\Rightarrow$, defined as

$$x \Rightarrow y = \max\{z \in [0,1] \mid x * z \leq y\}.$$

Structures of this kind were found to be relevant examples over which one could define a propositional language (with connectives $\wedge, \vee$ interpreted by the truth degree functions min and max, and connectives $\&, \to$ interpreted by a left-continuous t-norm and its residuum) to develop fuzzy set theory within –at least, as far as the complementation of fuzzy sets is not considered as an additional primitive operation but derived from the implication connective $\to$.[1]

Therefore, each (left continuous) t-norm $*$ uniquely determines a semantical (propositional) calculus $PC(*)$ over formulas defined in the usual way from a countable set of propositional variables, connectives $\wedge$, $\&$ and $\to$ and the truth-constant $\overline{0}$ [39]. Further connectives are defined as follows:

$$\begin{array}{rcl} \varphi \vee \psi & \text{is} & ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi), \\ \neg\varphi & \text{is} & \varphi \to \bar{0}, \\ \varphi \leftrightarrow \psi & \text{is} & (\varphi \to \psi) \wedge (\psi \to \varphi)^2. \end{array}$$

Evaluations of propositional variables are mappings e assigning each propositional variable p a truth-value $e(p) \in [0,1]$, which extend in a unique way to compound formulas as follows:

$$\begin{array}{rcl} e(\overline{0}) & = & 0 \\ e(\varphi \wedge \psi) & = & \min(e(\varphi), e(\psi)) \\ e(\varphi \& \psi) & = & e(\varphi) * e(\psi) \\ e(\varphi \to \psi) & = & e(\varphi) \Rightarrow e(\psi). \end{array}$$

Note that, from the above definitions, $e(\varphi \vee \psi) = \max(e(\varphi), e(\psi))$, $\neg\varphi = e(\varphi) \Rightarrow 0$ and $e(\varphi \leftrightarrow \psi) = e(\varphi \to \psi) * e(\psi \to \varphi)$.

A worthwhile comment at this point is that, due to the fact that the adjointness condition (1) holds for the pair $(*, \Rightarrow)$, these calculi admit a *graded form of modus ponens*. Indeed, for any evaluation e, any lower bound $\alpha \in [0,1]$ for the value $e(\varphi \to \psi)$ and any lower bound $\beta \in [0,1]$ for the value $e(\varphi)$, a lower bound for the value $e(\psi)$ of the consequent ψ can be calculated according to the following rule:

$$\frac{e(\varphi) \geq \beta, \quad e(\varphi \to \psi) \geq \alpha}{e(\psi) \geq \alpha * \beta}.$$

Two prominent many-valued logics that fall in this class of calculi, namely Łukasiewicz and Gödel infinitely-valued logics, [52, 35], denoted Ł and G respectively, were defined long before fuzzy logic was born. They indeed correspond to the calculi defined by Łukasiewicz and min t-norms T_L and T_G respectively. Łukasiewicz logic Ł has received a lot of attention since the fifties. By the early 1960's Łukasiewicz infinite-valued logic had reached its mathematical maturity through (several proofs of) its completeness among which it is worth mentioning the

[1] Notice that in the setup considered by Gehrke, Nguyen, Walker and Walker in the aforementioned [32, 57], the authors call *classical* or *standard* fuzzy logic the one based on the algebra of truth-values $\mathbb{I} = ([0,1], \min, \max, 1-x)$. Note that in $\mathbb{I}$ the negation $n(x) = 1 - x$ is a primitive connective while the residuum of the t-norm min is not definable. missing.

[2] Equivalently, $\varphi \leftrightarrow \psi$ might be defined by $(\varphi \to \psi)\&(\psi \to \varphi)$.

first one proved by Rose and Rosser [61] and that algebraic one provided by Chang [11, 12] that shows Łukasiewicz logic to be complete with respect to a class of algebraic structures, which were to become established by the name of *MV-algebras*. Many results about Łukasiewicz logic and MV-algebras can be found in the books [13, 56]. On the other hand, a completeness theorem for Gödel logic was already given in the fifties by Dummett [24]. It is worth to recall, that are nowadays known as *Gödel algebras* that the algebraic structures related to Gödel logic are linear Heyting algebras (known as Gödel algebras in the context of fuzzy logics), that have been studied in the setting of intermediate or superintuitionistic logics, i.e. logics between intuitionistic and classical logic.

The type of logical setting based on left-continuous t-norms for fuzzy set theory was considered in the 1980s and beginning of the 1990s, although in a naive way: these early approaches were mainly semantically oriented calculi and what was in general missing, with the exception of infinite-valued Łukasiewicz logic Ł and Gödel logic G,[3] was a systematic investigation of suitable, adequately axiomatized, syntactic calculi.

The first proposal to fill this gap was made by Höhle [42, 43, 44], who introduced his *monoidal logic* ML in 1994. This is a common generalization of Łukasiewicz logic Ł, intuitionistic logic IPC and the additive fragment of Girard's integral, commutative linear logic aMALL [33]. The algebraic semantics for ML is given by the class of ML-algebras, namely the variety of commutative, integral and bounded residuated lattices. At this point, it is interesting to notice that Höhle's monoidal logic ML belongs to the family of substructural logics, in fact the variety of ML-algebras coincides with the variety of algebras associated to the logic FL_{ew}, i.e. Full Lambek calculus with exchange and weakening. Adequate axiomatizations for the propositional as well as for the first-order version of this logic were given in [42, 44].

Monoidal logic intended to articulate the relationship between fuzzy set theory and the t-norm based setting of its algebraic set-theoretic operations. However, this relation was not strongly enough tight as we discuss below. Indeed, in contrast to Höhle's general approach, Hájek's proposal was to restrict the scope of the analysis and he devoted himself to the task of axiomatizing the *common tautologies* of all *continuous t-norm* based calculi. In short, he aimed at defining the logic of all continuous t-norms [40, 39], called *Basic fuzzy logic* and usually denoted BL.

There are two crucial properties which pave the way to the original algebraic semantics for BL and that mark a difference w.r.t. the ML-algebras in general.

- The first one is that for any (left-continuous) t-norm $*$ and their residuation operation $\Rightarrow$ one has
$$(a \Rightarrow b) \vee (b \Rightarrow a) = 1\,, \tag{2}$$
with $\vee$ denoting the lattice join here, i.e. the max-operation for a linearly ordered carrier. This equation is known as the *prelinearity* condition, and it is not in general satisfied in ML-algebras. Indeed, this condition is crucial for the usual assumption that fuzzy logics are logics of comparative truth, that is, truth degrees are always comparable. Algebraically, this amounts to the requirement that fuzzy logics should be complete with respect to linearly ordered algebras of truth-values. For instance, if this condition is imposed upon the Heyting algebras, which form an adequate algebraic semantics for intuitionistic logic, the resulting class of prelinear Heyting algebras is an adequate algebraic semantics for the infinite-valued Gödel logic.

- The second observation is that the continuity condition of a t-norm can be given in algebraic terms: for any left-continous t-norm $*$ and its residuum $\Rightarrow$ it holds that the *divisibility* condition
$$a * (a \Rightarrow b) = a \wedge b \tag{3}$$
is satisfied if and only if $*$ is a continuous t-norm, see [43]. In the condition above $*$ denotes the semigroup operation and $\wedge$ the lattice meet (i.e. the min operation on $[0, 1]$).

[3] In 1996 Product logic [41] was added to this list.

BL-algebras were then defined as ML-algebras further satisfying the prelinearity and the divisibility conditions. In particular all BL-algebras over the real unit interval $[0,1]$ are those defined by a continuous t-norm and its residuum, and they are known as *standard* BL-algebras. In his highly influential monograph [39], Hájek characterized his basic (propositional) fuzzy logic BL as the logic whose algebraic semantics is the class of BL-algebras, and gave an axiomatization. Moreover, soon after, it was shown by Cignoli *et al.* in [14] that, as Hájek intended, BL indeed captures the tautologies common to the logical calculi $PC(*)$ for all continuous t-norms $*$. In other words, BL is complete, not only with respect to the whole class of BL-algebras, but with respect to the subclass of BL-algebras defined on the real unit interval $[0,1]$, i.e. $Th(BL) = \bigcap\{Taut(PC(*)) \mid * \text{ is a continuous t-norm}\}$, where $Th(BL)$ denotes the set of theorems of BL and $Taut(PC(*))$ the set of tautologies of $PC(*)$.

Drawing on the fundamental facts that every left-continuous t-norm $*$ has a residuum and that the prelinearity axiom is a common tautology of all $PC(*)$ calculi, the logic BL was generalized in [28] in order to capture the common tautologies of left-continuous t-norms. Indeed, the *monoidal t-norm based logic* MTL was first defined and axiomatized by Esteva and Godo in [28] with respect to the class of MTL-algebras, i.e., the class of prelinear Höhle's ML-algebras. Then, Jenei and Montagna in [47] proved that MTL was indeed the logic of all left-continuous t-norms, in the sense that $Th(MTL) = \bigcap\{Taut(PC(*)) \mid * \text{ is a left-continuous t-norm}\}$.

The study of formal systems of fuzzy logic has been developed continually since then. In fact, the book by Hájek [39] has been considered as a sort of official start of the discipline nowadays known as Mathematical Fuzzy Logic, with the goal of providing rigorous, logical foundations to fuzzy logic in the *narrow sense*, i.e. in the sense of logical systems of many-valued logic aiming at the formalization of Zadeh's model of approximate reasoning based on fuzzy sets. This branch of mathematical logic has thrived over the past twenty-five years and from many points of view (logical calculi, algebraic, predicate calculi, proof-theory, functional representation, complexity analysis, etc.), as witnessed by a number of important monographs that have appeared around the beginning of this century see [39, 38, 58, 53], and culminated more recently in a series of three handbooks [15, 16, 17].

3 Three-valued fuzzy logics

From what we presented in the above section, it is clear that our setting for fuzzy logic is the formal setup put forward by Hájek and colleagues. More precisely, in this chapter, by fuzzy logic we will understand any extension of MTL. In this setting, three-valued fuzzy logics can be approached along at least two different paths:

(i) as systems arising from the family of fuzzy logics briefly described above when one restricts the intended set of truth-values to three values in a linearly ordered way, say $\{0, 1/2, 1\}$ with the natural order; or

(ii) as three-valued systems that provide meaningful accounts of the most basic notion of fuzziness or gradualness, and whose generalizations to settings with more truth-values ultimately lead to the $[0,1]$-valued systems of MFL.

The first path amounts to consider the extensions of fuzzy logics L (i.e. an axiomatic extension of MTL) with the axiom

$$(3V)\ (\varphi_1 \to \varphi_2) \vee (\varphi_2 \to \varphi_3) \vee (\varphi_3 \to \varphi_4)$$

that forces the linearly ordered members of its corresponding variety of algebras, call it $\mathbb{L}_3$, to have at most three different values. The second path amounts to consider a 3-valued approach to fuzziness and their associated logics. Both approaches are discussed in the rest of this section.

3.1 Three-valued fuzzy logics obtained by restriction from MTL

Before properly dealing with three-valued fuzzy logics, it is necessary to recall some basic definitions and properties, in particular for the logic MTL [28, 47].

As we recalled in the previous sections, the logic MTL can be considered as the ground formalism from which systems of Mathematical Fuzzy Logic based on t-norms have been mainly developed. MTL is usually presented over the signature $\{\&, \to, \wedge, \bot\}$ of type $(2, 2, 2, 0)$, where & is called the strong conjunction, $\to$ is an implication, $\wedge$ is the weak or lattice conjunction, and $\bot$ is the truth-constant falsum. Formulas, which will be denoted by lower-case Greek letters, are defined as usual by induction from a countable set of propositional variables $p_1, p_2, \ldots$. Other useful connectives and constants are definable as follows: $\neg\varphi := \varphi \to \bot$, $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$, $\varphi \vee \psi := ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$, $\top := \neg\bot$. Further symbols and abbreviations will be defined when they are needed.

Axioms of MTL are:

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,

(A2) $(\varphi \& \psi) \to \varphi$,

(A3) $(\varphi \& \psi) \to (\psi \& \varphi)$,

(A4) $(\varphi \wedge \psi) \to \varphi$,

(A5) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$,

(A6) $(\varphi \& (\varphi \to \psi)) \to (\varphi \wedge \psi)$,

(A7) $(\varphi \to (\psi \to \chi)) \leftrightarrow ((\varphi \& \psi) \to \chi)$,

(A8) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$,

(A9) $\bot \to \varphi$.

The only inference rule is modus ponens.

The logic MTL is algebraizable in the sense of [8] and its equivalent algebraic semantics is the variety $\mathbb{MTL}$ of MTL-algebras. Those are commutative, integral, bounded residuated lattices $\mathbf{A} = (A, *, \Rightarrow, \wedge, 0)$ further satisfying the prelinearity equation (2) we already discussed in Section 2. In every MTL-algebra $\mathbf{A}$ the lattice-order can be defined as follows: for all $a, b \in A$, $a \leq b$ iff $a \Rightarrow b = 1$.

The fact that every MTL-algebra satisfies prelinearity, is enough to show that every subdirectly irreducible MTL-algebra is linearly ordered, ie., is an MTL-chain. As a consequence, the logic MTL is sound and complete with respect to the totally ordered algebras of $\mathbb{MTL}$. Indeed, MTL is *standard complete*, meaning that it is complete with respect to those MTL-chains whose universe is the real unit interval $[0, 1]$, $*$ is a left-continuous t-norm, and $\Rightarrow$ is the residuum of $*$.

Starting from MTL, one can hence obtain, as schematic extensions, several remarkable system that today constitute the core of Mathematical Fuzzy Logic. Their equivalent algebraic semantics are those subvarieties of MTL-algebras further satisfying the algebraic version of the corresponding logical axioms. In particular:

- Hájek logic BL is MTL+$\{$(Div) $(\varphi \wedge \psi) \to (\varphi \& (\varphi \to \psi))\}$. In algebraic terms this means that, in every BL-algebra, the operator $\wedge$ is definable from $*$ and $\Rightarrow$ as $a \wedge b = (a * (a \Rightarrow b))$.

- Involutive MTL (IMTL) is the logic MTL+$\{$(Inv) $\neg\neg\varphi \to \varphi\}$. The formula $\varphi \to \neg\neg\varphi$ is a theorem of MTL. Thus, in IMTL-algebras, the equation $\neg\neg x = x$ holds.

- Łukasiewicz logic (Ł) is hence the schematic extension of IMTL+(Div). That is to say, Ł is BL+(Inv). Conforming to a standard terminology, we will call MV-algebras the algebraic semantics of Ł.

- Strict MTL (SMTL) is MTL+$\{$(Pseudo) $\varphi \wedge \neg\varphi \to \bot\}$. In every SMTL-chain, then, the negation operator is such that $\neg 0 = 1$, and for all $x > 0$, $\neg x = 0$. SMTL+(Div) is a logical system known in the literature as Strict BL (SBL).

- The two most relevant axiomatic extensions of SBL are Gödel logic (G) axiomatized by SBL+{(Con) $\varphi \to (\varphi \& \varphi)$} forcing the connective $\wedge$ to be idempotent and Product logic Π given by SBL+{(Π1) $\neg\neg\chi \to (((\varphi \& \chi) \to (\psi \& \chi)) \to (\varphi \to \psi))$} that expresses the cancellation property for the product operation between non-zero real numbers.

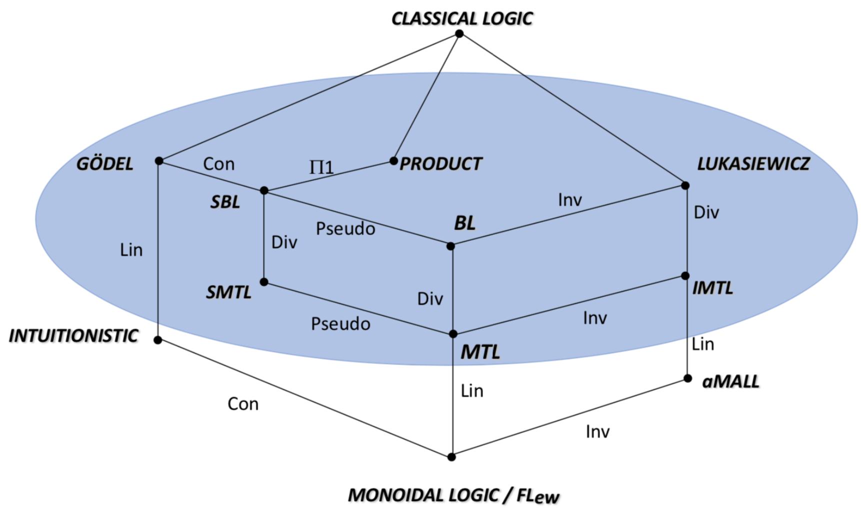

Figure 1: Main systems of mathematical fuzzy logic (within the circle) and their relations to well-known substructural logics. An upward link from a logic L_1 to a logic L_2 with a label Ax means that L_2 is the axiomatic extension of L_1 with axiom Ax. aMALL stands for the affine additive-multiplicative fragment of Linear logic.

The following schematic extension of MTL will play a central role in the rest of the present chapter.

Definition 1. For every axiomatic extension L of MTL, the logic L_3 is obtained by adding to L the following axiom:

(3V) $(\varphi_1 \to \varphi_2) \vee (\varphi_2 \to \varphi_3) \vee (\varphi_3 \to \varphi_4)$.

Since it is an axiomatic extension of MTL, MTL_3 remains being algebraizable, and its equivalent algebraic semantics is given by variety $\mathbb{MTL}_3$ of MTL_3-algebras, that is, MTL-algebras satisfying the equation

(3V) $(x \Rightarrow y) \vee (y \Rightarrow z) \vee (z \Rightarrow v) = 1$.

Moreover, analogously to $\mathbb{MTL}$, since the pre-linearity equation

(Lin) $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$

also holds in the variety $\mathbb{MTL}_3$, the variety itself is generated by the subclass of its linearly ordered algebras, or chains. In fact, the equation (3V) forces $\mathbb{MTL}_3$-chains to have, at most, three different values.

The same clearly applies to the cases of $Ł_3$ and G_3. We will henceforth denote by $\mathbb{MV}_3$ and $\mathbb{G}_3$ the corresponding varieties of algebras. Moreover, we will denote by $\mathbf{Ł}_3$ and $\mathbf{G}_3$ the standard algebras (i.e., those structures on $\{0, 1/2, 1\}$) of the respective varieties. More precisely, we make the following definitions.

Definition 2. (1) The 3-element Łukasiewicz (or MV) chain $\mathbf{Ł}_3$ is the system $(\{0, 1/2, 1\}, *_L, \Rightarrow_L, \wedge, 0)$ where, for all $x, y \in \{0, 1/2, 1\}$,

$$x *_L y = \max\{0, x + y - 1\},\ x \Rightarrow_L y = \min\{1, 1 - x + y\} \text{ and } x \wedge y = \min\{x, y\}.$$

Moreover the definable negation $\neg_L x = x \Rightarrow_L 0 = 1 - x$.
(2) The 3-element Gödel chain $\mathbf{G}_3$ is the system $(\{0, 1/2, 1\}, *_G, \Rightarrow_G, \wedge, 0)$ where, for all $x, y \in \{0, 1/2, 1\}$,

$$x *_G y = x \wedge y = \min\{x, y\} \text{ and } x \Rightarrow_G y = 1 \text{ if } x \leq y \text{ and } x \Rightarrow y = y \text{ otherwise.}$$

In this case the corresponding negation $\neg_G x = x \Rightarrow_G 0$ yields $\neg_G x = 1$ if $x \leq y$ and $\neg_G x = 0$ otherwise.

It should be noted that these definitions, in particular the one for the algebra $\mathbf{Ł}_3$, make use of the numerical nature of the third value $1/2$ to show that the connecxtives are the restriction of the ones in the unit real interval $[0, 1]$ to its subset $\{0, 1/2, 1\}$. But we could have also defined isomorphic algebras on a more abstract domain $\{0, u, 1\}$ by means of the following truth tables:

1) $\mathbf{Ł}_3$:

$*_L$	0	u	1
0	0	0	0
u	0	0	u
1	0	u	1

$\rightarrow_L$	0	u	1
0	1	1	1
u	u	1	1
1	0	u	1

$\wedge$	0	u	1
0	0	0	0
u	0	u	u
1	0	u	1

$\neg_L$	
0	1
u	u
1	0

2) $\mathbf{G}_3$:

$*_G = \wedge$	0	u	1
0	0	0	0
u	0	u	u
1	0	u	1

$\rightarrow_G$	0	u	1
0	1	1	1
u	0	1	1
1	0	u	1

$\neg_G$	
0	1
u	0
1	0

Proposition 1. *The only linearly ordered algebras in $\mathbb{MTL}_3$ are, up to isomorphism, the two-element boolean algebra, the three-element MV-chain $\mathbf{Ł}_3$ and the three-element Gödel chain $\mathbf{G}_3$.*

Proof. The proof amounts to showing that the only three-element MTL-chains are $\mathbf{Ł}_3$ and $\mathbf{G}_3$. Then it will be immediate to show that the two-element boolean algebra belongs to $\mathbb{MTL}_3$, as it is a subalgebra of both $\mathbf{Ł}_3$ and $\mathbf{G}_3$ and $\mathbb{MTL}_3$ is closed under subalgebras.

Thus, let $\mathbf{A}$ be any MTL-chain with three-values, i.e., on $\{0, a, 1\}$ with $0 < a < 1$. It is enough to check that the implication operation $\Rightarrow$ in $\mathbf{A}$ is either Łukasiewicz implication or Gödel implication. Note that, as a binary operation $\Rightarrow: \{0, a, 1\} \times \{0, a, 1\} \to \{0, a, 1\}$, the only unknown value is $a \Rightarrow 0$ since the rest of values come fixed by MTL properties, in particular by the residuation property. Indeed, we have:

- $x \Rightarrow y = 1$, for all values x, y such that $x \leq y$
- $1 \Rightarrow x = x$, for all x

These rules determine all the values except for $a \Rightarrow 0$, i.e. for $\neg a$. Moreover, note that the value 1 is not allowed for $a \Rightarrow 0$. Indeed, since the condition $x \leq \neg\neg x$ is valid in MTL, if we set $a \Rightarrow 0 = 1$, we get $a \leq \neg\neg a = \neg 1 = 0$, contradiction. Therefore, either $a \Rightarrow 0 = 1/2$, in which case $\Rightarrow$ is Łukasiewicz implication, or $a \Rightarrow 0 = 0$, in which case $\Rightarrow$ is Gödel implication. □

Remark 1. (1) Notice that the equation/axiom (3V) is stronger than the equation/axiom (Lin). Indeed, it is enough to take in (3V) $z = x$ and $v = y$ to recover (Lin). Also notice that (3V) also forces the divisibility equation to hold in every MTL-chain. Indeed, and skipping the trivial case, let us take $x = 1$, $y = a$ and show that $\mathbf{A} \models x \wedge y \leq (x * (x \Rightarrow y))$, i.e., $a = 1 \wedge a \leq (1 * (1 \Rightarrow a)) = (1 \Rightarrow a)$ which amounts to showing that $1 \Rightarrow a \geq a$. By the residuation property, $1 \Rightarrow a \geq a$ iff $a * 1 \leq a$ and in fact $a * 1 = a$, thus the claim is settled.

(2) Also notice that the logic obtained by adding (3V) to product logic Π is classical logic CPC. Indeed, and slightly more in general, the unique finite product chain is the two-element Boolean algebra. This is indeed easy to see: if $\mathbf{A}$ is any product chain that contains an element $0 < a < 1$, then it also contains all the distinct elements $a^2, a^3, \ldots$ none of them being 0. Thus, $\mathbf{A}$ is necessarily infinite.

As an immediate consequence of the above remark, many systems collapse when being extended with the axiom (3V). Indeed the following equalities among logics hold:

- FL_{ew} + (3V) = ML + (3V) = MTL_3 = BL_3;
- IPC + (3V) = SMTL_3 = SBL_3 = G_3;
- aMALL+ (3V) = IMTL_3 = Ł_3;
- Π + (3V) = CPC.

All these relations are summarised in Figure 2 below.

From an algebraic point of view, note that analogous equalities hold between the corresponding varieties of algebras. Moreover, it turns out that the only subvarieties of $\mathbb{MTL}_3$, besides the variety of Boolean algebras, are:

- $\mathbb{MTL}_3$ itself, generated as a variety and as quasi-variety by $\mathbf{Ł}_3$ and $\mathbf{G}_3$,
- $\mathbb{MV}_3$, the subvariety of MV-algebras generated as a variety and as quasi-variety by $\mathbf{Ł}_3$, and
- $\mathbb{G}_3$, the subvariety of Gödel algebras generated as a variety and as quasi-variety by $\mathbf{G}_3$,

Let us denote by $\Omega_{\text{Ł}_3}$ the set of 3-valued evaluations of the language on the 3-element MV-chain $\mathbf{Ł}_3$, and by Ω_{G_3} the set of 3-valued evaluations of the language on the 3-element Gödel-chain $\mathbf{G}_3$. Then the second and third items above express the well-known fact that the logics Ł_3 and G_3 are complete with respect the semantics given by the set of 3-valued evaluations from $\Omega_{\text{Ł}_3}$ and Ω_{G_3} respectively, see e.g. [13, 38]

On the other hand, the first item tells us that the (semantic) entailment relation determined by the variety $\mathbb{MTL}_3$ coincides with the one defined as follows: for any set of formulas $\Gamma \cup \{\varphi\}$:

- $\Gamma \models_{\mathbb{MTL}_3} \varphi$ if for any evaluation $e \in \Omega_{\text{Ł}_3} \cup \Omega_{G_3}$, if $e(\psi) = 1$ for all $\psi \in \Gamma$, then $e(\varphi) = 1$.

Therefore, we get the following completeness result for the logic MTL_3.

Theorem 1. *If we denote by $\vdash_{MTL_3}$ the relation of provability in MTL_3 defined from the set of its axioms and modus ponens as inference rule, then the following completeness property holds: for any set of formulas $\Gamma \cup \{\varphi\}$, $\Gamma \vdash_{\mathbb{MTL}_3} \varphi$ iff $\Gamma \models_{\mathbb{MTL}_3} \varphi$.*

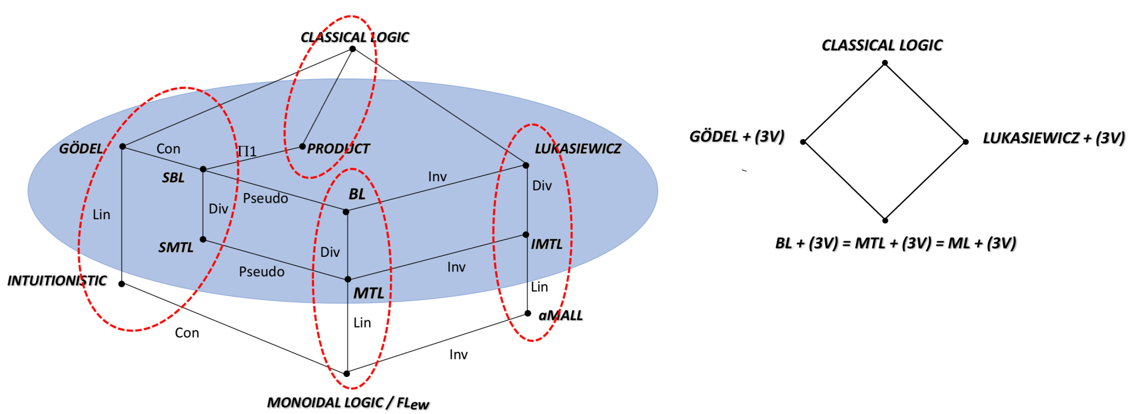

Figure 2: The three-valued fuzzy logics resulting from extending the logics from Fig. 1 with the axiom (3V). In the left-hand side, each dashed ovals contains those logics that become equivalent under the added three-valued axiom (3V), that boils down to the graph of logics in the left-hand side.

The three-valued logics Ł_3 and G_3, as well the relations between them, will be explored in greater detail in Section 4.

Remark 2. To finish this section we would like to draw the reader attention on the fact that, besides our choice of framing fuzzy logics within MTL, there are other approaches to fuzzy logic in the literature. In particular it is worth mentioning the one we already recalled above developed by Gehrke, Nguyen, Walker and Walker in [32, 57], where the authors call (*classical*) fuzzy logic the one having as algebraic semantics the variety generated by the De Morgan algebra $\mathbb{I} = ([0,1], \min, \max, 1-x)$. Under this latter interpretation then they show (see Theorem 4.4.1 of [57]) that "*The propositional calculus for three-valued Łukasiewicz logic and the propositional calculus for fuzzy logic are the same*". This result needs to be correctly contextualised as it is in apparent contradiction with what we claim in this section. First of all, what they call three-valued Łukasiewicz logic is only the $(\wedge, \vee, \neg)$-fragment ***(i.e. the $\to$-free fragment)*** of the original three-valued Łukasiewicz logic Ł_3; secondly, what this result indeed states, in algebraic terms, is that the variety generated by $\mathbb{I}$ and the variety generated by $(\{0, 1/2, 1\}, \min, \max, 1-x)$ are the same (see [32, Theorem 12] and [48]). A similar result in our setting cannot hold because $\mathbb{MV}_3$ and $\mathbb{G}_3$ are different and both are proper subvarieties of $\mathbb{MTL}_3$.

3.2 A 3-valued approach to fuzziness

A fuzzy (in the sense of gradual) property is characterised by the existence of borderline cases, that is, objects or situations for which the property only partially applies, see e.g. [23]. Given a set of situations Ω, the most elementary description of a vague property or concept α is in terms of:

- a set of prototypical situations or *examples* $[\alpha^+] \subseteq \Omega$, where α definitely applies, and
- a set of *counterexamples* $[\alpha^-] \subseteq \Omega$, where α definitely does not apply.

In this section we will assume to work with *complete* descriptions of this kind, that is, for each concept α,

- the remaining set of situations $[\alpha^\sim] = \Omega \setminus ([\alpha^+] \cup [\alpha^-])$ will be those where α only partially applies to, that is, the *borderline cases*.

To be in a consistent scenario, we will require that there is no situation where α both fully applies and does not apply to, in other words, the constraint $[\alpha^+] \cap [\alpha^-] = \emptyset$ will always be satisfied.

Such a scenario clearly leads to a three-valued logical framework, see e.g. [63], where for each situation $w \in \Omega$, the degree $w(\alpha)$ to which α applies at w (or, equivalently, the truth degree of the assertion "w is α") can be naturally defined as follows:

$$w(\alpha) = \begin{cases} 1, & \text{if } w \in [\alpha^+] \\ 0, & \text{if } w \in [\alpha^-] \\ 1/2, & \text{otherwise} \end{cases}$$

It is worth emphasizing that in this 3-valued model, the third value $1/2$ is not meant to represent ignorance about whether a concept applies or not to a situation, rather it is meant to represent that the concept only partially applies to a situation, or equivalently, that the situation is a borderline case for the concept, see [18] for a discussion of this topic.

In this framework, interpretations of atomic concepts α are in terms of disjoint pairs (i.e. orthopairs) of examples and counterexamples. That is, if we let $(\Omega \times \Omega)^* = \{(A, B) \mid A, B \subseteq \Omega, A \cap B = \emptyset\}$ to be the set of orthopairs from Ω, then an interpretation for atomic concepts is a mapping $e : Var \to (\Omega \times \Omega)^*$, where for each $\alpha \in Var$, we will write $e(\alpha) = ([\alpha^+]_e, [\alpha^-]_e)$, but we will omit the subscript e when clear from the context. The key question is then how such an evaluation can be extended in a meaningful way to compound concepts.

For our purposes, we consider a language with four connectives: conjunction ($\wedge$), disjunction ($\vee$), negation ($\neg$) and implication ($\to$). Given $e(\alpha) = ([\alpha^+], [\alpha^-])$ and $e(\beta) = ([\beta^+], [\beta^-])$, the rules for $\wedge$ and $\vee$ seem to be clear enough and are given as:

- $e(\alpha \wedge \beta) = ([\alpha^+] \cap [\beta^+], [\alpha^-] \cup [\beta^-])$
- $e(\alpha \vee \beta) = ([\alpha^+] \cup [\beta^+], [\alpha^-] \cap [\beta^-])$

The first condition expresses that, according to e, the set of examples for $\alpha \wedge \beta$ are those examples of α that are also examples of β, while the counterexamples for $\alpha \wedge \beta$ are those that are either a counterexample of α or of β. The condition for the disjunction $\alpha \vee \beta$ is the dual one.

On the other hand, the natural rule for the negation seems to be

- $e(\neg\alpha) = ([\alpha^-], [\alpha^+])$

expressing that the examples for α are the counterexamples for $\neg\alpha$ and viceversa, that is, $[(\neg\alpha)^+] = [\alpha^-]$ and $[(\neg\alpha)^-] = [\alpha^+]$. However, one might also consider the following slightly different forms of non-involutive negation:

- $e(\neg_D\alpha) = ([\alpha^-] \cup [\alpha^\sim], [\alpha^+])$
- $e(\neg_G\alpha) = ([\alpha^-], [\alpha^+] \cup [\alpha^\sim])$

where $[\alpha^\sim] = \Omega \setminus ([\alpha^+] \cup [\alpha^-])$, that would correspond to a broader understanding of, respectively, the examples of $\neg\alpha$ (those that are not examples of α) and the counterexamples of $\neg\alpha$ (those that are not counterexamples of α).[4]

The case for the implication $\rightarrow$ is somehow more involved. The most direct option could be generalizing the classical definition of material implication and take $\alpha \rightarrow \beta := \neg\alpha \vee \beta$, and hence

$$e(\neg\alpha \vee \beta) = ([\alpha^-] \cup [\beta^+], [\alpha^+] \cap [\beta^-]).$$

In this case, the logical framework turns out to correspond to the well-known strong Kleene's three-valued logic [63]. However, it is also well-known that in Kleene's logic the interpretation of the intermediate value 1/2 is usually considered as *ignorance*.[5] This makes it natural to claim that if it is not known whether w is an example or counterexample of both α and β, it remains unknown whether it is an example or counterexample of the compound concept $\alpha \rightarrow \beta$. Nonetheless, if 1/2 is assumed to denote a borderline case, it is perfectly natural to consider, in that case, that w is an example of $\alpha \rightarrow \beta$. Similarly, if w is a borderline case for α and a counterexample for β, then one can find sound reasons to consider w as a borderline case for $\alpha \rightarrow \beta$, as in Kleene logic, or else as a counterexample for $\alpha \rightarrow \beta$. Namely, the latter case arises when we tolerantly consider w as mainly not being a counter-example of α, and then w can be taken as a counter-example for $\alpha \rightarrow \beta$.

These small changes amount to moving from Kleene's three-valued logic implication $\rightarrow_K$ to either Łukasiewicz's three-valued implication $\rightarrow_L$ or to Gödel three-valued implication $\rightarrow_G$ [25, 29]. Indeed, in such cases, we have:

- $e(\alpha \rightarrow_L \beta) = ([\alpha^-] \cup [\beta^+] \cup ([\alpha^\sim] \cap [\beta^\sim]), [\alpha^+] \cap [\beta^-])$,
- $e(\alpha \rightarrow_G \beta) = ([\alpha^-] \cup [\beta^+] \cup ([\alpha^\sim] \cap [\beta^\sim]), ([\alpha^+] \cup [\alpha^\sim]) \cap [\beta^-])$,

where we use the notation $[\gamma^\sim] = \Omega \setminus ([\gamma^+] \cup [\gamma^-])$, which, in terms of truth-functions, amounts to require $1/2 \rightarrow 1/2 = 1$ and then either $1/2 \rightarrow 0 = 1/2$ in the former case or $1/2 \rightarrow 0 = 0$ in the latter. Below we introduce the truth-functions of the implication in Kleene, Łukasiewicz and Gödel three-valued logics, where we can observe the small differences among them:

$\rightarrow_K$	0	1/2	1
0	1	1	1
1/2	1/2	1/2	1
1	0	1/2	1

$\rightarrow_L$	0	1/2	1
0	1	1	1
1/2	1/2	1	1
1	0	1/2	1

$\rightarrow_G$	0	1/2	1
0	1	1	1
1/2	0	1	1
1	0	1/2	1

[4] In fact, $\neg_G$ corresponds to the intuitionistic negation on the chain $\{0, 1/2, 1\}$, also known as Gödel negation, while $\neg_D$ corresponds to the so-called dual intuitionistic negation.

[5] See e.g. Priest's chapter in this handbook [60] for a discussion on this issue.

Therefore, if we stick to the condition $1/2 \to 1/2 = 1$, which guarantees a residuated implication, from the above analysis we basically recover again the two three-valued fuzzy logics of the approach (i), with the proviso that in the case of Gödel three-valued logic the negation is not the involutive $\neg$ but the negation $\neg_G$ considered above.

Notice that, with the above definitions of $\vee$ and $\neg$, the connectives $\to_L$ and $\to_G$ are in fact inter-definable. Indeed, we have the following relations:

- $e(\alpha \to_L \beta) = e((\neg\alpha \vee \beta) \vee (\alpha \to_G \beta))$,
- $e(\alpha \to_G \beta) = e((\alpha \to_L \beta)^2 \vee \beta)$,

where $(\alpha \to_L \beta)^2 = \neg((\alpha \to_L \beta) \to_L \neg(\alpha \to_L \beta))$. This reflects the well-known fact that the logic resulting from the addition of an involutive negation to the 3-valued Gödel logic is identical to the 3-valued Łukasiewicz logic, see e.g. [20, Prop. 2].

In Section 6 we will formalize several notions of logical consequence behind these intuitive and simple semantics and relate them to the 3-valued Łukasiewicz and Gödel logics described in the next section.

4 Three-valued Łukasiewicz and Gödel logics

In this section we will focus on those logics that, in the previous section, were shown to be the unique three-valued fuzzy logics (in the sense of being schematic extensions of MTL), namely Łukasiewicz and Gödel logics.

Actually, although these formalisms can be defined as extensions of MTL (and, in fact, of BL), it is worth remarking that both three-valued Łukasiewicz and Gödel logics can be axiomatized in a reduced language. This will be discussed in the next Subsections 4.1 and 4.2. Furthermore, in Subsection 4.3, we will comment on remarkable expansions of Łukasiewicz and Gödel logics and we will make clear the existing relations between these three-valued logical systems.

4.1 Three-valued Łukasiewicz logic and its algebraic semantics

Three-valued Łukasiewicz logic was originally introduced by Jan Łukasiewicz in [52] motivated by the desire to formalize reasoning with possibly undetermined truth values. Besides the presentation of Łukasiewicz logic as axiomatic extension of the logic MTL, as in Subsection 3.1, and the original formulation of this calculus, nowadays it is common to introduce it in a simplified language made of a countable set $V = \{p_1, p_2, \ldots\}$ of variables, along with the binary connective $\to$ and the unary connective $\neg$.

We will denote by $\mathfrak{F}(V)$ the class of formulas defined, as usual, from the set of variables V. Further connectives and constants are definable from $\to$ and $\neg$ as follows:

$\top$	is	$\varphi \to \varphi$
$\bot$	is	$\neg\top$
$\varphi \oplus \psi$	is	$\neg\varphi \to \psi$
$\varphi \& \psi$	is	$\neg(\neg\varphi \oplus \neg\psi)$
$\varphi \vee \psi$	is	$(\varphi \to \psi) \to \psi$
$\varphi \wedge \psi$	is	$\neg(\neg\varphi \vee \neg\psi)$
$\varphi \leftrightarrow \psi$	is	$(\varphi \to \psi) \wedge (\psi \to \varphi)$

We will henceforth also adopt the following abbreviation: for every $n \in \mathbb{N}$ and for every $\varphi \in \mathfrak{F}(V)$, $n\varphi$ will stand for $\varphi \oplus \cdots \oplus \varphi$ (n-times).

The propositional three-valued Łukasiewicz logic, denoted by $Ł_3$, is defined as the Hilbert style system of axioms and rule for the infinitely-valued propositional Łukasiewicz logic, denoted by Ł (cf. [39]):[6]

[6] In fact this Hilbert system is shown to be equivalent in [39] to the axiomatic extension of BL logic with the axiom (Inv) $\neg\neg\varphi \to \varphi$, forcing the negation to be involutive.

(Ł1) $\varphi \to (\psi \to \varphi)$,

(Ł2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,

(Ł3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$,

(Ł4) $(\varphi \vee \psi) \to (\psi \vee \varphi)$,

(MP) The rule of modus ponens: $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$.

together with the axiom[7]

(3V) $(\varphi_1 \to \varphi_2) \vee (\varphi_2 \to \varphi_3) \vee (\varphi_3 \to \varphi_4)$.

The syntactical notions of *deduction* and *proof* are the usual ones (see [39]). A *theory* is any subset of $\mathfrak{F}(V)$, and for every theory Γ and for every formula φ we will write $\Gamma \vdash_{\text{Ł}_3} \varphi$ if φ can be proved from Γ in the logic Ł_3.

The algebraic counterpart of the three-valued Łukasiewicz calculus is the class of three-valued MV-algebras that we will now briefly present. To start with, MV-algebras (cf. [13, 39, 56]) are systems that, as customary, will be presented in the algebraic language $(\oplus, \neg, 0)$ of type $(2, 1, 0)$. Then, an MV-algebra $M = (M, \oplus, \neg, 0^M)$ is such that the reduct $(M, \oplus, 0^M)$ is a commutative monoid, and the following equations hold:

(MV1) $x \oplus 1^M = 1^M$,

(MV2) $\neg\neg x = x$,

(MV3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

As we already abbreviated before, in (MV1), 1^M stands for $\neg 0^M$. In every MV-algebra, the operation $\otimes$ that coincides with the Łukasiewicz t-norm on the MV-algebra over $[0, 1]$, can hence be defined as follows: $x \otimes y = \neg(\neg x \oplus \neg y)$. The residuum of $\otimes$ is here definable as $x \Rightarrow y = \neg x \oplus y$. In what follows we will also adopt the following notation: for every integer $n > 0$, and for every x in an MV-algebra $\mathbf{A}$, x^n stands for $x \otimes \ldots \otimes x$ (n-times) and nx stands for $x \oplus \ldots \oplus x$ (n-times).

MV-algebras clearly form a variety that will be denoted by $\mathbb{MV}$. Thus, *MV_3-algebras* are those MV-algebras in the proper subvariety $\mathbb{MV}_3$ of $\mathbb{MV}$ captured by the next equation:

(3V) $(x_1 \Rightarrow x_2) \vee (x_2 \Rightarrow x_3) \vee (x_3 \Rightarrow x_4) = 1$

Notice that the equation (3V) can be equivalently formulated in $\mathbb{MV}$ as $3x = 2x$, or $x^3 = x^2$.

An *evaluation* e of formulas of $\mathfrak{F}(V)$ into an MV_3-algebra M is defined as in Subsection 3.1. The variety $\mathbb{MV}_3$ of MV_3-algebras constitutes the equivalent algebraic semantics for three-valued Łukasiewicz logic.

We anticipated in Subsection 3.1 that $\mathbb{MV}_3$ is generated, as a variety and as a quasi-variety, by the standard algebra $\mathbf{Ł}_3$ and hence Ł_3 is strongly complete with respect to this standard algebra.

Theorem 2. *Three-valued Łukasiewicz logic is* strong standard complete*, i.e. for every theory $\Gamma \subseteq \mathfrak{F}(V)$, and for every formula φ, $\Gamma \vdash_{Ł_3} \varphi$ iff every evaluation into $\mathbf{Ł}_3$ that satisfies Γ, satisfies φ as well.*

In addition, the fact that $\mathbf{Ł}_3$ is indeed generic for the variety $\mathbb{MV}_3$ implies that, for every finite k, the k-generated free algebras of $\mathbb{MV}_3$ can be represented, up to isomorphism and by general results, as the MV_3-subalgebras of $(\mathbf{Ł}_3)^{(\mathbf{Ł}_3)^k}$ generated by the projection maps. Although it is not difficult to present finitely generated free algebras of $\mathbb{MV}_3$ in general, in the next example we will only deal with the 1-generated case. The other cases can be found in [1, 13]

[7]The chosen presentation for the Hilbert-style calculus of Ł_3, is motivated by the fact that up to here, the axioms (Ł_1)-(Ł_4) and the modus ponens rule, capture the infinite-valued Łukasiewicz logic Ł_∞ (see [39]), while the following axiom (Ł_5) indeed forces truth-values to be only three. Thus, in a sense, our presentation for Ł_3 follows the idea developed in Subsection 3.1, where three-valued logics are presented by restricting the truth-value set from $[0, 1]$ to $\{0, 1/2, 1\}$.

Example 1 (1-generated free MV$_3$-algebra)**.** According to what we just stated above, the free MV$_3$ algebra on one generator $\mathrm{Free}_{\mathbb{MV}_3}(1)$ is the subalgebra of $\mathbf{Ł}_3^{\mathbf{Ł}_3}$ generated by x. Thus, each of its elements can be seen as a function $f : \{0, 1/2, 1\} \to \{0, 1/2, 1\}$ such that: $f(0), f(1) \in \{0, 1\}$, while $f(1/2) \in \{0, 1/2, 0\}$. Therefore $\mathrm{Free}_{\mathbb{MV}_3}(1)$ can be displayed as the direct product of $\mathbf{Ł}_2 \times \mathbf{Ł}_3 \times \mathbf{Ł}_2$, where, in fact, $\mathbf{Ł}_2$ indicates the two-valued MV-algebra, i.e., the Boolean chain on $\mathbb{2} = \{0, 1\}$. The Hasse diagram of $\mathrm{Free}_{\mathbb{MV}_3}(1)$ is hence displayed in Figure 3.

Therefore, in particular, $\mathrm{Free}_{\mathbb{MV}_3}(1)$ contains the following 12 elements: the two constant functions $f_\top : x \mapsto 1$ and $f_\bot : x \mapsto 0$, together with x; $\neg x$; $x \wedge \neg x$; $x \vee \neg x$; $2x$; x^2; $\neg 2x$; $\neg x^2$; $2x \wedge \neg x^2$ and $\neg(2x \wedge \neg x^2)$.

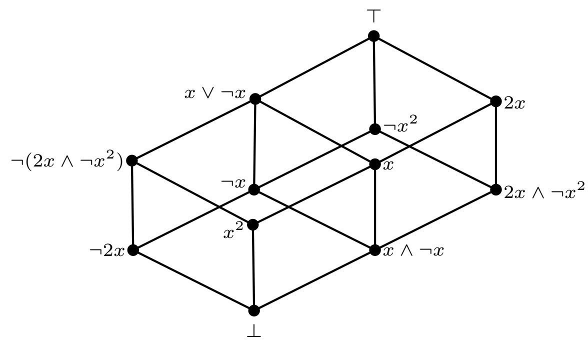

Figure 3: The Hasse diagram of the free MV$_3$-algebra over one generator.

We end this first subsection with a remark about the definability in Ł$_3$ of the Baaz-Monteiro projection operator Δ, that will be used later in Subsection 4.3. Indeed, for each formula φ, if we denote by $\Delta\varphi$ the formula $\varphi \& \varphi$, it is easy to check that, for any $\mathbf{Ł}_3$-evaluation e, we have:

$$e(\Delta\varphi) = \begin{cases} 1 & \text{if } e(\varphi) = 1 \\ 0 & \text{if } e(\varphi) < 1. \end{cases}$$

That is, the connective Δ so defined is actually the restriction of the well-known Baaz-Monteiro operator [55, 4] to the domain $\{0, 1/2, 1\}$.

4.2 Three-valued Gödel logic

In the same way as Łukasiewicz logic was originally introduced and studied outside the frame of what today is the discipline of Mathematical Fuzzy Logic, also Gödel logic was originally presented by Horn in [45] as the *prelinear* extension of Intuitionistic propositional calculus. From this historical perspective, Gödel logic can be presented in the same language of Intuitionistic logic, and its corresponding algebraic strutuctures, *Gödel algebras*, as prelinear Heyting algebras. Hence, formulas of three-valued Gödel logic, G$_3$, are inductively defined from a countable set $V = \{p_1, p_2, \ldots\}$ of variables, along with the binary connectives $\to$ and $\wedge$ and the constant $\bot$. We will denote by $\mathfrak{F}(V)$ the class of formulas defined from the set of variables V.

Further connectives are definable from $\to$, $\wedge$ and $\bot$ as follows:

$\top$	is	$\varphi \to \varphi$
$\varphi \vee \psi$	is	$((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$
$\neg\varphi$	is	$\varphi \to \bot$
$\varphi \leftrightarrow \psi$	is	$(\varphi \to \psi) \wedge (\psi \to \varphi)$

The propositional Gödel logic, denoted by G, is defined as the following Hilbert style system of axioms and rule (cf. [39]):[8]

[8]This axiomatization comes from adding axiom (A7) to the axioms of Hájek's BL logic [39]. Later it was shown that axioms (A2) and (A3) were in fact redundant, see [6] for a detailed exposition and the references therein.

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$;

(A2) $(\varphi \wedge \psi) \to \varphi$;

(A3) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$;

(A4a) $(\varphi \to (\psi \to \chi)) \to ((\varphi \wedge \psi) \to \chi)$;

(A4b) $((\varphi \wedge \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$;

(A5) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$;

(A6) $\bot \to \varphi$;

(A7) $\varphi \to (\varphi \wedge \varphi)$;

(MP) The rule of modus ponens: $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$.

The logic G_3 is the extension of G by the axiom:

(3V) $(\varphi_1 \to \varphi_2) \vee (\varphi_2 \to \varphi_3) \vee (\varphi_3 \to \varphi_4)$

As in the case of the logic Ł_3, the syntactical notions of *deduction* and *proof* in G_3 are the usual ones (see [39]). A *theory* is any subset of $\mathfrak{F}(V)$. For every theory Γ and for every formula φ we will write $\Gamma \vdash_{\mathrm{G}_3} \varphi$ if φ can be proved from Γ in the logic G_3.

The algebraic counterpart of Gödel propositional logic is the variety of *Gödel algebras*, that will be denoted by $\mathbb{G}$. A Gödel algebra (cf. [39]) is a system $A = (A, \wedge, \vee, \Rightarrow, 0)$ of type $(2, 2, 2, 0)$ such that the reduct $(A, \wedge, \vee, 0)$ is a distributive lattice with minimum 0, and the following equations hold:

(G1) $\Rightarrow$ and $\wedge$ form an adjoint pair satisfying the residuation condition $x \Rightarrow y \geq z$ iff $x \wedge z \leq y$;

(Lin) $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$.

That is, Gödel algebras are, in particular, commutative, integral, idempotent and bounded, residuated lattices. Further operations are definable from $\wedge$, $\vee$, $\Rightarrow$ and 0 as follows:

1	is	$x \Rightarrow x$
$\neg x$	is	$x \Rightarrow 0$
$x \Leftrightarrow y$	is	$(x \Rightarrow y) \wedge (y \Rightarrow x)$.

As we anticipated at the beginning of this subsection, Gödel algebras are indeed Heyting algebras that further satisfy the prelinearity identity (2). By a *three-valued Gödel algebra*, we hence understand a Gödel algebra that further satisfies the equation

(3V) $(x_1 \Rightarrow x_2) \vee (x_2 \Rightarrow x_3) \vee (x_3 \Rightarrow x_4) = 1$.

The class of three-valued G-algebras constitutes a variety that will be denoted $\mathbb{G}_3$ and that forms the equivalent algebraic semantics of the three-valued Gödel logic G_3. Moreover, $\mathbb{G}_3$ is generated as a variety and as a quasi-variety by the standard algebra $\mathbf{G}_3$.

An *evaluation* e of formulas of $\mathfrak{F}(V)$ into a G_3-algebra A is any map $e : V \to A$ that extends to compound formulas by truth functionality using the operations in A. We say that e is a model of (or satisfies) a formula $\varphi \in \mathfrak{F}(V)$ when $e(\varphi) = 1$. As for the case of Ł_3, recall in Theorem 2, G_3 is strongly complete w.r.t. to the standard algebra $\mathbf{G}_3$.

Theorem 3. *The logic G_3 is* strong standard complete*, i.e.: for every theory $\Gamma \subseteq \mathfrak{F}(V)$, and for every formula φ, $\Gamma \vdash_{G_3} \varphi$ (iff $\Gamma \vdash_{\mathbb{G}_3} \varphi$) iff every evaluation into the algebra $\mathbf{G}_3$ that satisfies Γ, satisfies φ as well.*

As we already observed for the case of Ł$_3$, also the k-generated free Gödel algebra is the subalgebra of $(\mathbf{G}_3)^{(\mathbf{G}_3)^k}$ generated by the projection maps. The following example gives a hint on the 1-generated free algebra of $\mathbb{G}_3$. More complex cases can be found in [1].

Example 2 (1-generated free G_3-algebra)**.** As we recalled at the beginning of this subsection, three-valued Gödel algebras can be regarded as those Heyting algebras that further satisfy (Lin) and (G3) identities. Remarkably, (Lin) has the effect that the variety of Gödel algebras (and hence $\mathbb{G}_3$ a fortiori) is locally finite, whilst the same property does not hold for Heyting structures. In particular, while the free Heyting algebra over one generator is the well-known infinite Rieger-Nishimura lattice, the free three-valued Gödel algebra over one-generator, $\mathrm{Free}_{\mathbb{G}_3}(1)$, only contains six elements: $\bot$; x; $\neg x$; $x \vee \neg x$; $\neg\neg x$ and $\top$ (see its Hasse diagram depicted in Figure 4). Indeed, $\mathrm{Free}_{\mathbb{G}_3}(1) \cong \mathbf{G}_3 \times \mathbf{G}_2$ where, again $\mathbf{G}_2$ is the Boolean chain on $2 = \{0, 1\}$.

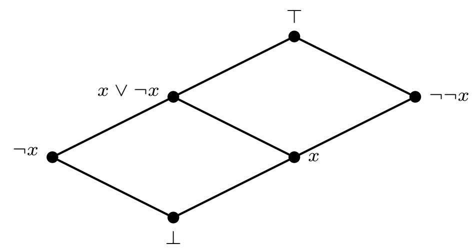

Figure 4: The Hasse diagram of the free G_3-algebra over one generator.

As it is evident by a visual comparison with Figure 3 in Example 1, $\mathrm{Free}_{\mathbb{G}_3}(1)$ contains less elements than $\mathrm{Free}_{\mathbb{MV}_3}(1)$ and in fact all the elements of $\mathrm{Free}_{\mathbb{G}_3}(1)$ appear in $\mathrm{Free}_{\mathbb{MV}_3}(1)$. Thus, G_3 has a *lower expressive power* (in this sense) than Ł$_3$. Notice that, in particular, no functions of $\mathrm{Free}_{\mathbb{G}_3}(1)$ correspond to the following operations: $\sim(x) = 1 - x$; $\Delta(x) = 1$ if $x = 1$ and $\Delta(x) = 0$ otherwise; $D(x) = 0$ if $x = 1$ and $D(x) = 1$ otherwise.

4.3 Relation between Łukasiewicz and Gödel three-valued logics

As we have already observed in the last parts of Subsection 4.1 the Monteiro-Baaz operator Δ is definable in $\mathbf{Ł}_3$ as $\Delta x = x * x$. Notice that, by using Δ we can hence define Gödel implication on $\mathbf{Ł}_3$ as $x \Rightarrow_G y = \Delta(x \Rightarrow_{\text{Ł}} y) \vee y$. However, as we pointed out at the end of Example 2, while all the operations of $\mathbf{G}_3$ are definable in $\mathbf{Ł}_3$, the converse is not true because, for instance, Δ is not definable in $\mathbf{G}_3$. In this section we will hence consider three main expansions of G_3 and show their relation with three-valued Łukasiewicz logic. To ease the reading, we will use subscripts to distinguish Łukasiewicz from Gödel connectives, as we did above. The expansion we will deal with have the following standard semantics:

- Let $\mathbf{G}_3^{\sim}$ be the expansion of $\mathbf{G}_3$ with an involutive negation $\sim$ that on the chain $\{0, 1/2, 1\}$ is $\sim x = 1 - x$. Notice that, in $\mathbf{G}_3^{\sim}$, the operator Δ is hence definable as $\Delta x = \neg{\sim}x$, and the Łukasiewicz implication is definable too as $x \Rightarrow_{\text{Ł}} y = (x \Rightarrow_G y) \vee (\sim x \vee y)$.

- The algebra $\mathbf{G}_3^{\Delta}$ is obtained by expanding $\mathbf{G}_3$ by the unary Baaz-Monteiro operation Δ that maps 1 to 1 and 0 and 1/2 to 0.

- Finally, let $\mathbf{G}_3^{D}$ be the algebra obtained by expanding $\mathbf{G}_3$ by the unary operator D that maps 1 to 0 and 0 and 1/2 to 1. This map is the dual of the intuitionistic negation, and it has been axiomatized by Moisil in [54] in the frame of positive intuitionistic logic.

The next result shows a quite strong property, i.e. on $\mathbf{G}_3$ the above defined operations are inter-definable.

	Δ	D	$\sim$
0	0	1	1
1/2	0	1	1/2
1	1	0	0

Figure 5: Operations Δ, D and $\sim$ on the algebra $\mathbf{G_3}$.

Lemma 1. *In the algebra* $\mathbf{G}_3 = (\{0, 1/2, 1\}, \wedge, \Rightarrow_G, 0, 1)$*, the three unary operations* $\Delta, D, \sim$ *are mutually inter-definable.*

Proof. Recall the definable Gödel negation $\neg_G x = x \Rightarrow_G 0$. Then, by direct inspection of their definition, one can easily check that the following equalities among operations on $\{0, 1/2, 1\}$ hold true: for all $x \in \{0, 1/2, 1\}$,

- $\Delta(x) = \neg_G D(x) = DD(x)$, whence Δ can be defined from D;
- $D(x) = \neg_G \Delta(x)$, and hence D can be defined from Δ;
- $\Delta x = \neg_G {\sim} x$, that is to say $\sim$ defines Δ;
- ${\sim} x = \neg_G x \vee (\neg_G \Delta x \wedge x)$, and therefore Δ defines $\sim$.

This settles the claim. □

Remark 3. Łukasiewicz negation is involutive, whence $\sim$ can be trivially defined in $\mathbf{Ł}_3$. Also, at the end of Subsection 4.1, we observed that Δ is definable in the same standard algebra. As for D, it is a direct computation to show that it can be defined in $\mathbf{Ł}_3$ by the term $\neg(x * x)$, i.e., $\neg\Delta(x)$.

Besides the algebraic descriptions for the three operators $\sim$, Δ and D, at the logical level, they can be captured, on G_3, in a slightly simplified form compared with the original axioms needed to capture them on the whole real unit interval. Those are the following:

- The logic $G_3^{\sim}$ is obtained by expanding the language of G_3 with the unary connective $\sim$ and by adding to G_3 the following axioms:

  ($\sim$1) ${\sim}{\sim}\varphi \leftrightarrow \varphi$;

  ($\sim$2) ${\sim}\bot$.

- The logic G_3^{Δ} is obtained by expanding the language of G_3 with the unary connective Δ and by adding to G_3 the following axioms and rule:

  (Δ1) $\Delta\varphi \vee \neg\Delta\varphi$;

  (Δ2) $\Delta\varphi \to \varphi$;

  (N) from φ derive $\Delta\varphi$.

- The logic G_3^{D} is obtained by expanding the language of G_3 with the unary connective D and by adding to G_3 the following axiom and rule:

  (D1) $\varphi \vee D\varphi$;

  (D2) from $\varphi \vee \psi$ derive $\neg D\varphi \vee \psi$.

The following result collects some completeness results for the above logics with respect to their algebraic semantics (see [27] and [30] for details).

Proposition 2. *The following strong completeness results hold:*

(1) *$G_3^{\sim}$ is strongly complete with respect to the variety of $G_3^{\sim}$-algebras. More precisely, $G_3^{\sim}$ is strongly complete with respect to the algebra $\mathbf{G_3^{\sim}}$.*

(2) *G_3^{Δ} is strongly complete with respect to the variety of G_3^{Δ}-algebras. Indeed, $G_{\sim}$ is strongly complete with respect to the algebra $\mathbf{G_3^{\Delta}}$.*

(3) *G_3^{D} is strongly complete with respect to the variety of G_3^{D}-algebras. More precisely, G_3^{D} is strongly complete with respect to the algebra $\mathbf{G_3^{D}}$.*

Proof. Completeness results with respect to the corresponding varieties of algebras is a direct consequence of the algebraizability of the logics. To prove completeness with respect to the corresponding standard algebras it is enough to observe the following:

- The only unary operation $\sim$ on $\mathbf{G_3}$ such that $\sim\sim x = x$ and $\sim 0 = 1$ is the one defined as $\sim x = 1 - x$;
- The only unary operation Δ on $\mathbf{G_3}$ such that $\max(\Delta x, \neg_G \Delta x) = 1$, $\Delta x \leq x$ and $\Delta 1 = 1$ is the Baaz-Monteiro operation; and
- The only unary operation D on $\mathbf{G_3}$ such that $\max(x, Dx) = 1$ and $\neg_G D1 = 1$ (i.e. $D1 = 0$) is the dual of the intuitionistic negation.

□

The above observations at the algebraic level have the following direct consequences at the logical level.

Proposition 3. *The following relations among logics hold:*

(i) *Gödel Logic G_3 is interpretable in Lukasiewicz logic L_3, in the sense that the G_3 connectives are definable from the L_3 connectives respecting the semantics. But the converse relation does not hold, in particular Lukasiewicz implication is not definable in G_3.*

(ii) *The four logics $Ł_3$, $G_3^{\sim}$, G_3^{D} and G_3^{Δ} are pairwise equivalent, in the sense of being mutually interpretable.*

Proof. (i) It is enough to recall that in $\mathbf{Ł}_3$ we can define Gödel implication as $x \Rightarrow_G y = \Delta(x \Rightarrow_{Ł} y) \vee y$, where $\Delta x = x * x$. On the other hand, as we pointed out at the end of Example 2, Δ is not definable in $\mathbf{G}_3$.

(ii) The proof of the equivalence between the three extensions of G_3 is based on the completeness result and the interdefinability of the three connectives $\Delta, D, \sim$ in the corresponding standard algebras proved in Lemma 1.

Due to Remark 3 it suffices to show the definability of Łukasiewiz implication, for example, in $\mathbf{G}_3^{\sim}$. Indeed we can easily prove that $x \Rightarrow_{Ł} y = (x \Rightarrow_G y) \vee (\sim x \vee y)$. □

Finally, let us observe that the connectives $\sim, \Delta$ and D can be directly added to the language of MTL_3 and axiomatized by the same axioms we recalled above for the case of G_3. The logics so defined will be denoted by $MTL_3^{\sim}$, MTL_3^{Δ} and MTL_3^{D} respectively.

The completeness of MTL_3 with respect to the variety $\mathbb{MTL}_3$ generated by both $\mathbf{Ł}_3$ and $\mathbf{G}_3$ together with what we observed in Remark 3, hence shows that the algebraic semantics for $MTL_3^{\star}$ (for $\star = \sim, \Delta, D$) is made of expansions by $\star$ of MTL_3-algebras. Now, if we focus on MTL_3-chains, we observe the following:

- $\mathbf{Ł}_3$ defines each $\star = \sim, \Delta, D$, whence each term t in the language expanded by $\star$ can be translated in the language of MV_3 algebras by $t^{\bullet}$ in such a way that t holds in $\mathbf{Ł}_3^{\star}$ iff $t^{\bullet}$ holds in $\mathbf{Ł}_3$.
- $\mathbf{G}_3^{\sim} = \mathbf{G}_3^{\Delta} = \mathbf{G}_3^{D}$ by Proposition 3. Moreover each of the above chains is equivalent to $\mathbf{Ł}_3$ that, in turn, allows to interpret each term of $\mathbf{Ł}_3^{\star}$.

Thus, we can conclude with the following.

Corollary 1. $MTL_3^{\sim} = MTL_3^{\Delta} = MTL_3^{D} = Ł_3$.

5 Degree-preserving and matrix logic companions of three-valued fuzzy logics

The usual consequence relation for any fuzzy logic system L that is complete with respect to an algebra on the real unit interval $[0,1]_L$ is defined as follows: for every set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \models_L \varphi \text{ iff } \quad \text{for every evaluation } e \text{ over } [0,1]_L, \text{ if } e(\gamma) = 1 \text{ for all } \gamma \in \Gamma, \text{ then } e(\varphi) = 1$$

By definition, this notion of consequence relation is truth-preserving, that is, it only requires the truth-value 1 to be preserved from the premises to the conclusion. In other words, this corresponds to the logic defined by the logical matrix $\langle [0,1]_L, \{1\} \rangle$, with 1 as the only designated value.

Of course, one can consider other notions of consequence relation besides the truth-preserving one, for instance the logics corresponding to logical matrices $\langle [0,1]_L, F \rangle$, where F is a lattice filter of $[0,1]$ acting as the set of designated values.

There is also a particularly interesting notion of consequence relation that preserves not only the value 1 but all the lower bounds of truth-degrees from the premises to the conclusion, and it is defined as follows: for every set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \models_L^{\leq} \varphi \text{ iff } \quad \text{for every evaluation } e \text{ over } [0,1]_L \text{ and every } a \in [0,1], \text{ if } e(\gamma) \geq a \text{ for all } \gamma \in \Gamma, \text{ then } e(\varphi) \geq a$$

This variant of the logic L, denoted $L^{\leq}$, is known in the literature as the *degree-preserving companion*[9] of L, see e.g. [31, 7]. Note that $L^{\leq}$ is the logic corresponding to the the intersection of the family of matrix logics $\langle [0,1]_L, F_a \rangle$ for all $a \in [0,1]$, where $F_a = [a,1]$.

It is not difficult to show that L and $L^{\leq}$ have the same theorems and also that, for every finite set of formulas $\Gamma \cup \{\varphi\}$:

$$\Gamma \models_L^{\leq} \varphi \text{ iff } \models_L \Gamma^{\wedge} \to \varphi$$

where $\Gamma^{\wedge}$ stands for $\gamma_1 \wedge \ldots \wedge \gamma_k$ if $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$. If $\Gamma = \emptyset$ then one takes $\Gamma^{\wedge} = \top$.

Moreover let us observe that the (truth-preserving logic) L is *explosive*, i.e. for every formula φ, since $\varphi \& \neg\varphi \to \bot$ is a theorem/tautology in any fuzzy logic L, it follows that from φ and its negation $\neg\varphi$ we can derive in L any formula, in other words, the explosion rule

$$(\text{exp}) \quad \frac{\varphi \quad \neg\varphi}{\psi}$$

is sound in L. However, this is not always the case with $L^{\leq}$. Namely, if L is not an extension of SMTL = MTL + $\{\varphi \wedge \neg\varphi \to \bot\}$, there exist formulas φ and ψ different from $\bot$ and L-evaluations e such that $\min(e(\varphi), e(\neg\varphi)) > e(\psi) > 0$, and hence, $\{\varphi, \neg\varphi\} \not\models_L^{\leq} \psi$. Therefore, in such a case, $L^{\leq}$ is a *paraconsistent* logic with respect to the negation $\neg$ (see e.g. [10] for definitions and results about paraconsistent logics).

As regards axiomatization, if L has a Hilbert style axiomatization with Modus Ponens as the only inference rule, then the logic $L^{\leq}$ admits a Hilbert-style axiomatization as well having the same axioms of L and the following deduction rules [7]:

(Adj-∧) $\dfrac{\varphi \quad \psi}{\varphi \wedge \psi}$

(MP-*r*) if $\vdash_L \varphi \to \psi$ (i.e. if $\varphi \to \psi$ is a theorem of L), then from φ derive ψ.

[9] Although commonly used, strictly speaking, the term *degree-preserving* might not be the most suitable one to refer to a logic defined by the above condition on truth-values, as this condition actually expresses the preservation, not of exact truth-degrees, but of lower bounds for them: if the truth-degrees of all the premises are greater or equal than a given (arbitrary) value, then so is the truth-degree of the conclusion.

Note that (Adj-$\wedge$) is the usual adjunction rule, while (MP-r) is a restricted form of Modus Ponens rule, since one can derive ψ from φ and $\varphi \to \psi$ only when the implication $\varphi \to \psi$ is a theorem of the logic L.

When we come to the three-valued case, all the above definitions and observations for the infinite-valued case easily transfer. In particular, observe that Łukasiewicz and Gödel three-valued logics, $Ł_3$ and G_3 resp., correspond to the logics of the matrices $\langle \mathbf{Ł}_3, \{1\}\rangle$ and $\langle \mathbf{G}_3, \{1\}\rangle$ resp. As for the degree-preserving logics, since we now have only three truth-values, the logics $Ł_3^{\leq}$ and $G_3^{\leq}$ are respectively the intersection of the logics defined by the matrices $\langle \mathbf{Ł}_3, \{1\}\rangle$ and $\langle \mathbf{Ł}_3, \{1/2, 1\}\rangle$, and by the matrices $\langle \mathbf{G}_3, \{1\}\rangle$ and $\langle \mathbf{G}_3, \{1/2, 1\}\rangle$.

In the next sections we will describe in more detail the matrix logics defined by lattice filters over $\mathbf{Ł}_3$ and $\mathbf{G}_3$.

5.1 Matrix logics defined by lattice filters over $\mathbf{Ł}_3$

As we have already indicated above, we have three logics determined by lattice filters of the algebra $\mathbf{Ł}_3$:

1. The logic determined by the matrix $\langle \mathbf{Ł}_3, F_1\rangle$ is the usual truth-preserving logic $Ł_3$.
2. The logic determined by the matrix $\langle \mathbf{Ł}_3, F_{1/2}\rangle$ is in fact the (paraconsistent) Da Costa-D'Ottaviano's 3-valued logic J_3 [46, 21]. We will henceforth use J_3 to indicate this matrix logic.
3. The logic determined by the two matrices $\langle \mathbf{Ł}_3, F_1\rangle$ and $\langle \mathbf{Ł}_3, F_{1/2}\rangle$ is the degree-preserving logic $Ł_3^{\leq}$, that is therefore the intersection of the two preceding logics $Ł_3$ and J_3.

Let us observe that the explosion rule (exp) is not valid in $\langle \mathbf{Ł}_3, F_{1/2}\rangle$, because if a $\mathbf{Ł}_3$-evaluation e is such that $e(\varphi) = 1/2$ then $e(\varphi \wedge \neg\varphi) = 1/2 \in F_{1/2}$. Thus, as a consequence, the explosion rule is not valid either in $Ł_3^{\leq}$, in accordance with what has been noted above, as Łukasiewicz logic is not an extension of SMTL.

Moreover, it is interesting to note that J_3 is maximally paraconsistent, in the sense that if we add a classical propositional logic theorem that is not a theorem in J_3, then we obtain classical propositional logic (cf. [19]).

Lemma 2. *Basic properties of these logics are:*

- *The logic $Ł_3$ is explosive while J_3 and $Ł_3^{\leq}$ are paraconsistent.*
- *The matrix logic J_3 is not comparable (as consequence relation) with $Ł_3$.*

The first item has been already justified above and the second item is a consequence of the fact that, e.g. $\varphi \vee \neg\varphi$ is a tautology of J_3 but not of $Ł_3$, and on the other hand, the explosion rule is valid in $Ł_3$ but not in J_3.

A natural question that arises then is the following: since the explosion rule (exp) is not valid in $Ł_3^{\leq}$ but it is valid in $Ł_3$, it makes sense to consider the logic obtained by adding the explosion rule to $Ł_3^{\leq}$, that we will denote $Ł_3^{\leq}$(exp). The following are some insights into this logic $Ł_3^{\leq}$(exp):

- The logic $Ł_3^{\leq}$(exp) is not comparable to J_3.
- The logic $Ł_3^{\leq}$(exp) strictly contains $Ł_3^{\leq}$ and it is strictly contained in $Ł_3$.

The first property can be proved in the same way as we have shown the non-comparability of $Ł_3$ and J_3 in the previous lemma. For one direction of the second item, observe that the explosion rule is valid in $Ł_3^{\leq}$(ex) but not in $Ł_3^{\leq}$. For the other direction we need the following result that is easy to prove (See [20, Lemma 1.2]). Denote by $\vdash^{\leq}_{exp}$ the deduction relation in the logic $Ł_3^{\leq}$(ex).

Lemma 3. *The following condition holds:*

- $\Gamma \vdash^{\leq}_{exp} \varphi$ *iff* *either* *for every evaluation* e *over* $[0,1]_{MV}$, $e(\Gamma^{\wedge}) \leq 1/2$,
  or *for every evaluation* e *over* $[0,1]_{MV}$, $e(\Gamma^{\wedge}) \leq e(\varphi)$.

Based on this characterization, it is easy to check that

$$p \wedge q \vdash_{\text{Ł}_3} p\&q \qquad \text{but} \qquad p \wedge q \not\vdash^{\leq}_{exp} p\&q,$$

the latter holds because there are evaluations e such that $e(p \wedge q) = 1 \not\leq 1/2$ (e.g. such that $e(p) = e(q) = 1$) and there are evaluations e such that $e(p \wedge q) > e(p\&q)$ (e.g. such that $e(p) = e(q) = 1/2$)).

A diagram of these logics is depicted in Figure 6.

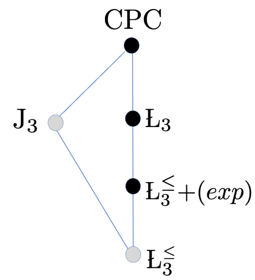

Figure 6: Graph of the three matrix logics defined by lattice filters over $\mathbf{Ł}_3$ ($Ł_3$, J_3 and $Ł_3^{\leq}$) together with the logic $L_3^{\leq}(exp)$ and classical logic. Grey nodes stand for paraconsistent logics.

5.2 Matrix logics defined by lattice filters over $\mathbf{G}_3$

Finally, we describe the three matrix logics defined by lattice filters over $\mathbf{G}_3$:

1. The logic determined by the matrix $\langle \mathbf{G}_3, F_1 \rangle$ is the usual truth-preserving logic G_3.
2. The logic determined by the matrix $\langle \mathbf{G}_3, F_{1/2} \rangle$ boilds down to classical propositional logic CPC since the excluded middle axiom $\varphi \vee \neg\varphi$ takes always value 1/2 or 1, while $\varphi \wedge \neg\varphi$ always takes value 0.
3. The logic determined by the two matrices $\langle \mathbf{G}_3, F_1 \rangle$ and $\langle \mathbf{G}_3, F_{1/2} \rangle$ is the degree-preserving logic $G_3^{\leq}$, that is therefore the intersection of the two preceding logics G_3 and CPC, that obviously coincides with G_3

Therefore, in the case of Gödel three-valued logic we do not get any new logic by considering the companion logic of the filter $F_{1/2}$ and the degree-preserving logic $G_3^{\leq}$.

6 Logics of examples and counterexamples

In this final section we provide a logical formalization of reasoning with fuzzy concepts based on the the intuitive semantics in terms of examples and counterexamples described in Section 3.2 and its relation to 3-valued logics.

An interpretation on a set X is a tuple $v = (X, v^+, v^-)$, where $v^+, v^- : Var \to 2^X$ are mappings assigning a pair of subsets of X to every atomic concept α such that $v^+(\alpha) \cap v^-(\alpha) = \emptyset$. For a less cluttered notation, we will write $v(\alpha) = (v^+(\alpha), v^-(\alpha))$. These mappings extend to compound concepts according to some given prescriptions for the different connectives. Here, in order to consider a concrete semantics, we will consider the following ones:

$(R_\wedge)$ $v(\varphi \wedge \psi) = (v^+(\varphi) \cap v^+(\psi), v^-(\varphi) \cup v^-(\psi))$

$(R_\vee)$ $v(\varphi \vee \psi) = (v^+(\varphi) \cup v^+(\psi), v^-(\varphi) \cap v^-(\psi))$

$(R_\neg)$ $v(\neg\varphi) = (v^-(\varphi), v^+(\varphi))$

$(R_\to)$ $v(\varphi \to \psi) = (v^-(\varphi) \cup v^+(\psi) \cup (v^\sim(\varphi) \cap v^\sim(\psi)), v^+(\varphi) \cap v^-(\psi))$

where we write $v^\sim(\chi)$ for $X \setminus \{v^+(\chi) \cup v^-(\chi)\}$. Then, given this set $\mathbf{R} = \{R_\wedge, R_\vee, R_\neg, R_\to\}$ of semantical definitions, one can consider three corresponding logical consequence relations: the one that preserves examples, the one that preserves counterexamples, and the one that preserves both. For any subset of formulas $\Gamma \cup \{\varphi\}$, we define:

- $\Gamma \models^+_\mathbf{R} \varphi$ if, for every interpretation v, $\bigcap_{\psi\in\Gamma} v^+(\psi) \subseteq v^+(\varphi)$,
  that is, when every example of all formulas in Γ is an example for φ as well.
- $\Gamma \models^-_\mathbf{R} \varphi$ if, for every interpretation v, $v^-(\varphi) \subseteq \bigcup_{\psi\in\Gamma} v^-(\psi)$,
  that is, when every counterexample of φ is a counterexample for some formula in Γ.
- $\Gamma \models^\pm_\mathbf{R} \varphi$ if $\Gamma \models^+_\mathbf{R} \varphi$ and $\Gamma \models^-_\mathbf{R} \varphi$

As already suggested, these logics can, in fact, be shown to faithfully correspond to well-known three-valued logical calculi.

Theorem 4. *The following equivalences hold:*

(i) $\Gamma \models^+_\mathbf{R} \varphi$ *iff* $\Gamma \vdash_{L_3} \varphi$

(ii) $\Gamma \models^-_\mathbf{R} \varphi$ *iff, for any* $Ł_3$*-evaluation* e*, if* $e(\varphi) = 0$ *then* $e(\psi) = 0$ *for some* $\psi \in \Gamma$*, i.e. iff, for any* $Ł_3$*-evaluation* e*, if* $e(\psi) \geq 1/2$ *for all* $\psi \in \Gamma$*, then* $e(\varphi) \geq 1/2$

(iii) $\Gamma \models^\pm_\mathbf{R} \varphi$ *iff* $\Gamma \vdash^\leq_{Ł_3} \varphi$

Proof. (i) $\Gamma \models^+_\mathbf{R} \varphi$ iff $\Gamma \vdash_{Ł_3} \varphi$

Suppose $\Gamma \not\vdash_{Ł_3} \varphi$. Then there exists a $Ł_3$-evaluation e such that $e(\psi) = 1$ for all $\psi \in \Gamma$ and $e(\varphi) < 1$. Let $X = \Omega$ be the set of all $Ł_3$-evaluations over the language and, for any formula ψ let the interpretation $v_e(\varphi) = (v_e^+(\varphi), v_e^-(\varphi))$, where $v_e^+(\varphi) = \{e \in \Omega \mid e(\varphi) = 1\}$ and $v_e^-(\varphi) = \{e \in \Omega \mid e(\varphi) = 0\}$. Then it is clear that $e \in \bigcap_{\psi\in\Gamma} v_e^+(\psi)$ while $e \notin v_e^+(\varphi)$, that is, $\Gamma \not\models^+_\mathbf{R} \varphi$.

Conversely, suppose $\Gamma \not\models^+_\mathbf{R} \varphi$, that is, there exists an interpretation $v = (X, v^+, v^-)$ such that $\bigcap_{\psi\in\Gamma} v^+(\psi) \not\subseteq v^+(\varphi)$. Let $x \in X$ such that $x \in \bigcap_{\psi\in\Gamma} v^+(\psi)$ and $x \notin v^+(\varphi)$. Define a $\{0, 1/2, 1\}$-valued mapping e_x on formulas as follows: for any formula ψ, $e_x(\psi) = 1$ if $x \in v^+(\psi)$, $e_x(\psi) = 0$ if $x \in v^-(\psi)$, and $e_x(\psi) = 1/2$ otherwise. It is easy to check that this is in fact a $Ł_3$-evaluation. Then, it is clear that $e_x(\psi) = 1$ for every $\psi \in \Gamma$ and $e_x(\varphi) < 1$, that is $\Gamma \not\vdash_{Ł_3} \varphi$.

(ii) $\Gamma \models^-_\mathbf{R} \varphi$ iff, for any $Ł_3$-evaluation e, if $e(\varphi) = 0$ then $e(\psi) = 0$ for some $\psi \in \Gamma$

Suppose there exists a $Ł_3$-evaluation e such that $e(\varphi) = 0$ and $e(\psi) > 0$ for all $\psi \in \Gamma$. As above, let $X = \Omega$ be the set of all $Ł_3$-evaluation and recall the definition of the interpretation v_e. Then $e \in v_e^-(\varphi)$ but $e \notin \bigcup_{\psi\in\Gamma} v_e^-(\psi)$, that is, $\Gamma \not\models^-_\mathbf{R} \varphi$.

(iii) It is a direct consequence of (i) and (ii) and the definition of $\vdash^\leq_{Ł_3}$.

□

It is easy to check that if we replace the prescriptions $(R_\neg)$ and $(R_\to)$ by the following ones:

$(R'_\neg)$ $v(\neg\alpha) = (v^-(\alpha), v^+(\alpha) \cup v^\sim(\alpha))$

$(R'_\to)$ $v(\varphi \to \psi) = (v^-(\varphi) \cup v^+(\psi) \cup (v^\sim(\varphi) \cap v^\sim(\psi)), (v^+(\varphi) \cup v^\sim(\varphi)) \cap v^-(\psi))$

then we get an equivalent to Theorem 4 for Gödel three-valued logic G_3, where we let $\mathbf{R}' = \{R_\wedge, R_\vee, R'_\neg, R'_\to\}$.

Theorem 5. *The following equivalences hold:*

(i) $\Gamma \models^{+}_{\mathbf{R}'} \varphi$ *iff* $\Gamma \vdash_{G_3} \varphi$

(ii) $\Gamma \models^{-}_{\mathbf{R}'} \varphi$ *iff, for any* G_3*-evaluation* e*, if* $e(\psi) \geq 1/2$ *for all* $\psi \in \Gamma$*, then* $e(\varphi) \geq 1/2$

(iii) $\Gamma \models^{\pm}_{\mathbf{R}'} \varphi$ *iff* $\Gamma \vdash_{G_3} \varphi$*, that is,* $\models^{+}_{\mathbf{R}'} = \models^{\pm}_{\mathbf{R}'} = \vdash_{G_3}$.

Therefore, as a sort of summary and conclusion of this last section, we can highlight the following observations:

- In the case of adopting the semantical prescriptions $\mathbf{R} = \{R_\wedge, R_\vee, R_\neg, R_\rightarrow\}$ to interpret concepts in terms of their examples and counterexamples, the corresponding examples-preserving logic is just Łukasiewicz three-valued logic $Ł_3$, the logic preserving counterexamples is Jaśkowski logic J_3, while the logic preserving both is the degree-preserving logic $Ł_3^{\leq}$.
- In the case of adopting the prescriptions $\mathbf{R}' = \{R_\wedge, R_\vee, R'_\neg, R'_\rightarrow\}$, the corresponding examples-preserving logic is Gödel three-valued logic G_3, the logic preserving counterexamples is classical logic CPC, while the logic preserving both is the degree-preserving logic is again G_3.

Consequently, in this context, the logical framework based on the three-valued Łukasiewicz logic $Ł_3$ looks more suitable than that based on Gödel three-valued logic G_3 as the most elementary setting to reason about concepts with borderline cases. In particular the logic $Ł_3^{\leq}$ appears as specially suitable whenever examples and counterexamples are given a symmetric role.

7 Conclusions

In this paper we have dealt with three-valued fuzzy logics. First, following Hájek's approach behind the [0, 1]-valued systems of mathematical fuzzy logic as logics for gradual properties, we have considered the logics resulting from the main extensions of the logic MTL when we restrict the set of truth-values to the chain $\{0, 1/2, 1\}$. We have seen that, essentially, only two three-valued logics remain, namely Łukasiewicz and Gödel three-valued logics. Second, we have also seen that these two three-valued fuzzy logics are the ones arising, under some assumptions, when we start from the crudest view of fuzziness in the sense of modelling a fuzzy concept in terms of its examples, counterexamples and borderline cases.

After summarizing the basic definitions and results about Łukasiewicz and Gödel three-valued logics, we have observed that G_3 is definable in $Ł_3$, but not conversely, and have identified three minimal expansions of G_3 with an additional unary operation that make the logic equivalent to $Ł_3$. In fact, we prove that the expansions of G_3 by the Monteiro-Baaz operator Δ, by an involutive negation $\sim$ or by the dual of intuitionistic negation D are pairwise equivalent and also equivalent to $Ł_3$. We have also studied the degree preserving logics corresponding to these three-valued logics and the matrix logics defined by the standard algebra and lattice filters. Finally, the logic of prototypes and counterexamples are defined and studied. This study makes clear that these logics make only sense when the underlying logic is the three-valued Łukasiewicz logic $Ł_3$.

The reader might wonder whether the relations between three-valued Łukasiewicz and Gödel logics described in Section 4.3 keep being valid in the n-valued (with $n > 3$) or even in the infinite-valued cases. The answer is that only some of the results remain valid (see e.g. [20] for the case of the relations of $G_n^{\sim}$ and $Ł_n$ for $n > 3$). For example, G_n keeps being definable in $Ł_n$ and not vice versa, but the extensions of n-valued Gödel logic G_n either with Δ, or with an involutive negation $\sim$ or with the dual of intuitionistic negation D, although pairwise equivalent, are not equivalent to $Ł_n$. Moreover, the infinite-valued logic G is not even definable in Ł because Δ is not definable in Ł.

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