Compound conditionals as random quantities and Boolean algebras

Tommaso Flaminio[1], **Angelo Gilio**[2,*], **Lluis Godo**[1], **Giuseppe Sanfilippo**[3]
[1]IIIA - CSIC, Campus de la UAB, Barcelona, Spain
[2]Department SBAI, University of Rome "La Sapienza", Rome, Italy (*retired)
[3] Department of Mathematics and Computer Science, University of Palermo, Palermo, Italy
{tommaso, godo}@iiia.csic.es, angelo.gilio@sbai.uniroma1.it, giuseppe.sanfilippo@unipa.it

Abstract

Conditionals play a key role in different areas of logic and probabilistic reasoning, and they have been studied and formalised from different angles. In this paper we focus on the de Finetti's notion of conditional as a three-valued object, with betting-based semantics, and its related approach as random quantity as mainly developed by two of the authors. Compound conditionals have been studied in the literature, but not in full generality. In this paper we provide a natural procedure to explicitly attach conditional random quantities to arbitrary compound conditionals that also allows us to compute their previsions. By studying the properties of these random quantities, we show that, in fact, the set of compound conditionals can be endowed with a Boolean algebraic structure. In doing so, we pave the way to build a bridge between the long standing tradition of three-valued conditionals and a more recent proposal of looking at conditionals as elements from suitable Boolean algebras.

1 Introduction

Conditional expressions are pivotal in representing knowledge and reasoning abilities of intelligent agents. Conditional reasoning features in a wide range of areas spanning non-monotonic reasoning (Adams 1975; Dubois and Prade 1994; Benferhat, Dubois, and Prade 1997; Kern-Isberner 2001; Gilio 2002; Gilio and Sanfilippo 2013b; Beierle et al. 2018), causal inference (Halpern 2016; van Rooij and Schulz 2019), and more generally reasoning under uncertainty (Halpern 2003; Coletti and Scozzafava 2002; Pfeifer and Sanfilippo 2017; Sanfilippo et al. 2018) or conditional preferences (Ghirardato 2002; Vantaggi 2010; Coletti, Petturiti, and Vantaggi 2019). Bruno de Finetti was one of the first who put forward an analysis of conditionals beyond the realm of conditional probability theory arguing that they cannot be described within the bounds of classical logic (de Finetti 1936; 1937). He expressed this by referring to them as *trievents*: a conditional, denoted as $(a|b)$, is a basic object to be read "a given b" that turns out to be true if both a and b are true, false if a is false and b is true, and *void* if b is false.

The vast literature on conditionals also includes the study of *compound* conditionals, that is to say, those expressions obtained by combining basic conditionals like $(a|b)$ by usual logical operations of "and", "or", "negation" and so forth, see, e.g., (Schay 1968; Calabrese 1987; McGee 1989; Goodman, Nguyen, and Walker 1991; Jeffrey 1991; Edgington 1995; Milne 1997; Nguyen and Walker 1994; Stalnaker and Jeffrey 1994; Kaufmann 2009; Ciucci and Dubois 2013).

In this line, and based on de Finetti's original conception, (Gilio and Sanfilippo 2013a; 2014; 2019; 2020; 2021a; 2021b) propose and develop an approach to interpret both basic and compound conditionals as random quantities. This approach has been proved to allow for a suitable *numerical representation* of conditionals and some families of compound conditional expressions. Indeed, as we will recall in Section 2, starting from trivents $(a|b)$ regarded as random quantities taking values in a three-element set $\{0, 1, x\}$, where x is a real value in $[0, 1]$ representing a conditional probability, one can extend such representation also to cover more complex conditional formulas.

An alternative, more logically-oriented approach to conditionals has been put forward in e.g. (van Fraassen 1976) or (Goodman and Nguyen 1994)'s Conditional Event algebras, and more recently in (Flaminio, Godo, and Hosni 2020) in a finitary context. These papers formalise conditional expressions in an algebraic setting, and therefore provide a *symbolic representation* of them. In this approach, as it is common in logico-algebraic representations, neither basic expressions $(a|b)$ nor compound conditional formulas necessarily have a numerical counterpart, as their interpretation remains at the symbolic level. However, this does not forbid, as shown in (Flaminio, Godo, and Hosni 2020), to consider e.g. a fully compatible probabilistic layer on top of it.

In the present paper we will put forward an analysis that takes inspiration from both of the above settings and proposes a numerical representation of conditionals that, in addition, also allows for a logico-symbolic representation. In particular, we will:

- present a natural and uniform procedure to interpret compound conditionals as random quantities;
- investigate the numerical and logical properties of such a representation for compound conditionals via

their associated random quantities; and

- prove that, at the logical level, the present approach, leads to the same results of (Flaminio, Godo, and Hosni 2020), where the authors showed that compound symbolic conditionals can be endowed with a Boolean algebra structure.

Let us further remark that the apparent contradiction between the perspective that looks at three-valued conditionals as numerical random quantities as done, e.g., in (Gilio and Sanfilippo 2014; 2021a), and the Boolean algebraic perspective on conditionals used in (Flaminio, Godo, and Hosni 2020) to reason about them, actually dissolves once we precisely set at which level the numerical and the symbolic representation intervene. Proofs of main results can be found in the Arxiv version (Flaminio et al. 2022).

The present paper is organized as follows. In the next Section 2 we will recall in some more details the original approach of (de Finetti 1936) (Subsection 2.1) and the one that followed of (Gilio and Sanfilippo 2014) (Subsection 2.2). The interpretation of compound conditionals in terms of conditional random quantities will be the subject of Section 3. There we will also provide examples in order for our basic construction to be clear. In Section 4 we will then prove numerical and also logical properties of the random quantities that represent compound conditionals, and in Section 5 we will examine some probabilistic aspects. The comparison between the algebras arising from the random quantities studied here and the Boolean algebras of conditionals of (Flaminio, Godo, and Hosni 2020) will be the topic of Section 6, where we prove that they are indeed isomorphic. In the last Section 7 we will gather some conclusions and remarks about future work on this subject.

2 Some preliminary comments on conjoined conditionals

In this section, in order to better understand the formalism and results of this paper, we recall some notions given in the approach by Gilio-Sanfilippo, and their relation with the notions and results given here.

We will denote here events by capital letters, such as $A, B, H, K, \ldots$. Moreover, we denote random quantities by capital letters, such as $X, Y, Z \ldots$. In particular, X_A will denote the indicator function of an event A.

2.1 The conditional prevision in the approach of de Finetti

Given a random quantity Z and an event $H \neq \bot$, in the approach of de Finetti (1935) the conditional prevision $\mathbb{P}(Z|H)$ can be assessed by applying the following conditional bet:

1) you are asked to assess the value $z = \mathbb{P}(Z|H)$, by knowing that if H is true then the bet is in effect;
2) if the bet has effect, then you pay z and you receive the random amount Z; otherwise, if H is false, the bet is null;
3) the checking of coherence for the assessment $z = \mathbb{P}(Z|H)$ is made by only considering the cases where the bet has effect.

We observe that for the random quantity $Z{\cdot}X_H + zX_{\overline{H}}$, by linearity of the prevision, it holds that

$$\mathbb{P}(Z{\cdot}X_H + zX_{\overline{H}}|H) = \mathbb{P}(Z{\cdot}X_H|H) + zP(\bar{H}|H) = \\ = \mathbb{P}(Z|H) + z \cdot 0 = \mathbb{P}(Z|H).$$

On the other hand,

$$\mathbb{P}(Z{\cdot}X_H + zX_{\overline{H}}) = \mathbb{P}(Z{\cdot}X_H) + zP(\bar{H}) = \\ = \mathbb{P}(Z|H)P(H) + zP(\bar{H}) = zP(H) + zP(\bar{H}) = z.$$

In other words, if $\mathbb{P}(Z|H) = z$, we have the following equalities:

$$\mathbb{P}(Z{\cdot}X_H + zX_{\overline{H}}) = \mathbb{P}(Z{\cdot}X_H + zX_{\overline{H}}|H) = \mathbb{P}(Z|H) = z. \tag{1}$$

Now, consider the random quantity $Z{\cdot}X_H + yX_{\overline{H}}$, under the assumptions that:

(i) $y = \mathbb{P}(Z{\cdot}X_H + yX_{\overline{H}})$;

(ii) in order to check the coherence of the assessment y, we discard the cases where you receive back the paid amount y (whatever y be), that is we discard the case where H is false (bet called off).

Of course, $y = z$ satisfies both (i) and (ii).

Question: is it coherent to assess $y \neq z$?

We show below that the answer is NO.
Indeed, if we make a bet on the random quantity

$$(Z{\cdot}X_H + zX_{\overline{H}}) - (Z{\cdot}X_H + yX_{\overline{H}}) = (z - y)X_{\overline{H}},$$

we should agree to pay $z - y$ by receiving $(z - y)X_{\overline{H}} \in \{0, z - y\}$. As when H is false we receive back the paid amount $z - y$ (whatever $z - y$ be), this case must be discarded for checking the coherence of the assessment $z - y$. Then, we should pay $z - y$ by knowing that (when the bet is not called off) we receive 0. Therefore, by coherence, $z - y = 0$, that is $y = z = \mathbb{P}(Z|H)$.

In other words, to assess the conditional prevision $\mathbb{P}(Z|H)$ is equivalent to assess the prevision $\mathbb{P}(Z{\cdot}X_H + zX_{\overline{H}})$. Thus, the conditional random quantity $Z|H$ can be defined as

$$Z|H = ZX_H + zX_{\overline{H}}, \text{ where } z = \mathbb{P}(ZX_H + zX_{\overline{H}}). \tag{2}$$

In this way we can look at z both as the conditional prevision $\mathbb{P}(Z|H)$ *à la de Finetti*, and as the prevision of the conditional random quantity $Z|H = ZX_H + zX_{\overline{H}}$. In particular, given two events A and $H \neq \emptyset$, if $x = \mathbb{P}(X_{AH} + xX_{\overline{H}})$, then $x = P(A|H)$ and hence

$$X_{A|H} = X_{AH} + P(A|H)X_{\overline{H}}. \tag{3}$$

2.2 The conjunction in the approach by Gilio & Sanfilippo

We recall that in the approach by Gilio and Sanfilippo the compound conditionals, like conjunctions and disjunctions, are defined as conditional random quantities, in the setting of coherence, see e.g. (Gilio and

Sanfilippo 2014; 2019; 2021b). In this section, in order to make explicit the numerical aspects, we recall these notions by using the notations of the current paper. Then, the indicator of an event A, or a conditional event $A|H$ is denoted (not by the same symbol, but) by X_A, or $X_{(A|H)}$, respectively. Likewise, the conjunction $(A|H) \wedge (B|K)$ is denoted by $X_{(A|H)\wedge(B|K)}$, and so on.

In the setting of coherence, given a probability assessment $P(A|H) = x, P(B|K) = y$, the conjunction of $A|H$ and $B|K$ is defined as

$$X_{(A|H)\wedge(B|K)} = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \bar{A}H \vee \bar{B}K \text{ is true,} \\ x, & \text{if } \bar{H}BK \text{ is true,} \\ y, & \text{if } AH\bar{K} \text{ is true,} \\ z, & \text{if } \bar{H}\bar{K} \text{ is true,} \end{cases} \quad (4)$$

where z is the prevision of $X_{(A|H)\wedge(B|K)}$, which (in the framework of the betting scheme) is the amount to be paid in order to receive the random amount $X_{(A|H)\wedge(B|K)}$. We observe that

$$X_{(A|H)\wedge(B|K)} = X_{AHBK} + xX_{\overline{H}BK} + yX_{AH\overline{K}} + zX_{\overline{H}\overline{K}}, \quad (5)$$

where $z = \mathbb{P}(X_{AHBK} + xX_{\overline{H}BK} + yX_{AH\overline{K}} + zX_{\overline{H}\overline{K}})$. Then, by applying (2), where H is replaced by $H \vee K$ and Z is replaced by $X_{AHBK} + xX_{\overline{H}BK} + yX_{AH\overline{K}}$, we have

$$Z \cdot X_{H\vee K} = (X_{AHBK} + xX_{\overline{H}BK} + yX_{AH\overline{K}})X_{H\vee K} = \\ = X_{AHBK} + xX_{\overline{H}BK} + yX_{AH\overline{K}},$$

and it follows that

$$X_{AHBK} + xX_{\overline{H}BK} + yX_{AH\overline{K}} + zX_{\overline{H}\overline{K}} = \\ = (X_{AHBK} + xX_{\overline{H}BK} + yX_{AH\overline{K}})|(H \vee K).$$

Hence, from (5) we obtain

$$X_{(A|H)\wedge(B|K)} = (X_{AHBK} + xX_{\overline{H}BK} + yX_{AH\overline{K}})|(H \vee K),$$

with

$$z = \mathbb{P}(X_{AHBK} + xX_{\overline{H}BK} + yX_{AH\overline{K}})|H \vee K) = \\ = P(AHBK|H \vee K) + xP(\bar{H}BK|H \vee K) + \\ + yP(AH\bar{K}|H \vee K). \quad (6)$$

We observe that, if $P(H \vee K) > 0$,

$$z = \frac{P(AHBK) + xP(\bar{H}BK) + yP(AH\bar{K})}{P(H \vee K)},$$

which is the well known formula given in (Kaufmann 2009) and in (McGee 1989).

In addition, by observing that

$$X_{(AHBK|H\vee K)} = X_{AHBK} + P(AHBK|H \vee K)X_{\overline{H}\overline{K}}, \\ xX_{(\overline{H}BK|H\vee K)} = xX_{\overline{H}BK} + xP(\bar{H}BK|H \vee K)X_{\overline{H}\overline{K}}, \\ yX_{(AH\overline{K}|H\vee K)} = yX_{AH\overline{K}} + yP(AH\bar{K}|H \vee K)X_{\overline{H}\overline{K}},$$

from (6) we obtain

$$X_{(AHBK|H\vee K)} + xX_{(\overline{H}BK|H\vee K)} + yX_{(AH\overline{K}|H\vee K)} \\ = X_{AHBK} + xX_{\overline{H}BK} + yX_{AH\overline{K}} + zX_{\overline{H}\overline{K}}.$$

Finally, $X_{(A|H)\wedge(B|K)}$ coincides with the random quantity $X_{(AHBK|H\vee K)} + xX_{(\overline{H}BK|H\vee K)} + yX_{(AH\overline{K}|H\vee K)}$.

In the next example we illustrate an application related to compound conditionals obtained when we consider a *double-bet* in soccer betting.

Example 2.1. Consider two soccer matches: (1) Barcelona–Real Madrid; (2) Juventus–Napoli. For each (uncancelled) match the possible outcomes are: *Home win*, *Draw*, and *Away win*. Let us consider two single bets on the events "Barcelona wins" and "Juventus draws". In a single bet, if a match is cancelled your stake will be refunded (the bet is called off). Then, actually, we have to consider two conditional bets: "Barcelona wins, given that match 1 is not cancelled" and "Juventus draws, given that match 2 is not cancelled". Define the events $A =$"Barcelona wins", $B =$"Juventus draws", $H =$ "the match 1 is not cancelled", and $K =$ "the match 2 is not cancelled". In the bet on $A|H$, with $P(A|H) = x$, you pay r and you receive $rQ_1X_{AH} + rX_{\overline{H}}$, where $Q_1 = \frac{1}{x}$. Similarly, in the bet on $B|K$, with $P(B|K) = y$, you pay r and you receive $rQ_2X_{BK} + rX_{\overline{K}}$, where $Q_2 = \frac{1}{y}$. A *double-bet* on "Barcelona wins and Juventus draws" is a linked series of the two single bets, where the return from one bet is automatically staked on the other bet. More precisely, you pay r by receiving the return on one bet, for instance $r(Q_1X_{AH} + X_{\overline{H}})$, which then is staked on the other bet, by receiving the global return $r(Q_1X_{AH} + X_{\overline{H}})(Q_2X_{BK} + X_{\overline{K}}) = rQ_1Q_2X_{AHBK} + rQ_2X_{\overline{H}BK} + rQ_1X_{AH\overline{K}} + rX_{\overline{H}\overline{K}}$. Notice that, if a match is cancelled, the bet will revert to a single bet on the remaining match, and if both matches are cancelled the double-bet is void and your stake will be refunded. In the particular case where $r = \frac{1}{Q_1Q_2} = xy$, it holds that $rQ_1Q_2 = 1$, $rQ_2 = x$, and $rQ_1 = y$; then, the random amount you receive becomes

$$X_{AHBK} + xX_{\overline{H}BK} + yX_{AH\overline{K}} + xyX_{\overline{H}\overline{K}},$$

which, from (5), coincides with $X_{(A|H)\wedge(B|K)}$, where $z = xy$. Then, the double-bet is a bet on the conjoined conditional $(A|H) \wedge (B|K) =$ "(Barcelona wins, given that match 1 is not cancelled) and (Juventus draws, given that match 2 is not cancelled)", where the prevision of the conjunction coincides with $P(A|H)P(B|K)$.

Notice that, under logical independence of the events A, H, B, K, the assessment $P(A|H) = x$, $P(B|K) = y$ and $\mathbb{P}[X_{(A|H)\wedge(B|K)}] = xy$ is coherent. Indeed, $xy \in [z', z'']$, where $z' = \max\{x+y-1, 0\}$ and $z'' = \min\{x, y\}$ are the tightest lower and upper bounds for the coherent extensions $z = \mathbb{P}[X_{(A|H)\wedge(B|K)}]$ of the assessment (x, y) (Gilio and Sanfilippo 2014, Theorem 7). Likewise double-bets, multiple-bets can be formalized by exploiting the notion of conjunction of n conditional events.

2.3 Relation with the present approach

For notational convenience, in the rest of the paper we will use lower case letters for events involved in (compound) conditionals; hence, for instance, we will denote by $(a|b)$ a conditional event and by $(a|b) \wedge (c|d)$ the conjunction of $(a|b)$ and $(c|d)$. Moreover, in order to distinguish the logical aspects from the numerical ones, with each compound conditional t we will associate a suitable random quantity X_t, whose numerical values are the conditional previsions of some intermediate objects, called reducts. In particular, with the conjunction $(a|b) \wedge (c|d)$ we will associate a random quantity $X_{(a|b)\wedge(c|d)}$, whose numerical values are conditional previsions associated with the reducts determined by the following partition which corresponds to the one in (4): $\{abcd\,,\ \bar{a}b \vee \bar{c}d\,,\ \bar{b}cd\,,\ ab\bar{d}\,,\ \bar{c}\bar{d}\}$. As we will see more in general later, the reducts associated with the elements of the partition above are denoted by $1\,,\ 0\,,\ (a|b)\,,\ (c|d)\,,\ (a|b) \wedge (c|d)$, respectively. From them we obtain the respective numerical values $1, 0, x, y, z$ of $X_{(a|b)\wedge(c|d)}$, which are interpreted by the following conditional previsions:

$$1 = \mathbb{P}(1|\Omega)\,,\ 0 = \mathbb{P}(0|\Omega)\,,\ x = \mathbb{P}(X_{(a|b)}|b)\,,$$
$$y = \mathbb{P}(X_{(c|d)}|d),\ z = \mathbb{P}[X_{(a|b)\wedge(c|d)}|(c \vee d)]\,.$$

We observe that, based on (2), it could be verified that

$$x = \mathbb{P}(X_{(a|b)}|b) = P(a|b)\,,\ y = \mathbb{P}(X_{(c|d)}|d) = P(c|d),$$
$$z = \mathbb{P}[X_{(a|b)\wedge(c|d)}|(c \vee d)] = \mathbb{P}[X_{(a|b)\wedge(c|d)}].$$

Thus, $X_{(a|b)\wedge(c|d)}$ is nothing else than the conjunction between the conditional events $(a|b)$ and $(c|d)$ defined in (4). We remark that, by this approach, each compound conditional t is not explicitly defined, but we operate on it by means of the associated random quantity X_t. Notice that, given a probability assessment $(x_1, \ldots, x_n)$ on a family $\{(a_1|b_1), \ldots, (a_n|b_n)\}$, where $x_i = P(a_i|b_i)$, the possible values of the random vector $(X_{(a_1|b_1)}, \ldots, X_{(a_n|b_n)})$ coincide with the so-called points Q_h's. By these points a geometrical approach for checking coherence and for propagation can be developed (Gilio and Sanfilippo 2020; 2021b).

3 General compound conditionals as conditional random quantities

From now on we will be considering a fix *finite* Boolean algebra of ordinary events $\mathbf{A} = (A, \wedge, \vee, \neg, \bot, \top)$. For the sake of a lighter notation, we will also use ab for the conjunction $a \wedge b$ of events a and b, $\bar{a}$ for $\neg a$ of the event a, while we will keep denoting the disjunction of a and b by $a \vee b$.

In this setting, the set of the atoms $\mathrm{at}(\mathbf{A})$ of the algebra of events $\mathbf{A}$, can be identified with the set Ω of interpretations for $\mathbf{A}$, i.e. the set of Boolean algebra homomorphisms $w : \mathbf{A} \to \{0, 1\}$. Indeed, we will say that an event $a \in \mathbf{A}$ is *true* (resp. *false*) under an interpretation (or possible world) $w \in \Omega$ when $w(a) = 1$ (resp. $w(a) = 0$), also denoted as $w \models a$ (resp. $w \not\models a$).

As we anticipated in the previous section, conditional events of the form "if a then b", or "a given b", where a and b are events from $\mathbf{A}$ with b different from $\bot$, will be denoted by pairs $(a|b)$. We will also let $A|A' = \{(a|b) : a \in A, b \in A'\}$, where $A' = A\backslash\{\bot\}$, be the set of all conditionals that can be built from $\mathbf{A}$, that will be also called *basic conditionals*. *Compound conditionals*, then, are combinations of basic ones by operations of negation, conjunction and disjunction, that we will keep denoting them as $\neg$, $\wedge$ and $\vee$ without danger of confusion. Denote by $\mathbb{T}(A)$ the term algebra of type $(\wedge, \vee, \neg, \bot, \top)$ over the set $A|A'$, so that its support $\mathbb{T}(A)$ is the set of arbitrary terms generated from $A|A'$ (taken as variables) over the signature $(\neg, \wedge, \vee, \bot, \top)$. For instance, if $a, c, e \in A$ and $b, d, f \in A'$, then $(a|b) \wedge (c|d)$ or $(a|b) \vee ((c|d) \wedge \neg(e|f))$ are examples of compound conditionals from $\mathbb{T}(A)$. One of our ultimate aims will be to present a Boolean algebra obtained from $\mathbb{T}(A)$ that extends $\mathbf{A}$.

In the rest of the section we will further extend the random quantity-based approach to conditionals and propose an unambiguous procedure to interpret any compound conditional as a suitable random quantity.

For every $t \in \mathbb{T}(A)$, let us denote by $Cond(t) = \{(a_1|b_1), \ldots, (a_n|b_n)\}$ the set of basic conditionals appearing in t, and by $\mathbf{b}(t) = b_1 \vee \ldots \vee b_n$ the disjunction of the antecedents in $Cond(t)$.

Definition 3.1. Let $w \in \Omega$ be a classical interpretation and let $t \in \mathbb{T}(A)$ be a term. The *w-reduct* of t, denoted t^w, is the term in $\mathbb{T}(A)$, called the *w-reduct of* t, obtained from t by the following procedure:

(1) replace each $(a_i|b_i) \in Cond(t)$ by 1 if $w \models a_ib_i$, and by 0 if $w \models \bar{a}_ib_i$,

(2) apply the following reduction rules to subterms of t until no further reduction is possible:
$\neg 1 := 0,\quad \neg 0 := 1,$
$r \wedge 1 = 1 \wedge r := r,\quad r \wedge 0 = 0 \wedge r := 0,$ where the
$r \vee 1 = 1 \vee r := 1,\quad r \vee 0 = 0 \vee r := r,$
r denotes a subterm of t.

This symbolic reduction procedure has some interesting properties. First of all, $w \models \neg\mathbf{b}(t)$, then no reduction is possible and hence $t^w = t$. Second, the reduction commutes with the operation symbols, in the following sense:

Lemma 3.2. *For every terms $t \in \mathbb{T}(A)$ and for every $w \in \Omega$, the following hold: (i) $(\neg t)^w = \neg t^w$; (ii) $(t \wedge s)^w = t^w \wedge s^w$; (iii) $(t \vee s)^w = t^w \vee s^w$.*

We will denote by $Red(t) = \{t^w | w \in \Omega$ the set of w-reducts of t, and by $Red^0(t) = Red(t)\backslash\{t\}$, the set of its *proper w-reducts*.

Example 3.3. Let $t = (a|b) \wedge ((c|d) \vee \neg(e|f))$ and let w such that $w(a) = 1, w(b) = 0, w(c) = 0, w(d) = 0, w(e) = 1, w(f) = 1$, i.e. $w \models a\bar{b}\bar{c}\bar{d}ef$. Then

$$t^w = (a|b) \wedge ((c|d) \vee \neg 1) = (a|b) \wedge ((c|d) \vee 0) = (a|b) \wedge (c|d).$$

Let w' such that $w' \models abc\bar{d}ef$. Then

$$t^{w'} = 1 \wedge ((c|d) \vee \neg 1) = (c|d) \vee 0 = (c|d).$$

Further, if w'' is such that $w'' \models abcdef$, then $t^{w''} = 1 \wedge (1 \vee \neg 1) = 1 \wedge (1 \vee 0) = 1$. In fact, one can check that $Red^0(t) = \{1, 0, (a|b), (c|d), \neg(e|f), (a|b) \wedge (c|d), (a|b) \wedge \neg(e|f), (c|d) \vee \neg(e|f)\}$.

Now, we recall the notion of conditional prevision of a random quantity.

Definition 3.4. Let $X : \Omega \to [0,1]$ be a random quantity, and let $b \in A$ be an event. Then, given a conditional probability $P : A \times A' \to [0,1]$, the *conditional prevision of X given b* is defined as:

$$\mathbb{P}(X|b) = \sum_{w\in\Omega} X(w) \cdot P(w|b) = \sum_{w\in\Omega: w\models b} X(w) \cdot P(w|b).$$

The next definition presents a suitable way to associate a random quantity to every compound conditional $t \in \mathbb{T}(A)$.

Definition 3.5. Let $\mathbf{A}$ be a finite Boolean algebra and $P : A \times A' \to [0,1]$ a conditional probability on $\mathbf{A}$. For every term t in $\mathbb{T}(A)$, we define the random quantity $X_t : \Omega \to [0,1]$ as follows: for every $w \in \Omega$,

$$X_t(w) := \mathbb{P}(X_{t^w}|\mathbf{b}(t^w)).$$

If $t^w = 1$ or $t^w = 0$, we take $\mathbf{b}(t^w) = \top$, and hence $X_1 = 1$ or $X_0 = 0$ respectively. Thus, $X_1(w) = 1$ or $X_0(w) = 0$.

Regarding this definition, some observations are in order here:
(i) As we have observed before, if $w \models \bar{\mathbf{b}}(t)$ then $t^w = t$, and thus $X_t(w) = \mathbb{P}(X_t|\mathbf{b}(t))$, and hence $\mathbb{P}(X_t|\bar{\mathbf{b}}(t)) = \mathbb{P}(X_t|\mathbf{b}(t))$.
(ii) The above definition of X_t strongly depends on the assumed conditional probability P of $A \times A'$. Actually, once we fix the initial conditional probability P on $A \times A'$, all random quantities X_t are fully determined. Indeed, the above Definition 3.5 is in fact a recursive definition, since

$$\mathbb{P}(X_{t^w}|\mathbf{b}(t^w)) = \sum_{w'} X_{t^w}(w') \cdot P(w'|\mathbf{b}(t^w)),$$

and in turn, $X_{t^w}(w') = \mathbb{P}(X_{(t^w)^{w'}}|\mathbf{b}((t^w)^{w'}))$, and so on, until reaching basic conditionals.
(iii) As a consequence, in general different conditional probabilities P and P' on $\mathbf{A}$ will define different random quantities X_t for the same term t.
(iv) In the case t is of the form $t = (a|\top)$, the associated random quantity X_t coincides with the indicator function of a, that is, $X_t(w) = 1$ whenever $w \models a$, and $X_t(w) = 0$ otherwise. This shows that $(a|\top)$ can indeed be indentified with the plain event a. In this case, for the sake of a lighter notation, we will write X_a instead of $X_{(a|\top)}$.

Notation 1. From now on, for simplicity, for any $t \in \mathbb{T}(A)$, we will write $\mathbb{P}^c(X_t)$ for $\mathbb{P}(X_t|\mathbf{b}(t))$.

Displayed in another way, the random quantity X_t can be specified as follows: let $Red^0(t) = \{t^w | w \in \Omega\} = \{t_1, t_2, ..., t_k\}$ and let $\{E_1, E_2, ..., E_k\}$ be the corresponding subsets of interpretations leading to a same element of $Red^0(t)$, then

$$X_t(w) = \mathbb{P}(X_{t^w}|\mathbf{b}(t^w)) = $$
$$= \begin{cases} \mathbb{P}^c(X_{t_1}), & \text{if } w \models E_1 \\ \ldots, & \ldots \\ \mathbb{P}^c(X_{t_k}), & \text{if } w \models E_k \\ \hdashline \mathbb{P}^c(X_t), & \text{if } w \models \neg(E_1 \vee \ldots \vee E_k) \end{cases}$$

where the dashed line separates those cases where the interpretation w belongs to $\mathbf{b}(t)$ from those which do not.

In this setting, it is clear that X_t is in fact the following linear combination of the indicator functions of the events defining an associated partition:

$$X_t = \mathbb{P}^c(X_{t_1})X_{E_1} + \ldots + \mathbb{P}^c(X_{t_k})X_{E_k} + \mathbb{P}^c(X_t)X_{E_{k+1}}$$

where $E_{k+1} = \bar{E}_1 \wedge \ldots \wedge \bar{E}_k = \bar{\mathbf{b}}(t)$, and hence

$$\mathbb{P}^c(X_t) = \mathbb{P}^c(X_{t_1}) \cdot P(E_1|\mathbf{b}(t)) + \ldots + \mathbb{P}^c(X_{t_k}) \cdot P(E_k|\mathbf{b}(t)).$$

We remark that, from (i) above and from (1), it holds that

$$\begin{aligned} \mathbb{P}(X_t) &= \mathbb{P}(X_t \cdot X_{\mathbf{b}(t)}) + \mathbb{P}(X_t \cdot X_{\bar{\mathbf{b}}(t)}) = \\ &= \mathbb{P}(X_t|\mathbf{b}(t))P(\mathbf{b}(t)) + \mathbb{P}(X_t|\bar{\mathbf{b}}(t))P(\bar{\mathbf{b}}(t)) = \\ &= \mathbb{P}^c(X_t)P(\mathbf{b}(t)) + \mathbb{P}^c(X_t)P(\bar{\mathbf{b}}(t)) = \mathbb{P}^c(X_t). \end{aligned} \tag{7}$$

In other words, formula (7) shows that the prevision of X_t coincides with the conditional prevision $\mathbb{P}^c(X_t)$.

We end this section by exemplifying the above definition of X_t for selected known cases of compound conditionals t that will be helpful in next sections.

Example 3.6. Let $t = (a|b)$. Then, by applying the above definition, we get

$$t^w = \begin{cases} 1, & \text{if } w \models ab \\ 0, & \text{if } w \models \bar{a}b \\ (a|b), & \text{if } w \models \bar{b} \end{cases} \qquad \mathbf{b}(t^w) = \begin{cases} \top, & \text{if } w \models ab \\ \top, & \text{if } w \models \bar{a}b \\ b, & \text{if } w \models \bar{b} \end{cases}$$

and thus we have:

$$X_{(a|b)}(w) = \mathbb{P}(X_{t^w}|\mathbf{b}(t^w)) =$$
$$= \begin{cases} \mathbb{P}(X_1|\top) = 1, & \text{if } w \models ab \\ \mathbb{P}(X_0|\top) = 0, & \text{if } w \models \bar{a}b \\ \mathbb{P}(X_{(a|b)}|b), & \text{if } w \models \bar{b} \end{cases}.$$

Now, since $P(w|b) = 0$ whenever $w \models \bar{b}$, we have

$$\begin{aligned} &\mathbb{P}(X_{(a|b)}|b) = \\ &= 1 \cdot P(ab|b) + 0 \cdot P(\bar{a}b|b) + \mathbb{P}(X_{(a|b)}|b) \cdot 0 = \\ &= P(ab|b) = P(a|b). \end{aligned}$$

Therefore we get the following well-known three-valued representation of a conditional $(a|b)$:

$$X_{(a|b)}(w) = \begin{cases} 1, & \text{if } w \models ab \\ 0, & \text{if } w \models \bar{a}b \\ P(a|b), & \text{if } w \models \bar{b} \end{cases}.$$

Equivalently, in agreement with (3), $X_{(a|b)}$ can be expressed as the following linear combination of the indicator functions of the events ab and $\bar{b}$:

$$X_{(a|b)} = 1X_{ab} + 0X_{\bar{a}b} + P(a|b)X_{\bar{b}} = X_{ab} + P(a|b)X_{\bar{b}}.$$

Example 3.7. Now let $t = \neg(a|b)$, the negation of $(a|b)$. Then, by applying the above definition, we get:

$$t^w = \begin{cases} \neg 1 := 0, & \text{if } w \models ab \\ \neg 0 := 1, & \text{if } w \models \bar{a}b \\ \neg(a|b), & \text{if } w \models \bar{b} \end{cases},$$

$$\mathbf{b}(t^w) = \begin{cases} \top, & \text{if } w \models ab \\ \top, & \text{if } w \models \bar{a}b \\ b, & \text{if } w \models \bar{b} \end{cases},$$

and thus we have: $X_{\neg(a|b)}(w) =$

$$= \mathbb{P}(X_{t^w}|\mathbf{b}(t^w)) = \begin{cases} 0, & \text{if } w \models ab \\ 1, & \text{if } w \models \bar{a}b \\ \mathbb{P}(X_{\neg(a|b)}|b), & \text{if } w \models \bar{b} \end{cases}.$$

Now, since $P(w|b) = 0$ whenever $w \models \neg b$, we have

$$\begin{aligned} &\mathbb{P}(X_{\neg(a|b)}|b) = \\ &= 0 \cdot P(ab|b) + 1 \cdot P(\bar{a}b|b) + \mathbb{P}(X_{\neg(a|b)}|b) \cdot 0 \\ &= P(\bar{a}b|b) = P(\bar{a}|b) = 1 - P(a|b). \end{aligned}$$

Therefore the final expression for $X_{\neg(a|b)}$ is

$$X_{\neg(a|b)}(w) = \begin{cases} 0, & \text{if } w \models ab \\ 1, & \text{if } w \models \bar{a}b \\ 1 - P(a|b), & \text{if } w \models \bar{b} \end{cases}.$$

That is to say, $X_{\neg(a|b)} = 1 - X_{(a|b)} = X_{(\bar{a}|b)}$.

Example 3.8. Let us examine again the case of a conjunction of two conditionals $t = (a|b) \wedge (c|d)$ from the current perspective. Here we have $\mathbf{b}(t) = b \vee d$, and

$$X_t(w) = \begin{cases} 1, & \text{if } w \models abcd \\ 0, & \text{if } w \models (\bar{a}b) \vee (\bar{c}d) \\ \mathbb{P}^c(a|b), & \text{if } w \models \bar{b}cd \\ \mathbb{P}^c(c|d), & \text{if } w \models ab\bar{d} \\ \hdashline \mathbb{P}^c((a|b) \wedge (c|d)), & \text{if } w \models \bar{b}\bar{d} \end{cases}$$

Now, we know that $\mathbb{P}^c(a|b) = P(a|b)$ and $\mathbb{P}^c(c|d) = P(c|d)$. Then, using the above definition we get:

$$\begin{aligned} &\mathbb{P}^c((a|b) \wedge (c|d)) = \mathbb{P}(X_{(a|b) \wedge (c|d)}|b \vee d) \\ &\quad = P(abcd|b \vee d) + P(a|b) \cdot P(\bar{b}cd|b \vee d) + \\ &\qquad + P(c|d) \cdot P(ab\bar{d}|b \vee d), \end{aligned}$$

that coincides, when $P(b \vee d) > 0$, with the formula given in (McGee 1989; Kaufmann 2009).

Let us now consider two particular cases:

- First, consider the case $a \leqslant b = c \leqslant d$. Then $t = (a|b) \wedge (b|d)$, $\mathbf{b}(t) = b \vee d = d$, and moreover:
  - $abcd = abd = ad = a$
  - $(\bar{a}b) \vee (\bar{c}d) = (\bar{a}b) \vee (\bar{b}d) = (\bar{a}bd) \vee (\bar{a}\bar{b}d) = \bar{a}d$
  - $\bar{b}cd = \bar{b}b = \perp$
  - $ab\bar{d} = a\bar{d} = \perp$

  Hence, the above general definition reduces to:

  $$X_{(a|b) \wedge (b|d)}(w) = \begin{cases} 1, & \text{if } w \models a \\ 0, & \text{if } w \models \bar{a}d \\ \mathbb{P}^c((a|b) \wedge (b|d)), & \text{if } w \models \bar{d} \end{cases}$$

  where $\mathbb{P}^c((a|b) \wedge (b|d)) = 1 \cdot P(a|d) = P(a|d) = \mathbb{P}^c((a|d))$ and thus $X_{(a|b) \wedge (b|d)}(w) = X_{(a|d)}(w)$ for all w. Thus, from the numerical point of view, the compound conditional $(a|b) \wedge (b|d)$ behaves as the basic conditional $(a|d)$. See also (Gilio and Sanfilippo 2020, formulas (55) and (56)) where it is also observed that $P(a|d) = P(a|b)P(b|d)$ (*compound probability theorem*).

- Consider now the case $b = d$. Then $t = (a|b) \wedge (c|b)$ and $\mathbf{b}(t) = b$. In this case the above general expression reduces to the following one:

  $$X_t(w) = \begin{cases} 1, & \text{if } w \models abc \\ 0, & \text{if } w \models (\bar{a} \vee \bar{c})b = \overline{ac}b \\ \mathbb{P}^c((a|b) \wedge (c|b)), & \text{if } w \models \bar{b}, \end{cases}$$

  where $\mathbb{P}^c((a|b) \wedge (c|b)) = 1 \cdot P(abc|b) = P(ac|b) = \mathbb{P}^c((ac|b))$, and therefore it is clear that $X_{(a|b) \wedge (c|b)}(w) = X_{(a \wedge c|b)}(w)$ for all w. Thus, in this case, we see that the compound conditional $(a|b) \wedge (c|b)$ actually behaves like the basic conditional $(a \wedge c|b)$.

Example 3.9. Finally, let us consider a more complex compound conditional $t = (a|b) \wedge ((c|d) \vee \neg(e|f))$. Again, by the above definition, we get its associated random quantity:

$$X_t(w) = \begin{cases} 1, & \text{if } w \models ab(cd \vee \bar{e}b) \\ 0, & \text{if } w \models \bar{a}b \vee \bar{c}def \\ \mathbb{P}^c((a|b)), & \text{if } w \models \bar{b}(cd \vee \bar{e}f) \\ \mathbb{P}^c((a|b) \wedge (c|d)), & \text{if } w \models \bar{b}\bar{d}ef \\ \mathbb{P}^c((a|b) \wedge \neg(e|f)), & \text{if } w \models \bar{b}\bar{c}d\bar{f} \\ \mathbb{P}^c((c|d)), & \text{if } w \models ab\bar{d}ef \\ \mathbb{P}^c(\neg(e|f)), & \text{if } w \models ab\bar{c}d \\ \mathbb{P}^c((c|d) \vee \neg(e|f)), & \text{if } w \models ab\bar{d}\bar{f} \\ \hdashline y, & \text{if } w \models \bar{b}\bar{d}\bar{f} \end{cases}$$

where $y = \mathbb{P}^c((a|b) \wedge ((c|d) \vee \neg(e|f)))$, and for simplicity, we have written $\mathbb{P}^c(s)$ for $\mathbb{P}^c(X_s)$, i.e. for $\mathbb{P}(X_s|\mathbf{b}(s))$.

We end this section with the following remark on a betting interpretation for compound conditionals $t \in \mathbb{T}(A)$. A bet between a gambler and a bookmaker on X_t is specified as follows:

- The gambler pays to the bookmaker: $\mathbb{P}^c(X_t)$
- In a situation $w \in \Omega$, the gambler receives from the bookmaker: $X_t(w) = \mathbb{P}^c(X_{t^w})$

Note that if $w \models \neg\mathbf{b}$ then $t^w = t$, and thus $\mathbb{P}^c(X_{t^w}) = \mathbb{P}^c(X_t)$, i.e. what the gambler receives is what he has payed, i.e. it represents the situation in which the bet is called off. We observe that coherence requires that the prevision of the random gain must be equal to zero. Indeed, given t, X_t is specified as:

$$X_t(w) = \begin{cases} \mathbb{P}^c(X_{E_1}), & \text{if } w \models E_1 \\ \dots & \dots \\ \mathbb{P}^c(X_{E_k}), & \text{if } w \models E_k \\ \hdashline \mathbb{P}^c(X_t), & \text{if } w \models \neg(E_1 \vee \ldots \vee E_k) \end{cases}.$$

In this setting, the balance of the betting for the bookmaker in a given situation w such that $w \models E_i$ is what he receives, $\mathbb{P}(X_t)$, minus what he pays to the gambler, X_t, that is:

$$G(w) = \mathbb{P}(X_t) - X_t(w) = \mathbb{P}^c(X_t) - \mathbb{P}^c(X_{E_i}).$$

Moreover, when $w \models \bar{\mathbf{b}}(t)$, it holds that $G(w) = \mathbb{P}^c(X_t) - \mathbb{P}^c(X_t) = 0$. Then, the prevision of the random

quantity G of the balance will be:
$\mathbb{P}(G) = \sum_{w\in\Omega} G(w)P(w|\mathbf{b}(t)) = \sum_{i=1}^{k}(\mathbb{P}^c(X_t) - \mathbb{P}^c(X_{E_i}))P(E_i|\mathbf{b}(t)) = \mathbb{P}^c(X_t) - \mathbb{P}^c(X_t) = 0.$

Remark 3.10. Example 2.1 can be generalized to compound conditionals. For instance, we can construct a bet on the disjunction $t = (a|b) \vee (c|d)$, or on the compound $t = (a|b) \vee (c|d) \wedge (e|f)$, and so on.

4 Properties of compound conditionals

In this section we show some key properties of the random quantities associated to certain forms of compound quantities that will allow us to show they can be endowed somehow with a Boolean algebraic structure

We start by showing that if two compound conditional terms t and t' are such that $\mathbf{b}(t) \equiv \mathbf{b}(t')$, to check whether they have the same associated random quantity, it is enough to check whether the random quantities take the same value over those worlds satisfying the disjunction $\mathbf{b}(t)$. A related result is given in (Gilio and Sanfilippo 2014, Theorem 4).

Lemma 4.1. *Let $t, t' \in \mathbb{T}(A)$ such that $\mathbf{b}(t) \equiv \mathbf{b}(t')$ and $X_t(w) = X_{t'}(w)$ for each $w \in \Omega$ such that $w \models \mathbf{b}(t)$. Then $X_t = X_{t'}$.*

Proof. It is enough to show that $X_t(w) = X_{t'}(w)$ when $w \models \neg\mathbf{b}(t)$. Due to the fact that if $w \models \neg\mathbf{b}(t)$ then $X_t(w) = \mathbb{P}(X_t|\mathbf{b}(t))$, this is equivalent to show that $\mathbb{P}(X_t|\mathbf{b}(t)) = \mathbb{P}(X_{t'}|\mathbf{b}(t))$. But these previsions only depend on the values of X_t and $X_{t'}$ on w's such that $w \models \mathbf{b}(t)$, and by hypothesis they coincide. □

Clearly, if $\mathbf{b}(t) \equiv \mathbf{b}(t')$ and $t^w = t'^w$ then $X_t(w) = \mathbb{P}(X_{t^w}|\mathbf{b}(t)) = \mathbb{P}(X_{t'^w}|\mathbf{b}(t)) = X_{t'}(w)$. Hence we have the following corollary.

Corollary 4.2. *If $\mathbf{b}(t) = \mathbf{b}(t')$ and $t^w = t'^w$ for each $w \in \Omega$ such that $w \models \mathbf{b}(t)$, then $X_t = X_{t'}$.*

Proposition 4.3. *Let $t \in \mathbb{T}(A)$ and let s be a subterm of t. Further let $s' \in \mathbb{T}(A)$ such that $\mathbf{b}(s) = \mathbf{b}(s')$ and $X_{s^w} = X_{s'^w}$ for all $w \models \mathbf{b}(s)$, and let t' be the term obtained by uniformly replacing s by s' in t. Then $X_t = X_{t'}$.*

Proof. It follows from the recursive definition of X_t and that, for any $w \in \Omega$, t^w can be computed from the w-reducts of its subterms (Lemma 3.2). □

Next we show that the random quantities for compound conditionals satisfy properties very familiar from a Boolean algebraic perspective.

Proposition 4.4. *For every $t, s, r \in \mathbb{T}(A)$ the following conditions hold:*

1. $X_t = X_{t\wedge t}$; 2. $X_{t\wedge s} = X_{s\wedge t}$;
3. $X_{t\wedge(s\wedge r)} = X_{(t\wedge s)\wedge r}$; 4. $X_{t\wedge\neg t} = 0$;
5. $X_{\neg(t\wedge s)} = X_{\neg t\vee\neg s}$; 6. $X_{t\wedge(s\vee r)} = X_{(t\wedge s)\vee(t\wedge r)}$;
7. $X_{t\vee s} = X_t + X_s - X_{t\wedge s}$;
8. $X_{\neg t} = 1 - X_t$ *and hence* $X_{\neg\neg t} = X_t$;
9. *If* $a \leqslant b$, *then* $X_{(a|b)\wedge(a|b\vee c)} = X_{a|b\vee c}$

The proof of Proposition 4.4 is not included due to lack of space, but we notice that items (1)-(8) are proved in a similar way by induction on the complexity of the terms involved, and where the base cases are those only involving basic conditionals. Many of these base cases are proved in previous examples.

Now we show that compound conditionals actually form a Boolean algebra. In order to do it we first properly arrange the compound conditionals in equivalence classes each of which contains terms from $\mathbb{T}(A)$ that provides the same random quantity under *any* given conditional probability.

More precisely, recalling observation (ii) after Def. 3.5, let us define the binary relation $\equiv$ on $\mathbb{T}(A)$ as follows: for all $t, s \in \mathbb{T}(A)$, $t \equiv s$ iff $X_t = X_s$ under any conditional probability P on $A \times A'$.

It is immediate to check that $\equiv$ is an equivalence relation. Thus, let $\mathbb{T}(A)$ to be the quotient $\mathbb{T}(A)/\equiv$. If we denote by $[t]$ the equivalence class of a generic term $t \in \mathbb{T}(A)$ under $\equiv$, define operations on $\mathbb{T}(A)$ as follows: for all $[t], [s] \in \mathbb{T}(A)$, $[t]\wedge^*[s] = [s\wedge t]$, $[t]\vee^*[s] = [s\vee t]$, $\neg^*[t] = [\neg t]$, $0 = [(\bot|\top)]$, $1 = [(\top|\top)]$.

By Proposition 4.3, the above operations are well defined. Moreover, the following holds:

Theorem 4.5. *For every finite Boolean algebra $\mathbf{A}$, the structure $\mathcal{T}(\mathbf{A}) = (\mathbb{T}(A)/\equiv, \wedge^*, \vee^*, \neg^*, 0, 1)$ is a Boolean algebra.*

Proof. By Proposition 4.4, the operations $\wedge^*, \vee^*, \neg^*$ and constants $0, 1$ satisfy all the required equations for $\mathcal{T}(\mathbf{A})$ being a Boolean algebra. □

Remark 4.6. Notice that, given $[t]$ and $[s]$ in $\mathcal{T}(\mathbf{A})$, $[t] \leqslant [s]$ in the lattice order of $\mathcal{T}(\mathbf{A})$ iff $[t] \wedge^* [s] = [t]$ iff $[t \wedge s] = [t]$ iff $X_{t\wedge s} = X_t$.

Next proposition shows that well-known properties of conditionals also hold in the setting of the present paper.

Proposition 4.7. *The following properties hold in $\mathcal{T}(\mathbf{A})$:*

(i) $[(a|a)] = 1$, *(ii)* $[(a|b) \wedge (c|b)] = [(a \wedge c|b)]$,
(iii) $[\neg(a|b)] = [(\bar{a}|b)]$, *(iv)* $[(a \wedge b|b)] = [(a|b)]$,
(v) $[(a|b) \wedge (b|c)] = [(a|c)]$, *if* $a \leqslant b \leqslant c$,
(vi) *if* $a \leqslant b$, *then* $[(a|b \vee c)] \leqslant [(a|b)]$ *for all* c.

Proof. For each one of the equalities above, of the form $[t] = [s]$, we proved in previous examples, that $X_t = X_s$. As for the last claim, apply Proposition 4.4 (9). Then the claim follows by the definition of $\mathcal{T}(\mathbf{A})$. □

5 Probability of compound conditionals

Since $\mathcal{T}(\mathbf{A})$ is a Boolean algebra, we are allowed to define probabilities on it. In this section we will show that the previsions $\mathbb{P}(X_t)$'s of the random quantities X_t's determine in fact a probability on $\mathcal{T}(\mathbf{A})$.

Definition 5.1. Given a conditional probability $P : A \times A' \to [0, 1]$, we define the mapping $P^* : \mathbb{T}(A) \to$

$[0,1]$ as follows: for every $t \in \mathbb{T}(A)$, $P^*(t) =_{def} \mathbb{P}(X_t)$. In other words, based on (7)

$$P^*(t) = \mathbb{P}(X_t) = \mathbb{P}^c(X_t) = \mathbb{P}(X_t|\mathbf{b}(t)) = \\ = \sum_w X_t(w) \cdot P(w|\mathbf{b}(t)) = \sum_w \mathbb{P}^c(X_{t^w}) \cdot P(w|\mathbf{b}(t)).$$

From the above definition, it is cleat that if t and t' are terms such $t \equiv t'$, then $P^*(t) = P^*(t')$.

Proposition 5.2. *Any given conditional probability* $P : A \times A' \to [0,1]$ *fully determines the probabilities* $P^*(t)$ *for all compound conditional* $t \in \mathbb{T}(A)$.

Proof. Let $t \in \mathbb{T}(A)$ such that $Cond(t) = \{(a_1|b_1), \ldots, (a_n|b_n)\}$. We have seen that $P^*((a_i|b_i)) = P(a_i|b_i)$. Suppose that for every $s \in Red^0(t)$, there is a family of conditionals $C_s \subseteq A \times A'$ such that $P^*(s)$ is of the form $P^*(s) = f_s(\{P(c_i|d_i) : (c_i|d_i) \in C_s\})$ for some function f_s. Then, by definition, we have

$$P^*(t) = \mathbb{P}(X_t) = \sum_{w \in W} \mathbb{P}^c(X_{t^w}) \cdot P(w|\mathbf{b}(t)) = \\ = \sum_{w \in W} P^*(t^w) \cdot P(w|\mathbf{b}(t)) = \\ = \sum_{w \in W} f_{t^w}(\{P(c|d) : (c|d) \in C_{t^w}\}) \cdot P(w|\mathbf{b}(t)),$$

the latter expression clearly only depending on P. □

Moreover, the following is a direct consequence of Proposition 4.4.

Corollary 5.3. *For every* $t, s, r \in \mathbb{T}(A)$ *the following conditions hold:*

1. $P^*(t) = P^*(t \wedge t)$
2. $P^*(t \wedge s) = P^*(s \wedge t)$
3. $P^*(t \wedge (s \wedge r)) = P^*((t \wedge s) \wedge r)$
4. $P^*(t \wedge \neg t) = 0$
5. $P^*(\neg(t \wedge s)) = P^*(\neg t \vee \neg s)$
6. $P^*(\neg t) = 1 - P^*(t)$
7. $P^*(t) = 1$ *if* $X_t = 1$
8. $P^*(t) = 0$ *if* $X_t = 0$
9. $P^*(t \vee s) = P^*(t) + P^*(s) - P^*(t \wedge s)$
10. $P^*(t \wedge (s \vee r)) = P^*((t \wedge s) \vee (t \wedge r))$.

Notice that the items 6,7,8, and 9 of the above corollary together with the above remarked fact that $t \equiv t'$ implies $P^*(t) = P^*(t')$, show that P^* naturally induces a probability on the algebra $\mathcal{T}(\mathbf{A})$ that we will still denote by the same symbol.

Corollary 5.4. *For every conditional probability* P *on* $A \times A'$ *the map* P^* *is a probability measure on* $\mathcal{T}(\mathbf{A})$.

Furthermore, as we have checked in Example 3.6, for every basic conditional $(a|b)$ we have $P^*((a|b)) = P(a|b)$, thus complying with *the Equation* (Edgington 1995), or *Stalnaker's hypothesis*, see, e.g., (Douven and Dietz 2011; Sanfilippo et al. 2020). The next example shows how to compute the probability P^* of a (more complex) compound conditional.

Example 5.5. Continuing the above Example 3.9 for $t = (a|b) \wedge ((c|d) \vee \neg(e|f))$, we have $\mathbf{b}(t) = b \vee d \vee f$ and using the above Definition 5.1 we get:

$$P^*(t) = \mathbb{P}(X_t) = \mathbb{P}(X_t|\mathbf{b}(t)) = P(abcd \vee \bar{e}f)|b \vee d \vee f) + \\ + P(a|b)P(\bar{b}(cd \vee \bar{e}f)|b \vee d \vee f) + \\ + P^*((a|b) \wedge (c|d))P(\bar{b}\bar{d}ef|b \vee d \vee f) + \\ + P^*((a|b) \wedge \neg(e|f))P(\bar{b}\bar{c}d\bar{f}|b \vee d \vee f) + \\ + P(c|d)P(ab\bar{d}ef|b \vee d \vee f) + \\ + (1 - P(e|f))P(ab\bar{c}d|b \vee d \vee f) + \\ + P^*((c|d) \vee \neg(e|f))P(abd\bar{f}|b \vee d \vee f).$$

6 Boolean algebras of random variables and Boolean algebras of conditionals

The above Theorem 4.5, tells us that the random quantities that represent conditionals form a Boolean algebra. Actually, in (Flaminio, Godo, and Hosni 2020), the authors presented a way to construct Boolean algebras of conditionals that also represent conditional statements, although following a different line of thoughts.

For the sake of clarity, let us briefly recall how a Boolean algebras of conditional $\mathcal{C}(\mathbf{A})$ is defined for every Boolean algebra $\mathbf{A}$. Consider the free Boolean algebra Free$(A|A)$ generated by the set $A|A$ of all pairs $(a, b) \in A \times A$ such that $b \neq \bot$, in the language $\wedge, \vee, \neg, \top^*$. Every pair $(a, b) \in A|A$ will be henceforth denoted by $(a|b)$. Then, one considers the following set of basic requirements the algebra $\mathcal{C}(\mathbf{A})$ should satisfy:

(R1) For all $b \in A'$, the conditional $(b|b)$ will be the top element of $\mathcal{C}(\mathbf{A})$ and $(\neg b|b)$ will be the bottom;

(R2) Given $b \in A'$, the set of conditionals $A|b = \{(a|b) : a \in A\}$ will be the domain of a Boolean subalgebra of $\mathcal{C}(\mathbf{A})$, and in particular when $b = \top$, this subalgebra will be isomorphic to $\mathbf{A}$;

(R3) In a conditional $(a|b)$ we can replace the consequent a by $a \wedge b$, that is, the conditionals $(a|b)$ and $(a \wedge b|b)$ represent the same element of $\mathcal{C}(\mathbf{A})$;

(R4) For all $a \in A$ and all $b, c \in A'$, if $a \leqslant b \leqslant c$, then the result of conjunctively combining the conditionals $(a|b)$ and $(b|c)$ must yield the conditional $(a|c)$.

Notice that (R4) encodes a sort of restricted chaining of conditionals and it is inspired by the chain rule of conditional probabilities: $P(a|b) \cdot P(b|c) = P(a|c)$ whenever $a \leqslant b \leqslant c$.

One then proceeds by considering the smallest congruence relation $\equiv_{\mathfrak{C}}$ on Free$(A|A)$ satisfying:

(C1) $(b|b) \equiv_{\mathfrak{C}} \top^*$, for all $b \in A'$;

(C2) $(a_1|b) \wedge (a_2|b) \equiv_{\mathfrak{C}} (a_1 \wedge a_2|b)$, for all $a_1, a_2 \in A$, $b \in A'$;

(C3) $\neg(a|b) \equiv_{\mathfrak{C}} (\bar{a}|b)$, for all $a \in A$, $b \in A'$;

(C4) $(a \wedge b|b) \equiv_{\mathfrak{C}} (a|b)$, for all $a \in A$, $b \in A'$;

(C5) $(a|b) \wedge (b|c) \equiv_{\mathfrak{C}} (a|c)$, for all $a \in A$, $b, c \in A'$ such that $a \leqslant b \leqslant c$.

Note that (C1)-(C5) faithfully account for the requirements R1-R4 where, in particular, (C2) and (C3) account for R2. Finally, the algebra $\mathcal{C}(\mathbf{A})$ is formally defined as follows.

Definition 6.1. For every Boolean algebra $\mathbf{A}$, the *Boolean algebra of conditionals* of $\mathbf{A}$ is the quotient structure $\mathcal{C}(\mathbf{A}) = \text{Free}(A|A)/_{\equiv_{\mathfrak{C}}}$.

By definition, the algebra $\mathcal{C}(\mathbf{A})$ is finite whenever so is $\mathbf{A}$. In particular, if $\mathbf{A}$ is atomic with atoms $\alpha_1, \ldots, \alpha_n$, in (Flaminio, Godo, and Hosni 2020) it is shown that the atoms of $\mathcal{C}(\mathbf{A})$ are all in the following form: let $\langle \beta_1, \ldots, \beta_{n-1} \rangle$ be a sequence of pairwise different atoms of $\mathbf{A}$. Then, $(\beta_1|\top) \wedge (\beta_2|\overline{\beta_1}) \wedge (\beta_{n-1}|\overline{\beta_1} \wedge$

$\ldots \wedge \overline{\beta_{n-2}})$ is an atom of $\mathcal{C}(\mathbf{A})$ and indeed all atoms of $\mathcal{C}(\mathbf{A})$ have that form for some sequence of atoms of $\mathbf{A}$ of length $n-1$. For any such sequence $\langle \alpha \rangle$, we will write $\omega_{\langle \alpha \rangle}$ to denote its associated atom of $\mathcal{C}(\mathbf{A})$.

For every sequence $\langle \alpha \rangle$, $\omega_{\langle \alpha \rangle}$ clearly belongs to $\mathbb{T}(\mathbf{A})$. Thus it makes sense to look at these terms inside $\mathcal{T}(\mathbf{A})$.

Theorem 6.2. *For every finite Boolean algebra $\mathbf{A}$ with atoms $\alpha_1, \ldots, \alpha_n$, the set $\mathrm{at}(\mathcal{T}(\mathbf{A}))$ of atoms of $\mathcal{T}(\mathbf{A})$ coincides with the set $\{[\omega_{\langle \alpha \rangle}] \in \mathcal{T}(\mathbf{A}) : \omega_{\langle \alpha \rangle}$ is an atom of $\mathcal{C}(\mathbf{A})\}$.*

Proof. Direct inspection on the proofs of Proposition 4.3 and Theorem 4.4 from (Flaminio, Godo, and Hosni 2020) where the authors proved that $\mathrm{at}(\mathcal{C}(\mathbf{A}))$ is the set of atoms of $\mathcal{C}(\mathbf{A})$, shows that the unique properties of $\mathcal{C}(\mathbf{A})$ needed in their proofs are the basic properties of Boolean algebras (in particular commutativity and associativity of $\wedge$ and distributivity), plus the following:

- If $a \leqslant b$, then for all c, $[(a|b \vee c)] \leqslant [(a|b)]$ (see part (ii) in the proof of (Flaminio, Godo, and Hosni 2020, Theorem 4.4));
- for all $a \neq \bot$, $[(a|a)] = 1$ and $[(\bot|a)] = 0$ (see part (b) in the proof of (Flaminio, Godo, and Hosni 2020, Proposition 4.3));
- $[(a|b)] = \bigvee_{\alpha \in \mathrm{at}(\mathbf{A}) : \alpha \leqslant a} [(\alpha|b)]$; (see (a) in the proof of (Flaminio, Godo, and Hosni 2020, Proposition 4.3)).

All these properties hold in $\mathcal{T}(\mathbf{A})$, whence the same proofs apply to this case. $\square$

Corollary 6.3. *For every finite Boolean algebra $\mathbf{A}$, $\mathcal{T}(\mathbf{A})$ is isomorphic to $\mathcal{C}(\mathbf{A})$.*

Proof. Two Boolean algebras with the same cardinality are isomorphic. $\square$

In (Flaminio, Godo, and Hosni 2020) it is proved that any (unconditional) positive probability P on $\mathbf{A}$ canonically extends to a positive probability μ_P on $\mathcal{C}(\mathbf{A})$ such that for every basic conditional $(a|b)$, $\mu_P(a|b) = P(a \wedge b)/P(b)$.

We can finally apply the latter result and the above Theorem 6.2 to show our final outcome, for which we still need a previous lemma on the factorization of the probability P^* on the atoms of $\mathcal{T}(\mathbf{A})$, that is in fact a particular case of (Gilio and Sanfilippo 2020, Thm 18).

Lemma 6.4. *Let $\mathbf{A}$ be a finite Boolean algebra with $\mathrm{at}(\mathbf{A}) = \{\alpha_1, \ldots, \alpha_n\}$ and let P be a conditional probability on $A \times A'$. Then,*

$$P^*((\alpha_1|\top) \wedge (\alpha_2|\overline{\alpha_1}) \wedge \ldots \wedge (\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n)) = \\ = P(\alpha_1) \cdot P(\alpha_2|\overline{\alpha_1}) \cdot \ldots \cdot P(\alpha_{n-1}|\alpha_{n-1} \vee \alpha_n).$$

We observe that, if P is a positive probability on $\mathbf{A}$, for each basic conditional $(a|b)$, $P^*(a|b) = P(a \wedge b)/P(b) = \mu_P(a|b)$. Our last result shows that P^* and μ_P coincide on every compound conditional as well.

Theorem 6.5. *Given a positive probability P on $\mathbf{A}$ and any $t \in \mathbb{T}(A)$, $P^*(t) = \mu_P(t)$, once we identify the elements of $\mathcal{T}(\mathbf{A})$ with those of $\mathcal{C}(\mathbf{A})$.*

Proof. By Lemma 6.4, P^* coincides with μ_P on the atoms of $\mathcal{T}(\mathbf{A})$, and hence on the whole algebra. $\square$

7 Conclusions

In the present paper we have put forward an investigation on compound conditionals that, starting from the original setting proposed by de Finetti, aims at representing them in terms of conditional random quantities. Technically speaking, we start by a finite Boolean algebra $\mathbf{A}$ of events and a (coherent) conditional probability P on $A \times A'$, where $A' = A \backslash \{\bot\}$. Then, to each term t written in the language having as variables basic conditionals of the form $(a|b)$ (for $a \in A$ and $b \in A'$), we first consider, for each interpretation $w \in \Omega$ the reduct t^w of t, and then we associate to t a conditional random quantity $X_t : \Omega \to [0,1]$, that assigns to each $w \in \Omega$ the value $X_t(w)$ given by the conditional prevision $\mathbb{P}(X_{t^w}|\mathbf{b}(t^w))$.

By doing this, we have presented a natural and uniform procedure to interpret compound conditionals as random quantities and we have investigated the numerical and logical properties of such representation for compound conditionals via their associated random quantities. Our main contribution concerns the possibility of defining operations among conditionals by an iterative procedure. Furthermore, we have proved that these operations allow us to regard the set of those numerical representations as a Boolean algebra $\mathcal{T}(\mathbf{A})$. This latter result provides in turn a numerical counterpart of the construction explored in (Flaminio, Godo, and Hosni 2020), where the authors showed that compound symbolic conditionals, satisfying certain identities, can be endowed with a suitable Boolean algebra structure $\mathcal{C}(\mathbf{A})$. In particular, we have shown that the two algebraic structures that arise from these numerical and the symbolic representations of conditionals turn out to be isomorphic. Moreover, any conditional probability P on $A \times A'$ extends in the same way to both algebras $\mathcal{T}(\mathbf{A})$ and $\mathcal{C}(\mathbf{A})$.

As for future work we plan to generalize Theorem 6.5 to the case in which P is a conditional probability on $A \times A'$. Moreover, we aim at studying the extension of the random quantity-based approach to compound conditionals developed in this paper to deal with general forms of iterated conditionals, see e.g. (Sanfilippo et al. 2020; Gilio and Sanfilippo 2021c). Finally, we plan to study possible applications of compound conditionals to non-monotonic reasoning from conditional bases and to conditional logics more in general, in the line of possible interrelationships with different areas as discussed in (Aucher et al. 2019). Another interesting area of application of compound conditionals to be further explored is the area of psychology of uncertain reasoning (Elqayam et al. 2020).

Acknowledgments

The authors thank the anonymous referees for their comments. Flaminio and Godo acknowledge partial support by the MOSAIC project (EU H2020-MSCA-RISE-2020 Project 101007627) and also by the Spanish project PID2019-111544GB-C21 funded by MCIN/AEI/10.13039/501100011033. Sanfilippo acknowledges support by the FFR2020-FFR2021 projects of University of Palermo, Italy.

References

Adams, E. W. 1975. *The logic of conditionals*. Dordrecht: Reidel.

Aucher, G.; Égré, P.; Kern-Isberner, G.; and Poggiolesi, F. 2019. Conditional logics and conditional reasoning: New joint perspectives (Dagstuhl seminar 19032). *Dagstuhl Reports* 9(1):47–66.

Beierle, C.; Eichhorn, C.; Kern-Isberner, G.; and Kutsch, S. 2018. Properties of skeptical c-inference for conditional knowledge bases and its realization as a constraint satisfaction problem. *Ann. Math. Artif. Intell.* 83(3-4):247–275.

Benferhat, S.; Dubois, D.; and Prade, H. 1997. Nonmonotonic reasoning, conditional objects and possibility theory. *Artificial Intelligence* 92:259–276.

Calabrese, P. 1987. An algebraic synthesis of the foundations of logic and probability. *Information Sciences* 42(3):187 – 237.

Ciucci, D., and Dubois, D. 2013. A map of dependencies among three-valued logics. *Information Sciences* 250:162 – 177.

Coletti, G., and Scozzafava, R. 2002. *Probabilistic logic in a coherent setting*. Dordrecht: Kluwer.

Coletti, G.; Petturiti, D.; and Vantaggi, B. 2019. Dutch book rationality conditions for conditional preferences under ambiguity. *Ann. Oper. Res.* 279(1-2):115–150.

de Finetti, B. 1936. La Logique de la Probabilité. In *Actes du Congrès International de Philosophie Scientifique, Paris, 1935*. Hermann et C^{ie} Éditeurs, Paris. IV 1 – IV 9.

de Finetti, B. 1937. La prévision: ses lois logiques, ses sources subjectives. *Annales de l'Institut Henri Poincaré* 7(1):1–68.

Douven, I., and Dietz, R. 2011. A puzzle about Stalnaker's hypothesis. *Topoi* 31–37.

Dubois, D., and Prade, H. 1994. Conditional objects as nonmonotonic consequence relationships. *IEEE Trans. on Syst. Man and Cybernetics,* 24(12):1724 –1740.

Edgington, D. 1995. On conditionals. *Mind* 104:235–329.

Elqayam, S.; Douven, I.; Evans, J. S. B. T.; and Cruz, N. 2020. *Logic and Uncertainty in the Human Mind: A Tribute to David E. Over*. Oxon: Routledge.

Flaminio, T.; Gilio, A.; Godo, L.; and Sanfilippo, G. 2022. Compound conditionals as random quantities and Boolean algebras (version with proofs). `https://arxiv.org/abs/2204.14131`.

Flaminio, T.; Godo, L.; and Hosni, H. 2020. Boolean algebras of conditionals, probability and logic. *Artificial Intelligence* 286:103347.

Ghirardato, P. 2002. Revisiting savage in a conditional world. *Economic Theory* 20(1):83–92.

Gilio, A., and Sanfilippo, G. 2013a. Conjunction, disjunction and iterated conditioning of conditional events. In *Synergies of Soft Computing and Statistics for Intelligent Data Analysis*, volume 190 of *AISC*. Berlin: Springer. 399–407.

Gilio, A., and Sanfilippo, G. 2013b. Quasi conjunction, quasi disjunction, t-norms and t-conorms: Probabilistic aspects. *Information Sciences* 245:146–167.

Gilio, A., and Sanfilippo, G. 2014. Conditional random quantities and compounds of conditionals. *Studia Logica* 102(4):709–729.

Gilio, A., and Sanfilippo, G. 2019. Generalized logical operations among conditional events. *Applied Intelligence* 49(1):79–102.

Gilio, A., and Sanfilippo, G. 2020. Algebraic aspects and coherence conditions for conjoined and disjoined conditionals. *International Journal of Approximate Reasoning* 126:98 – 123.

Gilio, A., and Sanfilippo, G. 2021a. On compound and iterated conditionals. *Argumenta* 6(2):241–266.

Gilio, A., and Sanfilippo, G. 2021b. Compound conditionals, Fréchet-Hoeffding bounds, and Frank t-norms. *International Journal of Approximate Reasoning* 136:168–200.

Gilio, A., and Sanfilippo, G. 2021c. Iterated conditionals and characterization of p-entailment. In Vejnarová, J., and Wilson, N., eds., *Symbolic and Quantitative Approaches to Reasoning with Uncertainty, ECSQARU 2021*, volume 12897 of *LNCS*. Springer International Publishing. 629–643.

Gilio, A. 2002. Probabilistic reasoning under coherence in System P. *Annals of Mathematics and Artificial Intelligence* 34:5–34.

Goodman, I. R., and Nguyen, H. T. 1994. A theory of conditional information for probabilistic inference in intelligent systems: II. product space approach. *Inf. Sci.* 76(1-2):13–42.

Goodman, I. R.; Nguyen, H. T.; and Walker, E. A. 1991. *Conditional Inference and Logic for Intelligent Systems: A Theory of Measure-Free Conditioning*. North-Holland.

Halpern, J. Y. 2003. *Reasoning about Uncertainty*. Cambridge, MA, USA: MIT Press.

Halpern, J. Y. 2016. *Actual Causality*. The MIT Press.

Jeffrey, R. 1991. Matter-of-Fact Conditionals. *Proceedings of the Aristotelian Society, Supplementary Volume* 65:161–183.

Kaufmann, S. 2009. Conditionals right and left: Probabilities for the whole family. *Journal of Philosophical Logic* 38:1–53.

Kern-Isberner, G. 2001. *Conditionals in Nonmonotonic Reasoning and Belief Revision – Considering Conditionals as Agents.* Lecture Notes in Artificial Intelligence, Springer.

McGee, V. 1989. Conditional probabilities and compounds of conditionals. *Philosophical Review* 98(4):485–541.

Milne, P. 1997. Bruno de Finetti and the Logic of Conditional Events. *British Journal for the Philosophy of Science* 48(2):195–232.

Nguyen, H. T., and Walker, E. A. 1994. A history and introduction to the algebra of conditional events and probability logic. *IEEE Transactions on Systems, Man, and Cybernetics* 24(12):1671–1675.

Pfeifer, N., and Sanfilippo, G. 2017. Probabilistic squares and hexagons of opposition under coherence. *International Journal of Approximate Reasoning* 88:282–294.

Sanfilippo, G.; Pfeifer, N.; Over, D.; and Gilio, A. 2018. Probabilistic inferences from conjoined to iterated conditionals. *International Journal of Approximate Reasoning* 93(Supplement C):103 – 118.

Sanfilippo, G.; Gilio, A.; Over, D.; and Pfeifer, N. 2020. Probabilities of conditionals and previsions of iterated conditionals. *International Journal of Approximate Reasoning* 121:150 – 173.

Schay, G. 1968. An algebra of conditional events. *Journal of Mathematical Analysis and Applications* 24:334–344.

Stalnaker, R., and Jeffrey, R. 1994. Conditionals as random variables. In Eells, E., and Skyrms, B., eds., *Probability and Conditionals: Belief Revision and Rational Decision.* New York, NY, USA: Cambridge University Press. 31–46.

van Fraassen, B. 1976. Probabilities of conditionals. *Foundations of Probability Theory, Statistical Inference, and Statistical Theories of Science* 1:261–308. cited By 89.

van Rooij, R., and Schulz, K. 2019. Conditionals, causality and conditional probability. *Journal of Logic, Language and Information* 28(1):55–71.

Vantaggi, B. 2010. Incomplete preferences on conditional random quantities: Representability by conditional previsions. *Math. Soc. Sci.* 60(2):104–112.