An approach to inconsistency-tolerant reasoning about probability based on Łukasiewicz logic

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Abstract. In this paper we consider the probability logic over Łukasiewicz logic with rational truth-constants, denoted FP(RPL), and we explore a possible approach to reason from inconsistent FP(RPL) theories in a non-trivial way. It basically consists of suitably weakening the formulas in an inconsistent theory T, depending on the degree of inconsistency of T. We show that such a logical approach is in accordance with other proposals in the literature based on distance-based and violation-based inconsistency measures.

1 Introduction

Nowadays, there are huge amounts of available data and information and it is likely to encounter inconsistencies among different pieces of information. Thus, finding a suitable way of handling inconsistent information has become a challenge for both logicians and computer scientists working on knowledge representation techniques and reasoning models, see e.g. [2, 5, 14] among many others.

From a logical point of view, inconsistency is ubiquitous in many contexts in which, regardless of the given information being contradictory, one is still expected to extract inferences in a sensible way. In this work we aim at exploring a fuzzy logic-based approach to the handling of conflicts when the information is of probabilistic nature. Reasoning with inconsistent probabilistic information is indeed a research topic that has received growing attention in the last years, in particular with respect to inconsistency measurement of probabilistic knowledge bases, and to how inconsistency measures can be used to devise paraconsistent inference methods, see e.g. the survey [10].

The approach we follow here is based on a logical formalization of probabilistic reasoning on classical propositions as a modal theory over Łukasiewicz fuzzy logic, called FP(Ł), as developed by Hájek et al. [16, 15]. The idea is to understand the probability of a classical proposition φ as the truth-degree of a fuzzy modal proposition $P\varphi$, standing for the statement "φ is probable", in such a way that the higher (resp. lower) is the probability of φ, the more (resp. less) true is $P\varphi$. Then, the $[0, 1]$-based semantics of Łukasiewicz connectives, heavily relying on the usual addition and subtraction operations, make it possible to capture the postulates of probability measures (in particular the additivity property) with formulas in the language of FP(Ł). By expanding Łukasiewicz logic with rational truth-constants, yielding the logic called Rational Pavelka logic (RPL), it is possible to encode purely quantitative expressions like "the probability of φ is at least 0.4" as the modal formula $\overline{0.4} \to P\varphi$ in the language of FP(RPL), the probability logic obtained by replacing Ł by RPL in FP(Ł).

In this paper we explore a possible approach to reason from inconsistent FP(RPL) theories in a non-trivial way by suitably weakening formulas of an inconsistent theory T depending on a logical degree of inconsistency of T. The paper is then structured as follows. After this introduction, in the next section we recall main definitions and properties of Łukasiewicz logic Ł, as well as Rational Pavelka logic (RPL), its expansion with rational truth-constants, while n Section 3 we recap the probability logics based on Ł and on RPL. Section 4 is devoted to our proposal to deal with inconsistent probability theories over FP(RPL), and in Section 5 we relate our approach based on measuring the consistency of theories to those in the literature based on distance-based and violation-based inconsistency measures. We end with some conclusions and ideas for future work.

2 Łukasiewicz logic and Rational Pavelka logic

Łukasiewicz infinite-valued logic is one of the most prominent systems falling under the umbrella of Mathematical Fuzzy Logic (see e.g. [7]) although it was defined much before fuzzy logic was born. The interested reader is referred to the monographs [15, 6, 20] for full details.

The language of Łukasiewicz logic is built in the usual way from a set of propositional variables, one binary connective $\to$ (that is, Łukasiewicz implication) and the truth constant $\bar{0}$, that we will also denote as $\perp$. An *evaluation* e maps every propositional variable to a real number from the unit interval $[0,1]$ and extends to all formulas in the following way:

$$e(\bar{0}) = 0, \qquad e(\varphi \to \psi) = \min(1 - e(\varphi) + e(\psi), 1).$$

Other interesting connectives can be defined from them,

$$\begin{array}{llllll}
\bar{1} & \text{is } \varphi \to \varphi, & \neg\varphi & \text{is } \varphi \to \bar{0}, & \varphi \oplus \psi & \text{is } \neg\varphi \to \psi, \\
\varphi \& \psi & \text{is } \neg(\neg\varphi \oplus \neg\psi), & \varphi \ominus \psi & \text{is } \varphi \& \neg\psi, & \varphi \equiv \psi & \text{is } (\varphi \to \psi)\&(\psi \to \varphi), \\
\varphi \wedge \psi & \text{is } \varphi\&(\varphi \to \psi), & \varphi \vee \psi & \text{is } \neg(\neg\varphi \wedge \neg\psi), & &
\end{array}$$

and they have the following interpretations:

$$\begin{array}{ll}
e(\neg\varphi) = 1 - e(\varphi), & e(\varphi \oplus \psi) = \min(1, e(\varphi) + e(\psi)), \\
e(\varphi\&\psi) = \max(0, e(\varphi) + e(\psi) - 1), & e(\varphi \ominus \psi) = \max(0, e(\varphi) - e(\psi)), \\
e(\varphi \equiv \psi) = 1 - |e(\varphi) - e(\psi)|, & e(\varphi \wedge \psi) = \min(e(\varphi), e(\psi)), \\
e(\varphi \vee \psi) = \max(e(\varphi), e(\psi)). &
\end{array}$$

An evaluation e is called a *model* of a set of formulas T whenever $e(\varphi) = 1$ for each formula $\varphi \in T$. Axioms and rules of Łukasiewicz Logic are the following [6, 15]:

(Ł1) $\varphi \to (\psi \to \varphi)$
(Ł2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(Ł3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$
(Ł4) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$
(MP) Modus ponens: from φ and $\varphi \to \psi$ derive ψ

From this axiomatic system, the notion of proof from a theory (a set of formulas), denoted $\vdash_{\text{Ł}}$, is defined as usual.

The above axioms are tautologies or valid (i.e., they are evaluated to 1 by any evaluation), and the rule of modus ponens preserves validity. Moreover, the following completeness result holds.

Theorem 1. *The logic Ł is complete for deductions from finite theories. That is, if T is a finite theory, then $T \vdash_{L} \varphi$ iff $e(\varphi) = 1$ for each Łukasiewicz evaluation e model of T.*

In the rest of this section we briefly recall the expansion of Łukasiewicz logic with rational truth-constants that will be used later on. Following Hájek [15], the language of *Rational Pavelka logic*, denoted RPL, is the language of Łukasiewicz logic expanded with countably-many truth-constants $\overline{r}$, one for each rational $r \in [0,1]$. The evaluation of RPL formulas is as in Łukasiewicz logic, with the proviso that evaluations evaluate truth-constants to their intended value, that is, for any rational $r \in [0,1]$ and any evaluation e, $e(\overline{r}) = r$. Note that, for any evaluation e, $e(\overline{r} \to \varphi) = 1$ iff $e(\varphi) \geqslant r$, and $e(\overline{r} \equiv \varphi) = 1$ iff $e(\varphi) = r$.

Axioms and rules of RPL are those of Ł plus the following countable set of bookkeeping axioms for truth-constants:

(BK) $\overline{r} \to \overline{s} \equiv \overline{\min(1, 1 - r + s)}$, for any rationals $r, s \in [0,1]$.

The notion of proof is defined as in Łukasiewicz logic, and the deducibility relation will be denoted by $\vdash_{RPL}$. Moreover, completeness of Łukasiewicz logic smoothly extends to RPL as follows: if T is finite theory over RPL, then $T \vdash_{\mathrm{R}PL} \varphi$ iff $e(\varphi) = 1$ for any RPL-evaluation e model of T.

It is customary in RPL to introduce the following notions: for any set of RPL formulas $T \cup \{\varphi\}$, define:

- the *truth degree* of φ in T: $\|\varphi\|_T = \inf\{e(\varphi) : e$ is a RPL-evaluation model of $T\}$,
- the *provability degree* of φ from T: $|\ \varphi\ |_T = \sup\{r \in [0,1]_{\mathbb{Q}} \mid T \vdash_{RPL} \overline{r} \to \varphi\}$.

Then, the so-called *Pavelka-style completeness* for RPL refers to the result that

$$|\ \varphi\ |_T = \|\varphi\|_T$$

holds for any arbitrary (non necessarily finite) theory T [15]. However, if T is finite, we can restrict ourselves to rational-valued Łukasiewicz evaluations and get the following result, proved in [15].

Proposition 1. *If T is a finite theory over RPL, then:*

– $\|\varphi\|_T = 1$ *iff* $T \vdash_{RPL} \varphi$.
– $\|\varphi\|_T$ *is rational, hence* $\|\varphi\|_T = r$ *iff* $T \vdash_{RPL} \overline{r} \to \varphi$.

3 FP(RPL): a logic to reason about probability as modal theories over RPL

In this section we first describe the fuzzy modal logic FP(RPL) to reason qualitatively about probability, built upon Łukasiewicz logic RPL described in the previous section. We basically follow [15]. The language of FP(RPL) is defined in two layers:

Non-modal formulas: built from a set V of propositional variables, that will be assumed here to be finite, using the classical binary connectives $\wedge$ and $\neg$. Other connectives like $\vee$, $\to$ and $\leftrightarrow$ are defined from them in the usual way.[1] Non-modal formulas, or Boolean propositions, will be denoted by lower case Greek letters φ, ψ, etc. The set of non-modal formulas will be denoted by $\mathcal{L}$.

Modal formulas: built from elementary modal formulas of the form $P\varphi$, where φ is a non-modal formula, using the connectives and truth constants of Rational Pavelka logic. We shall denote them by upper case Greek letters Φ, Ψ, etc. Notice that we do not allow nested modalities of the form $P(P(\psi) \oplus P(\varphi))$, nor mixed formulas of the kind $\psi \to P\varphi$.

Definition 1. *The axioms of the logic FP(RPL) are the following:*

(i) Axioms of classical propositional logic for non-modal formulas
(ii) Axioms of RPL for modal formulas
(iii) Probabilistic modal axioms:[2]

(FP0) $P\varphi$, *for* φ *being a theorem of CPL*
(FP1) $P(\varphi \to \psi) \to (P\varphi \to P\psi)$
(FP2) $P(\neg\varphi) \equiv \neg P\varphi$
(FP3) $P(\varphi \vee \psi) \equiv (P\varphi \to P(\varphi \wedge \psi)) \to P\psi$

The only deduction rule of FP(RPL) is that of L (i.e. modus ponens*)*

The notion of proof for modal formulas is defined as usual from the above axioms and rule. We will denote by the expression $T \vdash_{FP} \Phi$ that in FP(RPL) a modal formula Φ follows from a theory (set of modal formulas) T. Note that FP(RPL) preserves (classical) logical equivalence. Indeed, due to axioms (FP0) and (FP1), FP(RPL) proves the formula $P\varphi \equiv P\psi$ whenever φ and ψ are (classically) logically equivalent.

The semantics for FP(RPL) is basically given by probability functions on classical (i.e. non-modal) formulas of $\mathcal{L}$, or equivalently, assuming $\mathcal{L}$ is built up

[1] Although we are using the same symbols $\wedge, \neg, \vee, \to$ as in Łukasiewicz logic to denote the conjunction, negation, disjunction and implication, the context will help in avoiding any confusion. In particular classical logic connectives will appear only under the scope of the operator P, see below.

[2] An equivalent formulation of (FP3) is $P(\varphi \vee \psi) \equiv P\varphi \oplus (P\psi \ominus P(\varphi \wedge \psi))$.

from a finite set of variables V, by probability functions on the set Ω of classical interpretations of $\mathcal{L}$. If $\mu : 2^{\Omega} \to [0, 1]$ is a probability, we will simply write $\mu(\varphi)$ to denote $\mu(\{w \in \Omega \mid w(\varphi) = 1\})$. We will denote by $\mathcal{P}(\mathcal{L})$ the set of probabilities on $\mathcal{L}$. Then every probability $\mu \in \mathcal{P}(\mathcal{L})$ determines an evaluation e_μ of modal formulas as follows: for a basic modal $P\varphi$,

$$e_\mu(P\varphi) = \mu(\varphi),$$

and it is extended to arbitrary modal formulas according to the semantics of Rational Pavelka logic: $e_\mu(\overline{r}) = r$, $e_\mu(\Phi \to_L \Psi) = \min(1 - e_\mu(\Phi) + e_\mu(\Psi), 1)$. Then we say that a probability $\mu \in \mathcal{P}(\mathcal{L})$ is a *model* of a theory T of modal formulas if $e_\mu(\Phi) = 1$ for every $\Phi \in T$.

FP(RPL) can be used to reason in a purely qualitative way about comparative probability statements by exploiting the fact a FP(RPL)-formula of the form $P\psi \to P\varphi$ is 1-true in a model defined by a probability μ iff $\mu(\psi) \leqslant \mu(\varphi)$. However, FP(RPL) also allows one to explicitly reason about numerical statements, like "the probability of φ is 0.8", "the probability of φ is at least 0.8", or "the probability of φ is at most 0.8". Indeed, the above statements can be easily expressed in FP(RPL):

- "the probability of φ is 0.8" as $P\varphi \equiv \overline{0.8}$,
- "the probability of φ is at least 0.8" as $\overline{0.8} \to P\varphi$, and
- "the probability of φ is at most 0.8" as $P\varphi \to \overline{0.8}$.

The following general Pavelka-style completeness result for FP(RPL) was presented in [16, 15].

Theorem 2. (Probabilistic completeness of FP(RPL)) *Let T be a modal theory over FP(L) and Φ a modal formula. Then,*

$$|\ \Phi\ |_T = \|\Phi\|_T,$$

where now $|\ \Phi\ |_T = \sup\{r \in [0,1]_{\mathbb{Q}} \mid T \vdash_{FP} \overline{r} \to \Phi\}$ and $\|\varphi\|_T = \inf\{e_\mu(\Phi) : \mu \in \mathcal{P}(\mathcal{L})$ is a model of $T\}$.

As in the case of RPL, if the modal theory T is finite, we can get a standard completeness result, that follows from [15, Th. 8.4.14].

Theorem 3 (Probabilistic completeness of FP(RPL)). *Let T be a finite modal theory over FP(RPL) and Φ a modal formula. Then $T \vdash_{FP(RPL)} \Phi$ iff $e_\mu(\Phi) = 1$ for each probability $\mu \in \mathcal{P}(\mathcal{L})$ model of T.*

Similarly to RPL, for deductions from finite theories FP(RPL) is still complete for rational-valued probabilities.

Corollary 1. *Let T be a finite modal theory over FP(RPL) and Φ a modal formula. Then $T \vdash_{FP(RPL)} \Phi$ iff $e_\mu(\Phi) = 1$ for each rational-valued probability $\mu \in \mathcal{P}(\mathcal{L})$ model of T.*

Moreover, since deductions in FP(RPL) from a finite theory can be encoded as deductions from a (larger) finite theory in RPL, as a direct corollary of Proposition 1, we get the following.

Corollary 2. *If T is finite, for any FP(RPL)-formula Φ, $\|\Phi\|_T$ is rational.*

4 Reasoning with inconsistent probabilistic information in FP(RPL)

If we want to reason in a non-trivial way from inconsistent probabilistic theories over FP(RPL), we need to devise possible ways to define paraconsistent reasoning inference relations in a meaningful form. The way we approach this issue is to compute what we will call the "degree of inconsistency" of a modal theory T, and use that value to suitably weaken the formulas in T so that the obtained weaker theory is consistent.

Let us recall from Section 3 that, from a semantical point of view, the logic FP(RPL) is defined as follows: for any set of FP(RPL)-formulas $T \cup \{\Phi\}$,

$$T \models_{FP} \Phi \quad \text{if,} \quad \begin{array}{l} \text{for every probability } \mu \in \mathcal{P}(\mathcal{L}) \text{ on Boolean formulas,} \\ \text{if } \mu \text{ is a model of } T \text{ then } e_\mu(\Phi) = 1. \end{array}$$

We will denote by $[\![T]\!]$ the set probabilities that are models of T. In other words, $[\![T]\!] = \{\text{probability } \mu \in \mathcal{P}(\mathcal{L}) \mid \text{for all } \Psi \in T, e_\mu(\Psi) = 1\}$.

Of course, the above definition trivializes in the case T is inconsistent, i.e. when $[\![T]\!] = \emptyset$. However, in FP(RPL) one can take advantage of its fuzzy component and consider the notion of (in)consistency as being fuzzy as well. Indeed, if a probabilistic theory T has no models, it makes sense to distinguish, for instance, cases where: (1) for every probability μ there is a formula $\Phi \in T$ such that $e_\mu(\Phi) = 0$; and (2) there exists a probability μ such that, for all $\Phi \in T$, $e_\mu(\Phi)$ is close to 1. In the former case T is clearly inconsistent, while in the latter case one could say that T is *close* to being consistent.

This observation justifies to introduce, for a given threshold $\alpha \in [0, 1]$, the set of *α-generalised models* (or just α-models) of a theory T defined as follows:

$$[\![T]\!]_\alpha = \{\mu \in \mathcal{P}(\mathcal{L}) \mid \text{for all } \Psi \in T, e_\mu(\Psi) \geqslant \alpha\}.$$

Note that the set $[\![T]\!]_1$ coincides with the set $[\![T]\!]$ of usual models of T, while $[\![T]\!]_0 = \mathcal{P}(\mathcal{L})$. Moreover, for any α, $[\![T]\!]_\alpha$ is a convex set of probabilities.

This in turn allows us to define the degree of consistency of a theory as the highest value α for which T has at least one α-generalised model.

Definition 2. *Let T be a theory of FP(RPL). The consistency degree of T is defined as*

$$Con(T) = \sup\{\beta \in [0, 1] \mid [\![T]\!]_\beta \neq \emptyset\}.$$

Dually, the inconsistency degree of T is defined as

$$Incon(T) = 1 - Con(T) = \inf\{1 - \beta \in [0, 1] \mid [\![T]\!]_\beta \neq \emptyset\}.$$

For every finite modal theory T, by completeness of FP(RPL) with respect to probability models (Theorem 3), we can also express $Con(T)$ and $Incon(T)$ as follows:

$$Con(T) = \sup\{\beta \in [0, 1] \mid T_\beta \not\vdash \bot\} \quad \text{and} \quad Incon(T) = \inf\{1 - \beta \in [0, 1] \mid T_\beta \not\vdash \bot\},$$

where $T_\beta = \{\overline{\beta} \to \Phi \mid \Phi \in T\}$. A somewhat different, yet equivalent, formulation for the degrees of consistency and inconsistency is as follows:

$$Con(T) = \sup\{\bigwedge_{\Phi \in T} e_\mu(\Phi) \mid \mu \in \mathcal{P}(\mathcal{L})\}, \quad Incon(T) = \inf\{\bigvee_{\Phi \in T} e_\mu(\neg\Phi) \mid \mu \in \mathcal{P}(\mathcal{L})\}.$$

It can be shown that, if T is finite, the suprema and the infima in the above definition and expressions of $Con(T)$ and $Incon(T)$, are in fact maxima and minima. And not only this, in fact, $Con(T)$ and $Incon(T)$ are always rational numbers.

Lemma 1. *Let T be a finite theory of FP(RPL). Then:*

$$Con(T) = \max\{\bigwedge_{\Phi \in T} e_\mu(\Phi) \mid \mu \textit{ probability}\} = \max\{\beta \in [0,1] \mid T_\beta \nvdash \bot\},$$
$$Incon(T) = \min\{\bigvee_{\Phi \in T} e_\mu(\neg\Phi) \mid \mu \textit{ probability}\} = \min\{1-\beta \in [0,1] \mid T_\beta \nvdash \bot\}.$$

Moreover, $Con(T)$ and $Incon(T)$ are rational.

In particular, from the previous lemma it follows that for a finite theory T, if $Con(T) = \alpha$, then $[\![T]\!]_\alpha \neq \varnothing$. Moreover:

(i) If $Con(T) = 1$ then T has a model, while if $Con(T) = 0$ then, for any probability μ there is a formula $\Psi \in T$ such that $e_\mu(\Psi) = 0$.
(ii) If $T' \subseteq T$ then $Con(T') \geqslant Con(T)$.

Let us clarify what the degree of consistency represents in the case of some very simple examples.

Example 1. Let us consider the following theory of precise probability assignments $T = \{\overline{r_i} \equiv P\varphi_i\}_{i=1,\ldots,n}$ to a set of events $\mathcal{E} = \{\varphi_1, \ldots, \varphi_n\}$. If μ is a probability, then $e_\mu(\overline{r_i} \equiv P\varphi_i) = 1 - |\mu(\varphi_i) - r_i|$. Then,

$$Con(T) = \sup_\mu \bigwedge_{i=1,\ldots,n} 1 - |\mu(\varphi_i) - r_i|, \quad Incon(T) = \inf_\mu \bigvee_{i=1,\ldots,n} |\mu(\varphi_i) - r_i|.$$

That is to say, $Incon(T)$ is nothing but the Chebyshev distance of the point $(r_1, \ldots, r_n) \in [0,1]^n$ to the convex set of *consistent* probability assignments $\mathscr{C}_\mathcal{E}$ on the events $\mathcal{E}$, i.e., the set of values that probabilities assign to the events in $\mathcal{E}$.

For instance, consider the theory $T^{\mathbf{b}} = \{P(p) \equiv \overline{1/2}, P(\neg p) \equiv \overline{1/3}\}$ given by the inconsistent assignment $\mathbf{b} : p \mapsto 1/2; \neg p \mapsto 1/3$. The set of all consistent assignments on events p and $\neg p$ is the set $\mathscr{C}_{\{p,\neg p\}} = \{(x, 1-x) \mid x \in [0,1]\}$, i.e. the segment in $[0,1]^2$ with endpoints $(1,0)$ and $(0,1)$ (see Figure 1), and the inconsistent assignment $\mathbf{b}$ is displayed as the point $(1/2, 1/3) \notin \mathscr{C}_{\{p,\neg p\}}$. As mentioned above, $Con(T^{\mathbf{b}})$ can be computed as 1 minus the Chebyshev distance between $(1/2, 1/3)$ and $\mathscr{C}_{\{p,\neg p\}}$. This value is attained at the point of coordinates $(7/12, 5/12)$ and then we have:

$$1 - |\mathbf{b}(p) - 7/12| = 1 - |\mathbf{b}(\neg p) - 5/12| = 1 - 1/12 = 11/12 = Con(T^{\mathbf{b}}).$$

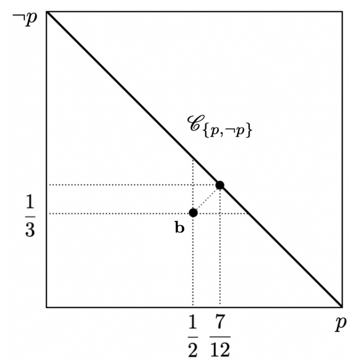

Fig. 1. $Con(T^{\mathbf{b}})$ is computed as 1 minus the Chebyshev distance between the point **b** that represents the partial assignment on p and $\neg p$, and the set of consistent assignments on p and $\neg p$.

Example 2. Let us now consider a theory representing an imprecise probability assignment to the set of events $\mathcal{E} = \{\varphi_1, \ldots, \varphi_n\}$:

$$T = \{(\overline{r_i - \epsilon_i} \to P\varphi_i) \wedge (P\varphi_i \to \overline{r_i + \epsilon_i})\}_{i=1,\ldots,n}$$

where, for each i, $r_i - \epsilon_i \geqslant 0$ and $r_i + \epsilon_i \leqslant 1$, that is $\epsilon_i \leqslant r_i \leqslant 1 - \epsilon_i$. Then, using that $\min((x - y) \to z, z \to (x + y)) = y \to (x \equiv z)$, if $y \leqslant x \leqslant 1 - y$ (where here and below we also use $\to$ and $\equiv$ to denote the truth-functions for Łukasiewicz implication and equivalence connectives), the degree of inconsistency of T can be computed as follows:

$$Incon(T) = 1 - Con(T) = 1 - \sup_{\mu} \bigwedge_{i=1,\ldots,n} e_\mu((\overline{r_i - \epsilon_i} \to P\varphi_i) \wedge (P\varphi_i \to \overline{r_i + \epsilon_i}))$$

$$= 1 - \sup_{\mu} \bigwedge_{i=1,\ldots,n} ((1 - \epsilon_i) \to |r_i - \mu(\varphi_i)| = \inf_{\mu} \bigvee_{i=1,\ldots,n} (1 - \epsilon_i) \otimes |r_i - \mu(\varphi_i)|.$$

As for paraconsistently reasoning from an inconsistent theory in FP(RPL), the idea we explore here is to use α-generalised models instead of usual models to define a context-dependent inconsistent-tolerant notion of probabilistic entailment.

Definition 3. *Let T be a theory such that $Con(T) = \alpha > 0$. We define:*

$$T \mathrel{|\!\!\approx^*} \Phi \text{ if } e_\mu(\Phi) = 1 \text{ for all probabilities } \mu \in [\![T]\!]_\alpha.$$

Note that for a finite theory T, $T \mathrel{|\!\!\not\approx^*} \perp$, hence $\mathrel{|\!\!\approx^*}$ does not trivialize even if T is inconsistent ($Con(T) < 1$). Moreover, observe that if $Con(T) = 0$, then $T \mathrel{|\!\!\approx^*} \Phi$ iff Φ is a theorem of FP(RPL). The following are some further interesting properties of the consequence relation $\mathrel{|\!\!\approx^*}$:

– Clearly, $\approx^*$ does not satisfy monotonicity. For instance, if $T' = \{P\varphi \equiv 0.4, P\varphi \rightarrow P\psi\}$, then $Con(T') = 1$ and trivially $T' \mid\!\approx^* 0.4 \equiv P\varphi$, but $T \not\mid\!\approx^* 0.4 \equiv P\varphi$, where $T = T' \cup \{0.3 \equiv P\varphi\}$.
– $\mid\!\approx^*$ is idempotent, that is, if $S \mid\!\approx^* \Phi$ and $T \mid\!\approx^* \Psi$ for all $\Psi \in S$, then $T \mid\!\approx^* \Phi$.

The next proposition shows that paraconsistent reasoning from an inconsistent theory T by means of the inference relation $\mid\!\approx^*$ can be reduced to usual reasoning in FP(RPL) by suitably weakening the formulas in the initial theory T.

Proposition 2. *Given a finite theory T, with $Con(T) = \alpha$, let $T_\alpha = \{\overline{\alpha} \rightarrow \Psi \mid \Psi \in T\}$. Then the following condition holds:*

$$T \mid\!\approx^* \Phi \textit{ iff } T_\alpha \vdash_{FP} \Phi.$$

Proof. If μ is a probability such that $e_\mu(\overline{\alpha} \rightarrow \Psi) = 1$, i.e. such that $e_\mu(\Psi) \geqslant \alpha$, for all $\Psi \in T$, then $\mu \in [\![T]\!]_\alpha$. But if we assume $T \mid\!\approx^* \Phi$, then it follows that $e_\mu(\Phi) = 1$. Hence $T_\alpha \vdash_{FP} \Phi$.

Conversely, assume $T_\alpha \vdash_{FP} \Phi$ with $Con(T) = \alpha$ and that $\mu \in \|T\|_\alpha$. The latter means that $e_\mu(\Psi) \geqslant \alpha$ for all $\Psi \in T$, i.e. $e_\mu(\alpha \rightarrow \Psi) = 1$ for all $\Psi \in T$. But then, since $T_\alpha \vdash_{FP} \Phi$, it follows that $e_\mu(\Phi) = 1$, that is, $T \mid\!\approx^* \Phi$.

The weakened theory T_α, that is consistent, can be seen as a *repair* of T. In the case the theory represents a precise probability assignment of the form $T = \{\overline{r_i} \equiv P\varphi_i\}_{i=1,\ldots,n}$, then $T_\alpha = \{(\overline{\alpha \otimes r_i} \rightarrow P\varphi_i) \wedge (P\varphi_i \rightarrow \overline{\alpha \Rightarrow r_i})\}_{i=1,\ldots,n}$, that is, it becomes a theory of an imprecise assignment. On the other hand, in the case the theory already represents an imprecise probability assignment of the form $T = \{(\overline{r_i} \rightarrow P\varphi_i) \wedge (P\varphi_i \rightarrow \overline{s_i})\}_{i=1,\ldots,n}$, then $T_\alpha = \{(\overline{\alpha \otimes r_i} \rightarrow P\varphi_i) \wedge (P\varphi_i \rightarrow \overline{\alpha \Rightarrow s_i})\}_{i=1,\ldots,n}$, that is, it represents a more imprecise assignment.

The consequence relation $\mid\!\approx^*$ introduced above has some nice features, but it may also have a counter-intuitive behaviour in some cases. For instance, let $T = \{\overline{0.3} \equiv P\varphi, \overline{0.4} \equiv P\varphi, \overline{0.6} \equiv P\psi\}$, where φ and ψ are assumed to be propositional variables. Then it is easy to check that $Con(T) = 0.95$, and hence $T \mid\!\approx^* \overline{0.35} \equiv P\varphi$. But strangely enough, $T \not\mid\!\approx^* \overline{0.6} \equiv P\psi$, since we can only derive $T \mid\!\approx^* \overline{0.95} \rightarrow (\overline{0.6} \equiv P\psi)$, even though the formula $\overline{0.6} \equiv P\psi$ is not involved in the conflict in T. The reason is that $Con(T)$ is a global measure that does not take into account individual formulas. Actually, if $T' = T \cup \{\overline{0.7} \equiv P\psi\}$, we still have $Con(T) = Con(T') = 0.95$.

The above example motivates the following iterative procedure to come up with a more suitable repair of an inconsistent theory T. The idea is to first identify minimal inconsistent subtheories S of T that are responsible for the degree of consistency of T, i.e. such that $Con(S) = Con(T) = \alpha$. Then we only repair the formulas of these subtheories by using the degree α. In a next step, one proceeds with the rest of the initial theory T by repeating the same process. This procedure stops when all the formulas have been dealt with in some previous step.

Step 1: Let $Con(T) = \alpha_1$. Then we know that the set of probabilities $[\![T]\!]_{\alpha_1}$ is non-empty. Hence, we can partition T in the following two disjoint subtheories:

- $T^{=} = \bigcup\{S \subseteq T \mid S$ minimal such that $Con(S) = \alpha_1\}$
- $T^{>} = T \backslash T^{=}$

Note that $T^{=} \neq \varnothing$ and if $T^{>} \neq \varnothing$ then $Con(T^{>}) > \alpha_1$. By definition $T^{=} \cap T^{>} = \varnothing$ and $T = T^{=} \cup T^{>}$.
Then we proceed to weaken only those formulas in $T^{=}$, so we define:

$$T^{(1)} = \{\overline{\alpha_1} \to \Phi \mid \Phi \in T^{=}\}.$$

If $T^{>} = \varnothing$, then we stop and we define the repaired theory as $T^{w} = T^{(1)}$. Otherwise we follow to the next step to repair $T^{>}$.

Step 2: Restrict the set of possible probability models to those of $[\![T]\!]_{\alpha_1}$ to compute the consistency degree of $T^{>}$.
Let $Con_T(T^{>}) = \max\{\beta \mid$ there exists $\mu \in [\![T]\!]_{\alpha_1}, e_\mu(\Phi) \geqslant \beta$ for all $\Phi \in T^{>}\} = \alpha_2$.
By definition, $\alpha_2 > \alpha_1$. And we proceed similarly as above, but restricting the set of models to those in $[\![T]\!]_{\alpha_1}$, and we partition $T^{>}$ into the following two subtheories:

- $(T^{>})^{=} = \bigcup\{S \subseteq T^{>} \mid S$ minimal such that $Con_T(S) = \alpha_2\}$
- $(T^{>})^{>} = T^{>} \backslash (T^{>})^{=}$

Again note that $(T^{>})^{=} \neq \varnothing$, and if $(T^{>})^{>} \neq \varnothing$ then $Con((T^{>})^{>}) > \alpha_2$. We proceed to the weakening of the subtheory $(T^{>})^{=}$ and define:

$$T^{(2)} = \{\overline{\alpha_2} \to \Phi \mid \Phi \in (T^{>})^{=}\}.$$

If $(T^{>})^{>} = \varnothing$, then we stop and we define the repaired theory as $T^{w} = T^{(1)} \cup T^{(2)}$. Otherwise we follow to the next step to repair $(T^{>})^{>}$.

Step 3: Restrict the set of possible probabilistic models to those of $[\![T]\!]_{\alpha_1} \cap [\![T^{>}]\!]_{\alpha_2}$ to compute the consistency degree of $(T^{>})^{>}$:
Let $Con_{T,T^{>}}((T^{>})^{>}) = \max\{\beta \mid$ there exists $\mu \in [\![T]\!]_{\alpha_1} \cap [\![T^{>}]\!]_{\alpha_2}, e_\mu(\Phi) \geqslant \beta$ for all $\Phi \in (T^{>})^{>}\} = \alpha_3$.
By definition, $\alpha_3 > \alpha_2 > \alpha_1$. we then follow the same procedure as above, but restricting the set of models to those in $[\![T]\!]_{\alpha_1} \cap [\![T^{>}]\!]_{\alpha_2}$, and we partition $(T^{>})^{>}$ into the following two subtheories:

- $((T^{>})^{>})^{=} = \cup\{S \subseteq (T^{>})^{>} \mid S$ minimal such that $Con_{T,T^{>}}(S) = \alpha_3\}$
- $((T^{>})^{>})^{>} = (T^{>})^{>} \backslash ((T^{>})^{>})^{=}$

Now we proceed to weaken the subtheory $((T^{>})^{>})^{=}$ and define:

$$T^{(3)} = \{\overline{\alpha_3} \to \Phi \mid \Phi \in ((T^{>})^{>})^{=}\}.$$

If $((T^{>})^{>})^{>} = \varnothing$, then we stop and we define the repaired theory as $T^{w} = T^{(1)} \cup T^{(2)} \cup T^{(3)}$. Otherwise we follow to the next step to repair $((T^{>})^{>})^{>}$.

... ...

This procedure goes on until, for a first m, $(...(T^{>})\overset{m}{\cdots})^{>}) = \emptyset$. Then the procedure stops and as a result we get a (finite) sequence of subtheories $T^{(1)}, T^{(2)}, \ldots, T^{(m)}$, with associated consistency values $\alpha_1 < \ldots < \alpha_m$. By construction, the theory

$$T^w = T^{(1)} \cup \ldots \cup T^{(m)}$$

is consistent.

This allows us to define a refined variant of the $\mid\!\approx^*$ consequence relation.

Definition 4. *Let T be a theory over FP(RPL). Then we define a refinement $\mid\!\approx^\circ$ of the consequence relation $\mid\!\approx^*$ as follows:*

$$T \mid\!\approx^\circ \Phi \text{ if } T^w \vdash_{FP} \Phi.$$

Compare this definition with the characterisation of $\mid\!\approx^*$ in Prop. 2. It is clear that $\mid\!\approx^\circ$ is stronger than $\mid\!\approx^*$ while still paraconsistent.

Example 3. Let $T = \{\overline{0.3} \equiv P\varphi, \overline{0.4} \equiv P\varphi, \overline{0.6} \equiv P\psi, \overline{0.8} \equiv P\psi, \overline{0.7} \equiv P\chi\}$, where φ, ψ, χ are propositional variables. Since $Con(T) = 0.9$, we have

$$T_{0.9} = \{\overline{0.9} \to (\overline{0.3} \equiv P\varphi), \overline{0.9} \to (\overline{0.4} \equiv P\varphi), \overline{0.9} \to (\overline{0.6} \equiv P\psi),\\ \overline{0.9} \to (\overline{0.8} \equiv P\psi), \overline{0.9} \to (\overline{0.7} \equiv P\chi)\}$$

Models of $T_{0.9}$ are probabilities μ such that $\mu(\varphi) \in [0.2, 0.4] \cap [0.3, 0.5] = [0.3, 0.4]$, $\mu(\psi) \in [0.5, 0.7] \cap [0.7, 0.9] = \{0.7\}$ and $\mu(\chi) \in [0.6, 0.8]$. However, using the refinement procedure, we get

$$T^w = \{\overline{0.9} \to (\overline{0.6} \equiv P\psi), \overline{0.9} \to (\overline{0.8} \equiv P\psi), \overline{0.95} \to (\overline{0.3} \equiv P\varphi),\\ \overline{0.95} \to (\overline{0.4} \equiv P\varphi), \overline{0.7} \equiv P\chi\},$$

that is equivalent to the theory

$$T'^w = \{\overline{0.5} \to P\psi, P\psi \to \overline{0.7}, \overline{0.7} \to P\psi, P\psi \to \overline{0.9}, \overline{0.25} \to P\varphi,\\ P\varphi \to \overline{0.35}, \overline{0.35} \to P\varphi, P\varphi \to \overline{0.45}, \overline{0.7} \equiv P\chi\}.$$

In this case, models of T^w are probabilities μ such that $\mu(\varphi) = 0.35$, $\mu(\psi) = 0.7$ and $\mu(\chi) = 0.7$, and hence the refined consequence relation $\mid\!\approx^\circ$ is such that:

$$T \mid\!\approx^\circ \overline{0.7} \equiv P\psi, \overline{0.35} \equiv P\varphi, \overline{0.7} \equiv P\chi.$$

5 Related approaches

In the literature there has been quite a lot of interest on measuring the inconsistency of probabilistic knowledge bases, see for instance [26, 19, 27, 22, 23, 9, 24, 25]. In particular, there is a nice overview by de Bona, Finger, Potyka and Thimm in [10] on which we will base the comparison with our approach.

First of all, by a probabilistic knowledge base it is usually understood a finite set of (conditional) probability constraints on classical propositional formulas (from a given finitely generated language $\mathcal{L}$), of the form $KB = \{(\varphi_i \mid \psi_i)[\underline{q}_i, \overline{q}_i] \mid i = 1, \ldots n\}$, where $\underline{q}_i$ and $\overline{q}_i$ are rational values from the unit interval $[0, 1]$. Such an expression $(\varphi_i \mid \psi_i)[\underline{q}_i, \overline{q}_i]$ intuitively expresses the constraint (or belief) that the conditional probability of φ_i given ψ_i lies in the interval $[\underline{q}_i, \overline{q}_i]$.

Then, a probability on formulas μ satisfies a conditional expression $(\varphi_i \mid \psi_i)[\underline{q}_i, \overline{q}_i]$, written $\mu \models (\varphi_i \mid \psi_i)[\underline{q}_i, \overline{q}_i]$, whenever $\mu(\varphi_i \wedge \psi_i) \geqslant \underline{q}_i \cdot \mu(\psi_i)$ and $\mu(\varphi_i \wedge \psi_i) \leqslant \overline{q}_i \cdot \mu(\psi_i)$. Such a probability is called a *model* of the formula. Of course, if $\mu(\psi_i) > 0$, these conditions amount to state that $\mu \models (\varphi_i \mid \psi_i)[\underline{q}_i, \overline{q}_i]$ when the conditional probability $\mu(\varphi_i \mid \psi_i)$ belongs to the interval $[\underline{q}_i, \overline{q}_i]$.

In the case a probabilistic knowledge base KB is inconsistent, a number of *inconsistency measures* have been proposed in the literature to measure how much inconsistent KB is, some of them generalising to the probabilistic case inconsistency measures already proposed for the propositional case, and some of them specifically tailored to deal with probabilistic expressions. Among these, there are the so-called distance-based measures and violation-based measures. Very roughly speaking, the former look for consistent knowledge bases that *minimize the distance* (for some suitable notion of distance) to the original inconsistent KB, while the latter look for probabilities that *minimize the violation* (for some suitable notion of violation) of the knowledge base [27, 22].

According to [10], when it comes to reasoning with an inconsistent probabilistic KB, there are two sensible ways to proceed: either repair the inconsistent knowledge base and then apply classical probabilistic reasoning, or apply paraconsistent reasoning models that can deal with inconsistent knowledge bases. For the first approach, distance-based measures are well-suited while for the second approach violation-based measures (together with so-called fuzzy-based measures) seem to be the most suitable ones.

We can show here that our approach to reason with inconsistent probabilistic theories over FP(RPL), when restricted to theories of the form $T = \{\overline{r_i} \equiv P\varphi_i\}_{i=1,\ldots,n}$, can be seen both as a distance-based approach and as violation-based approach. Note that here we do not deal with conditional probability expressions as most of the approaches in the literature, thus our case is simpler.

Indeed, in the distance-based approach, given a distance d on $\mathbb{R}^n$, and two theories $T = \{\overline{r_i} \equiv P\varphi_i\}_{i=1,\ldots,n}$ and $T' = \{\overline{r'_i} \equiv P\varphi_i\}_{i=1,\ldots,n}$, one can define the distance between T and T' as the distance between their corresponding vectors of truth-constants:

$$d(T, T') = d((r_1, \ldots, r_n), (r'_1, \ldots, r'_n))$$

Then, if $T = \{\overline{r_i} \equiv P\varphi_i\}_{i=1,\ldots,n}$ is an inconsistent theory, the aim is to look for a consistent theory (a *repair*), by minimally modifying the truth-constants r_i's such that the resulting new theory is at a minimum distance from T. Note that all possible repairs of T that are precise-assignments theories are of the form

$$T^\mu = \{\overline{\mu(\varphi_i)} \equiv P\varphi_i\}_{i=1,\ldots,n}$$

for μ being a rational-valued probability on formulas. In our approach, the degree of inconsistency of T can be seen as providing the minimum Chebyshev distance from T to the set of all its repairs, indeed we have:

$$Incon(T) = \inf_\mu \bigvee_{i=1,\ldots,n} |\mu(\varphi_i) - r_i| =$$
$$= \inf_\mu d_c((\mu(\varphi_1), \ldots, \mu(\varphi_n)), (r_1, \ldots, r_n)) = \inf_\mu d_c(T, T^\mu),$$

where d_c is the well-known Chebyshev distance in $\mathbb{R}^n$.

Suppose now that T represents an imprecise probability assignment

$$T = \{(\overline{r_i - \epsilon_i} \to P\varphi_i) \wedge (P\varphi_i \to \overline{r_i + \epsilon_i})\}_{i=1,\ldots,n}$$

where, for each i, $r_i - \epsilon_i \geqslant 0$ and $r_i + \epsilon_i \leqslant 1$, that is $\epsilon_i \leqslant r_i \leqslant 1 - \epsilon_i$. Then, as shown in Example 2, the degree of inconsistency of T is:

$$Incon(T) = \inf_\mu \bigvee_{i=1,\ldots,n} (1 - \epsilon_i) \otimes |r_i - \mu(\varphi_i)|.$$

Therefore, by defining $d_c^*(T, T^\mu) = \bigvee_{i=1,\ldots,n} (1 - \epsilon_i) \otimes |r_i - \mu(\varphi_i)|$, we can write

$$Incon(T) = \inf_\mu d_c^*(T, T^\mu).$$

Note that the definition of $d_c^*(T, T^\mu)$ is similar to the one of $d_c(T, T^\mu)$ that takes into account the width of the probability intervals assigned to the events in T. However, d_c^* is not symmetric in its arguments since T is in general an imprecise assignment theory, while T^μ is a precise assignment theory. The question is then whether d_c^* can still be considered as a kind of distance. What we can say in this respect is that: i) in the particular case T is a precise assignment theory, then all the ϵ_i's are zero, and thus $d_c^*(T, T^\mu) = d_c(T, T^\mu)$; and ii) it is not hard to check that the following restricted form of the triangle inequality holds for any $\mu, \sigma \in \mathcal{P}(\mathcal{L})$: $d_c^*(T, T^\mu) \leqslant d_c^*(T, T^\sigma) + d_c^*(T^\sigma, T^\mu)$. From all the above, we could claim that $Incon(\cdot)$ belongs (to a high degree) to the family of distance-based inconsistency measures.

On the other hand, in our setting, for a given inconsistent theory T over FP(RPL), a violation-based inconsistency measure should aim at, first, estimating how far every interpretation (i.e. every probability) is from satisfying every formula in T (violation degrees), and then, minimising a suitable aggregation of those degrees. We can show that $Incon(\cdot)$ can be seen as well as a violation-based measure in this sense. Indeed, given a probability μ, we define the violation degree of a formula $\Phi \in T$ by μ as the satisfaction degree of its negation, i.e.

$$vd_\mu(\Phi) = e_\mu(\neg\Phi) = 1 - e_\mu(\Phi),$$

and then we define the global violation degree of T as $vd_\mu(T) = \max_{\Phi \in T} vd_\mu(\Phi)$. Finally, according to Lemma 1, it is straightforward to check that

$$Incon(T) = \inf_\mu dv_\mu(T),$$

that is, $Incon(T)$ is nothing but the infimum of the violation degrees of T by all possible probabilities, and the set of *generalised models* of T are those probabilities yielding a minimum violation degree:

$$GMod(T) = \{\mu \in \mathcal{P}(\mathcal{L}) \mid dv_\mu(T) = Incon(T)\} = \|T\|_{Con(T)}.$$

Finally, we can show that, in our particular case, the set of consequences entailed by the set of generalised models in fact coincides with the common consequences of all theories in $Repairs(T)$. Namely, for a precise-assignment theory T, we have:

$$GMod(T) \subseteq [\![\Phi]\!] \text{ iff } \text{ for all } T^{\mu} \in Repairs(T),\ T^{\mu} \vdash_{RPL} \Phi.$$

6 Conclusions and future work

We have presented some initial steps towards an approach to reason with inconsistent probabilistic theories in the setting of a probabilistic logic defined on top of the $[0,1]$-valued Łukasiewicz fuzzy logic enriched with rational truth-constants, and have put it into relation with other approaches in the literature based on distance-based and violation-based inconsistency measures.

There is a lot of future work to be done, in particular to generalise the approach to deal with inconsistent theories about conditional probabilities. This would need to replace the underlying Łukasiewicz logic by a more powerful one like the $\text{Ł}\Pi\frac{1}{2}$ logic, which combines connectives from Łukasiewicz logic and Product fuzzy logics, as it was done in e.g. [13] to define a logic of conditional probability. Another venue to explore is to replace classical logic as a logic of events by a paraconsistent logic and then define probability on top of that paraconsistent logic, in the line of [4].

Acknowledgments

The authors are grateful to the anonymous reviewers for their helpful comments. They acknowledge partial support by the MOSAIC project (EU H2020- MSCA-RISE-2020 Project 101007627). Flaminio and Godo also acknowledge support by the Spanish project ISINC (PID2019-111544GB-C21) funded by MCIN/AEI/10.13039/501100011033, while Ugolini also acknowledges the Marie Sklodowska-Curie grant agreement No. 890616 (H2020-MSCA-IF-2019).

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