On the set of intermediate logics between the truth and degree preserving Łukasiewicz logics

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Abstract

The aim of this paper is to explore the class of intermediate logics between the truth-preserving Łukasiewicz logic Ł and its degree-preserving companion $Ł^{\leq}$. From a syntactical point of view, we introduce some families of inference rules (that generalize the explosion rule) that are admissible in $Ł^{\leq}$ and derivable in Ł and we characterize the corresponding intermediate logics. From a semantical point of view, we first consider the family of logics characterized by matrices defined by lattice filters in $[0,1]$, but we show there are intermediate logics falling outside this family. Finally, we study the case of finite-valued Łukasiewicz logics where we axiomatize a large family of intermediate logics defined by families of matrices $(\mathbf{A}, F)$ such that $\mathbf{A}$ is a finite MV-algebra and F is a lattice filter.

1 Introduction

In the last two decades, formal systems of fuzzy logic, nowadays under the umbrella of *mathematical fuzzy logic* (MFL) [10], have been proposed and studied as suitable tools for reasoning with propositions containing vague predicates. Their main feature is that they allow us to interpret formulas in a linearly ordered scale of truth values which makes them specially suited for representing gradual aspects of vagueness.

Particular deductive systems in MFL have been usually studied under the paradigm of (full) *truth-preservation* which, generalizing the classical notion of consequence, postulates that a formula follows from a set of premises if every algebraic evaluation that interprets the premises as true also interprets the conclusion as true. In other words, the defining requirement in the truth-preservation paradigm for an inference to be valid is, actually, that every algebraic evaluation that interprets the premises as completely true, will also interpret the conclusion as completely true. An alternative approach that has recently received some attention is based on the *degree-preservation* paradigm (see [15, 5]), in which a conclusion follows from a set of premises if, for all evaluations, the truth degree of the conclusion is not lower than those of the premises. It has been argued that this approach is more coherent with the commitment of many-valued logics to truth-degree semantics because all values play an equally important rôle in the corresponding notion of consequence (see e.g. [14]).

Recall that a logic with a negation $\neg$ is explosive (w.r.t. $\neg$) if from any theory containing a formula φ and its negation $\neg\varphi$ everything follows. That is, any $\neg$-contradictory theory is *explosive*. Paraconsistent logics, by its turn, are logics which contain a negation $\neg$ which is not explosive: that is, there is at least one theory containing some contradiction $\{\varphi, \neg\varphi\}$ which is not explosive (i.e., some formula is not derivable from such theory). As proved in two recent papers [13, 11], while the truth-preserving fuzzy logics are explosive w.r.t. the usual negation $\neg\varphi = \varphi \to \bot$, some (extensions of) degree-preserving fuzzy logics have been shown to exhibit some well-behaved paraconsistency properties. In particular, this is the case of the well-known Łukasiewicz logic Ł, whose degree preserving companion $Ł^{\leq}$ is not explosive, i.e. it is paraconsistent. Actually, the degree-preserving companions of finite-valued Łukasiewicz logics $Ł_n$ belong to the family of paraconsistent logics called *logics of formal inconsistency* (**LFI**s) [7].

Since, for instance, $Ł^{\leq}$ is included in Ł (in terms of their consequence operators), with $Ł^{\leq}$ being paraconsistent and Ł explosive, a natural question that arises in this setting is to ask about possible intermediate logics between $Ł^{\leq}$ and Ł. And in particular, to characterise them and also to study which of them are paraconsistent and which of them are explosive. In this paper we aim at answering these questions. To do this, one can follow two approaches.

From a syntactical point of view, since $Ł^{\leq}$ and Ł have the same theorems, intermediate logics will be necessarily defined as extensions of $Ł^{\leq}$ with inference rules admissible in $Ł^{\leq}$ and derivable in Ł. The problem is how to either find or at least give a characterization of inference rules satisfying these conditions. In this paper we begin with some examples of inference rules that are admissible in $Ł^{\leq}$ and derivable in Ł, but the main results come from the semantical approach.

From a semantical point of view, recall that Ł is complete with respect to all matrices $(\mathbf{A}, F)$ where $\mathbf{A}$ is an MV-algebra and F is the implicative filter $F = \{1^A\}$. Moreover, it is well known that, since Ł is standard complete, this family of matrices can be in fact reduced to only one, the matrix $([\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}, \{1\})$, where $[\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}$ is the MV-algebra over the real interval $[0, 1]$ (see Example 1). On the other hand, $Ł^{\leq}$ is complete with respect to the class of all matrices of type $(\mathbf{A}, F)$ where $\mathbf{A}$ is an MV-algebra and F is a lattice filter of $\mathbf{A}$, see e.g. [15, 5]. Besides, it is proved in [5] that $Ł^{\leq}$ is also complete with respect to the restricted set of all matrices $([\mathbf{0},\mathbf{1}]_{\mathrm{MV}}, F)$, where F is a lattice filter of $[0,1]$, i.e. intervals either of type $[a,1]$ with $a>0$ or of type $(a,1]$, for $a<1$. Therefore, from a semantical point of view, the intermediate logics we are interested in are logics defined from arbitrary sets of matrices of the type $(\mathbf{A}, F)$, where $\mathbf{A}$ is an MV-algebra and F is a lattice filter of $\mathbf{A}$, always including the matrix $([\mathbf{0},\mathbf{1}]_{\mathrm{MV}}, \{1\})$. The problem here is to study and characterize the logics defined by them.

Following a syntactical approach, in this paper we introduce some families of inference rules (inspired in the explosion rule) that are admissible in $Ł^{\leq}$ and derivable in Ł, and we characterize the corresponding intermediate logics. On the other hand, following a semantical approach, we then first study some families of logics characterized by families of matrices $([\mathbf{0},\mathbf{1}]_{\mathrm{MV}}, F)$ where $F \subseteq (0,1]$ is a lattice filter, and we prove that there are another intermediate logics (like the one defined by the explosion inference rule) that are not semantically defined by this type of matrices. Then we restrict ourselves to the case of finite-valued Łukasiewicz logics, where we define and axiomatize a large family of intermediate logics defined by families of matrices $(\mathbf{A}, F)$ with $\mathbf{A}$ being a finite MV-algebra and F is a lattice filter.

As far as we know, the only papers dealing with logics defined by matrices in the framework of the infinite-valued Łukasiewicz logic are [3, 4], where the author studies logics L_F defined by matrices $([\mathbf{0},\mathbf{1}]_{\mathrm{MV}}, F)$ with F being a principal lattice filter. However, these logics are out of the scope of this paper because they are not intermediate for $F \neq \{1\}$. Indeed, the condition to be intermediate is that the set of lattice filters defining the logic has to contain the filter $\{1\}$ in order to be contained in the truth-preserving Łukasiewicz Logic. Nevertheless some results that directly follow from the ones in [4] are included at the beginning of Section 4.

This paper is structured as follows. After this introduction, Section 2 contains some needed preliminaries about Łukasiewicz logics and their degree-preserving companion. Section 3 introduces some intermediate logics defined syntactically by adding to $Ł^{\leq}$ the explosion rule and some of its generalizations, and we characterize these logics semantically using evaluations. Section 4 deals with logics defined semantically by matrices. In its first part we define and axiomatize a family of intermediate logics defined semantically by some families of matrices of type $([\mathbf{0},\mathbf{1}]_{\mathrm{MV}}, F)$ where F is a lattice filter. The second part of Section 4 is devoted to prove that there are intermediate logics not defined by those families of matrices, and we give a general theorem characterizing intermediate logics as logics of matrices over general MV-algebras by lattice filters. In the last sections we study intermediate logics in the framework of finite-valued Łukasiewicz logics $Ł_n$. In Section 5 we give some results towards a general characterization of intermediate logics for finite-valued Łukasiewicz logics. In Section 6 we characterize and axiomatize intermediate logics defined by families of matrices of type $(\mathbf{A}, F)$ where $\mathbf{A}$ is a direct product of copies of $\mathbf{ŁV}_n$ (the MV-algebra associated to $Ł_n$) and F is a lattice filter. The lattices of these intermediate logics for $n=3$ and $n=4$ are described in Appendices A1 and A2. Finally, in Section 7 the case of $Ł_n$ when $n-1$ is a prime number is analyzed. The lattice of all intermediate logics for Ł$_3$ and Ł$_4$ are fully described in Appendices B1 and B2 respectively. The paper ends with some conclusions and further research proposals.

2 Preliminaries on Łukasiewicz logic Ł and the degree preserving companion Ł$^{\leq}$

2.1 Łukasiewicz logic and MV-algebras

The logical setting in which we frame our study is that of infinite-valued Łukasiewicz logic Ł, and its finite-valued axiomatic extensions Ł$_k$. Formulas of (any finite-valued) Łukasiewicz logic are inductively defined from a countable set $V = \{p_1, p_2, \ldots\}$ of variables, along with the binary connective $\to$ and the unary connective $\neg$. We will denote by $\mathfrak{F}(V)$ the class of formulas defined from the set of variables V.

Further connectives are definable from $\to$ and $\neg$ as follows:

$$
\begin{array}{lll}
\varphi \oplus \psi & \text{is} & \neg\varphi \to \psi \\
\varphi \otimes \psi & \text{is} & \neg(\neg\varphi \oplus \neg\psi) \\
\varphi \vee \psi & \text{is} & (\varphi \to \psi) \to \psi \\
\varphi \wedge \psi & \text{is} & \neg(\neg\varphi \vee \neg\psi) \\
\varphi \leftrightarrow \psi & \text{is} & (\varphi \to \psi) \wedge (\psi \to \varphi)
\end{array}
$$

The truth constant $\top$ is $\varphi \to \varphi$ and the truth constant $\perp$ is $\neg\top$, and we will henceforth use sometimes the following abbreviations: for every $n \in \mathbb{N}$ and for every $\varphi \in \mathfrak{F}(V)$, $n\varphi$ will stand for $\varphi \oplus \cdots \oplus \varphi$ (n-times), and φ^n will stand for $\varphi \otimes \cdots \otimes \varphi$ (n-times). When $n = 0$ we take $n\varphi = \varphi^n = \top$.

The propositional Łukasiewicz logic (Ł in symbols) is defined as the following Hilbert style system of axioms and rule (cf. [18]):

(Ł1) $\varphi \to (\psi \to \varphi)$,

(Ł2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,

(Ł3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$,

(Ł4) $(\varphi \vee \psi) \to (\psi \vee \varphi)$,

(MP) The rule of modus ponens: $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$.

For every $k \in \mathbb{N}$ with $k \geq 2$, the k-valued Łukasiewicz logic Ł$_k$ is the axiomatic extension of Ł defined by the following axioms (cf. [17, 18]):

(Ł5) $(k-1)\varphi \leftrightarrow k\varphi$,

(Ł6) $(n\varphi^{n-1})^k \leftrightarrow k\varphi^n$,
for every $n = 2, 3, \ldots, k-2$ that does not divide $k-1$.

The notion of *deduction* and *proof* in Ł or in $Ł_k$ are the usual ones (see e.g. [18]). A *theory* is any subset of $\mathfrak{F}(V)$, and for every theory Γ and for every formula φ we will write $\Gamma \vdash \varphi$ if φ can be proved from Γ in the logic $Ł_k$.

The algebraic counterpart of (resp. finite-valued) Łukasiewicz calculus is the class of (resp. finite-valued) *MV-algebras*. An MV-algebra (cf. [9, 18, 19]) is a system $\mathbf{M} = (M, \oplus, \neg, 0^M)$ of type $(2, 1, 0)$ such that the reduct $(M, \oplus, 0^M)$ is a commutative monoid, and the following equations hold:

(MV1) $x \oplus 1^M = 1^M$,

(MV2) $\neg\neg x = x$,

(MV3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

where in (MV1), 1^M stands for $\neg 0^M$.

For every $k \in \mathbb{N}$ with $k \geq 2$, an *MV_k-algebra* is an MV-algebra that also satisfies:

(MV4) $kx = (k-1)x$,

(MV5) $(nx^{n-1})^k = kx^n$, for every $n = 2, 3, \ldots, k-2$ not dividing $k-1$,

where, for every $n \geq 1$, $nx = x \oplus \cdots \oplus x$ (n-times), and $x^n = x \otimes \cdots \otimes x$ (n-times) [17]. When $n = 0$, $nx = x^n = 1^M$. As in the case of the logical language, here other operations can be defined as well, among them $x \to y$ is $\neg x \oplus y$ and $x \otimes y$ is $\neg(\neg x \oplus \neg y)$.

In every MV-algebra $\mathbf{M}$ we can define an order relation by the following stipulation: for every $x, y \in M$,

$$x \leq y \text{ iff } \neg x \oplus y = 1^M.$$

An MV-algebra is said to be linearly ordered, or an MV-chain, provided that the order $\leq$ is linear. The class of MV-algebras, $\mathbb{MV}$, constitutes a variety (i.e. an equational class [6]).

Example 1 (Standard Algebras)**.** (1) Equip the real unit interval $[0, 1]$ with the operations of

- truncated sum: for all $x, y \in [0, 1]$, $x \oplus y = \min(1, x + y)$,
- standard negation: for all $x \in [0, 1]$, $\neg x = 1 - x$.

Then the algebra $[\mathbf{0}, \mathbf{1}]_{\mathrm{MV}} = ([0, 1], \oplus, \neg, 0)$ is an MV-algebra called the *standard MV-algebra*. The variety of MV-algebras $\mathbb{MV}$ is generated, as a variety and as a quasi-variety, by $[0, 1]_{\mathrm{MV}}$ (cf. [8, 9]). This means that, in order to show that a given equality, or quasi-equality, written in the algebraic language of MV-algebras, holds in every MV-algebra, it is sufficient to check whether it holds in $[\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}$.

(2) For every $k \in \mathbb{N}$, let $ŁV_k = \{0, \frac{1}{k-1}, \ldots, \frac{k-2}{k-1}, 1\}$. Equip $ŁV_k$ with the restrictions to $ŁV_k$ of the above defined truncated sum and standard negation.

We will henceforth denote by $\mathbf{ŁV}_k$ the obtained structure, that is usually called the *standard* MV_k*-algebra*. The variety of MV_k-algebras is generated by $\mathbf{ŁV}_k$ (cf. [9]).

MV-algebras constitute the equivalent algebraic semantics for Łukasiewicz logic.[1] Similarly, for every k, MV_k-algebras form a variety, $\mathbb{MV}_k$, that is the equivalent algebraic semantics for $Ł_k$. Among other things, this implies that Łukasiewicz logic is (strongly) *complete* with respect to the class of MV-algebras, and that $Ł_k$ is (strongly) complete with respect to class of MV_k-algebras as well. This means the following. Let an *evaluation* e of formulas of $\mathfrak{F}(V)$ into an MV-algebra (MV_k-algebra) $\mathbf{M}$ be any map $e : V \to M$ that extends to compound formulas by truth functionality using the operations in $\mathbf{M}$. We say that e is a model of (or satisfies) a formula $\varphi \in \mathfrak{F}(V)$ when $e(\varphi) = 1^M$. Then, for any set of formulas $\Gamma \cup \{\varphi\} \subseteq \mathfrak{F}(V)$, $\Gamma \vdash \varphi$ iff for any MV-algebra $\mathbf{M}$ and any $\mathbf{M}$-evaluation e, if $e(\psi) = 1^M$ for any $\psi \in \Gamma$, then $e(\varphi) = 1^M$ as well.

But clearly, the above examples (and the results cited therein) show a stronger version of completeness for Ł and $Ł_k$ that we are going to make clear as follows.

Theorem 1. (1) *Łukasiewicz logic is* finitely strong standard complete*, i.e.: for every finite set of formulas* $\Gamma \cup \{\varphi\} \subseteq \mathfrak{F}(V)$*,* $\Gamma \vdash \varphi$ *in Ł iff every evaluation into the MV-algebra* $[\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}$ *that satisfies* Γ*, satisfies* φ *as well.*

(2) *For every* $k \in \mathbb{N}$*,* $Ł_k$ *is* strong real complete*, i.e.: for every set of formulas* $\Gamma \cup \{\varphi\} \subseteq \mathfrak{F}(V)$*,* $\Gamma \vdash \varphi$ *in* $Ł_k$ *iff every evaluation into the* MV_k*-algebra* $\mathbf{ŁV}_k$ *that satisfies* Γ*, satisfies* φ *as well.*

Remark 1. Every finite MV-algebra $\mathbf{M}$ can be represented as a finite direct product of finite MV-chains. In other words, for every finite MV-algebra $\mathbf{M}$, there exist a finite set of finite MV-chains $\mathbf{S}_1, \ldots, \mathbf{S}_k$, such that $\mathbf{M}$ is isomorphic to the direct product $\Pi_{i=1}^k \mathbf{S}_i$.

2.2 The degree-preserving companion of Łukasiewicz logic

Łukasiewicz logic Ł, and the main logics studied in Mathematical Fuzzy Logic, is a (full) truth-preserving fuzzy logic (in the sense that inference in these logics preserves the truth-value 1). But besides the truth-preserving paradigm so far considered, one can find an alternative approach in the literature, first introduced for Łukasiewicz logic by Wójcicki [22, 4.3.14] and then further explored in [15]. Based on the definitions in [15], we introduce the variant of Ł, that we shall denote by $Ł^{\leq}$, whose associated consequence relation is semantically defined as follows: for every finite set of formulas $\Gamma \cup \{\varphi\}$,

[1] Actually, the equivalent algebraic semantics of Ł, properly speaking, is the variety of Wajsberg algebras [16], although, as it is well-known, Wajsberg and MV-algebras are term-equivalent.

$$\Gamma \models_{\text{Ł}^{\leq}} \varphi \quad \text{iff} \quad \begin{array}{l}\text{for every evaluation } v \text{ over } [0,1]_{\text{MV}} \text{ and every } a \in [0,1],\\ \text{if } a \leq v(\gamma) \text{ for every } \gamma \in \Gamma, \text{ then } a \leq v(\varphi).^2\end{array}$$

If Γ is infinite, stipulate that $\Gamma \models_{\text{Ł}^{\leq}} \varphi$ when there exists a finite subset $\Gamma^0 \subset \Gamma$ such that $\Gamma^0 \models_{\text{Ł}^{\leq}} \varphi$. So defined, $\text{Ł}^{\leq}$ is known as the Łukasiewicz logic *preserving degrees of truth*, or the *degree-preserving companion* of Ł. Clearly, Ł and $\text{Ł}^{\leq}$ have the same theorems and, moreover, for every finite set of formulas $\Gamma \cup \{\varphi\}$:

$$\Gamma \models_{\text{Ł}^{\leq}} \varphi \text{ iff } \vdash_{\text{Ł}} \Gamma^{\wedge} \to \varphi,$$

where $\Gamma^{\wedge}$ means $\gamma_1 \wedge \ldots \wedge \gamma_k$ for $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ (when Γ is empty then $\Gamma^{\wedge}$ is $\top$).

As regards axiomatization, the logic $\text{Ł}^{\leq}$ admits a Hilbert-style axiomatization having the same axioms as Ł and the following deduction rules [5]:

(Adj-$\wedge$) $\quad \dfrac{\varphi \quad \psi}{\varphi \wedge \psi}$,

(MP-r) if $\vdash_{\text{Ł}} \varphi \to \psi$ (i.e. if $\varphi \to \psi$ is a theorem of Ł), then from φ derive ψ.

We will denote by $\vdash_{\text{Ł}^{\leq}}$ the corresponding consequence relation associated to the Hilbert calculus for $\text{Ł}^{\leq}$.

In [15] it is shown that the logic $\text{Ł}^{\leq}$ is not algebraizable in the sense of Blok and Pigozzi, but nevertheless it has a suitable semantics via logical matrices.

In general, by a logical matrix we understand a pair $(\mathbf{A}, F)$ where $\mathbf{A}$ is an algebra and F is a subset of *designated* elements of A. The logic L induced by the matrix $(\mathbf{A}, F)$ is defined as follows: for any subset of formulas $\Gamma \cup \{\varphi\}$,

$\Gamma \vdash_L \varphi$ if, for any evaluation e on $\mathbf{A}$, if $e(\psi) \in F$ for all $\psi \in \Gamma$, then $e(\varphi) \in F$.

The logic determined by a class of matrices is defined as the intersection of the logics defined by all the matrices in the family.

The matrices we will deal with in this paper will be pairs $(\mathbf{A}, F)$ where $\mathbf{A}$ is an MV-algebra and F is either an *implicative* or a *lattice filter* of A.[3] It is well-known [16, 21] that (infinite-valued) Łukasiewicz logic Ł is (strongly) complete with respect to the class of matrices

$$\{(\mathbf{A}, F) : \mathbf{A} \text{ is an MV-algebra and } F \text{ is an implicative filter of } \mathbf{A}\},$$

and also with respect to its subclass of matrices

$$\{(\mathbf{A}, \{1^A\}) : \mathbf{A} \text{ is an MV-algebra}\},$$

that are its reduced models. Moreover, Ł is finitely strong complete with respect to the single matrix $([\mathbf{0}, \mathbf{1}]_{\text{MV}}, \{1\})$, this is Theorem 1. On the other hand, the degree-preserving companion of Łukasiewicz logic $\text{Ł}^{\leq}$ is *complete* with respect to the class of matrices

[2] This condition is equivalent to require that for every evaluation v over $[0,1]_{\text{MV}}$, $\min\{v(\gamma) \mid \gamma \in \Gamma\} \leq v(\varphi)$.

[3] F is a *lattice filter* of a MV-algebra $\mathbf{A}$ if i) $1^A \in F$, ii) if $x \in F$ and $x \leq y$ then $y \in F$, and iii) if $x, y \in F$ then $x \wedge y \in F$. F is an *implicative filter* if it is a lattice filter and it is closed by modus ponens, that is, if $x, x \to y \in F$ then $y \in F$ as well.

$$\{(\mathbf{A}, F) : \mathbf{A} \text{ is an MV-algebra and } F \text{ is a lattice filter of } \mathbf{A}\},$$

see [15]. Moreover, in [15] it is also proved that $Ł^{\leq}$ is complete with respect to the smaller class of matrices over the standard MV-algebra:

$$\{([\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}, F) : F \text{ is a lattice filter of } [\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}\}.$$

Analogous results and relationships hold for the case of truth-preserving and degree-preserving finite-valued Łukasiewicz logics $Ł_k$ and $Ł_k^{\leq}$, replacing MV-algebras by MV_k-algebras and $[\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}$ by $\mathbf{ŁV}_k$.

3 Some syntactically defined intermediate logics

Recall (see, for instance, [7]) that a logic L containing a negation $\neg$ is said to be *explosive* (w.r.t. $\neg$) if, from any theory containing a formula φ and its negation $\neg\varphi$, any other formula can be derived: for every set of formulas $\Gamma \cup \{\varphi, \neg\varphi\}$,

$$\Gamma, \varphi, \neg\varphi \vdash_L \psi$$

for every formula ψ. On the other hand, L is said to be *paraconsistent* (w.r.t. $\neg$) if it is not explosive, that is: there is a set of formulas $\Gamma \cup \{\varphi, \neg\varphi\}$ such that

$$\Gamma, \varphi, \neg\varphi \not\vdash_L \psi$$

for some formula ψ.

As observed in [13, 11], while the truth-preserving fuzzy logics are explosive w.r.t. the usual negation ($\neg\varphi$ is defined as $\varphi \to \bot$), some (extensions of) degree-preserving fuzzy logics are paraconsistent. In particular, this is the case of Łukasiewicz logic Ł, which is explosive while its degree-preserving companion $Ł^{\leq}$ is paraconsistent.[4]

3.1 Adding the explosion rule to $Ł^{\leq}$

Given the aim of this paper, it seems very natural to begin with the study of the logic $Ł^{\leq}_{exp}$, the *weakest explosive intermediate logic*, defined syntactically as the extension of the usual Hilbert-style calculus for the logic $Ł^{\leq}$ with the *explosion inference rule*:

$$(exp)\ \ \frac{\varphi \quad \neg\varphi}{\bot}$$

It is clear that (exp) is admissible in $Ł^{\leq}$ (it does not add new theorems) and derivable in Ł, since it is a particular case of *modus ponens* rule (notice that $\neg\varphi = \varphi \to \bot$), but it is not weaker than the restricted modus ponens rule

[4]It is clear that in Ł, from $\{\varphi, \neg\varphi\}$ we can derive anything, since we have $\varphi, \neg\varphi \vdash_{Ł} \varphi \otimes \neg\varphi$, and $\varphi \otimes \neg\varphi \to \bot$ and $\bot \to \psi$ are theorems of Ł. On the contrary, in $Ł^{\leq}$ we have e.g. (for any propositional variable p) that $p, \neg p \not\vdash_{Ł^{\leq}} p \otimes \neg p$ and so $\{p, \neg p\}$ is not explosive. By the way, we have that $p, \neg p \vdash_{Ł^{\leq}} p \wedge \neg p$, but $\varphi \wedge \neg\varphi \to \bot$ is not a theorem of Ł.

(MP-r) used in the definition of $Ł^{\leq}$.

Notation. To simplify notation, from now on we will write $\vdash^{\leq}$ for $\vdash_{Ł^{\leq}}$, and $\vdash^{\leq}_{exp}$ for $\vdash_{Ł^{\leq}_{exp}}$.

Next lemmas show straightforward properties of the logic $Ł^{\leq}_{exp}$.

Lemma 1. *$\Gamma \vdash^{\leq}_{exp} \bot$ iff there exists φ such that $\Gamma \vdash^{\leq} \varphi \wedge \neg\varphi$.*

Proof. From right to left is immediate. Assume $\Gamma \vdash^{\leq}_{exp} \bot$ and consider the following two cases:

- if $\Gamma \vdash^{\leq} \bot$, then trivially $\Gamma \vdash^{\leq} \varphi \wedge \neg\varphi$.
- if $\Gamma \not\vdash^{\leq} \bot$, in any proof of $\bot$ from Γ in $Ł^{\leq}_{exp}$ there must be a first application of the rule (exp) to some pair of formulas φ and $\neg\varphi$. Therefore, both φ and $\neg\varphi$ have been proved without using the rule (exp), hence they are provable from Γ in the logic $Ł^{\leq}$. From this, $\Gamma \vdash^{\leq} \varphi \wedge \neg\varphi$, by rule (Adj-$\wedge$). □

Lemma 2. *If $\Gamma \not\vdash^{\leq}_{exp} \bot$ then, for every φ, it holds that $\left[\Gamma \vdash^{\leq}_{exp} \varphi \text{ iff } \Gamma \vdash^{\leq} \varphi\right]$.*

Proof. It is clear that $\Gamma \vdash^{\leq} \varphi$ implies $\Gamma \vdash^{\leq}_{exp} \varphi$. Therefore we have to prove that if $\Gamma \vdash^{\leq}_{exp} \varphi$ and $\Gamma \not\vdash^{\leq}_{exp} \bot$ then $\Gamma \vdash^{\leq} \varphi$. But this is easy, since if in a proof of φ from Γ in $Ł^{\leq}_{exp}$ the rule (exp) is applied, then we would have $\Gamma \vdash^{\leq}_{exp} \bot$, against the hypothesis. Thus, in no proof of φ from Γ in $Ł^{\leq}_{exp}$ the rule (exp) is applied, hence this means that $\Gamma \vdash^{\leq} \varphi$. □

Actually, the previous lemmas allow us to express $\vdash^{\leq}_{exp}$ only in terms of $\vdash^{\leq}$.

Proposition 1.

$$\begin{array}{llll} \Gamma \vdash^{\leq}_{exp} \varphi & \textit{iff} & \textit{either} & \textit{there exists } \psi \textit{ such that } \Gamma \vdash^{\leq} \psi \wedge \neg\psi, \\ & & \textit{or} & \Gamma \vdash^{\leq} \varphi. \end{array}$$

Proof. From left to right, suppose that $\Gamma \vdash^{\leq}_{exp} \varphi$. There are two case to analyze:

Case 1: $\Gamma \vdash^{\leq}_{exp} \bot$. Then, there exists ψ such that $\Gamma \vdash^{\leq} \psi \wedge \neg\psi$, by Lemma 1.

Case 2: $\Gamma \not\vdash^{\leq}_{exp} \bot$. Then, by Lemma 2, $\Gamma \vdash^{\leq} \varphi$ since, by hypothesis, $\Gamma \vdash^{\leq}_{exp} \varphi$.

From right to left, suppose first that $\Gamma \vdash^{\leq} \psi \wedge \neg\psi$ for some ψ. Then $\Gamma \vdash^{\leq}_{exp} \bot$, by Lemma 1, hence $\Gamma \vdash^{\leq}_{exp} \varphi$. On the other hand, if $\Gamma \vdash^{\leq} \varphi$ then obviously $\Gamma \vdash^{\leq}_{exp} \varphi$. □

As a consequence, since the semantics for $\vdash^{\leq}$ is clear and well-known, we can establish the exact semantics that characterizes the logic $Ł^{\leq}_{exp}$.

Notation. In the following, given a finite set of formulas Γ, we will use $\Gamma^{\wedge}$ to denote a $\wedge$-conjunction of all its formulas, $\bigwedge_{\varphi\in\Gamma} \varphi$ (if $\Gamma = \emptyset$ then $\Gamma^{\wedge}$ is $\top$).

Lemma 3. *There exists φ such that $\Gamma \vdash^{\leq} \varphi \wedge \neg\varphi$ iff for every $[\mathbf{0},\mathbf{1}]_{\mathrm{MV}}$-evaluation e, $e(\Gamma^{\wedge}) \leq 1/2$.*

Proof. Since, for any evaluation e we have $e(\varphi \wedge \neg\varphi) \leq 1/2$, the left to right direction is immediate. Assume now that for every evaluation e, $e(\Gamma^{\wedge}) \leq 1/2$. It is clear that then, for every evaluation e, $e(\neg\Gamma^{\wedge}) \geq 1/2$, and hence $e(\Gamma^{\wedge}) = e(\Gamma^{\wedge} \wedge \neg\Gamma^{\wedge})$. Take $\varphi = \Gamma^{\wedge}$. Therefore, by completeness of $\vdash^{\leq}$, this means that $\Gamma \vdash^{\leq} \varphi \wedge \neg\varphi$. $\square$

Proposition 2 (Soundness and Completeness of $Ł^{\leq}_{exp}$ w.r.t. $[\mathbf{0},\mathbf{1}]_{\mathrm{MV}}$)**.** *For any set of formulas $\Gamma \cup \{\varphi\}$, we have:*

$$\begin{array}{llll} \Gamma \vdash^{\leq}_{exp} \varphi & \textit{iff} & \textit{either} & \textit{for every } [\mathbf{0},\mathbf{1}]_{\mathrm{MV}}\textit{-evaluation } e,\ e(\Gamma^{\wedge}) \leq 1/2, \\ & & \textit{or} & \textit{for every } [\mathbf{0},\mathbf{1}]_{\mathrm{MV}}\textit{-evaluation } e,\ e(\Gamma^{\wedge}) \leq e(\varphi). \end{array}$$

Proof. It is a direct consequence of Proposition 1, Lemma 3 and the soundness and completeness of $Ł^{\leq}$ with respect to evaluations over $[\mathbf{0},\mathbf{1}]_{\mathrm{MV}}$. $\square$

This makes it clear that the corresponding notion of inconsistency in the logic $Ł^{\leq}_{exp}$ leading to explosion is somewhat more demanding than in the 1-preserving logic Ł: while the semantic condition for a set of formulas Γ to be inconsistent in Ł is that $e(\Gamma^{\wedge}) < 1$ for every evaluation e, in $Ł^{\leq}_{exp}$ the condition is strengthened to require $e(\Gamma^{\wedge}) \leq 1/2$ for any evaluation e.

3.2 There are infinitely-many paraconsistent and explosive intermediate logics

Once we have identified the weakest explosive logic, it is not difficult to define countable families of paraconsistent and explosive intermediate logics, only by slightly modifying the explosion rule (exp). Namely, let us consider, for each natural k, the following inference rules:

$$(exp^{-}_{k}) \quad \frac{\varphi \quad \neg(\varphi \oplus \overset{k}{\ldots} \oplus \varphi)}{\perp}$$

$$(exp^{+}_{k}) \quad \frac{\varphi \quad \neg(\varphi \otimes \overset{k}{\ldots} \otimes \varphi)}{\perp}$$

For $k = 1$ we recover the explosion rule: in fact, $(exp) = (exp^{-}_{1}) = (exp^{+}_{1})$. But for $k > 1$, it is clear that (exp^{-}_{k}) is strictly weaker than (exp) while (exp^{+}_{k}) is strictly stronger. This easily follows respectively from observing that, for $k > 1$, $\neg(\varphi \oplus \overset{k}{\ldots} \oplus \varphi) \to \neg\varphi$ and $\neg\varphi \to \neg(\varphi \otimes \overset{k}{\ldots} \otimes \varphi)$ are theorems in Łukasiewicz logic, while the converse implications are not.

Let us then consider the logics $Ł^{(k)}_{exp^{-}}$ and $Ł^{(k)}_{exp^{+}}$ to be the extensions of $Ł^{\leq}$ with the inference rules (exp^{-}_{k}) and (exp^{+}_{k}) respectively. From the above observations it follows that $Ł^{(k)}_{exp^{-}} \subset Ł^{\leq}_{exp} \subset Ł^{(k)}_{exp^{+}}$. Since $Ł^{\leq}_{exp}$ is the weakest explosive intermediate logic, it follows that all the logics $Ł^{(k)}_{exp^{-}}$ with $k > 1$ are paraconsistent while all the logics $Ł^{(k)}_{exp^+}$ are explosive. Moreover, they form a chain of intermediate logics with the following strict inclusions:[5]

$$Ł^{\leq} \subset \ldots \subset Ł^{(k)}_{exp^-} \subset \ldots \subset Ł^{(2)}_{exp^-} \subset Ł^{\leq}_{exp} \subset Ł^{(2)}_{exp^+} \subset \ldots \subset Ł^{(k)}_{exp^+} \subset \ldots \subset Ł$$

Therefore, as we can see, there are at least countably many paraconsistent and countably many explosive logics between $Ł^{\leq}$ and Ł. Moreover, as an easy generalization of the results for the logic $Ł^{\leq}_{exp}$ we can obtain the following result:

Proposition 3 (Soundness and Completeness)**.** *For any set of formulas* $\Gamma \cup \{\varphi\}$*, we have:*

$$\begin{array}{llll}
\Gamma \vdash^{(k)}_{exp^-} \varphi & \textit{iff} & \textit{either} & \textit{for every evaluation } e \textit{ over } [\mathbf{0},\mathbf{1}]_{\mathrm{MV}},\ e(\Gamma^{\wedge}) \leq 1/(k+1), \\
 & & \textit{or} & \textit{for every evaluation } e \textit{ over } [\mathbf{0},\mathbf{1}]_{\mathrm{MV}},\ e(\Gamma^{\wedge}) \leq e(\varphi). \\
\Gamma \vdash^{(k)}_{exp^+} \varphi & \textit{iff} & \textit{either} & \textit{for every evaluation } e \textit{ over } [\mathbf{0},\mathbf{1}]_{\mathrm{MV}},\ e(\Gamma^{\wedge}) \leq k/(k+1), \\
 & & \textit{or} & \textit{for every evaluation } e \textit{ over } [\mathbf{0},\mathbf{1}]_{\mathrm{MV}},\ e(\Gamma^{\wedge}) \leq e(\varphi).
\end{array}$$

Actually it is also very easy to further generalize the inference rules (exp^-_k) and (exp^+_k) by considering, for instance, the following rules:

$$(exp^-_{k,m}) \quad \frac{\varphi \quad \neg((\varphi \otimes \overset{m}{\cdots} \otimes \varphi) \oplus \overset{k}{\cdots} \oplus (\varphi \otimes \overset{m}{\cdots} \otimes \varphi))}{\bot}$$

$$(exp^+_{k,m}) \quad \frac{\varphi \quad \neg((\varphi \oplus \overset{m}{\cdots} \oplus \varphi) \otimes \overset{k}{\cdots} \otimes (\varphi \oplus \overset{m}{\cdots} \oplus \varphi))}{\bot}$$

It is obvious that, for each k we have $(exp^-_{k,1}) = (exp^-_k)$ and $(exp^+_{k,1}) = (exp^+_k)$, and for each m, we have $(exp^-_{1,m}) = (exp^+_m)$ and $(exp^+_{1,m}) = (exp^-_m)$. Hence, in particular, $(exp^-_{1,1}) = (exp^+_{1,1}) = (exp)$.

Now, let us define the logics $Ł^{(k,m)}_{exp^-}$ and $Ł^{(k,m)}_{exp^+}$ as the extensions of $Ł^{\leq}$ with the inference rules $(exp^-_{k,m})$ and $(exp^+_{k,m})$ respectively. Then for instance, we have that $Ł^{(1,1)}_{exp^-} = Ł^{(1,1)}_{exp^+} = Ł^{\leq}_{exp}$, $Ł^{(k,1)}_{exp^-} = Ł^{(k)}_{exp^-}$ and $Ł^{(k,1)}_{exp^+} = Ł^{(k)}_{exp^+}$. Therefore we get two doubly infinite families of intermediate logics, one between $Ł^{\leq}$ and $Ł^{\leq}_{exp}$ and another between $Ł^{\leq}_{exp}$ and Ł. Once again, in order to simplify notation from now on we will write $\vdash^{(k,m)}_{exp^-}$ and $\vdash^{(k,m)}_{exp^+}$ to denote their corresponding syntactic consequence relation.

Now we can proceed in an analogous way as the previous section to characterize the logics $Ł^{(k,m)}_{exp^-}$ and $Ł^{(k,m)}_{exp^+}$. We will omit proofs that are very similar.

[5]It is worth noting that, similarly, other generalizations of the explosion rule have been defined in [20] leading to a denumerable chain of logics, called $\mathcal{B}_n$, between the Belnap-Dunn logic and Kleene 3-valued logic.

Notation. In the following, to simplify notation we will write $\varphi^-_{k,m}$ and $\varphi^+_{k,m}$ as compact notations for the formulas $\varphi \wedge \neg((\varphi \otimes \overset{m}{\dots} \otimes \varphi) \oplus \overset{k}{\dots} \oplus (\varphi \otimes \overset{m}{\dots} \otimes \varphi))$ and $\varphi \wedge \neg((\varphi \oplus \overset{m}{\dots} \oplus \varphi) \otimes \overset{k}{\dots} \otimes (\varphi \oplus \overset{m}{\dots} \oplus \varphi))$ respectively.

Lemma 4. *For each set of formulas* Γ *we have:*

(i) $\Gamma \vdash^{(k,m)}_{exp^-} \perp$ *iff there exists* φ *such that* $\Gamma \vdash^{\leq} \varphi^-_{k,m}$.
$\Gamma \vdash^{(k,m)}_{exp^+} \perp$ *iff there exists* φ *such that* $\Gamma \vdash^{\leq} \varphi^+_{k,m}$.

(ii) If $\Gamma \nvdash^{(k,m)}_{exp^-} \perp$ *then* $\left[\Gamma \vdash^{(k,m)}_{exp^-} \varphi \text{ iff } \Gamma \vdash^{\leq} \varphi\right]$
If $\Gamma \nvdash^{(k,m)}_{exp^+} \perp$ *then* $\left[\Gamma \vdash^{(k,m)}_{exp^+} \varphi \text{ iff } \Gamma \vdash^{\leq} \varphi\right]$

From the previous lemma, it is possible to express $\vdash^{\leq}_{-(k,m)}$ and $\vdash^{\leq}_{+(k,m)}$ in terms of $\vdash^{\leq}$.

Proposition 4. *For any set of formulas* $\Gamma \cup \{\varphi\}$*, we have:*

(i) $\Gamma \vdash^{(k,m)}_{exp^-} \varphi$ *iff either there exists* ψ *such that* $\Gamma \vdash^{\leq} \psi^-_{k,m}$,
or $\Gamma \vdash^{\leq} \varphi$.

(ii) $\Gamma \vdash^{(k,m)}_{exp^+} \varphi$ *iff either there exists* ψ *such that* $\Gamma \vdash^{\leq} \psi^+_{k,m}$,
or $\Gamma \vdash^{\leq} \varphi$.

As a consequence, the semantics over $[\mathbf{0},\mathbf{1}]_{\mathrm{MV}}$ that characterize the logics $Ł^{(k,m)}_{exp^-}$ and $Ł^{(k,m)}_{exp^+}$ can now be established after two preliminary lemmas.

Lemma 5. *For any evaluation* e *on* $[\mathbf{0},\mathbf{1}]_{\mathrm{MV}}$ *and any formula* φ *we have:*

$$e(\varphi^-_{k,m}) \leq \frac{k(m-1)+1}{km+1} \qquad \text{and} \qquad e(\varphi^+_{k,m}) \leq \frac{k}{km+1}\ .$$

Proof. We prove the condition for $\varphi^+_{k,m}$, the one for $\varphi^-_{k,m}$ is analogous. Let $f(x) = 1 - (mx \otimes \overset{k}{\dots} \otimes mx)$. Then it is routine to check that

$$f(x) = \begin{cases} 1, & \text{if } x \leq \frac{1+k}{km} \\ k(mx-1), & \text{if } \frac{1+k}{km} < x < \frac{1}{m} \\ 0, & \text{otherwise.} \end{cases}$$

Since $f(x)$ is monotonically decreasing, $\min(x, f(x)) \leq y$, where y is such that $y = f(y)$, that is, $y = \frac{k}{km+1}$. Therefore, taking $x = e(\varphi)$, we have $e(\varphi^-_{k,m}) = \min(e(\varphi), f(e(\varphi))) \leq \frac{k}{km+1}$. □

Lemma 6. *For any set of formulas* Γ *we have:*

(i) There exists φ such that $\Gamma \vdash^{\leq} \varphi^{-}_{k,m}$ iff for every evaluation e, $e(\Gamma^{\wedge}) \leq \frac{k(m-1)+1}{km+1}$.

(ii) There exists φ such that $\Gamma \vdash^{\leq} \varphi^{+}_{k,m}$ iff for every evaluation e, $e(\Gamma^{\wedge}) \leq \frac{k}{km+1}$.

As in the previous section, from the above lemmas we can characterize the logics $Ł^{(k,m)}_{exp^-}$ and $Ł^{(k,m)}_{exp^+}$ with respect to semantics over the standard MV-algebra $[\mathbf{0},\mathbf{1}]_{\mathrm{MV}}$ as follows.

Proposition 5 (Soundness and completeness)**.** *For any subset of formulas $\Gamma \cup \{\varphi\}$, we have:*

$$\Gamma \vdash^{(k,m)}_{exp^-} \varphi \quad \text{iff} \quad \begin{array}{ll} \text{either} & \text{for every } [\mathbf{0},\mathbf{1}]_{\mathrm{MV}}\text{-evaluation } e,\ e(\Gamma^{\wedge}) \leq \frac{k(m-1)+1}{km+1}, \\ \text{or} & \text{for every } [\mathbf{0},\mathbf{1}]_{\mathrm{MV}}\text{-evaluation } e,\ e(\Gamma^{\wedge}) \leq e(\varphi). \end{array}$$

$$\Gamma \vdash^{(k,m)}_{exp^+} \varphi \quad \text{iff} \quad \begin{array}{ll} \text{either} & \text{for every } [\mathbf{0},\mathbf{1}]_{\mathrm{MV}}\text{-evaluation } e,\ e(\Gamma^{\wedge}) \leq \frac{k}{km+1}, \\ \text{or} & \text{for every } [\mathbf{0},\mathbf{1}]_{\mathrm{MV}}\text{-evaluation } e,\ e(\Gamma^{\wedge}) \leq e(\varphi). \end{array}$$

We omit the proof since it is a matter of routine to check the details.

Finally, as a consequence of this characterization, we can establish when these logics are paraconsistent or explosive.

Proposition 6. *For each natural k and m, the following hold:*

(i) The logics $Ł^{(k,m)}_{exp^-}$ are paraconsistent if $m = 1$, and explosive otherwise.

(ii) The logics $Ł^{(k,m)}_{exp^+}$ are explosive if $m = 1$, and paraconsistent otherwise.

Proof. By the previous Proposition 5, it reduces to check when the values $(k(m-1)+1)/(km+1)$ and $k/(km+1)$ are less than $1/2$ (paraconsistent) or greater or equal than $1/2$ (explosive). $\square$

In particular, this last proposition tells us that the only paraconsistent intermediate logics defined in this section are of the form $Ł^{(k)}_{exp^-}$ for $k > 1$.

4 Intermediate logics defined by matrices with lattice filters

In this section we begin by exploring the definition of intermediate logics defined by matrices over the standard MV-algebra $[\mathbf{0},\mathbf{1}]_{\mathrm{MV}}$. Then we show that these matrices are not enough to cover the logics syntactically defined in the previous section by adding to $Ł^{\leq}$ the explosion rule and some variants of them.

4.1 Intermediate logics defined by matrices with principal filters over [0, 1]

This small subsection contains some results that are easy consequences, into the framework of intermediate logics, of results originally obtained in [4]. Let $\bar{F}^x = [x, 1]$ be the principal lattice filter defined by $x \in (0, 1]$. In the cited paper, Bou proved that given $a, b \in (0, 1]$ with $a \neq b$, then the logics defined by the matrices $([\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}, \bar{F}^a)$ and $([\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}, \bar{F}^b)$ are incomparable. Note that these logics are not intermediate, and so they lie outside the scope of the present paper. But this result can be directly generalized into the framework of intermediate logics.

Proposition 7. *Let $a, b \in (0, 1]$ be such that $a \neq b$. Then the logics defined by the pairs of matrices $\{([\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}, \bar{F}^a),\ ([\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}, \{1\})\}$ and $\{([\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}, \bar{F}^b),$ $([\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}, \{1\})\}$ are incomparable.*

The proof is an easy extension of the proof of [4, Prop. 3.1].

Corollary 1. *The lattice of intermediate logics for Łukasiewicz infinite-valued logic has at least continuous width.*

Let us consider now the lattice filters defined by semi-open intervals $F^x = (x, 1]$ for all $x \in [0, 1)$. Then, also as extension of results in [4], the following results also hold:

- If a is rational, then the logic of the pair of matrices $\{([\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}, F^a), ([\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}, \{1\})\}$ is incomparable with the one defined by the pair of matrices $\{([\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}, \bar{F}^a), ([\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}, \{1\})\}$.
- If a is irrational, then the logics defined by the pairs of matrices $\{([\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}, F^a), ([\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}, \{1\})\}$ and $\{([\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}, \bar{F}^a), ([\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}, \{1\})\}$ coincide.

4.2 A family of intermediate logics $L(\mathcal{F}_a)$ parametrized by elements $a \in [0, 1)$

In this section we define and partially axiomatize a family of intermediate logics induced by sets of matrices over the standard MV-algebra $[\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}$ defined by the following families $\mathcal{F}_a$ of lattice filters parametrized by elements $a \in [0, 1)$:

$$\mathcal{F}_a = \{(b, 1] \ : \ b \geq a\} \cup \{[b, 1] \ : \ b > a\}.$$

Every filter in $\mathcal{F}_a$ is proper and the corresponding class of matrices $\mathcal{M}_a = \{([\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}, F) \ : \ F \in \mathcal{F}_a)$ define in the usual way a logic $L(\mathcal{F}_a)$, whose consequence relation will be denoted by $\models_a^{\leq}$.

Lemma 7. *The consequence relation $\models_a^{\leq}$ is equivalently defined as follows: if $\Gamma \cup \{\varphi\}$ is a finite set of formulas then $\Gamma \models_a^{\leq} \varphi$ iff, for every $[\mathbf{0}, \mathbf{1}]_{\mathrm{MV}}$-evaluation e,*

$$\textit{either} \ \ e(\Gamma^{\wedge}) \leq a \ \ \textit{or} \ \ e(\Gamma^{\wedge}) \leq e(\varphi).$$

Proof. By definition, $\Gamma \models^{\leq}_{a} \varphi$ iff, for every $F \in \mathcal{F}_a$ and every evaluation e in $[0,1]$, $e(\Gamma^{\wedge}) \in F$ implies $e(\varphi) \in F$, which is in fact equivalent to the condition: for every evaluation, and every $b > a$, $e(\Gamma^{\wedge}) \geq b$ implies $e(\varphi) \geq b$. That is, for every e, either $e(\Gamma^{\wedge}) \leq a$ or $e(\Gamma^{\wedge}) \leq e(\varphi)$. □

The axiomatization of the logics $L(\mathcal{F}_a)$ when $a = r$ is rational is quite easy. By McNaughton's theorem, there exists a formula $\Theta_r(p)$ depending exactly on the propositional variable p, whose associated function $f_r : [0,1] \to [0,1]$ is such that $f_r(x) = 1$ if, and only if, $x \in [0,r]$ (see e.g. [1]). That is, for every evaluation e, $e(\Theta_r(p)) = 1$ iff $e(p) \leq r$. In some cases it is easy to explicitly give the formula $\Theta_r(p)$, for example:

- If $r = 1/2$, then $\Theta_{1/2}(p) = p \to \neg p$,
- If $r = 1/(k+1)$ with $k \geq 2$, then $\Theta_{1/(k+1)}(p) = p \to \neg(p \oplus \overset{k}{\ldots} \oplus p)$,
- If $r = k/(k+1)$ with $k \geq 2$, then $\Theta_{k/(k+1)}(p) = p \to \neg(p \otimes \overset{k}{\ldots} \otimes p)$,

Using the formula $\Theta_r(p)$ next we define the logic Ł$^{\leq}_{r}$.

Definition 1. Let Ł$^{\leq}_{r}$ be the Hilbert calculus obtained from the one for Ł$^{\leq}$ adding the following inference rule:

$$(\bar{R}_r) \quad \frac{\varphi \qquad \vdash \Theta_r(\varphi) \vee (\varphi \to \psi)}{\psi}$$

The consequence relation of Ł$^{\leq}_{r}$ will be denoted by $\vdash^{\leq}_{r}$.

The adequacy of the proposed calculus with respect to the semantics of filters can be easily proved.

Proposition 8. *[Soundness and Completeness] For any rational $r \in [0,1)$, the logic Ł$^{\leq}_{r}$ is determined by the class of matrices $\mathcal{M}_r$. That is, for any finite set of formulas $\Gamma \cup \{\varphi\}$ we have:*

$$\Gamma \vdash^{\leq}_{r} \varphi \quad \textit{iff} \quad \Gamma \models^{\leq}_{r} \varphi.$$

Proof. It follows the same line as the proof of [5, Th. 2.12]. One direction is soundness and easily follows by taking into account Lemma 7 and the way the formula Θ_r is defined. For completeness, assume $\Gamma \models^{\leq}_{r} \varphi$ where $\Gamma = \{\psi_1, \ldots, \psi_n\}$. By Lemma 7, $\Gamma \models^{\leq}_{r} \varphi$ iff $\models_{Ł} \Theta_r(\Gamma^{\wedge}) \vee (\Gamma^{\wedge} \to \varphi)$, and by completeness of Łukasiewicz logic, we have $\vdash_{Ł} \Theta_r(\Gamma^{\wedge}) \vee (\Gamma^{\wedge} \to \varphi)$, so there is a proof in Ł of $\Theta_r(\Gamma^{\wedge}) \vee (\Gamma^{\wedge} \to \varphi)$. Finally, to get a proof of φ from Γ in Ł$^{\leq}_{r}$ it is enough to start with $n-1$ applications of the adjunction rule to get $\Gamma^{\wedge}$, followed with a proof of $\Theta_r(\Gamma^{\wedge}) \vee (\Gamma^{\wedge} \to \varphi)$, and finally an application of the inference rule $(\bar{R}_r)$. □

The hierarchy of the family of logics $L(\mathcal{F}_a)$ for $a \in [0,1)$ is stated in the following proposition.

Proposition 9. *The set of logics $L(\mathcal{F}_a)$ satisfies the following properties:*

(1) $L(\mathcal{F}_0) = Ł_0^{\leq} = Ł^{\leq}$.

(2) If $a < b$ *then* $L(\mathcal{F}_a) \subsetneq L(\mathcal{F}_b)$.

(3) $L(\mathcal{F}_a) \subsetneq Ł$, *for every element* a *in* $[0, 1)$.

(4) $L(\mathcal{F}_a)$ *is paraconsistent for* $a < 1/2$, *and is explosive for* $a \geq 1/2$.

Proof. (1) Immediate from Definition 7 and Proposition 8.

(2) If $a < b$ then $\mathcal{F}_a \supset \mathcal{F}_b$ and so $L(\mathcal{F}_a) \subset L(\mathcal{F}_b)$. It is clear from the axiomatization that the inclusion is strict when a, b are rational. And from them, taking into account that for any two elements $a, b \in [0, 1)$ with $a < b$, there exist rational numbers r, r' such that $a < r < r' < b$, the inclusion between $L(\mathcal{F}_a)$ and $L(\mathcal{F}_b)$ is also strict.

(3) Observe first that $L(\mathcal{F}_a)$ has the same theorems as Ł, and there is a rational r such that $a < r < 1$. Thus $L(\mathcal{F}_a) \subset Ł_r^{\leq}$. On the other hand, the rule $(\bar{R}_r)$ is clearly derivable in Ł: if e is an evaluation over $[0, 1]$ such that $e(\varphi) = 1$ and if $\Theta_r(\varphi) \vee (\varphi \to \psi)$ is a theorem of $Ł_r^{\leq}$ then it is also a theorem of Ł and so, $e(\Theta_r(\varphi) \vee (\varphi \to \psi)) = 1$. Then, either $e(\Theta_r(\varphi)) = 1$ or $e(\varphi \to \psi) = 1$. Since $e(\varphi) = 1 \not\leq 1/r$ then $e(\Theta_r(\varphi)) \neq 1$ and so $e(\varphi \to \psi) = 1$. From this, it follows that $e(\psi) = 1$. Therefore $L(\mathcal{F}_a) \subseteq Ł$.

In order to prove that the inclusion is proper, let $0 < r < 1$, and $\epsilon > 0$ such that $\epsilon < (1 - r)/2$ and $r - \epsilon > 0$. Let p and q two different propositional variables and e an evaluation such that $e(p) = r + \epsilon < 1$ and $e(q) = r - \epsilon > 0$. Then $e(p \to q) = 1 - 2\epsilon > r$ and $e(p) > r$ but $e(q) < r$. Then $p, p \to q \not\models_r^{\leq} q$. This shows that Modus Ponens is not derivable in $Ł_r^{\leq}$ and thus not derivable in $Ł_a^{\leq}$ either.

(4) If $a \geq 1/2$ and for an evaluation e, $e(\alpha) \in F$ for some $F \in \mathcal{F}_a$, then $e(\neg\alpha) \notin F$ and thus the explosion rule is valid. This is not true when $a < 1/2$ and in this case the explosion rule is not valid and thus the logic is paraconsistent. □

As a consequence, this proposition shows the existence of a (at least) continuous, linearly ordered set of intermediate logics $\{L(\mathcal{F}_a) \subsetneq Ł \ : \ a \in [0, 1)\}$, and we know which of them are paraconsistent and which are explosive. This, together with Corollary 1, leads to the following.

Corollary 2. *The lattice of intermediate logics between* $Ł^{\leq}$ *and* Ł *has at least continuous width and depth.*

Finally, remember that in Section 3.2 we have already studied the families of intermediate logics $Ł_{exp^-}^{(k,m)}$ and $Ł_{exp^+}^{(k,m)}$. Thus a natural question arises, what is the relation between them and the family of intermediate logics $Ł_r^{\leq}$ studied in this section? This is the content of the next subsection.

4.3 The explosion rules and semantics based on lattice filters of $[\mathbf{0},\mathbf{1}]_{\mathrm{MV}}$

Consider again the explosion rule:

$$(exp)\quad \frac{\varphi \quad \neg\varphi}{\bot}$$

and the logic $Ł^{\leq}_{exp}$ it defines as an extension of $Ł^{\leq}$. The question we address in this section is whether this logic has a semantics defined by a family of lattice filters in the standard MV-algebra $[\mathbf{0},\mathbf{1}]_{\mathrm{MV}}$.

Let $Fil([0,1])$ denote the set of proper lattice filters of $[0,1]$. It is easy to see that the lattice filters from $Fil([0,1])$ that are *compatible* with or *closed* under[6] the rule (exp) are exactly the set $\mathcal{F}_{1/2} = \{(a,1] \ : \ a \geq 1/2\} \cup \{[a,1] \ : \ a > 1/2\}$. Indeed, we have:

$$\begin{array}{ll}
\multicolumn{2}{l}{\{F \in Fil([0,1]) \ : \ \text{ for every } x \in [0,1], \text{ if } x \wedge \neg x \in F \text{ then } 0 \in F\}} \\
= & \{F \in Fil([0,1]) \ : \ \text{ for every } x \in [0,1], \, x \wedge \neg x \notin F\} \\
= & \{F \in Fil([0,1]) \ : \ \text{ for every } x \in [0,1], \text{ if } x \in F \text{ then } \neg x \notin F\} \\
= & \{F \in Fil([0,1]) \ : \ 1/2 \notin F\} \\
= & \{(a,1] \ : \ a \geq 1/2\} \cup \{[a,1] \ : \ a > 1/2\}. \\
= & \mathcal{F}_{1/2}.
\end{array}$$

Therefore the set of filters compatible with (exp) is $\mathcal{F}_{1/2}$, but its corresponding logic $Ł^{\leq}_{1/2}$ turns out to be different from $Ł^{\leq}_{exp}$. Remember that $Ł^{\leq}_{1/2}$ is defined by the Hilbert calculus by extending the one for $Ł^{\leq}$ with the inference rule

$$(\bar{R}_{1/2})\quad \frac{\varphi \qquad \vdash (\varphi \to \neg\varphi) \vee (\varphi \to \psi)}{\psi}.$$

Actually $Ł^{\leq}_{exp}$ is a weaker logic (in the sense of not having more consequences) than $Ł^{\leq}_{1/2}$, i.e. it holds that $\vdash^{\leq}_{exp} \subseteq \vdash^{\leq}_{1/2}$. This directly follows from a simple inspection of their semantic characterizations:

- $\Gamma \vdash^{\leq}_{exp} \varphi$ iff either $(\forall e)(e(\Gamma^{\wedge}) \leq \frac{1}{2})$ or $(\forall e)(e(\Gamma^{\wedge}) \leq e(\varphi))$;
- $\Gamma \vdash^{\leq}_{1/2} \varphi$ iff $(\forall e)($ either $e(\Gamma^{\wedge}) \leq \frac{1}{2}$ or $e(\Gamma^{\wedge}) \leq e(\varphi))$.

It is clear than the first condition implies the second, but not vice-versa in general. In particular the following example shows a derivation in $Ł^{\leq}_{1/2}$ that does not hold in $Ł^{\leq}_{exp}$.

Example 2. Let p and q be two different propositional variables. Then it is very easy to check that

$$(p \wedge \neg p) \vee q \vdash^{\leq}_{1/2} q \quad \text{but} \quad (p \wedge \neg p) \vee q \nvdash^{\leq}_{exp} q.$$

[6] A lattice filter $F \in Fil([0,1])$ is said to be compatible with or, equivalently, closed under a rule $\frac{\varphi_1,\ldots,\varphi_n}{\psi}$ whenever for every $[0,1]$-evaluation e, if $e(\varphi_i) \in F$ for every $i = 1,\ldots,n$, then $e(\psi) \in F$ as well.

Therefore, the logic $Ł^{\leq}_{exp}$ cannot be characterized by a family of lattice filters on $[\mathbf{0},\mathbf{1}]_{\mathrm{MV}}$. More generally, one can also show that the same situation holds with all the logics $Ł^{(k)}_{exp^-}$ and $Ł^{(k)}_{exp^+}$ as the following proposition shows.

Proposition 10. *The following conditions hold:*

- *The set of filters from $Fil([0,1])$ compatible with the rules $(exp^-_{k,m})$ and $(exp^+_{k,m})$ are respectively the sets $\mathcal{F}_{\frac{k(m-1)+1}{km+1}}$ and $\mathcal{F}_{\frac{k}{km+1}}$.*
- *If $\Gamma \vdash^{(k,m)}_{exp^-} \varphi$, then $\Gamma \vdash^{\leq}_{r} \varphi$ with $r = \frac{k(m-1)+1}{km+1}$, but the converse is not true in general.*
- *If $\Gamma \vdash^{(k,m)}_{exp^+} \varphi$, then $\Gamma \vdash^{\leq}_{r} \varphi$ with $r = \frac{k}{km+1}$, but the converse is not true in general.*

Proof. We only prove that the converse implications of the second and third items do not hold. Define a formula ψ depending on only one variable p such that its MacNaugton function f is given by the piecewise linear graph joining the points $a = (0,0), b = (r/2,0), c = ((r+1)/2,1)$ and $d = (1,1)$, where $r = 1/(k+1)$. Then it is clear that for $e(p) < 1/(k+1)$, $e(p) > e(\psi) = f(e(\psi))$, while $e(p) \leq e(\psi) = f(e(\psi))$, if $e(p) \geq 1/(k+1)$. Therefore we have that $p \vdash^{\leq}_{\frac{1}{k+1}} \psi$, but $p \not\vdash^{(k,1)}_{exp^-} \psi$. The case of $r = k/(k+1)$ is very similar, only special care has to be taken in choosing the point c. $\square$

These previous observations illustrate the fact that logics defined by families of lattice filters of $[\mathbf{0},\mathbf{1}]_{\mathrm{MV}}$ containing the filter $\{1\}$ do not cover the set of intermediate logics between $Ł^{\leq}$ and $Ł$, as one might have conjectured with a too simple an analysis of what happens with the logics $Ł^{\leq}$ and $Ł$. Actually, what one can easily show is that one needs to consider families of matrices defined by lattice filters over arbitrary MV-algebras. The following is a general result, adapted from well-known results in the literature.

Theorem 2. *Let L be a logic whose (Tarskian, finitary and structural) consequence relation $\models_L$ is such that $\vdash^{\leq}_{Ł} \subseteq \models_L \subseteq \vdash_{Ł}$. Then $\models_L$ is the logic induced by the family of matrices $(\mathbf{A}, F)$, where $\mathbf{A}$ is an MV-algebra and F is lattice filter of $\mathbf{A}$ compatible with $\models_L$ (i.e. if $\Gamma \models_L \varphi$ then for every $\mathbf{A}$-evaluation e, if $e(\Gamma^\wedge) \in F$ then $e(\varphi) \in F$).*

Proof. By definition, if $\Gamma \models_L \varphi$ then $\Gamma \models_{\mathcal{M}} \varphi$ for every $\mathcal{M} = (\mathbf{A}, F)$ such that $\mathbf{A}$ is an MV-algebra and F is lattice filter of $\mathbf{A}$ compatible with $\models_L$.

Conversely, suppose that $\Gamma \not\models_L \varphi$. Let $\mathfrak{F}(V)$ be the set of formulas and consider the Lindenbaum algebra $\mathbf{A} = \mathfrak{F}(V)/\equiv$, where $\varphi \equiv \psi$ iff $\vdash_{Ł} \varphi \leftrightarrow \psi$. Clearly, $\mathbf{A}$ is an MV-algebra. For each formula ψ, we will denote by $[\psi]$ the equivalence class of ψ, i.e. the set $\{\gamma \in \mathfrak{F}(V) \ : \ \psi \equiv \gamma\}$.

Now define $F = \{[\delta] \ : \ \Gamma \models_L \delta\}$. It is clear that F is a lattice filter of $\mathbf{A}$. Moreover F is compatible with $\models_L$. Indeed, we have to show that if $\Sigma \models_L \psi$ then for any evaluation $e : \mathfrak{F}(V) \to A$, $e(\Sigma^\wedge) \in F$ implies $e(\psi) \in F$.

But an evaluation $e : \mathfrak{F}(V) \to \mathfrak{F}(V)/\equiv$ can be turned into a substitution $\sigma : \mathfrak{F}(V) \to \mathfrak{F}(V)$ where $\sigma(\gamma) = \gamma'$ such that γ' is any formula in the equivalence class $e(\gamma) \in \mathfrak{F}(V)/\equiv$. In fact: consider a mapping $\sigma_0 : V \to \mathfrak{F}(V)$ such that $\sigma_0(p) \in e(p)$, for every propositional variable $p \in V$, and let $\sigma : \mathfrak{F}(V) \to \mathfrak{F}(V)$ be its unique extension to an homomorphism. Since e is an homomorphism then $e(\alpha \to \beta) = e(\alpha) \to e(\beta)$ and $e(\neg\alpha) = \neg e(\alpha)$. This means that, if $\alpha' \in e(\alpha)$ and $\beta' \in e(\beta)$ then $\neg\alpha' \in e(\neg\alpha)$ and $\alpha' \to \beta' \in e(\alpha \to \beta)$. From this, by induction on the complexity of the formula γ it can be proven that σ is a substitution such that $\sigma(\gamma) \in e(\gamma)$, for every formula γ.

Since $\models_L$ is structural, $\sigma(\Sigma) \models_L \sigma(\psi)$. But $e(\Sigma^\wedge) \in F$ implies that $\Gamma \models_L \sigma(\Sigma)$, therefore $\Gamma \models_L \sigma(\psi)$, hence $e(\psi) \in F$. This shows that F is compatible with $\models_L$.

Finally, let us check that $\Gamma \not\models_{\mathcal{M}} \varphi$ for $\mathcal{M} = (\mathbf{A}, F)$. Indeed, define the evaluation $h : \mathfrak{F}(V) \to A$ as follows: for every ψ, $h(\psi) = [\psi]$. It readily follows that $h(\Gamma^\wedge) = [\Gamma^\wedge] \in F$ since trivially $\Gamma \models_L \Gamma^\wedge$. However, since $\Gamma \not\models_L \varphi$, then $h(\varphi) = [\varphi] \notin F$. □

In the particular case the logic L is defined syntactically as an extension of $Ł^{\leq}$ with a set $\mathcal{R}$ of (structural) inference rules derivable in Ł, a matrix $(\mathbf{A}, F)$ (with $\mathbf{A}$ being an MV-algebra and F a lattice filter of A) is compatible with L whenever every rule in $\mathcal{R}$ is compatible with F.

The next example shows a matrix that distinguishes $Ł^{\leq}_{exp}$ from $Ł^{\leq}_{1/2}$.

Example 3. Consider the MV-algebra $\mathbf{A} = Ł_2 \times Ł_3$, thus,

$$A = \{(0,0), (0,1/2), (0,1), (1,0), (1,1/2), (1,1)\}$$

where $0_A = (0,0)$ and $1_A = (1,1)$. Let $a = (1,1/2)$, $b = (1,0)$ and $F = \{a, 1_A\}$. So defined, F is a lattice filter compatible with the explosion rule (exp). Indeed, $\neg a = (0,1/2) \notin F$, while $\neg 1_A = 0_A \notin F$. Now, let p and q be two different propositional variables, and let $\varphi = (p \wedge \neg p) \vee q$ and $\psi = q$.

As observed in Example 2, in the logic $Ł^{\leq}_{\frac{1}{2}}$ defined by all the lattice filters over $[0,1]$ compatible with (exp), we have $\varphi \models^{\leq}_{1/2} \psi$.

Let e be an evaluation over A such that $e(p) = a$ and $e(q) = b$. Then $e(\varphi) = (a \wedge \neg a) \vee b = \neg a \vee b = (0,1/2) \vee (1,0) = (1,1/2) = a$. Therefore, $e(\varphi) \in F$, but on the other hand, $e(\varphi) = a \not\leq b = e(\psi) \notin F$. This shows again that $\varphi \not\vdash^{\leq}_{exp} \psi$.

5 The case of finite-valued Łukasiewicz logics

As it has been made clear in the last section, one cannot restrict to families of lattice filters of $[\mathbf{0},\mathbf{1}]_{\mathrm{MV}}$ to account for all the intermediate logics between the infinite-valued logics $Ł^{\leq}$ and Ł: rather, one has to consider families of matrices with lattice filters over arbitrary MV-algebras. This apparently makes the task of identifying all the intermediate logics very hard, and we cannot offer so far satisfactory results. Therefore, we turn our attention in the rest of this paper to the case of finite-valued Łukasiewicz logics $Ł_n$, where the landscape appears to be more affordable. Indeed, in the finite-valued case, Theorem 2 can be specialized to this more concrete result. In the following, recall that for every $n \geq 2$ we denote by $ŁV_n$ the set $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ and by $\mathbf{ŁV}_n$ the corresponding MV-algebra.

Theorem 3. *Let L be a logic whose (Tarskian, finitary and structural) consequence relation $\models_L$ is such that $\vdash^{\leq}_{Ł_n} \subseteq \models_L \subseteq \vdash_{Ł_n}$. Then $\models_L$ is the logic induced by the family of matrices $(\mathbf{A}, F)$, where $\mathbf{A}$ is a direct product of finitely-many subalgebras of $\mathbf{ŁV}_n$ and F is a lattice filter of $\mathbf{A}$ compatible with $\models_L$.*

Proof. Assuming $\Gamma \not\models_L \varphi$, the crucial point here in contrast with Theorem 2 is to consider the Lindenbaum algebra $\mathbf{A'} = \mathfrak{F}(V_0)/\equiv$, where $\mathfrak{F}(V_0)$ is the set of formulas built from the finite set of variables V_0 appearing in $\Gamma \cup \{\varphi\}$, rather than the Lindenbaum algebra $\mathfrak{F}(V)/\equiv$ over all the formulas. The advantage is that $\mathbf{A'}$ is a finite MV-algebra (since the variety $\mathbb{MV}_n$ generated by the chain $\mathbf{ŁV}_n$ is locally finite), and moreover every finite MV-algebra is a direct product of finitely-many subalgebras of $\mathbf{ŁV}_n$. The rest of the proof runs analogously to that of Theorem 2. □

Given that a lattice filter F of a direct product of $Ł_n$-chains $\mathbf{A} = \prod_{i=1,k} \mathbf{S_i}$ is of the form $F = \prod_{i=1,k} F_i$, where each F_i is an lattice filter of $\mathbf{S}_i$, in order to study intermediate logics between $Ł_n^{\leq}$ and $Ł_n$ we need to study logics defined by matrices of the form $M = (\prod_{i=1,k} \mathbf{S_i}, \prod_{i=1,k} F_i)$, that we call *$Ł_n$-matrices.* Taking into account that each lattice filter F_i is of the form $F_i = [t_i, 1] = \{x \in S_i \ : \ x \geq t_i\}$ and each evaluation e over $\mathbf{A}$ is in fact given as a tuple $e = (e_1, \ldots, e_k)$ of evaluations e_i over the corresponding factors $\mathbf{S_i}$, the logic $L(M)$ given by the above matrix M is defined as follows:

$$\Gamma \vdash_M \varphi \ \text{ iff } \ (\forall e_1, \ldots, \forall e_k)\big(\text{ if } \bigwedge_{j=1}^{k} [e_j(\Gamma^\wedge) \geq t_j] \ \text{ then } \bigwedge_{j=1}^{k} [e_j(\varphi) \geq t_j] \,\big).$$

where each e_i ranges over evaluations on $\mathbf{S}_i$.[7] This expression makes it clear that having repeated pairs $(\mathbf{S}_i, F_i)$ in a matrix M is irrelevant to determine the corresponding logic, and so such repetitions could be eliminated without affecting the logic. Therefore, without loss of generality we can restrict ourselves to $Ł_n$-matrices $M = (\prod_{i=1,k} \mathbf{S_i}, \prod_{i=1,k} F_i)$ such that $F_i \neq F_j$ whenever $\mathbf{S}_i = \mathbf{S}_j$. A direct consequence of this is the fact that there are only finitely-many logics defined by $Ł_n$-matrices.

Moreover, since any such an $Ł_n$-matrix is determined by the set of factors $\mathbf{S}_i$ and the values $t_i \in S_i$ defining the filters F_i, we can equivalently describe matrices by means of non-empty sets $T = \{(t_i, \mathbf{S}_i) \ : \ i = 1, \ldots, k\}$ of pairs of

[7]Note that in this expression the $\bigwedge$'s denote metalinguistic conjunctions.

subalgebras and values such that $t_i > 0$ for all i, and $t_i \neq t_j$ whenever $\mathbf{S}_i = \mathbf{S}_j$. Such sets T will be called *matrix determination sets* for $Ł_n$.

For the sake of a more compact notation, in a determination set T we will only make explicit the subalgebras $\mathbf{S}_i$ that are different from $\mathbf{ŁV}_n$. For instance, when writing $T = \{t_1, (t_2, \mathbf{S}), t_3\}$ we will refer to the matrix $M_T = (\mathbf{ŁV}_n \times \mathbf{S} \times \mathbf{ŁV}_n, [t_1, 1] \times [t_2, 1] \times [t_3, 1])$. Abusing the notation once again, and without danger of confusion, we will also denote by $L(T)$ the logic $L(M_T)$ defined by the matrix M_T.

Finally, as usual, if $\mathcal{M}$ is a family of $Ł_n$-matrices, the logic $L(\mathcal{M})$ given by $\mathcal{M}$ is defined as the intersection of the logics $L(M)$, i.e. $\Gamma \vdash_{\mathcal{M}} \varphi$ iff $\Gamma \vdash_M \varphi$ for each $M \in \mathcal{M}$.

Therefore, if we denote by $\mathbf{Mat}(Ł_n)$ the set of L_n-matrices M_T defined by determination sets T, then the set of intermediate logics between $Ł_n^{\leq}$ and $Ł_n$ is exactly the set:

$$Int(Ł_n) = \{L(\mathcal{M}) \ : \ \mathcal{M} \subseteq \mathbf{Mat}(Ł_n) \text{ and } (1, \mathbf{ŁV}_n) \in T \text{ for some } T \in \mathcal{M}\}.$$

We are not able to provide a general full description of the whole set $Int(Ł_n)$ of intermediate logics: only partial results will be presented. Namely, in the next sections we provide the following:

- a full description of the set $Int_\Pi(Ł_n)$ of logics defined by (sets of) matrices from $\mathbf{Mat}(Ł_n)$ over direct products of the standard $Ł_n$-algebra $\mathbf{ŁV}_n$.
- an almost full description of the whole set $Int(Ł_n)$ when $n-1$ is a prime number.

6 Intermediate logics $Int_\Pi(\mathbf{Ł}_n)$ defined by matrices over direct products of $\mathbf{ŁV}_n$

As we have observed above, logics in $Int_\Pi(Ł_n)$ are given by sets of matrix determination sets of the form $T = \{t_1 > t_2 > \ldots > t_m\}$ with $t_i \in ŁV_n \setminus \{0\}$, corresponding to sets of matrices $M_T = ((\mathbf{ŁV}_n)^m, F_{t_1} \times \ldots \times F_{t_m})$, where $F_{t_i} = [t_i, 1]$ is an lattice filter of $\mathbf{ŁV}_n$. Recall that the logic $L(T)$ defined by the matrix M_T is defined as

$$\Gamma \vdash_{M_T} \varphi \ \text{ iff } \ (\forall e_1, \ldots, \forall e_k)\big(\text{ if } \bigwedge_{j=1}^{k} [e_j(\Gamma^\wedge) \geq t_j] \ \text{ then } \bigwedge_{j=1}^{k} [e_j(\varphi) \geq t_j] \ \big).$$

where each e_i ranges over all $\mathbf{ŁV}_n$-evaluations. Since these logics are totally determined by the lattice filters F_{t_i}, we will also use sometimes the more explicit notation $L(F_{t_1,\ldots,t_m})$ to denote the logic $L(T)$, to emphasize that it is defined by the lattice filter $F_{t_1} \times \ldots \times F_{t_m}$. Note that $L(F_1) = Ł_n$.

Given T, one can also consider the family of matrices $\mathcal{M}_T = \{(\mathbf{ŁV}_n, [t_i, 1]) \ : \ t_i \in T\}$ determined by each of the lattice filters $F_{t_i} = [t_i, 1]$. The corresponding logic $L(\mathcal{M}_T)$ is defined as:

$$\Gamma \vdash_{\mathcal{M}_T} \varphi \ \text{ iff } \ (\forall t \in T, \forall e)(\text{if } e(\Gamma^\wedge) \geq t \text{ then } e(\varphi) \geq t)$$

where e ranges over all $\mathbf{ŁV}_n$-evaluations. As a matter of fact, this logic is different from the above logic $L(T)$ and it holds that $L(\mathcal{M}_T) = \bigcap_{i=1,m} L(F_{t_i})$.[8] Note that, in particular, if $T = ŁV_n \setminus \{0\}$, then $L(\mathcal{M}_T) = Ł_n^{\leq}$.

Actually these two kinds of logics will play a distinguished role in our analysis. In what follows,

- $Int_\Pi^{LF}(Ł_n)$: will denote the set of of logics $L(T) = L(\Pi_{t \in T} F_t)$, with $1 \in T$, defined by the lattice filter $\Pi_{t \in T} F_t$ of the direct product $(\mathbf{ŁV}_n)^{|T|}$.
- $Int_\Pi^{OF}(Ł_n)$: will denote the set of logics $L(\mathcal{M}_T) = \bigcap_{t \in T} L(F_t)$, with $1 \in T$, defined by the set of (linearly ordered) lattice filters $\{F_t\}_{t \in F}$ of $\mathbf{ŁV}_n$.

Although the logics $L(T)$ and $L(\mathcal{M}_T)$ are different, they are closely related.

Proposition 11. *Let $\Gamma \cup \{\varphi\}$ be a finite set of formulas, T a matrix determination set for $Ł_n$, and let $t_0 = \max(T)$. Then:*

$$\Gamma \vdash_{M_T} \varphi \ \text{ iff } \ \text{either } (\forall e)\big(e(\Gamma^\wedge) < t_0\big) \ \text{ or } \ \Gamma \vdash_{\mathcal{M}_T} \varphi.$$

Proof. Consider the condition involved in the definition of $\Gamma \vdash_T \varphi$:

$$(\forall e_1, \ldots, \forall e_k)\Big(\text{ if } \bigwedge_{j=1}^{k} [e_j(\Gamma^\wedge) \geq t_j] \text{ then } \bigwedge_{j=1}^{k} [e_j(\varphi) \geq t_j]\Big)$$

This condition is in fact equivalent to

$$\begin{array}{ll} (\forall e_1, \ldots, \forall e_k) & \big[\ \big(\text{ if } \bigwedge_{j=1}^{k} [e_j(\Gamma^\wedge) \geq t_j] \text{ then } e_1(\varphi) \geq t_1\big) \\ & \ \text{and } \big(\text{ if } \bigwedge_{j=1}^{k} [e_j(\Gamma^\wedge) \geq t_j] \text{ then } e_2(\varphi) \geq t_2\big) \\ & \ldots \\ & \ \text{and } \big(\text{ if } \bigwedge_{j=1}^{k} [e_j(\Gamma^\wedge) \geq t_j] \text{ then } e_k(\varphi) \geq t_k\big)\ \big] \end{array}$$

and in turn to:

$$(\forall e)(\forall t' \in T)[\text{ if } e(\Gamma^\wedge) \geq t' \text{ and } (\forall t \neq t', \exists e')(e'(\Gamma^\wedge) \geq t) \text{ then } e(\varphi) \geq t'\big)]$$

and finally to:

$$(\forall e)(\forall t' \in T)[\text{ if } e(\Gamma^\wedge) \geq t' \text{ and } (\exists e')(e'(\Gamma^\wedge) \geq t_0) \text{ then } e(\varphi) \geq t'\big)]$$

and to:

$$\text{either } (\forall e)(e(\Gamma^\wedge) < t_0) \text{ or } (\forall e, \forall t' \in T)[\text{ if } e(\Gamma^\wedge) \geq t' \text{ then } e(\varphi) \geq t'\big)])$$

But the latter condition is nothing but

[8] Here $\bigcap_{i=1,m} L(F_{t_i})$ means the intersection of the the logics $L(F_{t_i})$ understood them as consequence relations.

$$\text{either } (\forall e)\big(e(\Gamma^\wedge) < t_0\big) \text{ or } \Gamma \vdash_{\mathcal{M}_T} \varphi.$$

□

Notation: From now on, given a generic logic L extension of $Ł_n^{\leq}$ and with associated consequence relation $\vdash$, $t \in ŁV_n$, and a set of formulas $\Gamma \cup \{\varphi\}$, consider the following conditions:

$K_t(\Gamma)$: $(\forall e)(e(\Gamma^\wedge) < t)$

$C_t(\Gamma, \varphi)$: $(\forall e)(\text{if } e(\Gamma^\wedge) \geq t \text{ then } e(\varphi) \geq t)$

We will say that L is characterized by the condition:

- K_t if, for any $\Gamma \cup \{\varphi\}$, $\Gamma \vdash \varphi$ iff $K_t(\Gamma)$
- C_t if, for any $\Gamma \cup \{\varphi\}$, $\Gamma \vdash \varphi$ iff $C_t(\Gamma, \varphi)$.

Given $\Gamma \cup \{\varphi\}$, any combination by conjunctions and disjunctions of the conditions $K_t(\Gamma)$ and $C_{t'}(\Gamma, \varphi)$ (for $t, t' \in ŁV_n$) can also be considered, in order to characterize other logics. For instance, using this notation, the last lemma for $T = \{t_1, \ldots, t_m\}$, with $t_1 > t_2 > \ldots > t_m$, says that, while the logic $L(\mathcal{M}_T)$ is characterized by the condition $C_{t_1} \wedge C_{t_2} \wedge \ldots \wedge C_{t_m}$, the logic $L(T)$ is characterized by the condition $K_{t_1} \vee (C_{t_1} \wedge C_{t_2} \wedge \ldots \wedge C_{t_m})$.[9]

As a direct consequence of Proposition 11, the following results hold.

Corollary 3. *Let $T, R \subseteq ŁV_n$ be two determination sets such that $max(T) = max(R)$. Then*

$$L(T) \cap L(R) = L(T \cup R).$$

Corollary 4. *Let $T, R \subseteq ŁV_n$ be two determination sets. Then*

$$L(\mathcal{M}_T) \cap L(\mathcal{M}_R) = L(\mathcal{M}_T \cup \mathcal{M}_R) = L(\mathcal{M}_{T \cup R})$$

Corollary 5. *Let $T \subseteq ŁV_n$ be a determination set. Then*

$$\bigcap_{t \in T} L(F_t) = L(\mathcal{M}_T) \ \subset \ L(T).$$

6.1 Lattice structures of intermediate logics from $Int_\Pi(Ł_n)$

From the previous results we can derive the lattice structure of some subsets of intermediate logics.

[9] The meaning of this notation, generalizing the notation for a logic being characterized by condition K_t or C_t, should be obvious. For instance, a logic being characterized by condition $K_{t_1} \vee (C_{t_1} \wedge C_{t_2})$ means that, for any $\Gamma \cup \{\varphi\}$, $\Gamma \vdash \varphi$ iff $K_{t_1}(\Gamma) \vee (C_{t_1}(\Gamma, \varphi) \wedge C_{t_2}(\Gamma, \varphi))$.

Lemma 8. *The set of intermediate logics* $Int_{\Pi}^{LF}(Ł_n)$ *equipped with the order defined by set inclusion of their consequence relations forms a Boolean lattice, denoted* $\mathbf{Int_{\Pi}^{LF}}(Ł_n)$*, that is anti-isomorphic to the Boolean lattice of subsets of* $ŁV_n \setminus \{0\}$*. The maximum of this Boolean lattice is* $Ł_n$*, the minimum is* $L(ŁV_n \setminus \{0\})$*, and the coatoms are the logics* $L(F_{1,t})$ *for each* $t \in ŁV_n \setminus \{0,1\}$*.*

Proof. Since $L(T) \subset L(R)$ if $R \subset T$, it is clear that the coatoms are the logics $L(F_{1,t})$, for each $t \in ŁV_n \setminus \{0,1\}$. Then, from Corollary 3 we obtain that $L(F_{1,t_1,\ldots,t_k}) \cap L(F_{1,r_1,\ldots,r_m}) = L(F_{1,t_1,\ldots,t_k,r_1,\ldots,r_m})$. Therefore, any element of $Int_{\Pi}^{LF}(Ł_n)$ is obtained by making intersections of the coatoms, one for each subset of $ŁV_n \setminus \{0\}$. This determines an structure isomorphic to the lattice of subsets of $ŁV_n \setminus \{0\}$ with the reverse order. $\square$

Observe that the lattice $\mathbf{Int_{\Pi}^{LF}}(Ł_n)$ is not a sublattice of the lattice $\mathbf{Int}(Ł_n)$ of all the intermediate logics, since its minimum $L(ŁV_n \setminus \{0\})$ is strictly greater than $Ł_n^{\leq}$.

Lemma 9. *The set of intermediate logics* $Int_{\Pi}^{OF}(Ł_n)$ *equipped with the order defined by set inclusion of their consequence relations forms a Boolean lattice, denoted* $\mathbf{Int_{\Pi}^{OF}}(Ł_n)$*, that is anti-isomorphic to the Boolean lattice of subsets of* $ŁV_n \setminus \{0\}$*. The maximum of this Boolean lattice is* $Ł_n$*, the minimum is* $Ł_n^{\leq}$ *and the coatoms are the logics* $L(\mathcal{M}_{\{1,t\}}) = L(F_1) \cap L(F_t)$ *for each* $t \in ŁV_n \setminus \{0,1\}$*.*

Proof. It is analogous to that of Lemma 8, replacing T's by $\mathcal{M}_T$'s and only noticing that now, from Corollary 4, we have $L(\mathcal{M}_T) \cap L(\mathcal{M}_R) = L(\mathcal{M}_{T\cup R})$. $\square$

Unlike the previous case, the lattice $\mathbf{Int_{\Pi}^{OF}}(Ł_n)$ is a sublattice of $\mathbf{Int}(Ł_n)$. Notice also that the last results prove that the lattice $\mathbf{Int_{\Pi}}(Ł_n)$ of intermediate logics defined by families of lattice filters of direct products of $\mathbf{ŁV}_n$, contains both the logics of the Boolean lattices $\mathbf{Int_{\Pi}^{LF}}(Ł_n)$ and $\mathbf{Int_{\Pi}^{OF}}(Ł_n)$ and their intersections.

Next lemma gives the relative position in $\mathbf{Int_{\Pi}}(Ł_n)$ of the logics belonging to the Boolean lattices $\mathbf{Int_{\Pi}^{LF}}(Ł_n)$ and $\mathbf{Int_{\Pi}^{OF}}(Ł_n)$.

Lemma 10. *The following properties hold:*

- *The logics* $L(F_{1,t})$ *are the coatoms of* $\mathbf{Int_{\Pi}}(Ł_n)$*,*
- $L(\mathcal{M}_{\{1,t\}}) \subset L(F_{1,t})$ *and there is no logic from* $Int_{\Pi}(Ł_n)$ *in between,*
- *If* $t \neq r$*, then* $L(\mathcal{M}_{\{1,t\}})$ *and* $L(F_{1,r})$ *are not comparable.*

Proof. Observe first that from Lemma 11, if $r, t \in ŁV_n \setminus \{0,1\}$ and $r \neq t$ then $L(F_{1,t})$ and $L(F_{1,r})$ are not comparable. Now the first item is an obvious consequence of the fact that if $L(\mathcal{M})$ is contained in the interval bounded by $Ł_n$ and $L(F_{1,t})$ then there must exist an element r such that $L(F_{1,r})$ has to be contained in $L(F_{1,t})$. And this is only true if $r = t$. The second and third items are obvious consequences of Lemma 11 as well, since it follows that $L(\mathcal{M}_{\{1,t\}}))$ is defined by conditions $C_1 \wedge C_t$, while $L(F_{1,t})$ is defined by conditions $K_1 \vee (C_1 \wedge C_t)$. $\square$

With all the preceding results, we can provide an informal description of the lattice of intermediate logics $\mathbf{Int}_{\mathbf{\Pi}}(\text{Ł}_n)$:

- The top of the lattice is Ł_n.
- In the second layer, we have the coatoms of the lattice, the logics $L(F_{1,t})$, also coatoms of the sublattice $\mathbf{Int}^{\mathbf{LF}}_{\mathbf{\Pi}}(\text{Ł}_n)$.
- In the third layer, just below the logics $L(F_{1,t})$, we have the logics $L(\mathcal{M}_{\{1,t\}})$, the coatoms of $\mathbf{Int}^{\mathbf{OF}}_{\mathbf{\Pi}}(\text{Ł}_n)$, and then we have as well as the pairwise intersection of the logics of the second layer, that is, the logics of the form $L(F_{1,t_1,t_2})$.
- By repeating the same process, the rest of logics of the sublattices $\mathbf{Int}^{\mathbf{LF}}_{\mathbf{\Pi}}(\text{Ł}_n)$ and $\mathbf{Int}^{\mathbf{OF}}_{\mathbf{\Pi}}(\text{Ł}_n)$ appear in lower layers together with all their intersections.
- Finally, there also appear logics resulting from intersections of the previous logics with some non-intermediate logics (hence outside $\mathbf{Int}(\text{Ł}_n)$). These new logics are of the form $L(\mathcal{M})$ where the set $\mathcal{M}$ contains at least some M_T with $|T| > 1$ and $\max(T) < 1$.

Some interesting examples of logics belonging to the latter class can be obtained by adding to $\text{Ł}_n^{\leq}$ inference rules like the explosion rule and its generalizations defined in previous sections. Thanks to Lemma 11, these logics can be easily characterized as logics of some lattice filters over direct products of L_n. Indeed:

- The logic $\text{Ł}_n^{\leq}+(exp)$ is characterized by the condition

$$K_{\frac{r}{n-1}} \vee (C_{\frac{1}{n-1}} \wedge \ldots \wedge C_{\frac{n-2}{n-1}} \wedge C_1),$$

  with r being the first natural such that $\frac{r}{n-1} > 1/2$. This condition defines the logic $L(\mathcal{M})$ where $\mathcal{M} = \{F_1, F_{\frac{n-2}{n-1}}, \ldots, F_{\frac{r+1}{n-1}}, F_{\frac{r}{n-1},\ldots,\frac{1}{n-1}}\}$. For $n > 3$, this logic belongs neither to $\mathbf{Int}^{\mathbf{LF}}_{\mathbf{\Pi}}(\text{Ł}_n)$ nor to $\mathbf{Int}^{\mathbf{OF}}_{\mathbf{\Pi}}(\text{Ł}_n)$. This is not true if $n = 3$, since the logic $\text{Ł}_3^{\leq}+(exp)$ is in fact the logic $L(F_{1,1/2})$, that belongs to $\mathbf{Int}^{\mathbf{LF}}_{\mathbf{\Pi}}(\text{Ł}_n)$.
- The logic $\text{Ł}_n^{\leq}+(exp_k^-)$ is characterized by condition

$$K_{\frac{r}{n-1}} \vee (C_{\frac{1}{n-1}} \wedge \ldots \wedge C_{\frac{n-2}{n-1}} \wedge C_1),$$

  with r the first natural such that $\frac{r}{n-1} > 1/k+1$. This condition defines the logic $L(\mathcal{M})$ where $\mathcal{M} = \{F_1, F_{\frac{n-2}{n-1}}, \ldots, F_{\frac{r+1}{n-1}}, F_{\frac{r}{n-1},\ldots,\frac{1}{n-1}}\}$.
- The logic $\text{Ł}_n^{\leq}+(exp_k^+)$ is characterized by conditions

$$K_{\frac{r}{n-1}} \vee (C_{\frac{1}{n-1}} \wedge \ldots \wedge C_{\frac{n-2}{n-1}} \wedge C_1),$$

  with r being the first natural such that $\frac{r}{n-1} > k/k+1$. This condition defines the logic $L(\mathcal{M})$ where $\mathcal{M} = \{F_1, F_{\frac{n-2}{n-1}}, \ldots, F_{\frac{r+1}{n-1}}, F_{\frac{r}{n-1},\ldots,\frac{1}{n-1}}\}$.

- The logics $\text{Ł}^{\leq}_{n,r} = \text{Ł}^{\leq}_n + (\bar{R}_r)$ are characterized by the condition

$$C_{\frac{t}{n-1}} \wedge \ldots \wedge C_{\frac{n-2}{n-1}} \wedge C_1,$$

with t being the first natural such that $\frac{t}{n-1} \geq r$. This condition defines the logic $L(\mathcal{F})$ where $\mathcal{M} = \{F_1, F_{\frac{n-2}{n-1}}, \ldots, F_{\frac{t}{n-1}}\}$.

The logics $\text{Ł}^{\leq}_{n,r}$ belong to the lattice $\mathbf{Int}^{\mathbf{OF}}_{\mathbf{\Pi}}(\text{Ł}_n)$, while the logics $\text{Ł}^{\leq}_n + (exp^-_k)$ and $\text{Ł}^{\leq}_n + (exp^+_k)$ belongs neither to $\mathbf{Int}^{\mathbf{LF}}_{\mathbf{\Pi}}(\text{Ł}_n)$ nor to $\mathbf{Int}^{\mathbf{OF}}_{\mathbf{\Pi}}(\text{Ł}_n)$, and moreover they cannot be obtained as intersection of logics belonging to these Boolean lattices. Actually, the logics $\text{Ł}^{\leq}_n + (exp^-_k)$ and $\text{Ł}^{\leq}_n + (exp^+_k)$ are obtained as intersection of logics of $\mathbf{Int}^{\mathbf{OF}}_{\mathbf{\Pi}}(\text{Ł}_n)$ with logics defined by lattice filters $F_{\{\frac{r}{n-1}, \ldots, \frac{1}{n-1}\}}$ with $r < n-1$, and thus not belonging to $\mathbf{Int}^{\mathbf{LF}}_{\mathbf{\Pi}}(\text{Ł}_n)$, since the latter are logics defined by filters F_T where $\max(T) = 1$.

Moreover, the logics belonging to $\mathbf{Int}_{\mathbf{\Pi}}(\text{Ł}_n)$ that are paraconsistent are characterized in the following proposition.

Proposition 12. *The paraconsistent logics of* $\mathbf{Int}_{\mathbf{\Pi}}(\text{Ł}_n)$ *are the logics* $L(F_1, F_t)$ *with* $t < 1/2$ *and all those contained in them.*

Proof. Observe first that, since the logics $L(F_{1,t})$ are semantically defined by condition $K_1 \vee (C_1 \wedge C_t)$, it is obvious that these logics are explosive for any $t \in \text{Ł}V_n \setminus \{0, 1\}$. Indeed for any evaluation e, $e(p \wedge \neg p) \leq 1/2 < 1$ and thus the explosion rule is valid in that logic. On the other hand, the logics $L(F_1, F_t)$ are semantically defined by the condition $C_1 \wedge C_t$, and if $t \leq 1/2$, there is at least one evaluation e such that $e(p \wedge \neg p) \geq t$. Since $e(\perp) = 0$, the explosion rule is not compatible with the lattice filter F_t, hence it is not sound in $L(F_1, F_t)$. So, the logic $L(F_1, F_t)$ is paraconsistent. Finally since any logic contained in a paraconsistent one is also paraconsistent, the proposition is proved. $\square$

In the examples of Appendices A1 and A2 we can see that in $\mathbf{Int}_{\mathbf{\Pi}}(\text{Ł}_3)$ the only paraconsistent logic is $\text{Ł}^{\leq}_3$ (since the only intermediate value, different from 0 and 1, is 1/2) while in $\mathbf{Int}_{\mathbf{\Pi}}(\text{Ł}_4)$ the paraconsistent logics are $L(F_{1,1/3})$ and those below it, $L(F_{1/3}, F_{1,2/3,1/3})$ and $\text{Ł}^{\leq}_4$.

6.2 About the axiomatization of logics of $\mathbf{Int}_{\mathbf{\Pi}}(\text{Ł}_n)$

In order to obtain the desired axiomatizations, notice first that as a consequence of Lemma 11, any logic $L(\mathcal{M})$ defined by a family $\mathcal{F}$ of lattice filters over direct products of $\mathbf{ŁV}_n$ is determined by conjunctions and disjunctions of conditions K_t and C_r. Thus, for every logic in $\mathbf{Int}_{\mathbf{\Pi}}(\text{Ł}_n)$ we can obtain a corresponding condition in a *simplified* disjunctive normal form (DNF). Here, simplified is in the sense that we remove disjuncts containing other disjuncts and that, due to their semantics, conjunctions $K_t \wedge C_r$ are simplified to K_t when $t \leq r$.[10]

[10]It holds since, if $t \leq r$, then $K_t(\Gamma)$ implies $C_r(\Gamma, \varphi)$, for every Γ and φ.

Moreover, each atomic condition, either of the form K_t or C_r, determines a set of pairs of values $(v_1, v_2) \in ŁV_n \times ŁV_n$ *satisfying* the condition in the following sense: (v_1, v_2) satisfies K_t if $v_1 < t$, and it satisfies C_r if $\min(v_1, v_2) \geq r$. Then the set of pairs satisfying a disjunct of a DNF (a conjunction of atomic conditions) will be the intersection of the sets satisfying each of its conditions.

On the other hand, it is clear that for any set $A \subset (ŁV_n)^2$ there is a McNaughton function f on two variables over $[0,1]^2$ such that $f(x,y) = 1$ for all $(x,y) \in A$ and $f(x,y) \neq 1$ for all $(x,y) \notin A$.

Having these observations in mind, we propose a method to axiomatize the logics of $\mathbf{Int_\Pi}(Ł_n)$. Let $\mathcal{F}$ be a family of lattice filters over direct products of $\mathbf{ŁV}_n$. Then the method to axiomatize the logic $L(\mathcal{F})$ can be sketched in the following steps:

1. Take the "and" of the conditions that semantically determine the logics $L(F_T)$ for each $F_T \in \mathcal{F}$. Compute their simplified disjunctive normal form, namely $D_1 \vee \ldots \vee D_k$.

2. For each disjunct D_i, compute the set $A_i \subseteq (ŁV_n)^2$ of pairs of values satisfying the condition D_i, and build a McNaughton function $MN_i(x,y)$ such that its restriction to $(ŁV_n)^2$ has value 1 on the points of A_i and a value less than 1 in points outside A_i.

3. The logic $L(\mathcal{F})$ is axiomatized by the axioms and rules of $Ł_n^{\leq}$ plus restricted inference rules of the form

$$\frac{\varphi \quad \vdash MN_i(\varphi, \psi)}{\psi}$$

one for each D_i.

Proposition 13 (Soudness and completeness)**.** *The method described above provides an effective way to come up with a sound and complete axiomatization of the logic $L(F_T)$.*

The proof is rather similar to the proof of Proposition 8 and thus is not repeated here. In the Appendix A we will illustrate the above method with the examples for $Ł_3$ and $Ł_4$.

7 Towards the description of the full lattice $\mathbf{Int}(\mathbf{Ł}_n)$: the case of $n-1$ prime

The introduction of lattice filters whose components are defined in different subalgebras of $\mathbf{ŁV}_n$ makes the study of their logics much more complicated. In this section we consider a relatively easy case, when $n-1$ is a prime number and hence when $\mathbf{ŁV}_n$ has a unique proper subalgebra, the two element Boolean algebra $\mathbf{ŁV}_2$.

Throughout this section we assume $n-1$ to be a prime number. Taking into account that the unique proper filter of $\mathbf{ŁV}_2$ is $\{1\}$, we only need to consider two types of $Ł_n$-determination sets T of lattice filters, depending of whether $(1, \mathbf{ŁV}_2) \in T$ or not. In the last section we have already studied logics $L(F_T)$ when $(1, \mathbf{ŁV}_2) \notin T$, i.e, when taking lattice filters over direct products of $\mathbf{ŁV}_n$. Now, next lemma gives a basic result in order to study logics $L(F_T)$ when $(1, \mathbf{ŁV}_2) \in T$.

Lemma 11. *Let T be a determination set for $Ł_n$ (with $n-1$ prime) such that $(1, \mathbf{ŁV}_2) \in T$. Then we have:*

- *if $T = \{(1, \mathbf{ŁV}_2), (1, \mathbf{ŁV}_n))\}$, then $L(F_T)$ strictly contains $Ł_n$;*
- *otherwise the logic $L(F_T)$ is not comparable with $Ł_n$, i.e. it is not an intermediate logic.*

Proof. We begin with the proof of the second item. Suppose that $T = \{(1, \mathbf{ŁV}_2), t_1, \ldots, t_k\}$.[11] By definition,

$$\Gamma \vdash_{M_T} \varphi \text{ iff } (\forall e_0, \forall e_1, \ldots, \forall e_k)\big(\text{ if } \bigwedge_{j=0}^{k} [e_j(\Gamma^\wedge) \geq t_j] \text{ then } \bigwedge_{j=0}^{k} [e_j(\varphi) \geq t_j]\big)$$

where e_0 ranges over evaluations over $ŁV_2$ and every e_i for $i > 0$ ranges over evaluations over $ŁV_n$. By splitting it for each component of the filter F_T, this is equivalent to: $\Gamma \vdash_T \varphi$ iff

- $\forall e_0$, if $[e_0(\Gamma^\wedge) = 1$ and $(\forall i > 0, \exists e_i : e_i(\Gamma^\wedge) \geq t_i)]$ then $e_0(\varphi) = 1$, and
- $\forall e_i$, if $[e_i(\Gamma^\wedge) \geq t_i$ and $(\forall j > 0$ and $j \neq i, \exists e_j : e_j(\Gamma^\wedge) \geq t_j)$ and $\exists e_0 : e_0(\Gamma^\wedge) = 1]$ then $e_i(\varphi) \geq t_i$, for all $i = 1, \ldots, k$

where again, e_0 ranges over evaluations over $ŁV_2$ and every e_i for $i > 0$ ranges over evaluations over $ŁV_n$. Taking into account that the existence of an evaluation over $ŁV_2$ with value 1 implies the existence of an evaluation over $ŁV_n$ with value 1, the former conditions can be simplified to:

- $\forall e_0$, if $e_0(\Gamma^\wedge) = 1$ then $e_0(\varphi) = 1$, and
- $\forall e_i$, if $[e_i(\Gamma^\wedge) \geq t_i$ and $\exists e_0 : e_0(\Gamma^\wedge) = 1]$ then $e_i(\varphi) \geq t_i$, for all $i = 1, \ldots, k$.

Which in turn are equivalent to:

- φ follows from Γ in classical logic (i.e. under Boolean semantics), noted $\Gamma \vdash_{CL} \varphi$, and
- either $\forall e_0 : e_0(\Gamma^\wedge) < 1$, or $\forall i = 1, \ldots, k$ and $\forall e_i$: if $e_i(\Gamma^\wedge) \geq t_i$ then $e_i(\varphi) \geq t_i$.

[11] Recall that this is a shortcut for $T = \{(1, \mathbf{ŁV}_2), (t_1, \mathbf{ŁV}_n), \ldots, (t_k, \mathbf{ŁV}_n)\}$.

The first item makes explicit that $\vdash_{L(F_T)} \subseteq \vdash_{CL}$. Taking into account that e_0 ranges over evaluations over $\mathbf{ŁV}_2$, then the first part of the disjunction of the second item implies that $\Gamma \vdash_T \varphi$ when for any classical evaluation v, $v(\Gamma^\wedge) = 0$, i.e., $\neg(\Gamma^\wedge)$ is a classical tautology. From this observation it is easy to prove that $L(F_T)$ is not comparable with $Ł_n$:

- clearly $\varphi \vdash_{Ł_n} \varphi^2$ but $p \nvdash_{L(F_T)} p^2$ for p being a propositional variable. Indeed neither $\neg p$ is a classical tautology nor for any evaluation e over $\mathbf{ŁV}_n$, if $t \neq 1$, $e(p) \geq t$ implies $e(p^2) \geq t$.
- on the other hand, we have $3(p \wedge \neg p) \nvdash_{Ł_n} (p \wedge \neg p)$ but $3(p \wedge \neg p) \vdash_{L(F_T)} (p \wedge \neg p)$. Indeed let $e(p) = k/(n-1)$ with k being the biggest natural such that $\frac{k}{n-1} \leq \frac{1}{2} \leq \frac{k+1}{n-1}$. Then it is clear that $e(p \wedge \neg p) = k/(n-1)$, but $e(3(p \wedge \neg p)) = 1$, and so $3(p \wedge \neg p) \nvdash_{Ł_n} (p \wedge \neg p)$. Moreover it is clear that $\neg(3(p \wedge \neg p))$ is a classical tautology[12] and thus $3(p \wedge \neg p) \vdash_{L(F_T)} (p \wedge \neg p)$.

In order to prove the first item of the lemma we reason in the same way and at the end we conclude that $\Gamma \vdash_{L(F_T)} \varphi$ iff either $\neg\Gamma^\wedge$ is a classical tautology or $\Gamma \vdash_{Ł_n} \varphi$. Therefore it is obvious that $Ł_n \subseteq L(F_T)$. Finally, as proven before, $3(p \wedge \neg p) \vdash_{L(F_T)} (p \wedge \neg p)$ and $3(p \wedge \neg p) \nvdash_{Ł_n} (p \wedge \neg p)$, hence the inclusion is strict. □

Before going further we introduce a new notation.

Notation: In what follows, consider the following condition:

$K_1^2(\Gamma)$: for every evaluation v over $\mathbf{ŁV}_2$, $v(\Gamma^\wedge) < 1$

In other words, condition $K_1^2(\Gamma)$ is equivalent to require that $\neg\Gamma^\wedge$ is a classical tautology.

Corollary 6. *If $(1, \mathbf{ŁV}_2) \in T \cap R$, then $L(F_T) \cap L(F_R) = L(F_{T \cup R})$*

Proof. We need only to take into account that $L(F_T) \cap L(F_R)$ is determined by the condition $(K_1^2 \vee \bigwedge\{C_t \ : \ (t, \mathbf{ŁV}_n) \in T\}) \wedge (K_1^2 \vee \bigwedge\{C_t \ : \ (t, \mathbf{ŁV}_n) \in R\}) = K_1^2 \vee \bigwedge\{C_t \ : \ (t, \mathbf{ŁV}_n) \in (T \cup R)\}$. □

This means that the family of logics $L(F_T)$ when$(1, \mathbf{ŁV}_2) \in T$ forms a $\wedge$-semilattice. Moreover all these logics $L(F_T)$, except for $T = \{(1, \mathbf{ŁV}_2), (1, \mathbf{ŁV}_n)\}$, are incomparable to L_n. But by intersecting these logics with $Ł_n$, we can obtain new intermediate logics.

Lemma 12. *If $T = \{(1, \mathbf{ŁV}_2), (t, \mathbf{ŁV}_n)\}$ with $t \neq 1$, then $L(F_T) \cap Ł_n$ is a new logic that strictly contains $L(F_{1,t})$.*

Proof. The logic $L(F_T) \cap Ł_n$ is semantically defined by the condition

$$(K_1^2 \vee C_t) \wedge C_1 = (K_1^2 \wedge C_1) \vee (C_1 \wedge C_t),$$

[12] That is, a tautology over $\mathbf{ŁV}_2$.

while $L(F_{1,t})$ is defined by the condition $K_1 \vee (C_1 \wedge C_t)$. But clearly K_1 implies both K_1^2 and C_1, thus the inclusion is proved. $\square$

This proof can be easily generalized to obtain the following corollary.

Corollary 7. *If $T = \{(1, \textbf{\L V}_2), t_1, \ldots, t_n\}$, then $L(F_T) \cap Ł_n$ is a new logic that strictly contains $L(F_{1,t_1,\ldots,t_n})$.*

As a consequence of the previous lemmas, it follows that these new logics form a sublattice in $\mathbf{Int}(Ł_n)$.

Proposition 14. *The set of logics $L(F_T) \cap Ł_n$, with $(1, \textbf{\L V}_2) \in T$, forms a Boolean lattice that will be denoted $\mathbf{Int}^{\mathbf{2}}_{\mathbf{\Pi}}(Ł_n)$, whose maximum is $Ł_n$ and whose minimum is $L(F_{T_*}) \cap Ł_n$, where $T_* = \{(1, \textbf{\L V}_2), \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$.*

Similarly to what has been done with $\mathbf{Int_{\Pi}}(Ł_n)$ in Section 6.1, we can provide now an informal description of the full lattice $\mathbf{Int}(Ł_n)$ when $n-1$ is prime:

- The top of the lattice is $Ł_n$.
- In the second layer, we have the coatoms of $\mathbf{Int}^{\mathbf{2}}_{\mathbf{\Pi}}(Ł_n)$, the logics $L(F_{(1,\textbf{\L V}_2),t}) \cap Ł_n = L(\mathcal{F})$ with $\mathcal{F} = \{F_{(1,\textbf{\L V}_2),t}, F_1\}$.
- In the third layer, we have the coatoms of $\mathbf{Int}^{\mathbf{LF}}_{\mathbf{\Pi}}(Ł_n)$ (each coatom $L(F_{1,t})$ just below $L(F_{(1,\textbf{\L V}_2),t}) \cap Ł_n$), and the pairwise intersections of coatoms of $\mathbf{Int}^{\mathbf{2}}_{\mathbf{\Pi}}(Ł_n)$, that is, the logics $L(F_{(1,\textbf{\L V}_2),t,r}) \cap Ł_n$.
- In a fourth layer, we have the coatoms of $\mathbf{Int}^{\mathbf{OF}}_{\mathbf{\Pi}}(Ł_n)$ (each coatom $L(\mathcal{F}_{\{1,t\}})$ just below $L(F_{1,t})$), the intersection of the coatoms of $\mathbf{Int}^{\mathbf{LF}}_{\mathbf{\Pi}}(Ł_n)$ (the logics $L(F_{1,t_1,t_2})$), and the 3-place intersections of the coatoms of $\mathbf{Int}^{\mathbf{2}}_{\mathbf{\Pi}}(Ł_n)$, that are the logics of the form $L(F_{(1,\textbf{\L V}_2),t,r,s}) \cap Ł_n$ for different $t, r, s \in ŁV_n \setminus \{0\}$.
- By repeating the same process, the rest of logics of the three Boolean sublattices $\mathbf{Int}^{\mathbf{LF}}_{\mathbf{\Pi}}(Ł_n)$, $\mathbf{Int}^{\mathbf{OF}}_{\mathbf{\Pi}}(Ł_n)$ and $\mathbf{Int}^{\mathbf{2}}_{\mathbf{\Pi}}(Ł_n)$ appear in lower layers together with all their intersections.
- Finally, there also appear logics resulting from intersections of the previous logics with some non-intermediate logics (hence outside $\mathbf{Int}(Ł_n)$). These new logics are of the form $L(\mathcal{F})$ where the set $\mathcal{F}$ contains at least some F_T with $|T| > 1$ and $\max(T) < 1$.

As a final remark, observe that any logic L in $\mathbf{Int}^{\mathbf{2}}_{\mathbf{\Pi}}(Ł_n)$ satisfies condition K_1^2 (that is, $K_1^2(\Gamma)$ implies $\Gamma \vdash \varphi$, for every Γ and φ), and all such logics are explosive. Indeed, for any crisp evaluation e, $e(\varphi \wedge \neg\varphi) = 0$, i.e., the condition $K_1^2(\{\varphi, \neg\varphi\})$ is verified, and hence $\varphi, \neg\varphi \vdash_L \bot$ for any $L \in \mathbf{Int}^{\mathbf{2}}_{\mathbf{\Pi}}(Ł_n)$. Nevertheles, we can obtain new paraconsistent logics by intersecting them with paraconsistent logics of $\mathbf{Int_{\Pi}}(Ł_n)$. This is the case, for instance, of the logic $\Lambda_7 = L(F_{(1,L_2),2/3,1/3}) \cap L(F_1, F_{1/3})$ shown in the graph of the all intermediate logics for $Ł_4$ of Fig. 4 in Appendix B2.

In the Appendices B1 and B2, as a matter of example, we show the lattice of all intermediate logics for $Ł_3$ and $Ł_4$.

8 Concluding remarks

In this paper we have provided results towards a full study of intermediate logics between the degree-preserving and the truth-preserving Łukasiewicz logics, both for the cases of the finite-valued and the infinite-valued logics. In the infinite-valued case we have proved that there is at least a countable sequence of paraconsistent logics between the minimal paraconsistent intermediate logic $Ł^{\leq}$ and the minimal explosive logic $Ł^{\leq}_{exp}$. Similarly, we have proved that there is at least a countable sequence of explosive logics between the minimal explosive logic $Ł^{\leq}_{exp}$ and the maximal intermediate logic Ł. We have also proved that it is not possible to characterize these logics by matrices over $[0, 1]$ defined by lattice filters, with the exception of $Ł^{\leq}$ and Ł. This makes the characterization of the full set of intermediate logics very difficult. Even when we have restricted ourselves to the case to finite-valued Łukasiewicz logics $Ł_n$ we have not succeded in providing a full description of intermediate logics in general, only in the case that $n - 1$ is a prime number we have provided a much more complete insight. Nevertheless the examples of $Ł_3$ and $Ł_4$ are fully described and we show a way for a general study of the case where $n - 1$ is a prime number.

The paper leaves a number of interesting questions for further research, for instance:

1. When studying the intermediate logics for $Ł_n$ we have used some logics that are not intermediate, that would be interesting to analyze. For example, we have used the logics $L(F_t)$ being $1 \neq t \in ŁV_n$ (that are logics not comparable with $Ł_n$), as well as the intersection of them with $Ł_n$. In the case of $Ł_3$ we can find the logic $L(F_{1/3})$ that coincides with the well-known paraconsistent logic J_3 introduced by N. da Costa and I. D'Ottaviano in [12]. Therefore, their study is also interesting in order to obtain new paraconsistent logics from Łukasiewicz logics and, perhaps, like in the case of $L(F_{1/3})$, some ideal paraconsistent logic in the sense of [2].

2. When studying the intermediate logics defined by matrices whose algebras are not direct products of $\mathbf{ŁV}_n$, one needs to consider matrices defined over subalgebras of $\mathbf{ŁV}_n$ as well. In the case of $n-1$ prime, all the logics defined by filters of type $F_{(1,\mathbf{ŁV}_2),t_1,\ldots,t_k}$, for either $t_1 < 1$ or $k > 1$ are explosive, and the full set of these logics forms a Boolean lattice anti-isomorphic to the set of subsets of $ŁV_n$. Is it also true, or is there an analogous result in the general case?

As a general conclusion we can say that the study initiated in this paper has introduced a wide family of paraconsistent logics with nice semantics, that can be enlarged when studying the logics described in first item above.

Finally, let us remark that, in the general setting of abstract algebraic logic, the paper provides a large set of examples of admissible rules in the degree preserving (finite or infinite-valued) Łukasiewicz logics.

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Appendix A: The lattices $\mathbf{Int_\Pi}(\text{Ł}_3)$ and $\mathbf{Int_\Pi}(\text{Ł}_4)$ of intermediate logics

A1: The lattice of intermediate logics $\mathbf{Int_\Pi}(\text{Ł}_3)$

In $\mathbf{ŁV}_3$ the lattice filters are F_1 and $F_{\frac{1}{2}}$, defining the logics $L(F_1) = \text{Ł}_3$ and $L(F_1, F_{\frac{1}{2}}) = \text{Ł}_3^{\leq}$, and there is only one lattice filter defining a different logic, the lattice filter $F_{1,\frac{1}{2}}$:

$L(F_{1,1/2})$

- Semantic condition: $K_1 \vee (C_1 \wedge C_{1/2})$
- Axiomatization:

$$\text{Ł}_3^{\leq} + (exp) = \text{Ł}_3^{\leq} + \frac{\varphi \quad\quad \vdash \neg(\varphi \oplus \varphi))}{\bot}$$

The lattice $\mathbf{Int_\Pi}(\text{Ł}_3)$ is then the chain of three elements depicted in Figure 1, where the sublattice $\mathbf{Int_\Pi^{LF}}(\text{Ł}_3)$ is composed of the logics Ł_3 and $L(F_{1,\frac{1}{2}})$, while the sublattice $\mathbf{Int_\Pi^{OF}}(\text{Ł}_3)$ is composed of the logics Ł_3 and $\text{Ł}_3^{\leq}$. The only paraconsistent logic in $\mathbf{Int_\Pi}(\text{Ł}_3)$ is obviously $\text{Ł}_3^{\leq}$.

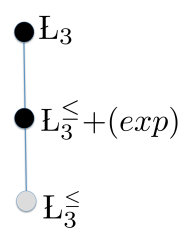

Figure 1: Logics between $\text{Ł}_3^{\leq}$ and Ł_3 in the lattice $\mathbf{Int_\Pi}(\text{Ł}_3)$.

A2: The lattice of intermediate logics $\mathbf{Int_\Pi}(\text{Ł}_4)$

The lattice $\mathbf{Int_\Pi}(\text{Ł}_4)$ contains the following logics:

- those belonging to the sublattice $\mathbf{Int_\Pi^{LF}}(\text{Ł}_4)$, i.e. the four logics Ł_4, $L(F_{1,2/3})$, $L(F_{1,1/3})$, and $L(F_{1,2/3,1/3})$;
- those in the sublattice $\mathbf{Int_\Pi^{OF}}(\text{Ł}_4)$, i.e. the four logics Ł_4, $L(F_1, F_{2/3})$, $L(F_1, F_{1/3})$, and $L(F_1, F_{2/3}, F_{1/3}) = \text{Ł}_4^{\leq}$;
- those obtained by intersection of logics in the two sublattices above, that is, the logics $L(F_1, F_{2/3}) \cap L(F_{1,2/3,1/3}) = L(\{F_1, F_{2/3}, F_{1,2/3,1/3}\})$, and $L(F_1, F_{1/3}) \cap L(F_{1,2/3,1/3}) = L(\{F_1, F_{1/3}, F_{1,2/3,1/3}\})$;

- those not appearing in the above items: in this case we only have the logic $L(F_1, F_{2/3,1/3}) = Ł_4^{\leq} + (exp)$.

The lattice $\mathbf{Int_\Pi}(Ł_4)$ is depicted in Figure 2, where the grey nodes correspond to paraconsistent logics. Next we describe the logics with the conditions characterizing them (we give the semantic conditions and their simplified disjunctive normal form if they are different from the original semantic conditions) and their axiomatization.[13] In the following description we omit the logics $Ł_4$ and $Ł_4^{\leq}$, whose axiomatizations are already well known.

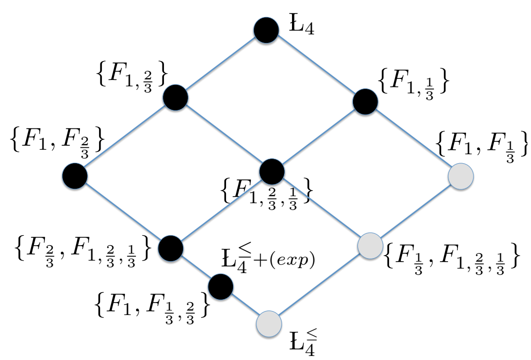

Figure 2: All intermediate logics between $Ł_4^{\leq}$ and $Ł_4$ in the lattice $\mathbf{Int_\Pi}(Ł_4)$.

1. Logics belonging to $\mathbf{Int_\Pi^{LF}}(Ł_4)$:

$L(F_{1,2/3})$

- Semantic condition: $K_1 \vee (C_1 \wedge C_{2/3})$
- Axiomatization:

$$Ł_4^{\leq} + \frac{\varphi \quad \vdash \neg(\varphi^3)}{\bot} + \frac{\varphi \quad \vdash (\varphi \to \neg\varphi) \vee (\varphi \to \psi)}{\psi}$$

$L(F_{1,1/3})$

- Semantic condition: $K_1 \vee (C_1 \wedge C_{1/3})$
- Axiomatization:

$$Ł_4^{\leq} + \frac{\varphi \quad \vdash \neg(\varphi^3)}{\bot} + \frac{\varphi \quad \vdash MN_{2/3,1/3}(\varphi, \psi) \vee (\varphi \to \psi)}{\psi}$$

$L(F_{1,2/3,1/3})$

[13] In the axiomatizations we use the abbreviations: φ^k for $\varphi \otimes \varphi \ldots \otimes \varphi$ and $k\varphi$ for $\varphi \oplus \varphi \ldots \oplus \varphi$.

- Semantic condition: $K_1 \vee (C_1 \wedge C_{2/3} \wedge C_{1/3})$
- Axiomatization:

$$Ł_4^{\leq} + \frac{\varphi \quad \vdash \neg(\varphi^3)}{\bot}$$

2. Logics belonging to $\mathbf{Int}_{\mathbf{\Pi}}^{\mathbf{OF}}(Ł_4)$:

$L(F_1, F_{2/3})$

- Semantic condition: $C_1 \wedge C_{2/3}$
- Axiomatization:

$$Ł_4^{\leq} + \frac{\varphi \quad \vdash (\varphi \to \neg\varphi) \vee (\varphi \to \psi)}{\psi}$$

$L(F_1, F_{1/3})$

- Semantic condition: $C_1 \wedge C_{1/3}$
- Axiomatization:

$$Ł_4^{\leq} + \frac{\varphi \quad \vdash MN_{2/3,1/3}(\varphi, \psi) \vee (\varphi \to \psi)}{\psi}$$

3. Logics obtained as intersection of logics of $\mathbf{Int}_{\mathbf{\Pi}}^{\mathbf{LF}}(Ł_4)$ and $\mathbf{Int}_{\mathbf{\Pi}}^{\mathbf{OF}}(Ł_4)$:

$L(F_{1,2/3,1/3}, F_1, F_{2/3})$

- Semantic condition: $(K_1 \vee (C_1 \wedge C_{2/3} \wedge C_{1/3})) \wedge (C_1 \wedge C_{2/3}) = (K_1 \wedge C_{2/3}) \vee (C_1 \wedge C_{2/3} \wedge C_{1/3})$
- Axiomatization:

$$Ł_4^{\leq} + \frac{\varphi \quad \vdash \neg\varphi^2 \vee (\neg\varphi^3 \wedge 2\psi)}{\psi}$$

$L(F_1, F_{1/3}, F_{1,2/3,1/3})$

- Semantic condition: $C_1 \wedge C_{1/3} \wedge (K_1 \vee (C_1 \wedge C_{2/3} \wedge C_{1/3})) = (K_{2/3} \wedge C_{1/3}) \vee (C_1 \wedge C_{2/3} \wedge C_{1/3})$
- Axiomatization:

$$Ł_4^{\leq} + \frac{\varphi \quad \vdash (\neg\varphi^3 \wedge 3\psi)}{\psi}$$

4. The remaining logic is related to the explosion inference rule:

$L(F_1, F_{2/3,1/3})$

- Semantic condition: $C_1 \wedge (K_{2/3} \vee (C_{2/3} \wedge C_{1/3})) = K_{2/3} \vee (C_1 \wedge C_{2/3} \wedge C_{1/3})$
- Axiomatization:

$$Ł_4^{\leq} + (exp) = Ł_4^{\leq} + \frac{\varphi \quad \vdash \neg(\varphi \oplus \varphi))}{\bot}$$

Appendix B: The lattice of all the intermediate logics for Ł$_3$ and Ł$_4$

B1: Intermediate logics for Ł$_3$

The Boolean lattice $\mathbf{Int}^{\mathbf{2}}_{\mathbf{\Pi}}(\text{Ł}_3)$ described in Section 7 contains only two intermediate logics, Ł$_3$ and $L(F_{(1,\mathbf{ŁV}_2),1/2}) \cap \text{Ł}_3$. As it can be proved, $L(F_{(1,\mathbf{ŁV}_2),1/2}) \cap \text{Ł}_3$ strictly contains the coatom of the Boolean lattice $\mathbf{Int}_{\mathbf{\Pi}}(\text{Ł}_3)$, i.e., it contains the logic $\text{Ł}_3^{\leq}+(exp) = L(F_{1,1/2})$. Indeed, the condition defining $L(F_{(1,\mathbf{ŁV}_2),1/2}) \cap \text{Ł}_3$ is $K_1^2 \vee C_{1/2}$, and the one defining $L(F_{1,1/2})$ is $K_1 \vee (C_1 \wedge C_{1/2})$. In order to show the inclusion it is now a simple computation to check that the derivation $\varphi \leftrightarrow \neg\varphi \vdash \perp$ holds in $L(F_{(1,\mathbf{ŁV}_2),1/2}) \cap \text{Ł}_3$ ($K_1^2(\{\varphi \leftrightarrow \neg\varphi\})$ is verified) but not in $L(F_{1,1/2})$ (neither condition K_1 nor C_1 is satisfied).

Therefore the lattice of all intermediate logics for Ł$_3$ is the chain of four elements depicted in Figure 3, where only $\text{Ł}_3^{\leq}$ is paraconsistent.

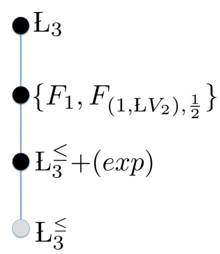

Figure 3: All intermediate logics between $\text{Ł}_3^{\leq}$ and Ł$_3$.

B2: Intermediate logics for Ł$_4$

The new intermediate logics obtained in Section 7 are the ones belonging to $\mathbf{Int}^{\mathbf{2}}_{\mathbf{\Pi}}(\text{Ł}_n)$ plus their intersection with the logics belonging to $\mathbf{Int}^{\mathbf{LF}}_{\mathbf{\Pi}}(\text{Ł}_n)$ and $\mathbf{Int}^{\mathbf{OF}}_{\mathbf{\Pi}}(\text{Ł}_n)$. For $n = 4$, they are described below together with the conditions defining them and their relation to the logics of matrices defined by lattice filters over direct products of copies of $\mathbf{ŁV}_4$ described in Appendix A2 and depicted in Figure 2.

In order to have a complete description of the intermediate logics for Ł$_4$ we have to consider the relation between the logics depicted in Figure 2 and the ones obtained by intersecting logics of $\mathbf{Int}^{\mathbf{2}}_{\mathbf{\Pi}}(\text{Ł}_4)$ with logics of either $\mathbf{Int}^{\mathbf{LF}}_{\mathbf{\Pi}}(\text{Ł}_4)$ or $\mathbf{Int}^{\mathbf{OF}}_{\mathbf{\Pi}}(\text{Ł}_4)$. The complete graph of the lattice $\mathbf{Int}(\text{Ł}_4)$ is depicted in Figure 4, where again the grey nodes correspond to paraconsistent logics and where the new logics obtained $(\Lambda_1, \ldots, \Lambda_7)$ are listed below together with their characterizing conditions and their relative position in the graph. At the end of this subsection we sketch a method to prove all the inclusions described below and we show the proof for two particular cases (which are not consequence of results of Appendix A2).

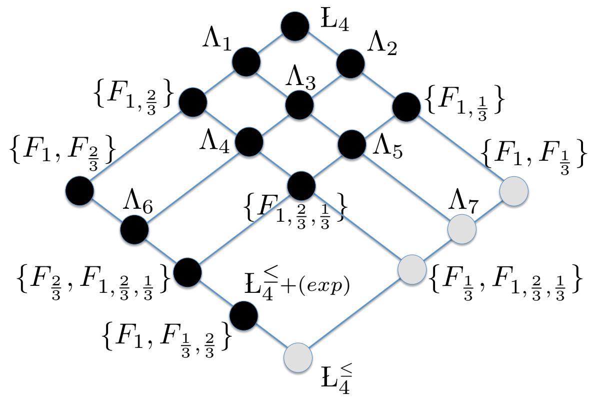

Figure 4: All intermediate logics between $Ł_4^{\leq}$ and $Ł_4$.

1. Logics belonging to $\mathbf{Int}^{\mathbf{2}}_{\mathbf{\Pi}}(Ł_4)$ with their defining conditions, and their relation to the logics of $\mathbf{Int}_{\mathbf{\Pi}}(Ł_4)$, depicted in Fig. 2:

   $\Lambda_1 = Ł_4 \cap L(F_{(1,\mathbf{ŁV}_2),2/3})$

   - Semantical condition: $C_1 \wedge (K_1^2 \vee C_{2/3}) = (K_1^2 \wedge C_1) \vee (C_1 \wedge C_{2/3})$
   - This logic strictly contains $L(F_{1,2/3})$ and it is not comparable with $L(F_{1,1/3})$.

   $\Lambda_2 = Ł_4 \cap L(F_{(1,\mathbf{ŁV}_2),1/3})$

   - Semantical condition: $C_1 \wedge (K_1^2 \vee C_{1/3}) = (K_1^2 \wedge C_1) \vee (C_1 \wedge C_{1/3})$
   - This logic strictly contains $L(F_{1,1/3})$ and it is not comparable with $L(F_{1,2/3})$.

   $\Lambda_3 = Ł_4 \cap L(F_{(1,\mathbf{ŁV}_2),2/3}) \cap L(F_{(1,\mathbf{ŁV}_2),1/3}) = Ł_4 \cap L(F_{(1,\mathbf{ŁV}_2),2/3,1/3})$

   - Semantical condition: $C_1 \wedge (K_1^2 \vee (C_{2/3} \wedge C_{1/3})) = (K_1^2 \wedge C_1) \vee (C_1 \wedge C_{2/3} \wedge C_{1/3})$
   - This logic is strictly contained in Λ_1 and in Λ_2, and it strictly contains $L(F_{1,2/3,1/3})$. Moreover, it is not comparable with $L(F_{1,2/3})$ and $L(F_{1,1/3})$.

2. Logics obtained by intersection with logics of $\mathbf{Int}^{\mathbf{LF}}_{\mathbf{\Pi}}(Ł_4)$:

   $\Lambda_4 = Ł_4 \cap L(F_{(1,\mathbf{ŁV}_2),2/3,1/3}) \cap L(F_{1,2/3}) = L(F_{(1,\mathbf{ŁV}_2),2/3,1/3}) \cap L(F_{1,2/3})$

   - Semantical condition: $((K_1^2 \wedge C_1) \vee (C_1 \wedge C_{2/3} \wedge C_{1/3})) \wedge (K_1 \vee (C_1 \wedge C_{2/3})) = (K_1 \wedge K_1^2 \wedge C_1) \vee (K_1 \wedge C_{2/3} \wedge C_{1/3}) \vee (K_1^2 \wedge C_1 \wedge C_{2/3}) \vee (C_1 \wedge C_{2/3} \wedge C_{1/3}) =$[14] $K_1 \vee (K_1^2 \wedge C_1 \wedge C_{2/3}) \vee (C_1 \wedge C_{2/3} \wedge C_{1/3})$

[14] Take into account that K_1 implies $K_1^2 \wedge C_1$, that is: $K_1(\Gamma)$ implies $K_1^2(\Gamma) \wedge C_1(\Gamma, \varphi)$, for every Γ, φ.

- This logic is strictly contained in $L(F_{1,2/3})$ and strictly contains $L(F_{1,2/3,1/3})$. Moreover, it is not comparable with $L(F_1, F_{2/3})$.

$\Lambda_5 = Ł_4 \cap L(F_{(1,\mathbf{ŁV}_2),2/3,1/3}) \cap L(F_{1,1/3}) = L(F_{(1,\mathbf{ŁV}_2),2/3,1/3}) \cap L(F_{1,1/3})$

- Semantical condition: $((K_1^2 \wedge C_1) \vee (C_1 \wedge C_{2/3} \wedge C_{1/3})) \wedge (K_1 \vee (C_1 \wedge C_{1/3})) = (K_1 \wedge K_1^2 \wedge C_1) \vee (K_1 \wedge C_{2/3} \wedge C_{1/3}) \vee (K_1^2 \wedge C_1 \wedge C_{1/3}) \vee (C_1 \wedge C_{2/3} \wedge C_{1/3}) = K_1 \vee (K_1^2 \wedge C_1 \wedge C_{1/3}) \vee (C_1 \wedge C_{2/3} \wedge C_{1/3})$
- This logic is strictly contained in $L(F_{1,1/3})$ and strictly contains $L(F_{1,2/3,1/3})$. Moreover, it is not comparable with $L(F_1 \cap F_{1/3})$.

3. Logics obtained by intersection with logics of $\mathbf{Int}_{\mathbf{\Pi}}^{\mathbf{OF}}(Ł_4)$:

$\Lambda_6 = L(F_{(1,\mathbf{ŁV}_2),2/3,1/3}) \cap L(F_{1,2/3}) \cap L(F_1, F_{2/3}) = L(F_{(1,\mathbf{ŁV}_2),2/3,1/3}) \cap L(F_1, F_{2/3})$

- Semantical condition: $(K_1 \vee (K_1^2 \wedge C_1 \wedge C_{2/3}) \vee (C_1 \wedge C_{2/3} \wedge C_{1/3})) \wedge (C_1 \wedge C_{2/3})) = (K_1^2 \wedge C_1 \wedge C_{2/3}) \vee (K_1 \wedge C_{2/3}) \vee (C_1 \wedge C_{2/3} \wedge C_{1/3})$
- This logic is strictly contained in $L(F_1, F_{2/3})$ and strictly contains $L(F_{1,2/3,1/3}) \cap L(F_1, F_{2/3})$.

$\Lambda_7 = L(F_{(1,\mathbf{ŁV}_2),2/3,1/3}) \cap L(F_{1,1/3}) \cap L(F_1, F_{1/3}) = L(F_{(1,\mathbf{ŁV}_2),2/3,1/3}) \cap L(F_1, F_{1/3})$

- Semantical condition: $(K_1 \vee (K_1^2 \wedge C_1 \wedge C_{1/3}) \vee (C_1 \wedge C_{2/3} \wedge C_{1/3})) \wedge (C_1 \wedge C_{1/3})) = (K_1^2 \wedge C_1 \wedge C_{1/3}) \vee (K_1 \wedge C_{1/3}) \vee (C_1 \wedge C_{2/3} \wedge C_{1/3})$
- This logic is strictly contained in $L(F_1, F_{1/3})$ and strictly contains $L(F_{1,2/3,1/3}) \cap L(F_1, F_{1/3})$.

Now we describe a method to prove that a logic of the family above is strictly contained in some other logic of that family. The basic idea is that, for any given function $f : ŁV_4 \longrightarrow ŁV_4$ such that $f(0), f(1) \in \{0, 1\}$, there exists a McNaugthon function $f_M : [0,1] \to [0,1]$ such that its restriction to $ŁV_4$ coincides with f, and thus there is a logical formula that corresponds to this function. Therefore to prove an inclusion it is enough to provide two functions f_4 and g_4 that 'satisfy' (abusing the language by identifying functions and formulas) one condition and not the other. Here is one example.

We have claimed that the logic $L(F_{(1,\mathbf{ŁV}_2),2/3,1/3}) \cap L(F_{1,2/3})$ is strictly contained in $L(F_{1,2/3})$ and strictly contains $L(F_{1,2/3,1/3})$. Take the conditions defining these logics:

- $L(F_{(1,\mathbf{ŁV}_2),2/3,1/3}) \cap L(F_{1,2/3})$ is defined by the condition $K_1 \vee (K_1^2 \wedge C_1 \wedge C_{2/3}) \vee (C_1 \wedge C_{2/3} \wedge C_{1/3})$,
- $L(F_{1,2/3})$ is defined by the condition $K_1 \vee (C_1 \wedge C_{2/3})$,
- $L(F_{1,2/3,1/3})$ is defined by the condition $K_1 \vee (C_1 \wedge C_{2/3} \wedge C_{1/3})$.

To prove that $L(F_{(1,\mathbf{ŁV}_2),2/3,1/3}) \cap L(F_{1,2/3})$ is strictly contained in $L(F_{1,2/3})$ we need to define functions f_4 and g_4 satisfying the condition of the second logic but not the condition of the first one. Take $f_4(x) = x$ and $g_4(x) = 2/3$ if $x = 2/3$

and $g_4(x) = 0$ otherwise. Obviously f_4 neither satisfies K_1 nor K_1^2, while both f_4 and g_4 satisfy C_1 and $C_{2/3}$, but not $C_{1/3}$. Thus f_4 and g_4 satisfy the condition of $L(F_{1,2/3})$ but not the condition of $L(F_{(1,\mathbf{ŁV}_2),2/3,1/3}) \cap L(F_{1,2/3})$.

On the other hand to prove that $L(F_{(1,\mathbf{ŁV}_2),2/3,1/3}) \cap L(F_{1,2/3})$ strictly contains $L(F_{1,2/3,1/3})$ we take the functions f_4 and g_4 defined as follows: $f_4(0) = f_4(1) = 0, f_4(1/3) = 1$ and $f_4(2/3) = 1/3$, and $g_4(1/3) = 1$ and $g_4(x) = 0$ otherwise. One can check that f_4 satisfies K_1^2 but not K_1. Moreover both functions satisfy C_1 and $C_{2/3}$ but not $C_{1/3}$. Thus f_4 and g_4 satisfy the condition of $L(F_{(1,\mathbf{ŁV}_2),2/3,1/3}) \cap L(F_{1,2/3})$ but not the condition of $L(F_{1,2/3,1/3})$.