NON AXIOMATIZABILITY OF MODAL ŁUKASIEWICZ LOGIC

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Abstract. In this work we study the decidability of the global modal logic arising from Kripke frames evaluated on certain residuated lattices (including all BL algebras), known in the literature as crisp modal many-valued logics. We exhibit a large family of these modal logics that are undecidable, in opposition to classical modal logic and to the propositional logics defined over the same classes of algebras. These include the global modal logics arising from the standard Łukasiewicz and Product algebras. Furthermore, it is shown that global modal Łukasiewicz and Product logics are not recursively axiomatizable. We conclude the paper by solving negatively the open question of whether a global modal logic coincides with the local modal logic closed under the unrestricted necessitation rule.

1. Introduction

Modal logic is one of the most developed and studied non-classical logics, exhibiting a beautiful equilibrium between complexity and expressivity. Generalizations of the concepts of necessity and possibility offer a rich setting to model and study notions from many different areas, including provability predicates, temporal and epistemic concepts, workflow in software applications, etc. On the other hand, substructural logics, defined by Gentzen systems lacking some structural rules, provide a formal framework to manage vague and resource sensitive information in a very general and adaptable fashion.

Modal many-valued logic is at the intersection of both modal and substructural logics. It is a field in ongoing development, that has been studied in the literature both from a purely theoretic point of view and also pursuing the development of frameworks suitable to model complex environments that require valued information. The notion of a modal many-valued logic studied in this paper follows the tradition initiated by Fitting [11, 12] and Hàjek [17, 15]. These logics are defined over valued Kripke models, i.e., Kripke structures where the accessibility relation and each variable (at each world of the model) take values in a certain algebra, and the interpretation of the modalities generalizes that of classical modal logic. Local and global deduction stands for the interpretation of the premises and conclusion in the derivability relation: whether they are to be considered respectively world-wise or globally. These two semantics (local and global) behave with respect to the First Order semantics of the corresponding many-valued logic in the analogous way to how they do in the classical case (i.e., modal formulas can be translated into certain first order formulas). This approach differs from another relevant framework of so-called modal substructural logics studied for instance in [22, 27, 19].

Over valued Kripke models, in contrast to classical modal logic, the usual modal operators ($\Box$ and $\Diamond$) are not inter-definable, and the K axiom is not true, in general. Apart from the minimal logics in terms of modal operators, the logics arising from valued Kripke models whose accessibility relation is two-valued (we will call them crisp) are also worth of special attention, having as underlying relational semantics usual (classical) Kripke frames. Thus, the systematic study of modal many-valued logics has focused in the combinations of the previous different characteristics over relevant classes of algebras. In particular, much of the work has been devoted to study the logics arising from the three main standard algebras associated with a continuous t-norm, and complete with respect to the three main fuzzy logics: Gödel -G-, Łukasiewicz -Ł- and Product -Π. Even if the problem of axiomatizing these modal logics has received much attention, so far only Gödel modal logics have been showed to be axiomatizable in the usual sense.

In [7, 6, 20, 28] all the minimal modal logics associated to the standard Gödel algebra are axiomatized, taking into account the different options for what concerns the modal operators and the accessibility relation. On the other hand, a general study of finite MTL algebras expanded with the $\Box$ operator is given in [3]. The axiomatic systems proposed there rely on the addition of canonical constants[1] which among other things, make $\Box$ and $\Diamond$ interdefinable in finite algebras ([33]). A similar situation happens in subsequent works concerning the study of the (standard) modal Product logic [32] or other infinite MTL chains, where in order to get a complete axiomatization it has been necessary to expand the language with a countable dense set of constants and the Monteiro-Baaz Δ operator. Indirectly, it occurs also in the case of modal expansions of (standard) modal Łukasiewicz logic [18], where there is a dense countable set of elements of the algebra which is syntactically definable in the logic, and serves the same purposes as the constants from the previous cases. This phenomenon points to the problematic of adapting the completeness proofs to more general cases (e.g. defining logics over classes of algebras, as opposed to a single algebra only), and to the presence of infinitary inference rules in their modal axiomatizations, which arise from the propositional requirements about the (infinitely many) constants [31]. Indeed, the axiomatic systems proposed in [18] and [32] are infinitary (they include an inference rule with infinitely many premises). Approaching this problem from a different perspective, in [10] a logic loosely related to the modal expansion of Ł (based on interpretations on closed intervals of the real line) is recursively axiomatized. Nevertheless, axiomatizing the (finitary) modal expansions of Łukasiewicz and Product logics remained an open problem.

One of the main contributions of this paper concerns this question. We prove that the finitary global modal deductions with crisp accessibility and both $\Box$ and $\Diamond$ operators over the standard Łukasiewicz and Product algebras are not recursively axiomatizable (Theorem 4.8 and Corollary 4.10).

These results can be seen in relation to a celebrated result by Scarpellini [29] that states that the set of tautologies of the infinitely-valued predicate Łukasiewicz logic is not recursively enumerable (later refined in [26, 25], showing that it is in fact Π_2-complete). In the context of [21], the present paper classifies as non-axiomatizable two relevant many-valued logics, that have a natural definition and a track of related works in the literature.

A related second question that is worth of attention is that of the decidability of modal many-valued logics. It is known that while first order (classical) logic is undecidable, modal logic is, as propositional logic, decidable. In many-valued logics, similarly, first order logics are undecidable, while the propositional cases are co-NP complete[2]. For what concerns modal many-valued logics, the results known about decidability are fairly partial.

Gödel modal logics do not enjoy in general the finite model property with respect to the intended semantics [7], but interestingly enough in [4] it is proven the decidability of the local consequence relation for these logics, and related results for S5 over Gödel logics (decidability over order-based modal logics is further developed in [5]). However, in relation to the ongoing work, we outline that the decidability of the global consequence over the previous classes of models still remains open. It is also known that local standard modal Łukasiewicz logic is decidable [30], following from the completeness of the logic with respect to so-called witnessed models. On the other hand, in [30] it is also shown the undecidability of the local deductions over transitive models with crisp accessibility, valued respectively over the standard Łukasiewicz and Product algebras. In the context of fuzzy description logics (equivalent to a fragment of multi-modal logics), some undecidability results for the satisfiability question have been proven [2] (see [8] for a modal logic presentation of the results) for Łukasiewicz and Product valued fragments of FDLs. These results translated to the many-valued multi-modal framework amount to undecidability of satisfiability over some fragments of multi-modal logics over the corresponding algebras (which include a strong negation $\sim x = 1 - x$, constants and a valued accessibility relation).

[1] Namely, one constant symbol for each element of the non-modal algebra.

[2] Nevertheless, while it is known that classical FO logic is Σ_1-complete, tautologies of FO over the standard Łukasiewicz algebra form a Π_2 complete set [26, 25], and those over the standard Product algebra are Π_2-hard [15].

In order to reach the results of non-axiomatizability pointed out before, we studied also the decidability of deduction in finitary global modal logics. For that reason, in this paper we contribute to this question too, and show that the deduction is undecidable for a large class of crisp modal logics whose algebras of evaluation hold certain basic conditions (Theorem 3.2). This class includes the modal logics over the standard Łukasiewicz and Product algebras. Thus, the two main problems left open concerning the decidability of minimal crisp bi-modal logics based on continuous t-norm logics are the decidability of the local modal Product logic and that of the global modal Gödel logic.

In the last section of the paper, we study the relation between the local and the global modal deductions, particularly motivated by the peculiarities of standard modal Łukasiewicz logics: while the local deduction (and so, the tautologies) is decidable (Δ_1), the global deduction is not recursively enumerable (Σ_1). We observe that as a result, the global deduction cannot be axiomatized by the local one extended with the usual necessitation rule $N_\Box$. This contrasts with all other known cases and allows us to close negatively this open question, posed in [3].

The paper is organized as follows. We start in Section 2 by introducing necessary preliminaries. In Section 3 we study the decidability of a large family of residuated lattice based modal logics, and prove they are undecidable by reducing to them the Post Correspondence Problem. A remarkable feature of the reduction is that not only the minimal logic, but also the logics over finite models of the corresponding classes are proven undecidable, opening the way for showing negative results on the enumerability of those logics. In Section 4 we obtain negative results concerning the axiomatization (in the usual finitary way) of some of the above logics, namely that the finitary standard crisp global modal Łukasiewicz logic and Product logics are not recursively enumerable. We conclude the paper with Section 5 by showing that a global modal logic might fail to be axiomatized by an axiomatization of its corresponding logical logic plus the necessitation rule (we prove this is the case for all logics from Section 3).

2. Preliminaries

In this work, logics are identified with consequence relations [13], as opposed to sets of only formulae. In the literature of modal logics it is common this second approach [9], but we opt for the former presentation since the differences between local and global modal logics are lost if only the tautologies of the logic are considered. Observe that the lack of deduction theorem makes the usual implication and the logical consequence not interchangeable.

Let us begin by introducing this very basic framework.

Given a set of variables $\mathcal{V}$ and an algebraic language $\mathtt{L}$, the set $Fm^{\mathtt{L}}(\mathcal{V})$ is the set of formulas build from $\mathcal{V}$ using the symbols from $\mathtt{L}$. Unless stated otherwise, $\mathcal{V}$ is a fixed denumerable set, and it will be omitted in the notation of the set of formulas, and if the language is clear from the context we will omit it as well. A ***rule*** in the language $\mathtt{L}$ is a pair $\langle \Gamma, \varphi\rangle \in \mathcal{P}(Fm^{\mathtt{L}}(\mathcal{V})) \times Fm^{\mathtt{L}}(\mathcal{V})$. A ***logic*** $\mathcal{L}$ over language $\mathtt{L}$ is a set of rules such that:

(1) $\mathcal{L}$ is reflexive, i.e., for every $\Gamma \subseteq Fm$ and every $\gamma \in \Gamma$, $\langle \Gamma, \gamma\rangle \in \mathcal{L}$,
(2) $\mathcal{L}$ satisfies cut, i.e., if $\langle \Gamma, \phi\rangle \in \mathcal{L}$ for all $\phi \in \Phi$, and $\langle \Phi, \varphi\rangle \in \mathcal{L}$ then $\langle \Gamma, \varphi\rangle \in \mathcal{L}$,
(3) $\mathcal{L}$ is substitution invariant, i.e., for any substitution σ, if $\langle \Gamma, \varphi\rangle \in \mathcal{L}$ then $\langle \sigma[\Gamma], \sigma(\varphi)\rangle \in \mathcal{L}$.

Whenever $\langle \Gamma, \varphi\rangle \in \mathcal{L}$ we will write $\Gamma \vdash_{\mathcal{L}} \varphi$. Given a set of rules R, we will write R^l to denote the minimal logic containing the rules in R (the language and set of variables are assumed fixed). We say that a set of rules R axiomatizes a logic $\mathcal{L}$ whenever $R^l = \mathcal{L}$. Observe that in this sense, any logic is axiomatized at least by itself.

With computational questions in mind, we are interested in logics determined by its deductions from finite sets of premises (called finitary rules), and in knowing when these logics are recursively enumerable (namely, whether there is a recursive procedure that enumerates all valid finitary deductions of the logic). A logic is ***finitary*** whenever $\Gamma \vdash_{\mathcal{L}} \varphi$ if and only if $\Gamma_0 \vdash_{\mathcal{L}} \varphi$ for some finite $\Gamma_0 \subseteq_\omega \Gamma$[3]. For a set

[3] As usual, $\subseteq_\omega$ denotes the relation of finite subset.

of finitary rules R, the logic R^l can be equivalently obtained through the usual notion of finite proof[4] in R. Given a finite set of formulas $\Gamma\cup\varphi$, a ***proof*** or ***derivation*** of φ from Γ in R is a finite list of formulas $\psi_1,\ldots,\psi_n$ such that $\psi_n=\varphi$ and for each ψ_i in the list, either $\psi_i\in\Gamma$ or there is a rule $\Sigma\vdash\phi$ and a substitution σ such that $\sigma(\phi)=\psi_i$ and $\sigma[\Sigma]$ (possibly empty) is a subset of $\{\psi_1,\ldots,\psi_{i-1}\}$ (or empty if $i=1$). It is well known that $R^l=\{\langle\Gamma,\varphi\rangle\colon\Gamma\cup\{\varphi\}\subseteq Fm$ and there is a proof of φ from Γ in $R\}$.

We will say that a logic is ***axiomatizable*** whenever it can be axiomatized with a recursive set of finitary rules. In this case, it is clear that the logic is finitary and RE. On the other hand, a finitary RE logic with an idempotent operation is always axiomatizable. This can be proven as it is done for Craig's Theorem, applying for each $\Gamma_n\vdash\varphi_n$ in the logic the idempotent operation n times both over each one the premises and over the consequence of the derivation (which produces a recursive set). Since in the rest of the paper we will work with logics having an idempotent operation, we can formulate the previous relations as follows.

Observation 2.1. *A finitary logic is recursively enumerable if and only if it is axiomatizable.*

Modal many-valued logics arise from Kripke structures evaluated over certain algebras, putting together relational and algebraic semantics in a fashion adapted to model different reasoning notions. Along the next section, the algebraic setting of these semantics will be the one of FL_{ew}-algebras, the corresponding algebraic semantics of the Full Lambek Calculus with exchange and weakening. This will offer a very general approach to the topic while relying in well-known algebraic structures. We will later focus on modal expansions of MV and product algebras.

Definition 2.2. A $\mathbf{FL_{ew}}$***-algebra*** is a structure $\mathbf{A}=\langle A;\wedge,\vee,\cdot,\rightarrow,\overline{0},\overline{1}\rangle$ such that

- $\langle A;\wedge,\vee,\overline{0},\overline{1}\rangle$ is a bounded lattice;
- $\langle A;\cdot,1\rangle$ is a commutative monoid;
- $\mathbf{A}$ satisfies $a\cdot b\leqslant c$ if and only if $a\leqslant b\rightarrow c$ for any $a,b,c\in A$.

We will usually write ab instead of $a\cdot b$, and abbreviate $\overbrace{x\cdot x\cdots x}^{n}$ by x^n. Moreover, as it is usual, we will define $\neg a$ to stand for $a\rightarrow 0$. A chain is a linearly ordered algebra.

In the setting of the previous definition, we will denote by $\mathbf{Fm}'$ the algebra of formulas built over a countable set of variables $\mathcal{V}$ using the language corresponding to the above class of algebras (i.e., $\langle\wedge/2,\vee/2,\cdot/2,\rightarrow/2,\neg/1,\overline{0}/0,\overline{1}/0\rangle$). We will refer to the bottom and top elements of the algebra, $\overline{0}^{\mathbf{A}}$ and $\overline{1}^{\mathbf{A}}$, simply by 0 and 1. Moreover, we will again write $\varphi\psi$ instead of $\varphi\cdot\psi$ and φ^n for the product of φ with itself n times, and we let, as usual

$$(\varphi\leftrightarrow\psi)\coloneqq(\varphi\rightarrow\psi)\cdot(\psi\rightarrow\varphi)\quad\text{and}\quad\neg\varphi\coloneqq\varphi\rightarrow\overline{0}.$$

For a set of formulas $\Gamma\cup\{\varphi\}$ and a class of $\mathbf{FL_{ew}}$-algebras $\mathbb{A}$, we write $\Gamma\models_{\mathbb{A}}\varphi$ if and only if, for each $\mathbf{A}\in\mathbb{A}$ and any $h\in Hom(\mathbf{Fm}',\mathbf{A})$, if $h(\gamma)=1$ for each $\gamma\in\Gamma$, then $h(\varphi)=1$ too. We will write $\models_{\mathbf{A}}$ instead of $\models_{\{\mathbf{A}\}}$.

$\mathbb{FL}$, the class of $\mathbf{FL_{ew}}$-algebras, is a variety thoughtfully studied [23], [14]. The logic $\mathbb{FL}_{ew}$, the Full Lambek Calculus with exchange and weakening, is complete (for finite sets of formulas) with respect to $\mathbb{FL}$ in the sense introduced above, i.e., for any $\Gamma\cup\{\varphi\}\subset_\omega Fm'$

$$\Gamma\vdash_{\mathbb{FL}_{ew}}\varphi\text{ iff }\Gamma\models_{\mathbb{FL}}\varphi$$

Let us introduce some examples of well-known varieties of $\mathbf{FL_{ew}}$-algebras. ***Heyting Algebras***, the algebraic counterpart of Intuitionistic logic, are $\mathbf{FL_{ew}}$-algebras where $\wedge=\cdot$. More in particular, the variety of ***Gödel algebras***, $\mathbb{G}$, (corresponding to intermediate Gödel-Dummett logic $\mathcal{G}$) is that of semilinear Heyting algebras, i.e., those satisfying $(a\rightarrow b)\vee(b\rightarrow a)=1$ for all a,b in the algebra. ***BL algebras***, the algebraic counterpart of Hàjek Basic Logic BL, are semilinear $\mathbf{FL_{ew}}$ algebras where $a\cdot(a\rightarrow b)=a\wedge b$ for any two elements in the algebra. The variety of ***MV algebras*** $\mathbb{MV}$, algebraic counterpart of Łukasiewicz logic Ł, is formed by the involutive BL algebras (i.e., satisfying $\neg\neg b\rightarrow b$),

[4]There exists also a more general notion of proof managing infinitary rules, based on wellfounded trees, that we will not use here.

and that of ***Product algebras*** (corresponding to Product Logic Π), $\mathbb{P}$ is formed by those BL algebras satisfying $\neg\neg a \to ((b \cdot a \to c \cdot a) \to (b \to c)$ and $a \wedge \neg a \to 0$.

Particular algebras in the previous classes are the so-called ***standard*** ones, whose universe is the standard unit real interval $[0,1]$.

- $[0,1]_G$, the **standard Gödel algebra**, puts
$$a \cdot b \coloneqq a \wedge b \qquad \text{and} \qquad a \to b \coloneqq \begin{cases} 1 & \text{if } a \leqslant b \\ b & \text{otherwise} \end{cases}$$
- $[0,1]_{\text{Ł}}$, the **standard MV algebra**, puts
$$a \cdot b \coloneqq \max\{0, a+b-1\} \qquad \text{and} \qquad a \to b \coloneqq \min\{1, 1-a+b\}$$
- MV_n, the n**-valued MV algebra** is the subalgebra of $[0,1]_{\text{Ł}}$ with universe $\{0, \frac{1}{n-1}, \dots, \frac{n-1}{n-1}\}$.
- $[0,1]_\Pi$, the **standard Product (Π) algebra**, puts
$$a \cdot b \coloneqq a \times b \qquad \text{and} \qquad a \to b \coloneqq \begin{cases} 1 & \text{if } a \leqslant b \\ b/a & \text{otherwise} \end{cases}$$
for $\times$ the usual product between real numbers;

It is known that the standard Gödel, MV and Product algebras generate their corresponding varieties. They do so also as quasi-varieties, which implies the completeness of the logics (understood as consequence relations) with respect to the logical matrices over the respective standard algebra. In the case of Gödel, it is also the case that the variety is generated as a generalized quasi-variety, while this fails for MV and Product algebras. The previous amounts to say that for any set of formulas $\Gamma \cup \{\varphi\}$, it holds that $\Gamma \vdash_{\mathcal{G}} \varphi$ if and only if $\Gamma \models_{[0,1]_G} \varphi$ (and if and only if $\Gamma \models_{\mathbb{G}} \varphi$). For a finite set of formulas $\Gamma \cup \{\varphi\}$ it holds that $\Gamma \vdash_{\text{Ł}} \varphi$ if and only if $\Gamma \models_{[0,1]_{\text{Ł}}} \varphi$ (if and only if $\Gamma \models_{\mathbb{MV}} \varphi$); and $\Gamma \vdash_\Pi \varphi$ if and only if $\Gamma \models_{[0,1]_\Pi} \varphi$ (if and only if $\Gamma \models_{\mathbb{P}} \varphi$);

Le us introduce some other families of $\mathbf{FL_{ew}}$-algebras that will be of use later on.

Definition 2.3. Let $\mathbf{A}$ be a $\mathbf{FL_{ew}}$-algebra.

- $\mathbf{A}$ is n***-contractive*** whenever $a^{n+1} = a^n$ for all $a \in A$.
- $\mathbf{A}$ is ***weakly-saturated*** if for any two elements $a, b \in A$, if $a \leqslant b^n$ for all $n \in \mathbb{N}$ then $ab = a$.

Observe that if $\mathbf{A}$ is n-contractive, the element a^n is idempotent (namely $a^n \cdot a^n = a^n$) for any $a \in A$. Simple examples of these algebras comprehend Heyting algebras (1-contractive), or MV_n algebras ($(n-1)$-contractive). On the other hand, the standard MV-algebra and product algebra are not n-contractive for any n. For what concerns weakly saturation, observe that if the element $\inf\{b^n : n \in \mathbb{N}\}$ exists in a weakly saturated algebra, then it is an idempotent element. Examples of weakly saturated algebras are the standard MV-algebra, the standard product algebra, as well as the algebras belonging to the generalised quasi-varieties generated by them.

The algebra of modal formulas $\mathbf{Fm}$ is built in the same way as $\mathbf{Fm}'$, expanding the language of $\mathbf{FL_{ew}}$-algebras with two unary operators $\Box$ and $\Diamond$. While it is clear how to lift an evaluation from the set of propositional variables $\mathcal{V}$ into an FL_{ew}-algebra to $\mathbf{Fm}'$, the semantic definition of the modal operators depends on the relational structure in the following way.

Definition 2.4. Let $\mathbf{A}$ be a $\mathbf{FL_{ew}}$-algebra. An $\mathbf{A}$***-Kripke model*** is a structure $\mathfrak{M} = \langle W, R, e\rangle$ such that

- $\langle W, R\rangle$ is a Kripke frame. That is to say, W is a non-empty set of so-called worlds and $R \subseteq W \times W$ is a binary relation over W, called accessibility relation. We will often write Rvw instead of $\langle v, w\rangle \in R$;
- $e\colon \mathcal{V} \times W \to A$. e is uniquely extended to Fm' by letting
$$e(v, \overline{c}) \coloneqq c \text{ for } c \in \{0,1\} \qquad e(v, \varphi \star \psi) \coloneqq e(v,\varphi) \star e(v,\psi) \text{ for } \star \in \{\wedge, \vee, \odot, \to\}$$
$$e(v, \Box\varphi) \coloneqq \bigwedge_{\langle v,w\rangle \in R} e(w,\varphi) \qquad e(v, \Diamond\varphi) \coloneqq \bigvee_{\langle v,w\rangle \in R} e(w,\varphi)$$

A model is ***safe*** whenever the values of $e(v, \Box\varphi)$ and $e(v, \Diamond\varphi)$ are defined for any formula at any world. We will call $\mathbf{FL_{ew}}$***-Kripke models*** to the class of all safe $\mathbf{A}$-Kripke models, for all $\mathbf{FL_{ew}}$-algebra $\mathbf{A}$.

We call a model $\mathfrak{M}$ ***directed*** whenever there is some world $u \in W$ in it such that, for any $v \in W$, there is some path[5] from u to v in $\mathfrak{M}$. For what concerns notation, given a class of models $\mathbb{C}$, we denote by $\omega\mathbb{C}$ the finite models in $\mathbb{C}$. On the other hand, for a class of algebras $\mathbb{A}$ (or a single algebra $\mathbf{A}$) we write $K\mathbb{A}$ ($K\mathbf{A}$) to denote the class of safe Kripke models over the algebras in the class (or over the single algebra specified). Finally, in order to lighten the reading, we will let $K\L$ and $K\Pi$ to denote respectively $K[0,1]_{\L}$ and $K[0,1]_{\Pi}$.

Towards the definition of modal logics over $\mathbf{FL_{ew}}$-algebras relying in the notion of $\mathbf{FL_{ew}}$-Kripke models, it is natural to preserve the notion of truth world-wise being $\{1\}$ (in order to obtain, world-wise, the propositional $\mathbb{FL}_{ew}$ logic). With this in mind, for any $\mathbf{A}$-Kripke model $\mathfrak{M}$ and $v \in W$ we say that $\mathfrak{M}$ ***satisfies a formula*** φ ***in*** v, and write $\mathfrak{M}, v \models \varphi$ whenever $e(v, \varphi) = 1$. Similarly, we simply say that $\mathfrak{M}$ ***satisfies a formula*** φ, and write $\mathfrak{M} \models \varphi$ whenever for all $v \in W$ $\mathfrak{M}, v \models \varphi$. The same definitions apply to sets of formulas.

As it happens in the classical case, the previous definition of satisfiability gives place to two different logics (where logic stands for logical consequence relation): the local and the global one. Along this work we will focus on the study of the global logic, but in Section 5 we will point out some results involving the local modal logic as well.

Definition 2.5. Let $\Gamma \cup \{\varphi\} \subseteq_\omega Fm$, and $\mathbb{C}$ be a class of $\mathbf{FL_{ew}}$-Kripke models.

- φ ***follows from*** Γ ***globally in*** $\mathbb{C}$, and we write $\Gamma \vdash_{\mathbb{C}} \varphi$, whenever for any $\mathfrak{M} \in \mathbb{C}$,
$$\mathfrak{M} \models \Gamma \text{ implies } \mathfrak{M} \models \varphi.$$
- φ ***follows from*** Γ ***locally in*** $\mathbb{C}$, and we write $\Gamma \vdash^l_{\mathbb{C}} \varphi$, whenever for any $\mathfrak{M} \in \mathbb{C}$ and any $v \in W$,
$$\mathfrak{M}, v \models \Gamma \text{ implies } \mathfrak{M}, v \models \varphi;$$

If $\mathbb{C}$ is clear from the context, we will simply write $\vdash$ and $\vdash^l$ instead.

The corresponding global and local modal logics arising from the previous derivation notions are the finitary ones, namely, for arbitrary Γ and φ,

$$\Gamma \vdash_{\mathbb{C}} \varphi \text{ if and only if there is } \Gamma_0 \subseteq_\omega \Gamma \text{ such that } \Gamma_0 \vdash_{\mathbb{C}} \varphi$$

and the analogous for the local logic.

For a single Kripke model $\mathfrak{M}$, we write $\Gamma \vdash_{\mathfrak{M}} \varphi$ instead of $\Gamma \vdash_{\{\mathfrak{M}\}} \varphi$. In a similar fashion, for a model $\mathfrak{M}$ and a world $u \in W$ we write $\Gamma \nvdash_{\langle \mathfrak{M}, u\rangle} \varphi$ to denote that $\mathfrak{M} \models \Gamma$ and $\mathfrak{M}, u \not\models \varphi$ (namely, φ does not follow globally from Γ in $\mathfrak{M}$, and world u witnesses this fact). In a more general setting, fixed a Kripke frame $\mathfrak{F}$ and an algebra $\mathbf{A}$, we write $\Gamma \vdash_{\mathfrak{F}_{\mathbf{A}}} \varphi$ whenever $\Gamma \vdash_{\mathfrak{M}} \varphi$ for any $\mathbf{A}$-Kripke model $\mathfrak{M}$ with underlying Kripke frame $\mathfrak{F}$.

Tautologies (formulas following from $\emptyset$) of $\vdash^l_{\mathbb{C}}$ and $\vdash_{\mathbb{C}}$ coincide, and $\vdash^l_{\mathbb{C}}$ is strictly weaker than $\vdash_{\mathbb{C}}$, a trivial separating case being the usual necessitation rule $\varphi \vdash \Box\varphi$, valid in the global case and not in the local one. Observe that from the definition of $\vdash^l_{\mathbb{C}}$ and $\vdash_{\mathbb{C}}$, these logics are determined by the directed models generated from the models in $\mathbb{C}$.

Also, the unraveling and filtration[6] techniques can be applied to any directed model, obtaining a directed tree with the exact same behavior as the original model. Even if the resulting tree might be infinite, all worlds in them are, by construction, at a finite distance from the root. Thus, $\vdash_{K\mathbb{C}} = \vdash_{K\mathbb{C}^T}$, for $K\mathbb{C}^T$ being the class of ***safe directed trees generated by models in*** $K\mathbb{C}$.

Some useful notions concerning Kripke models are the following ones.

[5] Sequence of worlds $\{w_i : i \in I\}$ such that $u = w_0, Rw_iw_{i+1}, w_k = v$.

[6] Identifying worlds v, w such that $e(v, \varphi) = e(w, \varphi)$ for any formula φ.

Definition 2.6. Given a Kripke model $\mathfrak{M}$ and $w \in W$, we let the ***height of*** w be the map $\mathtt{h}\colon W \to \mathbb{N} \cup \{\infty\}$ [7] given by

$$\mathtt{h}(w) \coloneqq sup\{k \in \mathbb{N} : \exists w_0, \ldots, w_k \text{ with } w_0 = w \text{ and } Rw_iw_{i+1} \text{ for all } 0 \leqslant i \leqslant k\}.$$

Observe that if there exists some cycle in the model, all worlds involved in it have infinite height.

Definition 2.7. Let φ be a formula of Fm. We let the ***subformulas of*** φ be the set inductively defined by

$$\begin{aligned} SFm(p) &\coloneqq \{p\}, \text{ for } p \text{ propositional variable or constant} \\ SFm(\nabla\varphi) &\coloneqq SFm(\varphi) \cup \{\nabla\varphi\} \text{ for } \nabla \in \{\neg, \Box, \Diamond\} \\ SFm(\varphi_1 \star \varphi_2) &\coloneqq SFm(\varphi_1) \cup SFm(\varphi_2) \cup \{\varphi_1 \star \varphi_2\} \text{ for } \star \in \{\wedge, \vee, \cdot, \to\} \end{aligned}$$

We let the ***propositional subformulas of*** φ be the set

$$\begin{aligned} PSFm(p) &\coloneqq \{p\}, \text{ for } p \text{ propositional variable or constant} \\ PSFm(\nabla\varphi) &\coloneqq \{\nabla\varphi\} \text{ for } \nabla \in \{\Box, \Diamond\} \\ PSFm(\neg\varphi) &\coloneqq SFm(\varphi) \cup \{\neg\varphi\} \\ PSFm(\varphi_1 \star \varphi_2) &\coloneqq SFm(\varphi_1) \cup SFm(\varphi_2) \cup \{\varphi_1 \star \varphi_2\} \text{ for } \star \in \{\wedge, \vee, \cdot, \to\} \end{aligned}$$

For Γ a set of formulas we let $(P)SFm(\Gamma) \coloneqq \bigcup_{\gamma \in \Gamma} (P)SFm(\gamma)$.

Let us finish the preliminaries by stating a well-known undecidable problem, that will be used in the next sections to show undecidability of some of the modal logics introduced above. Recall that given two numbers $\mathtt{x}, \mathtt{y}$ in base $s \in \mathbb{N}$, their concatenation $\mathtt{x} \smile \mathtt{y}$ is given by $\mathtt{x}s^{\|\mathtt{y}\|} + \mathtt{y}$, where $\| \mathtt{y} \|$ is the number of digits of $\mathtt{y}$ in base s.

Definition 2.8 (**Post Correspondence Problem (PCP)**)**.** An instance P of the PCP consists on a list $\langle \mathtt{x_1}, \mathtt{y_1} \rangle \ldots \langle \mathtt{x_n}, \mathtt{y_n} \rangle$ of pairs of numbers in some base $s \geqslant 2$. A solution for P is a sequence of indexes $i_1, \ldots, i_k$ with $1 \leqslant i_j \leqslant n$ such that

$$\mathtt{x}_{\mathtt{i_1}} \smile \ldots \smile \mathtt{x}_{\mathtt{i_k}} = \mathtt{y}_{\mathtt{i_1}} \smile \ldots \smile \mathtt{y}_{\mathtt{i_k}}.$$

The decision problem for PCP is, given a PCP instance, to decide whether such a solution exists or not. This question is undecidable [24].

3. Undecidability of global modal logics

Along this section, unless stated otherwise, we let $\mathbb{A}$ to be a class of weakly-saturated $\mathbf{FL_{ew}}$ chains such that for any $n \in \mathbb{N}$ there is some $\mathbf{A}_n \in \mathbb{A}$ such that $\mathbf{A}_n$ is non n-contractive. That is to say, there is some $a \in A_n$ such that $a^{n+1} < a^n$.

Examples of such classes of algebras are $\{[0,1]_Ł\}$ $\{MV_n\colon n \in \mathbb{N}\}$ and $\{[0,1]_\Pi\}$. Natural examples of classes of algebras that not satisfying the above conditions are $\{[0,1]_G\}$ (and the variety generated by it) and the varieties of MV and product algebras. The main result of this section is the undecidability of the logic $\vdash_{K\mathbb{A}}$ and that of $\vdash_{\omega K\mathbb{A}}$.

Theorem 3.1. *The problem of determining whether φ follows globally from Γ in $K\mathbb{A}$ is undecidable. Moreover, also the problem of determining whether φ follows globally from Γ in $\omega K\mathbb{A}$ is undecidable. More in particular, the three-variable fragment of both previous deductive systems is undecidable.*

The previous theorem follows as a direct consequence of the following result.

Theorem 3.2. *Let P be an instance of the Post Correspondence Problem.Then there is $\Gamma_P \cup \{\varphi_P\} \subset_\omega Fm$ in three variables for which the following are equivalent*

(1) P is satisfiable;
(2) $\Gamma_P \nvdash_{K\mathbb{A}} \varphi_P$;
(3) $\Gamma_P \nvdash_{\omega K\mathbb{A}} \varphi_P$.

[7] Where $x < \infty$ for any $x \in \mathbb{N}$.

Trivially, (3) $\Rightarrow$ (2). In what remains of this section we will first show that (1) $\Rightarrow$ (3), and afterwards, that both (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1). To this aim, let us begin by defining the set of formulas $\Gamma_P \cup \{\varphi_P\}$.

For $P = \{\langle \mathtt{x_1}, \mathtt{y_1}\rangle \dots \langle \mathtt{x_n}, \mathtt{y_n}\rangle\}$ we let Γ_P be the following formulas in variables $\mathcal{V} = \{x, y, z\}$:

(1) $\neg\Box\overline{0} \to (\Box p \leftrightarrow \Diamond p)$ for each $p \in \mathcal{V}$;

(2) $\neg\Box\overline{0} \to (z \leftrightarrow \Box z)$;

(3) $\bigvee_{1\leqslant i\leqslant n} (x \leftrightarrow (\Box x)^{s^{\|\mathtt{x_i}\|}} z^{\mathtt{x_i}}) \wedge (y \leftrightarrow (\Box y)^{s^{\|\mathtt{y_i}\|}} z^{\mathtt{y_i}})$;

Finally, let $\varphi_P = (x \leftrightarrow y)^2 \to (x \to xz) \vee z$.

Roughly speaking, variables x and y will store information on the concatenation of the corresponding elements of the PCP, while z will have a technical role.

Given a solution of P, it is not hard to construct a finite model globally satisfying Γ_P and not φ_P.

Proof. (of Theorem 3.2, (1) $\Rightarrow$ (3))
Let $i_1, \dots, i_k$ be a solution for P, so $\mathtt{x_{i_1}} \smile \dots \smile \mathtt{x_{i_k}} = \mathtt{y_{i_1}} \smile \dots \smile \mathtt{y_{i_k}} = r$ for some $r \in \mathbb{N}$. Pick some non r-contractive algebra $\mathbf{A} \in \mathbb{A}$ and $a \in A$ such that $a^{r+1} < a^r$, and define a finite $\mathbf{A}$-Kripke model $\mathfrak{M}$ as follows:

- $W := \{v_1, \dots v_k\}$;
- $R := \{\langle v_s, v_{s-1}\rangle : 2 \leqslant s \leqslant k\}$;
- For each $1 \leqslant j \leqslant k$ let
  - $e(v_j, z) = a$;
  - $e(v_j, x) = a^{\mathtt{x_{i_1}} \smile \dots \smile \mathtt{x_{i_j}}}$;
  - $e(v_j, y) = a^{\mathtt{y_{i_1}} \smile \dots \smile \mathtt{y_{i_j}}}$;

The formula $\neg\Box 0$ is evaluated to 0 in v_1, and to 1 in all other worlds of the model. Thus, since z is evaluated to the same value in all worlds of the model, and each world has exactly one successor except for v_1 (which has none), clearly the family of formulas in (1) and in (2) from Γ_P are satisfied in all worlds of the model.

To check that formula (3) from Γ_P is satisfied in all worlds of the model we reason by induction on the height of the world -in the sense of Definition 2.6. For v_1 (with height equal to 0), given that it does not have any successors, it is clear that

$$\begin{aligned} e(v_1, (3)) &= \bigvee_{1\leqslant j\leqslant n} (e(v_1, x) \leftrightarrow e(v_1, z)^{\mathtt{x_j}}) \wedge (e(v_1, y) \leftrightarrow e(v_1, z)^{\mathtt{y_j}}) \\ &= \bigvee_{1\leqslant j\leqslant n} (a^{\mathtt{x_{i_1}}} \leftrightarrow a^{\mathtt{x_j}}) \wedge (a^{\mathtt{y_{i_1}}} \leftrightarrow a^{\mathtt{y_j}}) \\ &\geqslant (a^{\mathtt{x_{i_1}}} \leftrightarrow a^{\mathtt{x_{i_1}}}) \wedge (a^{\mathtt{y_{i_1}}} \leftrightarrow a^{\mathtt{y_{i_1}}}) = 1 \end{aligned}$$

For any other v_r with $r > 1$, recall that its only successor is v_{r-1}. Applying the definition of concatenation , and the fact that for any $\mathbf{A} \in \mathbb{A}$ and any $a \in A, n, m \in \mathbb{N}$, trivially $a^n a^m = a^{n+m}$ and $(a^n)^m = a^{nm}$, we can prove that

$$\begin{aligned} e(v_r, (3)) &= \bigvee_{1\leqslant j\leqslant n} (e(v_r, x) \leftrightarrow e(v_r, \Box x)^{s^{\|\mathtt{x_j}\|}} e(v_r, z)^{\mathtt{x_j}}) \wedge (e(v_r, y) \leftrightarrow e(v_r, \Box y)^{s^{\|\mathtt{y_j}\|}} e(v_r, z)^{\mathtt{y_j}}) \\ &= \bigvee_{1\leqslant j\leqslant n} (a^{\mathtt{x_{i_1}} \smile \dots \smile \mathtt{x_{i_r}}} \leftrightarrow e(v_{r-1}, x)^{s^{\|\mathtt{x_j}\|}} a^{\mathtt{x_j}}) \wedge (a^{\mathtt{y_{i_1}} \smile \dots \smile \mathtt{y_{i_r}}} \leftrightarrow e(v_{r-1}, y)^{s^{\|\mathtt{y_j}\|}} a^{\mathtt{y_j}}) \\ &= \bigvee_{1\leqslant j\leqslant n} (a^{\mathtt{x_{i_1}} \smile \dots \smile \mathtt{x_{i_r}}} \leftrightarrow (a^{\mathtt{x_{i_1}} \smile \dots \smile \mathtt{x_{i_{r-1}}}})^{s^{\|\mathtt{x_j}\|}} a^{\mathtt{x_j}}) \wedge (a^{\mathtt{y_{i_1}} \smile \dots \smile \mathtt{y_{i_r}}} \leftrightarrow (a^{\mathtt{y_{i_1}} \smile \dots \smile \mathtt{y_{i_{r-1}}}})^{s^{\|\mathtt{y_j}\|}} a^{\mathtt{y_j}}) \\ &= \bigvee_{1\leqslant j\leqslant n} (a^{\mathtt{x_{i_1}} \smile \dots \smile \mathtt{x_{i_r}}} \leftrightarrow (a^{\mathtt{x_{i_1}} \smile \dots \smile \mathtt{x_{i_{r-1}}} \smile \mathtt{x_j}}) \wedge (a^{\mathtt{y_{i_1}} \smile \dots \smile \mathtt{y_{i_r}}} \leftrightarrow (a^{\mathtt{y_{i_1}} \smile \dots \smile \mathtt{y_{i_{r-1}}} \smile \mathtt{y_j}}) \\ &\geqslant (a^{\mathtt{x_{i_1}} \smile \dots \smile \mathtt{x_{i_r}}} \leftrightarrow (a^{\mathtt{x_{i_1}} \smile \dots \smile \mathtt{x_{i_{r-1}}} \smile \mathtt{x_{i_r}}}) \wedge (a^{\mathtt{y_{i_1}} \smile \dots \smile \mathtt{y_{i_r}}} \leftrightarrow (a^{\mathtt{y_{i_1}} \smile \dots \smile \mathtt{y_{i_{r-1}}} \smile \mathtt{y_{i_r}}}) = 1 \end{aligned}$$

With the above, we have proven that $\mathfrak{M} \models \Gamma_P$.

On the other hand, since $i_1, \dots i_k$ was a solution for P, $e(v_k, x) = e(v_k, y)$. Moreover, $e(v_k, z) = a < 1$, and $e(v_k, xz) = a^{r+1} < a^r = e(v_k, x)$, so $e(v_k, xz \to x) < 1$. This implies that $e(v_k, x \leftrightarrow y)^2 \to e(v_k, z) \vee e(v_k, xz \to x) < 1$, proving that $\Gamma_P \not\vdash_{\omega K\mathbb{A}} \varphi_P$. ⊠

In order to prove the other implications of Theorem 3.2, let us first show some technical characteristics of the models satisfying Γ_P and not φ_P.

A first easy observation is that in any model satisfying Γ_P, the variable z takes the same value in all connected worlds of the model. Relying in the completeness with respect to trees, we can prove that, in these models, z is evaluated to the same value in the whole model.

Lemma 3.3. *Let $\mathbf{A} \in \mathbb{A}$, and $\mathfrak{M} \in K\mathbf{A}^T$ with root u be such that $\Gamma_P \not\vdash_{\langle \mathfrak{M}, u\rangle} \varphi_P$. Then there is $\alpha_z \in A$ such that, for any world v in the model, $e(v, z) = \alpha_z$.*

Proof. Let $\alpha_z = e(u, z)$. It is easy to prove the lemma by induction on the separation of v from u, always finite because $K\mathbf{A}^T$ are directed trees.

If $v = u$ then the claim follows trivially. Otherwise, assume that there are $w_0, w_1, \dots, w_{k+1} \in W$ with $w_0 = u, w_{k+1} = v$ and such that Rw_iw_{i+1} for all $0 \leqslant i \leqslant k$. Since $e(w_k, (1)) = e(w_k, (2)) = 1$ and Rw_kw_{k+1}, then we know

$$e(w_k, \Box z) = e(w_k, \Diamond z) \qquad \text{and} \qquad e(w_k, z) = e(w_k, \Box z)$$

From the first equality we get that, for all $v_1, v_2 \in W$ such that Rw_kv_1 and Rw_kv_2, then $e(v_1, z) = e(v_2, z)$. In particular, this yields that $e(w_k, \Box z) = e(w_{k+1}, z)$. Together with the second equality, it follows that $e(w_k, z) = e(w_{k+1}, z) = e(v, z)$. Applying Induction Hypothesis, we conclude $e(u, z) = e(w_k, z) = e(v, z)$. ⊠

The fact that algebras in $\mathbb{A}$ are linearly ordered and weakly saturated allows also to prove that such models can be assumed to be of finite height.

Lemma 3.4. *Let $\mathbf{A} \in \mathbb{A}$, and $\mathfrak{M} \in K\mathbf{A}^T$ with root u be such that $\Gamma_P \not\vdash_{\langle \mathfrak{M}, u\rangle} \varphi_P$. Then u is of finite height.*

Proof. From Lemma 3.3 we know that in any world v of $\mathfrak{M}$ it holds that $e(u, z) = e(v, z) = \alpha_z$. Moreover, from (3) in Γ_P it follows that

$$e(u, x) \leqslant \alpha_z^n \qquad \text{for all } n \in \mathbb{N} \text{ such that } n \leqslant \mathtt{h}(u)$$

If u was to be of infinite height, by weakly saturation of $\mathbf{A}$, it would follow that $e(u, x)e(u, z) = e(u, x)$. However, since $e(u, \varphi_P) < 1$, necessarily $e(u, xz) < e(u, x)$, and thus u must be of finite height. ⊠

As a corollary, we get that the values of x and y at each world are powers of α_z.

Corollary 3.5. *Let $\mathbf{A} \in \mathbb{A}$, and $\mathfrak{M} \in K\mathbf{A}^T$ with root $u \in W$ be such that $\Gamma_P \not\vdash_{\langle \mathfrak{M}, u\rangle} \varphi_P$. Then for any $v \in W$ there are $a_v, b_v \in \mathbb{N}$ such that*

$$e(v, x) = \alpha_z^{a_v} \qquad \textit{and} \qquad e(v, y) = \alpha_z^{b_v}$$

Moreover, if $\mathtt{h}(v) < \mathtt{h}(w)$ then $a_v < a_w$ and $b_v < b_w$.

Proof. The first part follows easily by induction on the height of the model, from the previous lemma and formulas (1) and (3) in Γ_P. The second claim, is immediate for the case when Rvw, using (3), since $e(v, x) \leqslant e(w, x)\alpha_z$ (and the same for variable y). For arbitrary $\mathtt{h}(v) < \mathtt{h}(w)$, this process is iterated. ⊠

Another corollary can be proven after observing how the implication behaves between powers of the same element in FL_{ew} chains.

Lemma 3.6. *Let $A \in \mathbb{A}$. For any $m > n \in \mathbb{N}$, and any $a \in A$ such that $a^{m+1} < a^m$, it holds that $(a^n \to a^m)^2 \leqslant a$.*

Proof. If $n+1 < m$ (i.e., $m = n+1+k$ for some $k \geqslant 1$), we know that $a^{n+1} > a^m$: otherwise $a^m = a^{n+1+k} = a^{n+1}$ implying that $a^{m+1} = a^{n+2} = a^{n+1} = a^m$ too, which contradicts the assumptions. Thus, $a^{n+1} \to a^m < 1$. By residuation, this is equivalent to $a \to (a^n \to a^m) < 1$, which implies $a > a^n \to a^m$. In particular, this is also greater or equal than $(a^n \to a^m)^2$.
Otherwise, $n+1 = m$. Suppose $a^n \to a^{n+1} = b$. By residuation, $ba^n \leqslant a^{n+1}$, and so, $bba^n \leqslant ba^{n+1} \leqslant a^{n+2}$. Again by residuation, it follows that $b^2 \leqslant a^n \to a^{n+2}$. This is now an implication of the previous kind (with $n+1 < m' = n+2$), and so we have proven before that $a^n \to a^{n+2} < a$. This implies that $b^2 \leqslant a$. ⊠

Corollary 3.7. *Let* $\mathbf{A} \in \mathbb{A}$, *and* $\mathfrak{M} \in K\mathbf{A}^T$ *with root* $u \in W$ *be such that* $\Gamma_P \not\vdash_{\langle \mathfrak{M}, u\rangle} \varphi_P$. *Then* $e(u,x) = e(u,y)$.

Proof. Corollary 3.5 implies $e(u, x \leftrightarrow y) = \alpha_z^a \leftrightarrow \alpha_z^b$ for some $a, b \in \mathbb{N}$. From the previous lemma we get that either $e(u, x \leftrightarrow y) = 1$ or $e(u, x \leftrightarrow y)^2 \leqslant \alpha_z$. Since the second condition implies $e(u, \varphi_P) = 1$, and this is false, necessarily $e(u,x) = e(u,y)$. ⊠

We can now prove that if $\Gamma_P \not\vdash_{K\mathbb{A}} \varphi_P$ then it happens in a finite model with the structure depicted in Figure 1. Let us define

$$\widehat{K\mathbb{A}} \coloneqq \bigcup_{n \in \mathbb{N}} \{\langle W, R, e\rangle \in K\mathbb{A} \colon W = \{v_1, \ldots, v_n\} \text{ and } R = \{\langle v_i, v_{i-1}\rangle \colon 2 \leqslant i \leqslant n\}\}.$$

That is to say, $\widehat{K\mathbb{A}}$ is the restriction of the models in $K\mathbb{A}$ to the ones with the structure from Figure 1. In particular, it contains only finite models.

Lemma 3.8.

$$\Gamma_P \vdash_{K\mathbb{A}} \varphi_P \iff \Gamma_P \vdash_{\widehat{K\mathbb{A}}} \varphi_P$$

Proof. Soundness is immediate since $\widehat{K\mathbb{A}} \subset K\mathbb{A}$. Concerning the right-to-left direction, assume $\Gamma_P \not\vdash_{K\mathbb{A}} \varphi_P$. We know then there is a model $\mathfrak{M} \in K\mathbb{A}^T$ and $u \in W$ such that $\Gamma_P \not\vdash_{\langle \mathfrak{M}, u\rangle} \varphi_P$.

Let us define the submodel $\widehat{\mathfrak{M}}$ of $\mathfrak{M}$ by letting its universe be a set $\{v_i \colon i \in \mathbb{N}\}$ such that $v_0 \coloneqq u$ and for each $i \in \mathbb{N}$, either Rv_iv_{i+1} or v_i has no successors in $\mathfrak{M}$ and $v_{i+1} = v_i$.

Define the model $\widehat{\mathfrak{M}}$ by restricting to $\widehat{W}$ the accessibility relation and the evaluation from $\mathfrak{M}$. From Lemma 3.4 we know u has finite height in the original model, and so also $\widehat{\mathfrak{M}}$ is finite. Then, by construction, it belongs to $\widehat{K\mathbb{A}}$.

Restricting to a submodel does not change the value of propositional variables at each world, i.e., for any $p \in \mathcal{V}$ (and thus, also for any non-modal formula) and any $t \in \widehat{W}$ it holds that $\hat{e}(t,p) = e(t,p)$. For other formulas, we prove the analogous by induction on the formula and on the height of t in $\widehat{\mathfrak{M}}$.

If t is of height equal to 0 (i.e., there are no successors) is trivial to check, since by construction, t does not have successors in $\mathfrak{M}$ either. Thus, all formulas beginning with a modality contained in $SFm(\Gamma_P \cup \{\varphi_P\})$ are evaluated (both in $\mathfrak{M}$ and in $\widehat{\mathfrak{M}}$) to either $\overline{1}$ ($\Box$) or $\overline{0}$ ($\Diamond$). Since the values of the propositional variables are not modified by taking submodels, this concludes the proof of the step.

For $\mathtt{h}(t) = n+1$ in $\widehat{\mathfrak{M}}$, observe that t has successors both in $\mathfrak{M}$ and $\widehat{\mathfrak{M}}$, so $\hat{e}(t, \Box\overline{0}) = e(t, \Box\overline{0}) = 0$. On the other hand, $e(t, \Box p) = e(t, \Diamond p)$ for all $p \in \mathcal{V}$ (from formulas in (1)), and so, in all successors of t in $\mathfrak{M}$, each variable p takes the same value, say α_p. Then, in particular, in the world s chosen as the only successor of t in the construction of $\widehat{\mathfrak{M}}$, it also holds that $e(s,p) = \hat{e}(s,p) = \alpha_p$. Since by construction of $\widehat{\mathfrak{M}}, t$ has only one successor s, it holds that $\hat{e}(t, \Box p) = \hat{e}(t, \Diamond p) = \hat{e}(s,p)$. Then, $\hat{e}(t, \Box p) = \hat{e}(t, \Diamond p) = e(t, \Diamond p) = e(t, \Box p) = e(s,p) = \alpha_p$.

The only formulas beginning with a modality appearing in $SFm(\Gamma_P \cup \{\varphi_P\})$ are of the form $\Box\overline{0}$, $\Box p$ and $\Diamond p$ for $p \in \mathcal{V}$. Since the evaluation of all these formulas and of the propositional variables from $\mathcal{V}$ in world t coincides in $\mathfrak{M}$ and $\widehat{\mathfrak{M}}$ we conclude that the evaluation in t of formulas built from these ones using propositional connectives is also preserved from $\mathfrak{M}$ to $\widehat{\mathfrak{M}}$. This concludes the proof of the lemma.

⊠

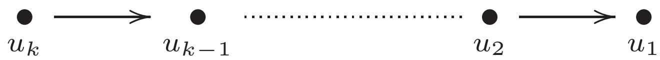

FIGURE 1. Frame for the Global logic proof

At this point, it is possible to obtain a useful characterization of x and y in terms of α_z at each world of a model from $\widehat{K\mathbb{A}}$ satisfying Γ_P and not φ_P in its root (u_k).

Lemma 3.9. *Let $\mathfrak{M} = \langle \{u_1, \ldots, u_k\}, \{\langle u_{i+1}, u_i\rangle \; 1 \leqslant i < k\}, e\rangle \in \widehat{K\mathbb{A}}$ be such that $\Gamma_P \not\vdash_{\langle \mathfrak{M}, u_k\rangle} \varphi_P$. Then there exist $i_1, \ldots, i_k$ with $1 \leqslant i_j \leqslant n$ for each $1 \leqslant j \leqslant k$, such that for each $1 \leqslant j \leqslant k$,*

$$e(u_j, x) = \alpha_z^{\mathtt{x}_{\mathtt{i}_1}\smile\cdots\smile\mathtt{x}_{\mathtt{i}_j}} \qquad \text{and} \qquad e(u_j, y) = \alpha_z^{\mathtt{y}_{\mathtt{i}_1}\smile\cdots\smile\mathtt{y}_{\mathtt{i}_j}}.$$

Moreover, for each $1 \leqslant j \leqslant k$,

$$e(u_j, x) = e(u_j, y) \text{ if and only if } \mathtt{x}_{\mathtt{i}_1}\smile \ldots \smile\mathtt{x}_{\mathtt{i}_j} = \mathtt{y}_{\mathtt{i}_1}\smile \ldots \smile\mathtt{y}_{\mathtt{i}_j}.$$

Proof. We will prove the first claim of the lemma by induction on j. The details are only given for the x case, the other one is proven in the same fashion.

For $j = 1$, u_1 does not have successors. From formula (3) in Γ_P (relying in the fact that the algebras in $\mathbb{A}$ are chains) and Lemma 3.3 it follows there is $i_1 \in \{1, \ldots, n\}$ for which

$$e(u_1, x) = e(u_1, \Box x)^{s^{\|\mathtt{x}_{\mathtt{i}_1}\|}} e(u_1, z)^{\mathtt{x}_{\mathtt{i}_1}} = 1^{s^{\|\mathtt{x}_{\mathtt{i}_1}\|}} \alpha_z^{\mathtt{x}_{\mathtt{i}_1}} = \alpha_z^{\mathtt{x}_{\mathtt{i}_1}}.$$

For $j = r+1$, observe the only successor of u_j in $\mathfrak{M}$ is u_r. Then, from (3) and Lemma 3.3 it follows that there is $i_j \in \{1, \ldots, n\}$ for which

$$e(u_j, x) = e(u_j, \Box x)^{s^{\|\mathtt{x}_{\mathtt{i}_j}\|}} e(u_j, z)^{\mathtt{x}_{\mathtt{i}_1}} = e(u_r, x)^{s^{\|\mathtt{x}_{\mathtt{i}_j}\|}} \alpha_z^{\mathtt{x}_{\mathtt{i}_j}}$$

By Induction Hypothesis the above value equals $(\alpha_z^{\mathtt{x}_{\mathtt{i}_1}\smile\cdots\smile\mathtt{x}_{\mathtt{i}_r}})^{s^{\|\mathtt{x}_{\mathtt{i}_j}\|}} \alpha_z^{\mathtt{x}_{\mathtt{i}_j}}$ and by simple properties of the monoidal operation, to $(\alpha_z^{\mathtt{x}_{\mathtt{i}_1}\smile\cdots\smile\mathtt{x}_{\mathtt{i}_r}})^{s^{\|\mathtt{x}_{\mathtt{i}_j}\|}+\mathtt{x}_{\mathtt{i}_j}}$. This value, by definition of the concatenation of numbers in base s, is exactly $\alpha_z^{\mathtt{x}_{\mathtt{i}_1}\smile\cdots\smile\mathtt{x}_{\mathtt{i}_j}}$, concluding the proof of the first claim.

Concerning the second claim, assume towards a contradiction that there is $1 \leqslant j \leqslant k$ such that $\mathtt{x}_{\mathtt{i}_1}\smile \ldots \smile\mathtt{x}_{\mathtt{i}_j} \neq \mathtt{y}_{\mathtt{i}_1}\smile \ldots \smile\mathtt{y}_{\mathtt{i}_j}$ and $e(u_j, x) = \alpha_z^{\mathtt{x}_{\mathtt{i}_1}\smile\cdots\smile\mathtt{x}_{\mathtt{i}_j}} = \alpha_z^{\mathtt{y}_{\mathtt{i}_1}\smile\cdots\smile\mathtt{y}_{\mathtt{i}_j}} = e(u_j, y)$. If $\mathtt{x}_{\mathtt{i}_1}\smile \ldots \smile\mathtt{x}_{\mathtt{i}_j} < \mathtt{y}_{\mathtt{i}_1}\smile \ldots \smile\mathtt{y}_{\mathtt{i}_j}$, it follows that $\alpha_z^{\mathtt{x}_{\mathtt{i}_1}\smile\cdots\smile\mathtt{x}_{\mathtt{i}_j}} \alpha_z^n = \alpha_z^{\mathtt{x}_{\mathtt{i}_1}\smile\cdots\smile\mathtt{x}_{\mathtt{i}_j}}$ for any $n \geqslant 0$. Thus, in particular, from Corollary 3.5 $e(u_k, x) = e(u_j, x)$, and also $e(u_k, x)\alpha_z = e(u_k, x) = e(u_j, x)$. However, $\mathfrak{M}, u_k \not\models \varphi_P$ implies that $e(u_k, x)\alpha_z < e(u_k, x)$, reaching a contradiction.

The proof is analogous if $\mathtt{x}_{\mathtt{i}_1}\smile \ldots \smile\mathtt{x}_{\mathtt{i}_j} > \mathtt{y}_{\mathtt{i}_1}\smile \ldots \smile\mathtt{y}_{\mathtt{i}_j}$. ⊠

All the previous technical lemmas lead to a simple proof of Theorem 3.2.

Proof. (of Theorem 3.2, (2) ⇒ (3) ⇒ (1))
Assume condition (2) of the lemma, i.e. $\Gamma_P \not\vdash_{K\mathbb{A}} \varphi_P$. Lemma 3.8 implies there is a model $\mathfrak{M} \in \widehat{K\mathbb{A}}$ and $u \in W$ such that $\Gamma_P \not\vdash_{\langle \mathfrak{M}, u_k\rangle} \varphi_P$. Since all models in $\widehat{K\mathbb{A}}$ are finite, this proves point (3). From here, from Corollary 3.7 we know that $e(u, x) = e(u, y)$. Then, by Lemma 3.9, it follows that that there exist indexes $i_1, \ldots, i_k$ in $\{1, \ldots, n\}$ for which $\mathtt{x}_{\mathtt{i}_1}\smile \ldots \smile\mathtt{x}_{\mathtt{i}_k} = \mathtt{y}_{\mathtt{i}_1}\smile \ldots \smile\mathtt{y}_{\mathtt{i}_k}$. This is a solution for the Post Correspondence Instance (P), concluding the proof of (3) ⇒ (1). ⊠

4. NON AXIOMATIZABILITY OF MODAL ŁUKASIEWICZ AND PRODUCT LOGICS

The undecidability of the previous family of modal logics over finite models reaches the question of thire axiomatizaiblity. In particular, it was an open problem how to axiomatize the finitary standard modal Łukasiewicz logic ([18],[10]) and standard modal Product logic ([32]). In these cases, an axiomatization for the logic with both $\Box$ and $\Diamond$ modalities over crisp-accessibility models had not been obtained. Instead, axiomatizations of related but different deductive systems had been proposed (for instance, their corresponding infinitary companion, or over extender languages).

We close this open problem for the standard Łukasiewicz and Product logics with a negative answer: these logics are in fact not axiomatizable, since their respective sets of valid consequences are not recursively enumerable. We will devote this section to prove the previous claims. For that, three properties turn out to be crucial: undecidability of the global consequence over finite models of the class, decidability of the propositional logic and completeness of the global consequence with respect to certain well-behaved models (in these cases, in terms of witnessing conditions). We will prove this negative result for the modal expansion of the standard Łukasiewicz logic, and then the analogous will follow for the Product logic relying on the known isomorphism between the stardard MV-algebra and a certain Product algebra.

The first one of the above properties was proven in Section 3. Let us show how decidability of the propositional underlying logic ($\models_{\mathbb{A}}$) implies that the set $\{\langle \Gamma, \varphi\rangle \colon \Gamma \not\vdash_{\omega K\mathbb{A}} \varphi\}$ is recursively enumerable, which will allow us to conclude there is no possible axiomatization for the logics of finite models over those classes of algebras.

We can first see that the global consequence relation over a finite frame is decidable as long as the underlying propositional consequence relation is decidable too.

Lemma 4.1. *Let $\mathfrak{F}$ be a finite frame, and $\mathbb{A}$ a class of residuated lattices for which $\Gamma \models_{\mathbb{A}} \varphi$ is decidable. Then the problem of determining whether $\Gamma \vdash_{\mathfrak{F}_{\mathbb{A}}} \varphi$ is decidable.*

Proof. We will define a translation between global consequences and the $\mathbb{A}$-propositional logic.

Let $\mathfrak{F} = \langle W, R\rangle$, ψ modal formula with variables in a finite set $\mathcal{V}$, $v \in W$ and x not in $\mathcal{V}$. Consider the extended set of propositional variables

$$\mathcal{V}^* := \{p^v \colon p \in \mathcal{V}, v \in W\} \cup \{x^v_{\nabla\varphi} \colon \nabla \in \{\Box, \Diamond\}, \nabla\varphi \in SFm(\psi), v \in W\}.$$

We define recursively the non-modal formula $\langle \psi, v\rangle^*$ over $\mathcal{V}^*$ as follows:

$$\begin{array}{rcl} \langle c, v\rangle^* & := & c \text{ for } c \in \{\overline{0}, \overline{1}\} \\ \langle p, v\rangle^* & := & p^v \text{ for } p \in \mathcal{V} \\ \langle \varphi \star \chi, v\rangle^* & := & \langle \varphi, v\rangle^* \star \langle \chi, v\rangle^* \text{ for } \star \in \{\&, \to\} \\ \langle \nabla\varphi, v\rangle^* & := & x^v_{\nabla\varphi} \text{ for } \nabla \in \{\Box, \Diamond\} \end{array}$$

For Σ a set of formulas we let $\langle \Sigma, v\rangle^* := \{\langle \sigma, v\rangle^* \colon \sigma \in \Sigma\}$, with set of original variables $\mathcal{V} := \bigcup\{\mathcal{V}ars(\sigma) \colon \sigma \in \Sigma\}$. Moreover, consider the formulas[8]

$$\delta^v_\Box(\psi) := x^v_{\Box\psi} \leftrightarrow \bigwedge_{w \in W : Rvw} \langle \psi, w\rangle^* \qquad \text{and} \qquad \delta^v_\Diamond(\psi) := x^v_{\Diamond\psi} \leftrightarrow \bigvee_{w \in W : Rvw} \langle \psi, w\rangle^*$$

And from there, define the set of formulas

$$\Delta^v(\Gamma) := \{\delta^v_\Box(\psi) \colon \Box\psi \in SFm(\Gamma, \varphi)\} \cup \{\delta^v_\Diamond(\psi) \colon \Diamond\psi \in SFm(\Gamma, \varphi)\}$$

We will now prove that $\Gamma \vdash_{\mathfrak{F}_{\mathbb{A}}} \varphi$ if and only if

$$\{\langle \Gamma, v\rangle^*, \Delta^v(\Gamma) \colon v \in W\} \models_{\mathbb{A}} \bigwedge_{v \in W} \langle \varphi, v\rangle^* \tag{1}$$

To prove the right-to-left direction, assume $\Gamma \not\vdash_{\mathfrak{F}_{\mathbb{A}}} \varphi$. Then there is $\mathbf{A} \in \mathbb{A}$, and an $\mathbf{A}$-Kripke model over $\mathfrak{F}$ in which $e(v, \Gamma) \subseteq \{1\}$ for all v and $e(v_0, \varphi) < 1$ for some $v_0 \in W$. Consider then the mapping $h \colon \mathcal{V}^* \to A$ defined by $h(p^v) = e(v, p)$, $h(x^v_{\Box\psi}) = e(v, \Box\psi)$ and $h(x^v_{\Diamond\psi}) = e(v, \Diamond\psi,)$. It is easy to see that the extension of this mapping to a homomorphism into $\mathbf{A}$ satisfies $h(\langle \psi, v\rangle^*) = e(v, \psi)$ for any $\psi \in SFm(\Gamma, \varphi)$. Thus, it satisfies the premises in equation (1), since $e(v, \Gamma) \subseteq \{1\}$ for all v and by the semantical definition of $\Box$ and $\Diamond$ in the model. On the other hand, it does not satisfy the consequence, since $e(v_0, \varphi) < 1$.

[8]These are proper formulas because W is a finite set.

Conversely, given a propositional homomorphism over some algebra $\mathbf{A} \in \mathbb{A}$ satisfying

$$\{\langle \Gamma, v\rangle^*, \Delta^v(\Gamma) \colon v \in W\}$$

and not satisfying $\bigwedge_{v\in W} \langle \varphi, v\rangle^*$, we can consider the $\mathbf{A}$-Kripke model over $\mathfrak{F}$ that lets $e(v,p) = h(p^v)$. Since $h[\Delta^v(\Gamma] = \{1\}$, then $e(v,\psi) = h(\langle \psi, v\rangle^*)$ for any $\psi \in SFm(\Gamma,\varphi)$, concluding the proof. ⊠

Corollary 4.2. *Let $j \in \mathbb{N}$, and let $\mathbb{A}$ be a class of residuated lattices for which $\Gamma \models_{\mathbb{A}} \varphi$ is decidable. Then the problem of determining whether φ follows globally from Γ in all $\mathbb{A}$-models of cardinality j (denoted by $\Gamma \vdash_{jK\mathbb{A}} \varphi$) is decidable.*

Proof. There is a finite number of frames of cardinality j, and so, for each one, we can run the decision procedure from the above lemma. ⊠

It is now natural how to exhibit a recursive procedure enumerating the elements not belonging to $\vdash_{\omega K\mathbb{A}}$.

Lemma 4.3. *Let $\mathbb{A}$ be a class of residuated lattices for which $\Gamma \models_{\mathbb{A}} \varphi$ is decidable. Then the set $\{\langle \Gamma, \varphi\rangle \in \mathcal{P}_\omega(Fm) \times Fm \colon \Gamma \not\vdash_{\omega K\mathbb{A}} \varphi\}$ is recursively enumerable.*

Proof. Let us enumerate all pairs $\langle \Gamma, \varphi\rangle \in \mathcal{P}_\omega(Fm) \times Fm$, and initialize P as the empty set. Now, for each $i \in \mathbb{N}$, store $\langle \Gamma_i, \varphi_i\rangle$ in P, and then check, for each $\langle \Gamma, \varphi\rangle \in P$ and for each $j \leqslant i$, whether $\Gamma \vdash_{jK\mathbb{A}} \varphi$. Whenever the answer is negative, return that pair and continue. This is a finite amount (since P is always finite) of decidable operations (from Corollary 4.2), thus recursive.

Suppose $\Gamma \not\vdash_{\omega K\mathbb{A}} \varphi$, and that this happens in some $\mathbf{A}$-model of cardinality k for some $\mathbf{A} \in \mathbb{A}$. At some step j, $\langle \Gamma, \varphi\rangle$ will be stored in P. Then, at step $\max\{k, j\}$, the pair $\langle \Gamma, \varphi\rangle$ will be tested against all models of cardinality k via the previous corollary, and so returned as output. ⊠

At this point, we can say that for any class of algebras $\mathbb{C}$ satisfying the premises of Theorem 3.1 and such that $\models_{\mathbb{C}}$ is decidable, the logic $\vdash_{\omega K\mathbb{C}}$ is not recursively enumerable. Otherwise, since the previous Lemma proves that $\{\langle \Gamma, \varphi\rangle \in \mathcal{P}_\omega(Fm)\times Fm \colon \Gamma \not\vdash_{\omega K\mathbb{C}} \varphi\}$ is recursively enumerable, the logic $\vdash_{\omega K\mathbb{C}}$ would be decidable, contradicting Theorem 3.1. Since Ł and Π are decidable logics, and thus by completeness, finitary $\models_{[0,1]_{Ł}}$ and $\models_{[0,1]_{\Pi}}$ are decidable, we get the following corollary.

Corollary 4.4. *The logics $\vdash_{\omega KL}$ and $\vdash_{\omega K\Pi}$ are not axiomatizable.*

However, since it is not a general fact that the logics $\vdash_{K\mathbb{C}}$ are complete with respect to finite models, the lack of axiomatization of the previous logics does still not close the problems commented in the beginning of this section.

4.1. **Modal Łukasiewicz Logic is not axiomatizable.** We can show that, even if the global modal Łukasiewicz logic might not enjoy the finite model property, if there existed a (recursive) axiomatization for $\vdash_{KŁ}$ then also $\vdash_{\omega KŁ}$ would be axiomatizable too.

For a frame $\mathfrak{F}$, and along this subsection, we will write $\vdash_{\mathfrak{F}_Ł}$ to denote the global consequence relation over the class of standard Łukasiewicz models built over $\mathfrak{F}$.

A result shown in [16] (Lemma 3) will allow us to prove here completeness of $\vdash_{\omega KŁ}$ with respect to witnessed models, in a similar fashion to how it is done for tautologies of FDL over Łukasiewicz logic in that same publication. We do not introduce details of first order (standard) Łukasiewicz logic here, we refer the interested reader to eg. [15]. Just recall that

- A standard Łukasiewicz first order model is a structure $\langle W, \{P_i\}_{i\in I}\rangle$ where W is a non-empty set and for each $i \in I$ and $ar(i)$ the arity of P_i, $P_i \colon W^{ar(i)} \to [0,1]$,
- An evaluation in a (F.O) model is a mapping $v \colon \mathcal{V} \mapsto W$. Moreover, we write $v[x \mapsto m]$ to denote the evaluation v where the mapping of the variable x is overwritten and x is mapped to m (and simply $[x \mapsto m]$ denotes that the evaluation of x is m and the other variables are irrelevant).
- The value of a formula φ in a (F.O) model $\mathfrak{M}$ under an evaluation v, denoted by $\|\varphi\|_{\mathfrak{M},v}$ is inductively defined by

– $\|P_i(x_1, \dots, x_{ar(i)})\|_{\mathfrak{M},v} = P_i(v(x_1), \dots, v(x_{ar(i)}))$;
– $\|\varphi_1 \star \varphi_2\|_{\mathfrak{M},v} = \|\varphi_1\|_{\mathfrak{M},v} \star \|\varphi_2\|_{\mathfrak{M},v}$ for $\star$ propositional (Ł) operation;
– $\|\forall x\varphi(x)\|_{\mathfrak{M},v} = \bigwedge_{m\in W} \|\varphi\|_{\mathfrak{M},v[x\mapsto m]}$,
– $\|\exists x\varphi(x)\|_{\mathfrak{M},v} = \bigvee_{m\in W} \|\varphi\|_{\mathfrak{M},v[x\mapsto m]}$.

Observe the value of a sentence (closed formula) in a model is constant under any evaluation, so we can simply write its value in a model. Moreover, we say that a model $\mathfrak{M}$ is **witnessed** whenever for any sentence $Qx\varphi(x)$ for $Q \in \{\forall, \exists\}$ there is some $m \in W$ such that

$$\|Qx\varphi(x)\|_{\mathfrak{M}} = \|\varphi(x)\|_{\mathfrak{M},[x\mapsto m]}$$

The consequence relation over standard Łukasiewicz first order models, $\models_{\forall[0,1]_{Ł}}$ is defined for sentences by stating $\Gamma \models_{\forall[0,1]_{Ł}} \varphi$ whenever for any standard Łukasiewicz first order model $\mathfrak{M}$, if $\|\Gamma\|_{\mathfrak{M}} \subseteq \{1\}$ then $\|\varphi\|_{\mathfrak{M}} = \{1\}$.

Lemma 4.5 ([16], Lemma 3)**.** *Let $\mathfrak{M}$ be a standard Łukasiewicz first order model. Then there is a (standard Łukasiewicz first order) witnessed model $\mathfrak{M}'$ such that $\mathfrak{M}$ is a submodel of $\mathfrak{M}'$ and for any sentence α it holds that*

$$\|\alpha\|_{\mathfrak{M}} = 1 \textit{ if and only if } \|\alpha\|_{\mathfrak{M}'} = 1.$$

From here, we can easily prove completeness of $\vdash_{KŁ}$ with respect to witnessed Kripke models, i.e., those for which, for any modal formula $\nabla\varphi$ (with $\nabla \in \{\Box, \Diamond\}$) and any world v there is some world w such that Rvw and

$$e(v, \nabla\varphi) = (w, \varphi).$$

Lemma 4.6. *If $\Gamma \nvdash_{KŁ} \varphi$ there is a witnessed standard Łukasiewicz Kripke model $\mathfrak{M}$ and $v \in W$ such that $\Gamma \nvdash_{\langle\mathfrak{M},v\rangle} \varphi$.*

Proof. We can use the usual translation from modal to F.O. logics in order to move from a Kripke model to a suitable FO model. For φ modal formula, consider the F.O. language $\{R/2, \{P_p/1\colon p \text{ variable in } \varphi\}\}$. For arbitrary $i \in \mathbb{N}$, let us define the translation $\langle\varphi, x_i\rangle^*$ recursively by letting

- $\langle p, x_i\rangle^* := P_p(x_i)$;
- $\langle\varphi \star \psi, x_i\rangle^* := \langle\varphi, x_i\rangle^* \star \langle\psi, x_i\rangle^*$ for $\star$ propositional connective;
- $\langle\Box\varphi, x_i\rangle^* := \forall x_{i+1} R(x_i, x_{i+1}) \to \langle\varphi, x_{i+1}\rangle^*$;
- $\langle\Diamond\varphi, x_i\rangle^* := \exists x_{i+1} R(x_i, x_{i+1}) \cdot \langle\varphi, x_{i+1}\rangle^*$;

It is a simple exercise that we do not detail here to check that

$$\Gamma \vdash_{KŁ} \varphi \iff \{\forall x_0\langle\gamma, x_0\rangle^*\}_{\gamma\in\Gamma}, \forall x\forall y(R(x,y) \vee \neg R(x,y)) \models_{\forall[0,1]_{Ł}} \forall x_0\langle\varphi, x_0\rangle^*$$

If $\Gamma \nvdash_{KŁ} \varphi$ there is some F.O. model satisfying the premises of the right side of the above equation and not $\forall x_0\langle\varphi, x_0\rangle^*$. From the previous lemma we know there is a witnessed (F.O.) model $\mathfrak{M}$ in which the same conditions hold. Then, there is some m in the universe for which $\|\langle\varphi, x_0\rangle^*\|_{\mathfrak{M},[x_0\mapsto m]} < 1$. At this point, it is only necessary to build a witnessed Kripke model $\widehat{\mathfrak{M}}$ from $\mathfrak{M}$ that is a global model for Γ but does not satisfy φ at some world. In order to do that, let the universe of the Kripke model be the same universe of $\mathfrak{M}$, and let the accessibility relation be given by the interpretation of the binary predicate R in $\mathfrak{M}$. Observe that, since $\forall x\forall y(R(x,y) \vee \neg R(x,y))$ is true in the model, necessarily $R(x,y) \in \{0,1\}$, and thus the resulting model will be crisp. Finally, let $e(v,p) = \|P_p(v)\|_{\mathfrak{M}}$ for each variable p and each world $v \in W$.

By induction on the complexity of the formula it is easy to check that for any $\psi \in SFm(\Gamma, \varphi)$ and any $v \in W$ it holds $e(v, \psi) = \|\langle\psi, x_i\rangle^*\|_{\mathfrak{M},[x_i\mapsto v]}$, we leave the details to the reader. Moreover, since the F.O. model is witnessed, the Kripke model is witnessed too. Also, $\widehat{\mathfrak{M}}$ is a global model of Γ, while $e(m, \varphi) = \|\langle\varphi, x_0\rangle\|_{\mathfrak{M},[x_0\mapsto m]} < 1$, concluding the proof of the lemma. ⊠

We can use the non idempotency of the Łukasiewicz t-norm to recursively reduce the global consequence relation over finite models to the unrestricted global consequence relation.

Lemma 4.7. *$\Gamma \vdash_{\omega KL} \varphi$ if and only if for arbitrary $p, q \notin \mathcal{V}(\Gamma, \varphi)$ it holds*

$$\Gamma, \Xi(p), \xi(p,q) \vdash_{KL} \varphi \vee \psi(p,q)$$

for

- $\Xi(p) := \{\Box\overline{0} \vee (p \leftrightarrow \Box p), \Box\overline{0} \vee (\Box p \leftrightarrow \Diamond p)\}$,
- $\xi(p,q) := q \leftrightarrow p \cdot \Box q$,
- $\psi(p,q) := p \vee \neg p \vee q \vee \neg q$.

Proof. $\Rightarrow$: if $\Gamma, \Xi(p), \xi(p,q) \not\vdash_{K\text{Ł}} \varphi \vee \psi(p,q)$, then due to Lemma 4.6, there is a witnessed Ł Kripke model $\mathfrak{M}$ and $v \in W$ such that $\mathfrak{M} \models \Gamma, \Xi(p), \xi(p,q)$ (i.e. $e(u, \Gamma, \Xi(p), \xi(p,q)) \subseteq \{1\}$ for all u) and $e(v, \varphi \vee \psi(p,q)) < 1$. We can assume that $\mathfrak{M}$ is the unraveled tree generated from v. We will now prove that we can define a finite model equivalent to this one for what concerns the formulas in $F = SFm(\Gamma \cup \Xi(p) \cup \{\xi(p,q)\} \cup \{\varphi \vee \psi(p,q)\})$.

First, observe that from $e(u, \Xi(p)) \subseteq \{1\}$ for each $u \in W$ we have that there is $x \in [0,1]$ such that for any $u \in W$, $e(u,p) = x$, as it was proven in Lemma 3.3. Moreover, from $e(v, p \vee \neg p) < 1$ we have that $x \in (0,1)$.

On the other hand, from $e(u, \xi(p,q)) = 1$ for any $u \in W$ we get that $e(u,q) = e(u, \Box q)x$. Thus, for a world u of height $k \in \mathbb{N} \cup \{\infty\}$ we have that $e(u,q) \leqslant x^s$ for all $s \leqslant k$. In particular, if there were to be some world u of infinite height, v would also be of infinite height, and so $e(v,q) \leqslant x^n$ for all $n \in \mathbb{N}$. Since $x \in (0,1)$, by the definition of the product in the standard MV algebra, the previous family of inequalities implies that $e(v,q) = 0$. Then it holds that $e(v, \neg q) = 1$, which is not possible by the assumption of $e(v, \varphi \vee \psi(p,q)) < 1$. Thus, v -and so, all worlds of the model- must be of finite height.

We can apply now a filtration-like transformation to $\mathfrak{M}$ with respect to the set of formulas F to obtain a finite directed model. To do this, let us denote by $wit(u, \triangledown\chi)$ a witnessing world for modal formula $\triangledown\chi$ from world u (i.e., $e(u, \triangledown\chi) = e(wit(u, \triangledown\chi), \chi)$. Then define the universe $W' := \bigcup_i \in \omega W_i$ with

$$W_0 := \{v\}$$
$$W_i + 1 := \{wit(u, \triangledown\chi) \colon \triangledown\chi \in SFm(F), u \in W_i\}$$

Observe the previous construction leads to empty sets, as soon as the worlds in some W_i do not have successors.

Since all worlds of the model were of finite height, and F is a finite set of formulas, the model $\mathfrak{M}'$ resulting from restring $\mathfrak{M}$ to the universe W' is a finite directed model with root v. Moreover, it is such that $e'(w, \Gamma, \Xi(p), \xi(p,q)) \subseteq \{1\}$ for each world $w \in W'$, and $e'(v, \varphi \vee \psi(p,q)) < 1$. In particular, $e(w, \Gamma) \subseteq \{1\}$ at each world w, and $e(v, \varphi) < 1$.

$\Leftarrow$: Assume $\Gamma \not\vdash_{\omega K\text{Ł}} \varphi$, so there is a finite model $\mathfrak{M}$ and a world $v \in W$ such that $\Gamma \not\vdash_{\langle \mathfrak{M}, u \rangle} \varphi$. Let $k < \mathbb{N}$ be the height of v, and define a new model from $\mathfrak{M}$ by preserving the evaluation of all variables except for p, q (that do not appear in Γ, φ and so can be changed without affecting the evaluation of the formulas in Γ, φ). Let a be an arbitrary element in $(\frac{k}{k+1}, 1)$ and put, for each $w \in W$:

- $e(w,p) = a$,
- $e(w,q) = e(w, \Box q)a$ (observe this is well defined since all worlds have finite height, so we can define q inductively from the worlds of height 0).

This evaluation satisfies in all worlds of the model all formulas from $\Xi(p)$ and $\xi(p,q)$, and it forces $e(v,p) \notin \{0,1\}$ and $e(v,q) \notin \{0,1\}$. Moreover, it satisfies the formulas from Γ, and $e(v, \varphi) < 1$, since the evaluation of all variables appearing in Γ and φ has been preserved. Thus, $\Gamma, \Xi(p), \xi(p,q) \not\vdash_{K\text{Ł}} \varphi \vee \psi(p,q)$ either. ⊠

The fact that the (finitary) Łukasiewicz global modal logic is not axiomatizable follows as a consequence of previous reduction (which is recursive) and the undecidability of $\vdash_{\omega K\text{Ł}}$.

Theorem 4.8. *$\vdash_{KL}$ is not axiomatizable.*

Proof. Assume $\vdash_{K Ł}$ is axiomatizable, and so, recursively enumerable. We can prove that then $\vdash_{\omega K Ł}$ is recursively enumerable too, contradicting Corollary 4.4. For that, take a recursive enumeration of all pairs $\langle \Gamma, \varphi\rangle \in \mathcal{P}_\omega(Fm) \times Fm$ such that $\Gamma \vdash_{K Ł} \varphi$.

For each pair, let $\mathcal{V} = \mathcal{V}ars(\Gamma, \varphi)$, and check whether there are some $p, q \in \mathcal{V}$ such that $\Gamma = \Gamma_0(\mathcal{V} \setminus \{p, q\}) \cup \Xi(p) \cup \{\xi(p, q)\}$ and $\varphi = \varphi_0(\mathcal{V} \setminus \{p, q\}) \vee \psi(p, q)$ for some Γ_0, φ_0. This is a decidable procedure because Γ is a finite set and the translation is recursive. If that is the case, output $\langle \Gamma_0, \varphi_0\rangle$, and don't output anything otherwise. From Lemma 4.7 this procedure enumerates $\vdash_{\omega K Ł}$.

However, Corollay 4.4 states that $\vdash_{\omega K Ł}$ is not RE. This contradicts the initial assumption that $\vdash_{K Ł}$ was axiomatizable. ⊠

4.2. **Modal Product Logic is not axiomatizable either.** In [1], the authors prove that the standard MV algebra is isomorphic to the standard product algebra restricted to $[a, 1]$ for arbitrary fixed $0 < a < 1$. Relying in the isomorphism provided, they also show that the tautologies of standard Łukasiewicz propositional and predicate logics[9] can be recursively reduced to those of the respective (standard) product logic. In [15, Lem. 4.1.14, Lem. 6.3.5] these results are exhibited for what concerns the corresponding logical deduction relations.

We can follow a similar path in this work, slightly modifying the reduction so it works in the modal case.[10] We include the details of the new proof of reduction in the appendix.

Given a finite set of variables $\mathcal{V}$, let x be a propositional variable not in $\mathcal{V}$. For each formula φ of $K Ł$ in variables $\mathcal{V}$, define its translation φ^x as follows:

- $(0)^x$ is x,
- $(q)^x$ is $q \vee x$ for each $q \neq x$,
- $(\varphi \to \psi)^x$ is $(\varphi^x \to \psi^x)$,
- $(\varphi \odot \psi)^x$ is $x \vee (\varphi^x \odot \psi^x)$,
- $(\Box\varphi)^x$ is $\Box\varphi^x$

Further, let $\Theta^x := \{\Box x \leftrightarrow \Diamond x, \Box x \leftrightarrow x, \neg\neg x\}$.

In the spirit of Lemmas 2 and 3 from [1] it is possible to prove the following result. For convenience of the reader, we provide the details of the proof in the appendix.

Lemma 4.9. *$\Gamma \vdash_{K L} \varphi$ if and only if $\Gamma^x, \Theta^x \vdash_{K\Pi} \varphi^x$, for any $x \notin \mathcal{V}ar(\Gamma \cup \{\varphi\})$.*

Since the reduction is recursive, together with Theorem 4.8 the following is immediate.

Corollary 4.10. *$\vdash_{K\Pi}$ is not axiomatizable.*

We have proven that $\vdash_{K Ł}$ and $\vdash_{K\Pi}$ are not in Σ_1 from the Arithmetical Hierarchy. We leave open the question of whether they are Π_2-complete, as it happens for the tautologies of their first order versions, or if they belong to some other level of the hierarchy. The proofs of Ragaz in [26, 25] rely heavily on the expressive power of first order logic, and also in proving the result directly for tautologies of the logic. In the present work, the results affect the logic itself, since for instance, the tautologies of $\vdash_{K Ł}$ are decidable: they coincide with those of $\vdash^l_{K Ł}$ and this logic is decidable ([30, Corollary 4.5]).

5. The necessitation rule

Recall that in (classical) modal logic, the global deduction is axiomatized as the local one plus the (unrestricted) necessitation rule $N_\Box \colon x \vdash \Box x$. It was formulated as an open question in [3] whether this was the case in general, or at least, for modal expansions of fuzzy logics. This was the case in the modal logics known up to now (eg. in modal Gödel logics, and in the infinitary modal Łukasiewicz and Product logics studied in the literature). However, we can close negatively that problem, first by a simple counter example over the modal expansions of Łukasiewicz logic (using the non-axiomatizability of $\vdash_{K_Ł}$ proven in Theorem 4.8) and later proving this is a more general fact.

[9]Standard Łukasiewicz propositional logic is indeed the Łukasiewicz propositional logic. However, the predicate logic over the standard MV algebra and the analogous logic over all chains in the variety differ [15].

[10]An alternative proof reducing $\vdash_{\omega K\Pi}$ to $\vdash_{K\Pi}$, similar to the one on the previous section, can be also done.

First, it is possible to prove that the local deduction is decidable using the version of Lemma 4.6 referring to the local logic. A detailed proof of the decidability of $\vdash^l_{K_{\mathrm{L}}}$ can be found in [30, Corollary 4.5]. Thus, $\vdash^l_{K_{\mathrm{L}}}$ has a recursive axiomatization (for instance, built by enumerating all possible $\langle \Gamma, \varphi \rangle$ for $\Gamma \cup \{\varphi\} \subset_\omega Fm$, and then returning the pairs such that $\Gamma \vdash^l_{K_{\mathrm{L}}} \varphi$). On the other hand, if the global consequence were to coincide with the local one plus the $N_\Box$ rule, the logic axiomatized by adding to the previous system the $N_\Box$ rule should produce a recursive axiomatization of $\vdash_{K_{\mathrm{L}}}$, contradicting Theorem 4.8.

As we said, it is possible to widen the scope of the previous result, and produce a constructive proof serving all modal logics built over classes of algebras like the ones in Theorem 3.1. This can be done following a different approach from the more direct one in the Łukasiewicz case, and rather providing a derivation that is valid in the global modal logics and not in the local ones extended by the necessitation rule.

For simplicity, allow us to fix a class of algebras $\mathbb{A}$ like the one from Section 3, and let $\vdash$ and $\vdash^l$ denote $\vdash_{K_\mathbb{A}}$ and $\vdash^l_{K_\mathbb{A}}$ respectively. Further, let $\vdash^l_{N_\Box}$ denote the logic $\vdash^l_{K_\mathbb{A}}$ plus the necessitation rule $x \vdash \Box x$. An natural way to understand this extension is by considering the (possibly non recursive) list of finite derivations valid in $\vdash^l_{K_\mathbb{A}}$ and add to this set the rule schemata $N_\Box$. Let us call this set R. The minimal logic containing R is the logic $\vdash^l_{N_\Box}$. Since all rules in R have finitely many premises, the resulting logic is finitary.

Considering R as a non recursive axiomatization for $\vdash^l_{N_\Box}$, all derivations valid in $\vdash^l$ have a proof in the extended system of length 0. Thus, the length of the proofs in the extended system only reflects the applications of the necessitation rule. Since by definition $\vdash^l$ is a finitary logic, only finitely many applications of the rule are used at each specific derivation. This means that a proof of φ from Γ (finite) in this axiomatic system is given simply as a finite list of pairs $\langle \Gamma_i, \varphi_i \rangle_{0 \leqslant i \leqslant N}$ such that

- $\Gamma_0 = \Gamma$ and $\varphi_N = \varphi$,
- For each $0 \leqslant i \leqslant N$, $\Gamma_i \vdash^l \varphi_i$ and,
- $\Gamma_{i+1} = \Gamma_i \cup \{\Box \varphi_i\}$.

From here, it is quite simple to prove the following characterization of $\vdash^l_{N_\Box}$.

Lemma 5.1. $\Gamma \vdash^l_{N_\Box} \varphi \Longleftrightarrow \{\Box^i \Gamma\}_{i \in \mathbb{N}} \vdash^l \varphi$.

Proof. Right to left direction is immediate. For the other direction, if $\Gamma \vdash^l_{N_\Box} \varphi$, since the logic is finitary, φ can be proven from Γ by using the $N_\Box$ rule a finite number of times, say n. It can be easily proven by induction in n [11] that $\{\Box^i \Sigma\}_{i \leqslant n} \vdash^l \chi$ if and only if $\Sigma \vdash^l_{n \cdot N_\Box} \chi$, where $n \cdot N_\Box$ stands for using the $N_\Box$ rule up to n times. This concludes the proof. ⊠

We can then produce a set of formulas that yields a valid derivation in the global logics, but it does not in the local logic plus necessitation.

Theorem 5.2. $\vdash$ *does not coincide with* $\vdash^l_{N_\Box}$.

Proof. We claim that both

$$y \leftrightarrow \Box y, y \leftrightarrow \Diamond y, x \leftrightarrow (\Box x)y, \neg\Box \bot \vdash x \to xy$$

$$y \leftrightarrow \Box y, y \leftrightarrow \Diamond y, x \leftrightarrow (\Box x)y, \neg\Box \bot \nvdash^l_{N_\Box} x \to xy$$

which proves the theorem.

For what concerns the first claim, consider any Kripke model satisfying globally the set of premises. In particular, from $\neg\Box \bot$ we get that any world in the model has a successor, and so, infinite height in the sense of Definition 2.6. Moreover, the value of y is constant inside each connected part of the model, as in Lemma 3.3. Consider each connected submodel $\mathfrak{M}$, and let α be the value of y in it. Then, at any point of the model, $e(u,x) \leqslant \alpha^i$ for all $i \in \mathbb{N}$. Then, since the algebras in the class are weakly saturated, we get that $e(u,x)\alpha = e(u,x)$, proving the formula in the right side.

[11] Using that $\Gamma \vdash^l \psi \Rightarrow \Box\Gamma \vdash^l \Box\psi$.

In order to prove the second claim, let us denote by Σ the set of premises. From the previous lemma we have that our claim holds if and only if

$$\{\Box^i \Sigma\}_{i\in\mathbb{N}} \nvdash^l x \to xy.$$

Being $\vdash^l$ finitary by definition, this holds if and only if $\{\Box^i \Gamma\}_{i\leqslant N} \nvdash^l \varphi$ for all $N \in \mathbb{N}$. We can produce a counter-model for each $N \in \mathbb{N}$.

Indeed, consider a model with universe $\{0,\dots,N+1\}$, and the accessibility given by $R = \{\langle i, i+1\rangle : i \leqslant N\}$. For what concerns the evaluation, pick any $\mathbf{A} \in \mathbb{A}$ that is not $(N+1)$-contractive, and chose $a \in \mathbf{A}$ such that $a^{N+2} < a^{N+1}$. Then let

$$e(i,y) = a \text{ for } 1 \leqslant i \leqslant N+1 \qquad e(N+1,x) = 1 \qquad e(i,x) = a^{N+1-i} \text{ for } 1 \leqslant i \leqslant N$$

It is a simple exercise to check that this evaluation satisfies $\{\Box^i \Gamma\}_{i\leqslant N}$ at the world 0, i.e., $e(0, \Box^i \Sigma) = 1$ for all $i \leqslant N$. On the other hand, observe that $e(0,x) = a^{N+1}$. Due to the way we chose a it holds that $e(0,xy) = a^{N+2} < a^{N+1} = e(0,x)$, thus falsifying the consequence.

⊠

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6. Appendix

The proof of Lemma 4.9 detailed below draws inspiration from the results in [1], and relies in the same isomorphic mappings introduced there. However, the approach and details are different here, since we formulate different intermediate results, and we propose a more explicit proof using basic arithmetics.

Proof of Lemma 4.9: ($\Gamma \vdash_{K Ł} \varphi$ if and only if $\Gamma^x, \Theta^x \vdash_{K\Pi} \varphi^x$, for any $x \notin \mathcal{V}ar(\Gamma \cup \{\varphi\})$.)
If $\Gamma \nvdash_{KŁ} \varphi$, there is some standard Łukasiewicz Kripke model $\mathfrak{M}$ such that $\mathfrak{M} \models \Gamma$ but $\mathfrak{M}, v \not\models \varphi$ for some v in the model. Chose any arbitrary $a \in (0,1)$, and let us define an standard product model $\mathfrak{M}'$ by letting the universe and accessibility relations be those of $\mathfrak{M}$, and further, for each $w \in W$, let

- $e'(w,x) \coloneqq a$ and
- $e'(w,q) \coloneqq a^{1-e(w,q)}$ for each variable $q \neq x$.[12]

Claim 1: *For any formula ψ in variables from $\mathcal{V}ar(\Gamma \cup \{\varphi\})$, and for any $w \in W$, it holds that*

$$e'(w,\psi^x) = a^{1-e(w,\psi)}$$

Thus, $e'(w,\gamma^x) = a^{1-e(w,\gamma)} = a^0 = 1$ for each $\gamma \in \Gamma$ and $w \in W$. Also it is clear that $e'(w,\Theta^x) = 1$, since x is evaluated to the same element a in all worlds of $\mathfrak{M}'$. On the other hand, $e'(v,\varphi^x) = a^{1-e(v,\varphi)} < 1$, since $e(v,\varphi) < 1$. Thus, $\mathfrak{M}' \models \Gamma^x, \Theta^x$ and $\mathfrak{M}', v \not\models \varphi^x$, and so, $\Gamma^x, \Theta^x \nvdash_{K\Pi} \varphi^x$.

Proof of Claim 1. We prove it by induction on the complexity of the formula.

- For variables it is immediate from the definition, since $a \leqslant a^q$ for any $q \in [0,1]$.
- For $\psi = \psi_1 \odot \psi_2$, we have the following chain of equalities

$$\begin{aligned} &e'(w,(\psi_1 \odot \psi_2)^x) &&= e'(w, x \vee (\psi_1^x \odot \psi_2^x)) &&= e'(w,x) \vee (e(w,\psi_1^x) \cdot_\Pi e(w,\psi_2^x)) \\ &\overset{I.H}{=} a \vee (a^{1-e(w,\psi_1)} \cdot_\Pi a^{1-e(w,\psi_2)}) &&= a^{1-(e(w,\psi_1)+e(w,\psi_2)-1)} &&= a^{1-(e(w,\psi_1)\cdot_Ł e(w,\psi_2))} \\ &= a^{1-e(w,\psi_1 \odot \psi_2)} &&= a^{1-e(w,\psi)} \end{aligned}$$

- For $\psi = \psi_1 \to \psi_2$, we have the following chain of equalities

$$\begin{aligned} &e'(w,(\psi_1 \to \psi_2)^x) && = e'(w, \psi_1^x \to \psi_2^x) \\ &\overset{I.H}{=} a^{1-e(w,\psi_1)} \to_\Pi a^{1-e(w,\psi_2)} && = \begin{cases} 1 & \text{if } a^{1-e(w,\psi_1)} \leqslant a^{1-e(w,\psi_2)} \\ \frac{a^{1-e(w,\psi_2)}}{a^{1-e(w,\psi_1)}} & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } e(w,\psi_1) \leqslant e(w,\psi_2) \\ a^{1-e(w,\psi_2)-1+e(w,\psi_1)} & \text{otherwise} \end{cases} && = \begin{cases} 1 & \text{if } e(w,\psi_1) \leqslant e(w,\psi_2) \\ a^{1-(e(w,\psi_1)\to_Ł e(w,\psi_2))} & \text{otherwise} \end{cases} \\ &= a^{1-e(w,\psi_1 \to \psi_2)} \end{aligned}$$

- For $\psi = \Box\psi_1$, we know that

$$e'(w,(\Box\psi_1)^x) = e'(w,\Box\psi_1^x) = \bigwedge_{Rwu} e'(u,\psi_1^x) \overset{I.H}{=} \bigwedge_{Rwu} a^{1-e(u,\psi_1)}$$

On the one hand, since $e(w,\Box\psi_1) = \bigwedge_{Rwu} e(u,\psi_1) \leqslant e(u,\psi_1)$ for each u with Rwu, it holds that $a^{1-e(w,\Box\psi_1)} \leqslant \bigwedge_{Rwu} a^{1-e(u,\psi_1)} = e'(w,\Box\psi_1)$.

[12] This is the isomorphism between the standard MV algebra and the product algebra restricted to $[a,1]$ used in [1].

On the other hand,

$$\bigwedge_{Rwu} a^{1-e(u,\psi_1)} \leqslant a^{1-e(u,\psi_1)} \ \forall u \text{ s.t } Rwu \qquad \Longrightarrow a^{e(u,\psi_1)} \leqslant \frac{a}{\bigwedge_{Rwu} a^{1-e(u,\psi_1)}} \ \forall u \text{ s.t } Rwu$$

$$\overset{a\in(0,1)}{\Longrightarrow} e(u,\psi_1) \geqslant log_a\Big(\frac{a}{\bigwedge_{Rwu} a^{1-e(u,\psi_1)}}\Big) \ \forall u \text{ s.t } Rwu \qquad \Longrightarrow \bigwedge_{Rwu} e(u,\psi_1) \geqslant log_a\Big(\frac{a}{\bigwedge_{Rwu} a^{1-e(u,\psi_1)}}\Big)$$

$$\Longrightarrow a^{\bigwedge_{Rwu} e(u,\psi_1)} \leqslant \frac{a}{\bigwedge_{Rwu} a^{1-e(u,\psi_1)}} \qquad \Longrightarrow \bigwedge_{Rwu} a^{1-e(u,\psi_1)} \leqslant \frac{a}{a^{\bigwedge_{Rwu} e(u,\psi_1)}}$$

$$\Longrightarrow e'(w,(\Box\psi_1)^x) \leqslant a^{1-\bigwedge_{Rwu} e(u,\psi_1)}$$

This concludes the proof of the claim.

The other direction of the Lemma is proven similarly, using the corresponding inverse of the isomorphism from [1]. Assume there is an standard product model $\mathfrak{P}$ such that $\mathfrak{P} \models \Gamma^x, \Theta^x$ and $\mathfrak{P}, v \not\models \varphi^x$ for some v in the model. As proven in Lemma 3.3, there is some element a such that $e(w,x) = a$ for all w in the universe of the model. Moreover, since $e(v, \neg\neg x) = 1$, necessarily $a > 0$.

We can first show that if $a = 1$ then $e(w, \psi^x) = 1$ for any ψ with variables in $\mathcal{V}ar(\Gamma \cup \{\varphi\})$ and any w in the universe. It follows easily by induction on the complexity of the formula. Since this would contradict the fact that $e(v, \varphi^x) < 1$, necessarily $a < 1$.

Let us define an standard Łukasiewicz model $\mathfrak{P}'$ as the model whose universe and accessibility relation are those of $\mathfrak{P}$ and for each variable q and each world w let $e'(w,q) \coloneqq 1 - log_a a \vee e(w,q)$.

Claim 2. *For any formula ψ in variables from $\mathcal{V}ar(\Gamma \cup \{\varphi\})$ and for any $w \in W$ it holds that*

$$e'(w,\psi) = 1 - log_a e(w,\psi^x)$$

Thus, $e'(w,\gamma) = 1 - log_a e(w,\gamma^x) = 1 - log_a 1 = 1$ for each $\gamma \in \Gamma$ and $w \in W$, and $e'(v,\varphi) = 1 - log_a e(v,\varphi^x) = 1 - log_a \alpha$ for some $a \leqslant \alpha < 1$. Since the logarithm in base a of elements in that interval is a value in $(0,1]$, necessarily $e'(v,\varphi) < 1$, concluding the proof of the lemma.

Proof of Claim 2.

We will prove it by induction on the complexity of the formula. Observe a consequence is that $e(w,\psi^x) \geqslant a$ for all ψ and w as before[13]. We will use this property in the modal step, as $I.H'$.

- $e'(w,q) = 1 - log_a a \vee e(w,q) = 1 - log_a e(w,q^x)$,
- $e'(w,\psi_1 \odot \psi_2) = e'(w,\psi_1) \cdot_{\text{Ł}} e'(w,\psi_2) \overset{I.H}{=} max\{0, 1 - log_a e(w,\psi_1^x) + 1 - log_a e(w,\psi_2^x) - 1\} = max\{0, 1 - log_a(e(w,\psi_1^x \odot \psi_2^x))\}$. Now, if $0 < e(w,\psi_1^x \odot \psi_2^x) < a$, it holds that $log_a(e(w,\psi_1^x \odot \psi_2^x)) > 1$, and thus, $max\{0, 1 - log_a(e(w,\psi_1^x \odot \psi_2^x))\} = 0 = 1 - log_a(e(w,\psi_1^x \odot \psi_2^x) \vee a)$. It follows that $max\{0, 1 - log_a(e(w,\psi_1^x \odot \psi_2^x))\} = 1 - log_a(e(w,\psi_1^x \odot \psi_2^x) \vee a) = 1 - log_a e(w,(\psi_1 \odot \psi_2)^x)$.
- $e'(w,\psi_1 \to \psi_2) = min\{1, 1 - e'(w,\psi_1) + e'(w,\psi_2)\} \overset{I.H}{=} min\{1, 1 - (1 - log_a e(w,\psi_1^x)) + 1 - log_a e(w,\psi_2^x)\} = min\{1, 1 - (log_a e(w,\psi_2^x) - log_a e(w,\psi_1^x))\} = min\{1, 1 - log_a e(w,\psi_2^x)/e(w,\psi_1^x)\} = 1 - log_a(min\{1, e(w,\psi_2^x)/e(w,\psi_1^x)\}) = 1 - log_a e(w,\psi_1^x \to \psi_2^x) = 1 - log_a e(w,(\psi_1 \to \psi_2)^x)$.
- $e'(w,\Box\psi_1) = \bigwedge_{Rwv} e'(v,\psi_1) \overset{I.H}{=} \bigwedge_{Rwv}(1 - log_a e(v,\psi_1^x)) = 1 - \bigvee_{Rwv} log_a e(v,\psi_1^x)$. Now, by $I.H'$, we know that $e(v,\psi_1^x) \geqslant a$ for all v in the model, so in particular $\bigwedge_{Rwv} e(v,\psi_1^x) \in [a,1]$. Since the function $log_a()$ is continuous and decreasing in $[a,1]$, it follows that $1 - \bigvee_{Rwv} log_a e(v,\psi_1^x) = 1 - log_a \bigwedge_{Rwv} e(v,\psi_1^x) = 1 - log_a e(w,\Box\psi_1^x) = 1 - log_a e(w,(\Box\psi_1)^x)$, which concludes the proof of the step. ⊠

[13] Since e' is defined inductively from the value of propostional variables, it always returns a value in $[0,1]$. Thus, $0 \leqslant 1 - log_a e(w,\psi^x)$, so $log_a e(w,\psi^x) \leqslant 1$, which is only possible if $e(w,\psi^x) \geqslant a$.