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Implicational (Semilinear) Logics I: A New Hierarchy

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Abstract In Abstract Algebraic Logic, the general study of propositional non-classical logics has been traditionally based on the abstraction of the Lindenbaum-Tarski process. In this process one considers the Leibniz relation of indiscernible formulae. Such approach has resulted in a classification of logics partly based on generalizations of equivalence connectives: the *Leibniz hierarchy*. This paper performs an analogous abstract study of non-classical logics based on the kind of generalized implication connectives they possess. It yields a new classification of logics expanding Leibniz hierarchy: the *hierarchy of implicational logics*. In this framework the notion of *implicational semilinear logic* can be naturally introduced as a property of the implication, namely a logic L is an implicational semilinear logic iff it has an implication such that L is complete w.r.t. the matrices where the implication induces a linear order, a property which is typically satisfied by well-known systems of fuzzy logic. The hierarchy of implicational logics is then restricted to the semilinear case obtaining a classification of implicational semilinear logics that encompasses almost all the known examples of fuzzy logics and suggests new directions for research in the field.

Keywords Abstract algebraic logic · hierarchy of implicational logics · implicative logics · Leibniz hierarchy · linearly ordered logical matrices · mathematical fuzzy logic · non-classical logics · semilinear logics.

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1 Introduction

Algebraic Logic is the branch of Mathematical Logic that studies logical systems by giving them a semantics based on some particular kind of algebraic structures. It can be traced back to George Boole and his study of classical propositional logic by means of a two-element algebra that became its canonical semantics. Thus, in a sense, it could be argued that Algebraic Logic is the oldest branch of Mathematical Logic. Tarski's refinement of the proof by Linbenbaum that classical logic is indeed complete with respect to the semantics given by Boolean algebras starts from a theory T and a formula φ such that $T \nvdash_{\mathrm{CPC}} \varphi$, i.e. T does not prove φ in the classical propositional calculus, and then it considers the following binary relation on the set of formulae:

$$\langle \alpha, \beta \rangle \in \Omega(T) \quad \text{iff} \quad T \vdash_{\mathrm{CPC}} \alpha \leftrightarrow \beta.$$

This relation is shown to be in fact a congruence in the algebra of formulae $\mathbf{Fm}_{\mathcal{L}}$; moreover the formulae of T constitute exactly one equivalence class. Thus it is enough to take the corresponding quotient, $\mathbf{Fm}_{\mathcal{L}}/\Omega(T)$, and show that it is a Boolean algebra such that the class of T is its top element, and hence in this algebra the elements of T are interpreted as true while φ is not (because $T \nvdash_{\mathrm{CPC}} \varphi$).

Analogous proofs were later used to show the completeness of non-classical logics with respect to their corresponding algebraic semantics (e.g. intuitionistic logic w.r.t. Heyting algebras); indeed, it became known in Algebraic Logic as the standard method called the *Lindenbaum-Tarski process*. The fact that it could be analogously repeated in many propositional logics led to more general studies where it was used to show completeness theorems for broad classes of logics such as Rasiowa's implicative logics (studied in her monograph [24]). Abstract Algebraic Logic (AAL) was born as the natural next step to be taken in this evolution: the abstract study of logical systems through the generalization of the Lindenbaum-Tarski process to arbitrary logics. The last decades have seen the florescence of this subfield of Algebraic Logic resulting in a deep theory of the correspondence between logics and classes of algebras (or logical matrices defined over the algebras). The generalization of the Lindenbaum-Tarski construction capitalizes on the realization that the congruence $\Omega(T)$ is actually the relation consisting of those pairs of formulae that, relatively to T, are substitutable in any context *salva veritate*, i.e.:

$\langle \alpha, \beta \rangle \in \Omega(T)$ if, and only if, for every formula in at least one variable $\chi(x)$, $\chi(\alpha)$ is true relatively to T iff $\chi(\beta)$ is true relatively to T.

Thus, $\Omega(T)$ is the relation of logically equivalent formulae modulo T, and the quotient $\mathbf{Fm}_{\mathcal{L}}/\Omega(T)$ can be seen as the identification of indiscernible propositions, i.e. a formalization of the ancient Leibniz's principle of equality of indiscernibles. Therefore, $\Omega(T)$ is called the *Leibniz congruence of* T. In classical logic this relation is easily defined by means of the connective $\leftrightarrow$, whereas in other logics the situation can be substantially more complicated even though the Leibniz congruence still can be definable by means of some set of formulae in two variables (possibly infinite and with parameters). It gave rise to the class of *protoalgebraic logics*: logics where there is a set $E(p, q, \overrightarrow{r})$ of formulae defining the Leibniz congruence. This set can be seen as a generalization of the equivalence connective, and indeed it has been given the name *equivalence set*. By imposing several extra conditions, a number of subclasses of protoalgebraic logics were defined yielding a classification of logics called the *Leibniz hierarchy*. As it was based on properties of the congruence Ω and the sets $E(p, q, \overrightarrow{r})$, it was essentially an equivalence-based classification.

Largely independent of these developments, another subfield of Algebraic Logic has been rapidly growing in recent times: the algebraic study of fuzzy logics. A number of logical systems have been proposed and intensively studied to deal with the reasoning with vagueness and the concept of graded truth. They include, among many others, Gödel-Dummett logic [10], Łukasiewicz infinitely-valued logic [21], Product logic [18], Hájek's BL logic [16], logics based on left-continuous t-norms such as MTL logic [11], and uninorm logic UL [22]. All of them are many-valued logics which enjoy an algebraic semantics that can be generated (as a variety) from linearly ordered algebras, and thus they enjoy a completeness theorem with respect to linearly ordered algebras. This common feature has led Běhounek and Cintula to argue in [2] that fuzzy logics are *the logics of chains*, i.e. the logics that have a complete semantics based on linearly ordered algebras. Moreover, these chains are typically ordered in a uniform way by means of some implication connective $\to$ in the sense that in every algebra $\mathcal{A}$ there is a set $F \subseteq A$ of designated elements such that for every $a, b \in A$:

$$a \leq b \text{ iff } a \to b \in F.$$

An implication connective like this, in addition, plays a central rôle in the Lindenbaum-Tarski process of these logics since the congruence $\Omega(T)$ can be defined in the following way:

$$\langle \alpha, \beta \rangle \in \Omega(T) \text{ iff } T \vdash \alpha \to \beta \text{ and } T \vdash \beta \to \alpha.$$

In other words, the symmetrized implication $\{p \to q, q \to p\}$ gives an equivalence set in these logics. Therefore, implication connectives play a doubly fundamental rôle in fuzzy logics: they define order in the algebras and, when symmetrized, they allow the definition of the Leibniz congruence. From this point of view, implications are much more useful than just plain equivalence connectives.

Therefore, it makes sense to develop a finer classification in AAL based on implications instead of equivalences. This approach is more general since any implication gives rise to an equivalence (just by symmetrizing), while equivalences do not have all the features of an implication (they define only the identity order). This is what we intend to do in this paper. We proceed in the pure AAL style aiming at the most general possible framework. Thus, we allow implications to be connectives definable by means of possibly infinite and parameterized sets of formulae. This new approach to logical systems results in an implication-based classification of logics that expands the Leibniz hierarchy and will be called the *hierarchy of implicational logics*. Its largest class coincides with the largest class in the Leibniz hierarchy, i.e. the class of protoalgebraic logics, but it allows us to distinguish more subclasses yielding a hierarchy finer than the traditional one. In particular, our approach also fits well with some previously defined classes of logics: Rasiowa's *implicative logics* [24], and Cintula's *weakly implicative logics* [6]. In this framework of implicational logics we introduce a very general notion of implicational semilinear logic in a natural way: an implicational logic is semilinear if it has a semilinear implication, i.e. a generalized implication such that the logic is complete w.r.t. the models where it defines a linear order. In symbols, if L is a logic,

$\Rightarrow$ is an implication set and $\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$ is the class of models where this implication induces a linear ordering, then:

$$\mathrm{L} \text{ is implicational semilinear w.r.t. } \Rightarrow \text{ iff } \vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})}.$$

The term 'semilinear' was introduced by Olson and Raftery in [23] (in a much more specific context of residuated lattices) and it refers to the fact that in finitary semilinear logics the subdirectly irreducible matrices are linear (following the tradition of Universal Algebra to call a class of algebras 'semi$X$' whenever its subdirectly irreducible members have the property X; e.g. as in 'semisimple'). This technical notion of implicational semilinear logic is a first (big) step towards a mathematical definition of what a fuzzy logic is in the sense of [2], because it describes the most usual way in which logical systems happen to be semantically based on chains. In principle, this definition might not be intrinsic in the sense that different implications could induce different classes of linear models. Nevertheless, we will prove that for every (finite) $\Rightarrow$, $\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$ coincides with the class of relatively finitely subdirectly irreducible models of L. Moreover, for every algebra $\mathcal{A}$ of a reduced model, all its logical filters are up-sets with respect to the order given by the implication set. When the order is total, the up-sets are linearly ordered by inclusion, and hence logical filters form also a chain. Thus, an easy method to show that an implicational logic is not semilinear consists in giving a subdirectly irreducible algebra with two incomparable filters.

On the other hand, in the fuzzy logic literature the completeness of logics with respect to chains is usually shown by means of *linear filters*, i.e. filters which induce total orders in the quotient. One usually proves the *Linear Extension Property* LEP:

for every filter F and every $a \notin F$, there is a linear filter $F' \supseteq F$ such that $a \notin F'$.

We study this and some related properties in our general framework showing that it is equivalent to the semilinearity metarule and also to the fact that matrix models are subdirect products of linear ones.

The outline of the paper is the following: after this introduction, Section 2 gives the necessary basic notions from AAL that will be needed. Section 3 presents our theory of implications and the hierarchy of implicational logics that they induce as an expansion of the Leibniz hierarchy. The problem of showing mutual differences between the classes is completely solved by a series of examples. Then we introduce the concept of semilinear implication and show some of its general characterizations and important consequences in finitary logics. Finally, Section 4 restricts the hierarchy of implicational logics to the semilinear case, thus obtaining a new hierarchy of implicational semilinear logics. Some classes are shown to collapse and others to be different. Well-studied classes of fuzzy logics are shown to lie on the top of the classification.

This paper is the first part of our investigation on implicational (semilinear) logics. We are currently preparing a follow-up paper [7] where we consider, in a similar fashion, generalized disjunction connectives, and study their rôle in implicational logics. In particular, it will show that the classical proof by cases property of disjunction allows to characterize semilinearity, axiomatize semilinear logics, improve the results about the intrinsic classes of linear models and characterize completeness w.r.t. particular semantics in terms of embedding properties.

2 Preliminaries

2.1 Basic notions

For the development of the paper we need to recall the basic definitions and results of Abstract Algebraic Logic.[1] We start with some syntactical definitions. The notion of *propositional language* $\mathcal{L}$ is defined in the usual way (a set of connectives with finite arity). By $\mathbf{Fm}_{\mathcal{L}}$ we denote the free term algebra over a denumerable set of variables in the language $\mathcal{L}$, by $\mathrm{Fm}_{\mathcal{L}}$ we denote its universe and we call its elements *$\mathcal{L}$-formulae* (we omit $\mathcal{L}$ when it is clear from the context; analogously with other notions defined in this section). The set of sequences (resp. finite sequences) of $\mathcal{L}$-formulae is denoted by $\mathrm{Fm}_{\mathcal{L}}^{\leq\omega}$ (resp. $\mathrm{Fm}_{\mathcal{L}}^{<\omega}$). We denote by $\mathrm{Eq}_{\mathcal{L}}$ the set of $\mathcal{L}$-equations, i.e. formal expressions of the form $\varphi \approx \psi$, where $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$. The endomorphisms of $\mathbf{Fm}_{\mathcal{L}}$ are traditionally called *$\mathcal{L}$-substitutions*.

An *$\mathcal{L}$-consecution*[2] is a pair $\Gamma \rhd \varphi$, where $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$ and $\varphi \in \mathrm{Fm}_{\mathcal{L}}$. A consecution $\Gamma \rhd \varphi$ is *finitary* if Γ is finite. Notice that a set of consecutions $\vdash_{\mathrm{L}}$ can be understood as a relation between sets of formulae and formulae; we often use an infix notation, i.e. we write $\Gamma \vdash_{\mathrm{L}} \varphi$ instead of $\Gamma \rhd \varphi \in \vdash_{\mathrm{L}}$. We also write $\Gamma \vdash_{\mathrm{L}} \Delta$ when $\Gamma \vdash_{\mathrm{L}} \varphi$ for every $\varphi \in \Delta$. Finally, we write $\Gamma \dashv\vdash_{\mathrm{L}} \Delta$ when $\Gamma \vdash_{\mathrm{L}} \Delta$ and $\Delta \vdash_{\mathrm{L}} \Gamma$.

A *propositional logic* (also called *sentential logic* or just *logic*) is a pair $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}} \rangle$ where $\mathcal{L}$ is a propositional language and $\vdash_{\mathrm{L}}$ is a set of L-consecutions satisfying the following conditions for every $\Gamma \cup \Delta \cup \{\varphi, \psi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$:

1. $\varphi \vdash_{\mathrm{L}} \varphi$;
2. if $\Gamma \vdash_{\mathrm{L}} \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_{\mathrm{L}} \varphi$;
3. if $\Gamma \vdash_{\mathrm{L}} \varphi$ and $\Delta \vdash_{\mathrm{L}} \Gamma$, then $\Delta \vdash_{\mathrm{L}} \varphi$.
4. if $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\varphi)$ for each $\mathcal{L}$-substitution σ.

A logic L is *finitary* if for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ such that $\Gamma \vdash_{\mathrm{L}} \varphi$ there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathrm{L}} \varphi$.

A *theory* of a logic L is a set T of formulae such that if $T \vdash_{\mathrm{L}} \varphi$ then $\varphi \in T$. By $Th(\mathrm{L})$ we denote the set of all theories of L. Each propositional logic L defines a closure system over the set $\mathrm{Fm}_{\mathcal{L}}$ whose closed sets are the theories of L and the corresponding closure operator C over $\mathrm{Fm}_{\mathcal{L}}$ is defined as:

$$C(\Gamma) = \bigcap\{T \in Th(\mathrm{L}) \mid \Gamma \subseteq T\} = \{\varphi \in \mathrm{Fm}_{\mathcal{L}} \mid \Gamma \vdash_{\mathrm{L}} \varphi\}.$$

A logic can be presented by means of several kinds of proof systems. In this paper, we consider mainly Hilbert-style systems. Given a logic $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}} \rangle$, we say that a set $\mathcal{AS}$ of $\mathcal{L}$-consecutions is a *presentation* of L if the relation $\vdash_{\mathrm{L}}$ coincides with the provability relation given by $\mathcal{AS}$, i.e., for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\Gamma \vdash_{\mathrm{L}} \varphi$ iff there is a proof of φ from Γ in $\mathcal{AS}$. A proof (assuming that $\mathcal{AS}$ consists of finitary consecutions only)[3] is just a finite sequence of formulae $\langle \psi_0, \psi_1, \ldots, \psi_n \rangle$ such that $\psi_n = \varphi$ and for every $i < n$ either $\psi_i \in \Gamma$ or for some $\Delta \rhd \alpha \in \mathcal{AS}$ there is a substitution σ such that $\sigma(\alpha) = \psi_i$ and $\sigma[\Delta] \subseteq \{\psi_0, \ldots, \psi_{i-1}\}$.

[1] The reader can find comprehensive presentations of the field in the monographs [8,12] and in the survey [13]. Any necessary background in Universal Algebra can be found in [5].

[2] This term is borrowed from [1]; however, we use it in a very simplified version. The term 'sequent' is sometimes used instead.

[3] In the infinitary case we would need to consider proofs as founded trees labeled by formulae satisfying analogues of the conditions required in the finitary case.

Traditionally, propositional logics are given a semantics in terms of matrices. Given a language $\mathcal{L}$, an $\mathcal{L}$-*matrix* is a pair $\mathbf{A} = \langle \mathcal{A}, D \rangle$ where $\mathcal{A}$ is an $\mathcal{L}$-algebra and D is a subset of A called the *filter* of $\mathbf{A}$. A matrix is *trivial* if its algebra has only one element and its filter is the singleton of this element. A homomorphism from $\mathbf{Fm}_{\mathcal{L}}$ to $\mathcal{A}$ is called an $\mathcal{A}$-*evaluation*. The semantical consequence with respect to a class of $\mathcal{L}$-matrices $\mathbb{K}$ is defined as:

$$\Gamma \models_{\mathbb{K}} \varphi \text{ iff for each } \mathbf{A} \in \mathbb{K} \text{ and each } \mathbf{A}\text{-evaluation } e,\ e[\Gamma] \subseteq D \text{ implies } e(\varphi) \in D.$$

Clearly, $\langle \mathcal{L}, \models_{\mathbb{K}} \rangle$ is a logic. We say that a matrix $\mathbf{A}$ is a *model* of L if $\vdash_{\mathrm{L}} \subseteq \models_{\mathbf{A}}$. Let $\mathbf{MOD}(\mathrm{L})$ be the class of all models of L. Each logic is complete with respect to the semantics given by all of its models:

Theorem 1 *Let* $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}} \rangle$ *be a logic. Then* $\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}(\mathrm{L})}$.

Given an $\mathcal{L}$-algebra $\mathcal{A}$, a subset $F \subseteq A$ is an L-*filter* if $\langle \mathcal{A}, F \rangle \in \mathbf{MOD}(\mathrm{L})$. Let $\mathcal{F}i_{\mathrm{L}}(\mathcal{A})$ be the set of all L-filters over $\mathcal{A}$. Observe that for every set T of formulae, we have $T \in Th(\mathrm{L})$ iff $\langle \mathbf{Fm}_{\mathcal{L}}, T \rangle \in \mathbf{MOD}(\mathrm{L})$, i.e. $\mathcal{F}i_{\mathrm{L}}(\mathbf{Fm}_{\mathcal{L}}) = Th(\mathrm{L})$; these models are called the *Lindenbaum matrices* for L. It is straightforward to check that $\mathcal{F}i_{\mathrm{L}}(\mathcal{A})$ is closed under arbitrary intersections and hence it is a closure system. Let us recall that a basis of a closure system $\mathcal{C}$ over a set A is a family $\mathcal{B} \subseteq \mathcal{C}$ satisfying one of the following equivalent conditions:

1. for every $X \in \mathcal{C} \setminus \{A\}$ there is a $\mathcal{D} \subseteq \mathcal{B}$ such that $X = \bigcap \mathcal{D}$,
2. for every $Y \in \mathcal{C}$ and every $a \in A \setminus Y$ there is $Z \in \mathcal{B}$ such that $Y \subseteq Z$ and $a \notin Z$.

A crucial notion for the classification of logical systems in Abstract Algebraic Logic is the so-called *Leibniz congruence* of a matrix. Given a matrix $\mathbf{A} = \langle \mathcal{A}, F \rangle$, a binary relation $\Omega_{\mathcal{A}}(F) \subseteq A \times A$ is defined as:

$$\langle a, b \rangle \in \Omega_{\mathcal{A}}(F) \text{ if, and only if, for every } \mathcal{L}\text{-formula } \varphi(x, \overrightarrow{z}), \text{ and } \overrightarrow{c} \in A^{<\omega} \text{ we have } \varphi^{\mathcal{A}}(a, \overrightarrow{c}) \in F \text{ iff } \varphi^{\mathcal{A}}(b, \overrightarrow{c}) \in F.$$

Thus, we have defined the indiscernibility relation in $\mathbf{A}$. This relation has an important characterization. For an algebra $\mathcal{A}$ and a subset $F \subseteq A$, a congruence $\theta \in \mathcal{C}o(\mathcal{A})$ is said to be *compatible* with F if for every $a, b \in A$ such that $a \in F$ and $\langle a, b \rangle \in \theta$, we have $b \in F$.

Theorem 2 $\Omega_{\mathcal{A}}(F)$ *is the maximum congruence of* $\mathcal{A}$ *compatible with* F.

Observe that when $\mathcal{A}$ is the algebra of formulae $\mathrm{Fm}_{\mathcal{L}}$, the Leibniz congruence of a theory T is given by the pairs $\langle \alpha, \beta \rangle$ such that for every formula in at least one variable $\chi(x)$, $\chi(\alpha) \in T$ iff $\chi(\beta) \in T$. Inspired by the famous Leibniz's principle of equality of indiscernibles, $\Omega_{\mathcal{A}}(F)$ is called the *Leibniz congruence* of $\langle \mathcal{A}, F \rangle$. A matrix is said to be *reduced* if its Leibniz congruence is the identity relation. Given an arbitrary matrix $\mathbf{A} = \langle \mathcal{A}, F \rangle$, one can always produce a reduced one by factorizing through the Leibniz congruence, i.e. $\mathbf{A}^* = \langle \mathcal{A}/\Omega_{\mathcal{A}}(F), F/\Omega_{\mathcal{A}}(F) \rangle$. Given a logic L, the class of its reduced models is denoted by $\mathbf{MOD}^*(\mathrm{L})$, and the class of algebraic reducts of $\mathbf{MOD}^*(\mathrm{L})$ is denoted by $\mathbf{ALG}^*(\mathrm{L})$. Reduced models are enough to provide a complete semantics for the logic:

Theorem 3 *Let* $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}} \rangle$ *be a logic. Then* $\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^*(\mathrm{L})}$.

Matrices can be regarded as first-order structures where the filter corresponds to a unary predicate. In this context, one can define the usual notions of substructure (now called *submatrix*), isomorphism, homomorphism, strict homomorphism, direct product, reduced product and ultraproduct for matrices. Given a class of matrices $\mathbb{K}$, we will denote by $\mathbf{S}(\mathbb{K})$, $\mathbf{I}(\mathbb{K})$, $\mathbf{H}(\mathbb{K})$, $\mathbf{H}_{\mathrm{S}}(\mathbb{K})$, $\mathbf{P}(\mathbb{K})$, $\mathbf{P}_{\mathrm{R}}(\mathbb{K})$ and $\mathbf{P}_{\mathrm{U}}(\mathbb{K})$ the closure of $\mathbb{K}$ under the mentioned operations.

The notion of subdirect product from Universal Algebra is also generalized to matrices. A matrix $\mathbf{M}$ is said to be *representable as a subdirect product* of the family of matrices $\{\mathbf{M}_i \mid i \in I\}$ if there is an injective homomorphism α from $\mathbf{M}$ into the direct product $\prod_{i\in I} \mathbf{M}_i$ such that for every $i \in I$, the composition of α with the i-th projection, $\pi_i \circ \alpha$, is surjective. In this case, α is called a *subdirect representation*, and it is called *finite* if I is finite. Let L be a logic and $\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})$. A non-trivial matrix $\mathbf{M} \in \mathbb{K}$ is *(finitely) subdirectly irreducible relatively to* $\mathbb{K}$ if for every (finite) subdirect representation α of $\mathbf{M}$ with a family $\{\mathbf{M}_i \mid i \in I\} \subseteq \mathbb{K}$ there is $i \in I$ such that $\pi_i \circ \alpha$ is an isomorphism. The class of all relatively (finitely) subdirectly irreducible matrices is denoted as $\mathbb{K}_{\mathrm{R(F)SI}}$. If L is a finitary logic, one can prove that every matrix in $\mathbf{MOD}^*(\mathrm{L})$ is representable as a subdirect product of matrices in $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$, which immediately gives the following completeness theorem:

Theorem 4 *Let* $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}} \rangle$ *be a finitary logic. Then* $\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}}$.

We will use the following characterizations for relatively (finitely) subdirectly irreducible reduced models:

Proposition 1 *Let* L *be a logic and* $\mathbf{A} = \langle \mathcal{A}, F \rangle \in \mathbf{MOD}^*(\mathrm{L})$. *Then:*

1. $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$ *if, and only if,* F *is finitely meet-irreducible in* $\mathcal{F}i_{\mathrm{L}}(\mathcal{A})$.
2. $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ *if, and only if,* F *is meet-irreducible in* $\mathcal{F}i_{\mathrm{L}}(\mathcal{A})$.

2.2 Leibniz hierarchy

Notice that $\Omega_{\mathcal{A}}$ can be seen as a mapping from $\mathcal{F}i_{\mathrm{L}}(\mathcal{A})$ to $\mathcal{C}o(\mathcal{A})$; then it is called the *Leibniz operator*. Some classes of logics are defined according to the behavior of this operator. Let L be a logic in a language $\mathcal{L}$. Then:

1. L is called *protoalgebraic* if $\Omega_{\mathbf{Fm}_{\mathcal{L}}}$ is monotone on $Th(\mathrm{L})$, i.e. $T_1 \subseteq T_2$ implies $\Omega_{\mathbf{Fm}_{\mathcal{L}}}(T_1) \subseteq \Omega_{\mathbf{Fm}_{\mathcal{L}}}(T_2)$ for every $T_1, T_2 \in Th(\mathrm{L})$.
2. L is called *equivalential* if $\Omega_{\mathbf{Fm}_{\mathcal{L}}}$ is monotone and commutes with inverse substitutions on $Th(\mathrm{L})$, i.e.$\Omega_{\mathbf{Fm}_{\mathcal{L}}}(\sigma^{-1}[T]) = \sigma^{-1}[\Omega_{\mathbf{Fm}_{\mathcal{L}}}(T)]$ for every $T \in Th(\mathrm{L})$ and every $\mathcal{L}$-substitution σ.
3. L is called *weakly algebraizable* if $\Omega_{\mathbf{Fm}_{\mathcal{L}}}$ is monotone and injective on $Th(\mathrm{L})$.
4. L is called *algebraizable* if $\Omega_{\mathbf{Fm}_{\mathcal{L}}}$ is monotone, injective and it commutes with inverse substitutions on $Th(\mathrm{L})$.

All of these classes have been intensively studied and several nice characterizations have been obtained.

Theorem 5 *Let* $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}} \rangle$ *be a logic. The following are equivalent:*

1. L *is protoalgebraic.*
2. *For every* $\mathcal{L}$*-algebra* $\mathcal{A}$, $\Omega_{\mathcal{A}}$ *is monotone on* $\mathcal{F}i_{\mathrm{L}}(\mathcal{A})$.

3. $\mathbf{MOD}^*(\mathrm{L})$ *is closed under formation of subdirect products.*
4. *There exists a set* $E(p, q, \overrightarrow{r})$ *of formulae in two variables and possibly with parameters* $\overrightarrow{r}$ *such that:*
   - $\vdash_{\mathrm{L}} E(p, p, \overrightarrow{r})$
   - $p, \bigcup_{\overrightarrow{\alpha} \in \mathrm{Fm}_{\mathcal{L}}^{\leq\omega}} E(p, q, \overrightarrow{\alpha}) \vdash_{\mathrm{L}} q$
   - $\bigcup_{\overrightarrow{\alpha} \in \mathrm{Fm}_{\mathcal{L}}^{\leq\omega}} E(p, q, \overrightarrow{\alpha}) \vdash_{\mathrm{L}} E(c(s_1, \ldots, s_i, p, \ldots, s_n), c(s_1, \ldots, s_i, q, \ldots, s_n), \overrightarrow{\beta})$

     for every $\overrightarrow{\beta} \in \mathrm{Fm}_{\mathcal{L}}^{\leq\omega}$*, each* $\langle c, n\rangle \in \mathcal{L}$ *and each* $i < n$*.*
5. *There exists a set* $E(p, q, \overrightarrow{r})$ *of formulae in two variables and possibly with parameters such that it defines the Leibniz congruence on every model of* L*, i.e., for every* $\mathbf{A} = \langle \mathcal{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$ *and every* $a, b \in A$*,* $\langle a, b\rangle \in \Omega_{\mathcal{A}}(F)$ *iff* $E^{\mathcal{A}}(a, b, \overrightarrow{c}) \subseteq F$ *for every* $\overrightarrow{c}$ *in* A*.*

Any set $E(p, q, \overrightarrow{r})$ satisfying part 4 also satisfies part 5 and vice versa. These sets are called *parameterized equivalence sets.*[4]

Theorem 6 *Let* $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}}\rangle$ *be a logic. The following are equivalent:*

1. L *is equivalential.*
2. *For every* $\mathcal{L}$*-algebra* $\mathcal{A}$*,* $\Omega_{\mathcal{A}}$ *is monotone and it commutes with inverse images by homomorphisms, that is, for every* $\mathcal{L}$*-algebra* $\mathcal{B}$*, every homomorphism* $h\colon \mathcal{A} \to \mathcal{B}$ *and every* $F \in \mathcal{F}i_{\mathrm{L}}(\mathcal{B})$*,* $\Omega_{\mathcal{A}}(\sigma^{-1}[F]) = \sigma^{-1}[\Omega_{\mathcal{B}}(F)]$*.*
3. $\mathbf{MOD}^*(\mathrm{L})$ *is closed under formation of submatrices and direct products.*
4. *There exists a set* $E(p, q)$ *of formulae in two variables such that:*
   - $\vdash_{\mathrm{L}} E(p, p)$
   - $p, E(p, q) \vdash_{\mathrm{L}} q$
   - $E(p, q) \vdash_{\mathrm{L}} E(c(s_1, \ldots, s_i, p, \ldots, s_n), c(s_1, \ldots, s_i, q, \ldots, s_n))$

     for each $\langle c, n\rangle \in \mathcal{L}$ *and each* $i < n$*.*
5. *There exists a set* $E(p, q)$ *of formulae in two variables such that it defines Leibniz congruence on every model of* L*.*

Again, any set $E(p, q)$ satisfying part 4 also satisfies part 5 and vice versa. These sets are called *equivalence sets*. It is clear that all equivalential logics are protoalgebraic. Moreover, all the possible (parameterized) equivalence sets are mutually interderivable:

Proposition 2 *Let* $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}}\rangle$ *be a logic. Then:*

- *If* L *is equivalential and* $E(p, q), E'(p, q) \subseteq \mathrm{Fm}_{\mathcal{L}}$ *are equivalence sets for* L*, then* $E(p, q) \dashv\vdash_{\mathrm{L}} E'(p, q)$*.*
- *If* L *is protoalgebraic and* $E(p, q, \overrightarrow{r}), E'(p, q, \overrightarrow{r}) \subseteq \mathrm{Fm}_{\mathcal{L}}$ *are parameterized equivalence sets for* L*, then*

$$\bigcup\{E(p, q, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in \mathrm{Fm}_{\mathcal{L}}^{\leq\omega}\} \dashv\vdash_{\mathrm{L}} \bigcup\{E'(p, q, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in \mathrm{Fm}_{\mathcal{L}}^{\leq\omega}\}.$$

For finitary protoalgebraic logics, the relatively finitely subdirectly irreducible models can be described in the following way:

[4] The remaining properties of equivalence (i.e. symmetry and transitivity) which are not explicitly required in the syntactical conditions in 4, follow either directly from 5 or from 4 by syntactical arguments.

Theorem 7 *Let* L *be a finitary protoalgebraic logic complete with respect to a class* $\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})$*. Then* $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}} \subseteq \mathbf{H}_{\mathrm{S}}\mathbf{SP}_{\mathrm{U}}(\mathbb{K}^+)$*, where* $\mathbb{K}^+$ *is the class* $\mathbb{K}$ *plus the trivial matrix.*

The equational consequence relative to a class of algebras $\mathcal{K}$ is defined in the following way: $\Pi \models_{\mathcal{K}} \varphi \approx \psi$ if, and only if, for every $\mathcal{A} \in \mathcal{K}$ and every $\mathcal{A}$-evaluation e, if $e(\alpha) = e(\beta)$ for every $\alpha \approx \beta \in \Pi$, then $e(\varphi) = e(\psi)$.

Given a collection $\Pi \subseteq \mathrm{Eq}_{\mathcal{L}}$ of equations and a parameterized set $E(p, q, \overrightarrow{r})$ of formulae in two variables, $E[\Pi]$ denotes the set $\bigcup\{E(\varphi, \psi, \overrightarrow{\alpha}) \mid \varphi \approx \psi \in \Pi, \overrightarrow{\alpha} \in \mathrm{Fm}_{\mathcal{L}}^{\leq\omega}\}$ of formulae. Using this notation, the following theorem shows that any (parameterized) equivalence set provides a translation of the equational consequence relative to $\mathbf{ALG}^*(\mathrm{L})$ into the logic L.

Theorem 8 *Let* $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}} \rangle$ *be a logic and* $E(p, q, \overrightarrow{r}) \subseteq \mathrm{Fm}_{\mathcal{L}}$*. The following are equivalent:*

1. $E(p, q, \overrightarrow{r})$ *is a parameterized equivalence set for* L.
2. $p, \bigcup\{E(p, q, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in \mathrm{Fm}_{\mathcal{L}}^{\leq\omega}\} \vdash_{\mathrm{L}} q$ *and for every* $\Pi \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}$ *we have:* $\Pi \models_{\mathbf{ALG}^*(\mathrm{L})} \varphi \approx \psi$ *if, and only if,* $E[\Pi] \vdash_{\mathrm{L}} E(\varphi \approx \psi)$.

Theorem 9 *Let* $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}} \rangle$ *be a logic. The following are equivalent:*

1. L *is weakly algebraizable.*
2. *For every* $\mathcal{L}$*-algebra* $\mathcal{A}$*,* $\Omega_{\mathcal{A}}$ *is monotone and injective on* $\mathcal{F}i_{\mathrm{L}}(\mathcal{A})$.
3. *For every* $\mathcal{L}$*-algebra* $\mathcal{A}$*,* $\Omega_{\mathcal{A}}$ *is a lattice isomorphism of* $\mathcal{F}i_{\mathrm{L}}(\mathcal{A})$ *and* $\mathcal{C}o_{\mathbf{ALG}^*(\mathrm{L})}(\mathcal{A})$ *(the complete sublattice of congruences giving a quotient in* $\mathbf{ALG}^*(\mathrm{L})$*).*
4. L *is protoalgebraic and for every* $\mathcal{L}$*-algebra* $\mathcal{A}$ *and every* $F \in \mathcal{F}i_{\mathrm{L}}(\mathcal{A})$*,* $F/\Omega_{\mathcal{A}}(F)$ *is the least* L*-filter on* $\mathcal{A}/\Omega_{\mathcal{A}}(F)$.
5. *There exists a parameterized set* $E(p, q, \overrightarrow{r})$ *of formulae in two variables and a set* $\mathcal{E}(p) \subseteq \mathrm{Eq}_{\mathcal{L}}$ *of equations in one variable such that:*
   - *For every* $\Pi \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}$*,* $\Pi \models_{\mathbf{ALG}^*(\mathrm{L})} \varphi \approx \psi$ *iff* $E[\Pi] \vdash_{\mathrm{L}} E(\varphi \approx \psi)$*,*
   - $p \dashv\vdash_{\mathrm{L}} E[\mathcal{E}(p)]$.

Given a set $\mathcal{E}(p)$ of equations in one variable and a set Γ of formulae, $\mathcal{E}[\Gamma]$ denotes the set $\bigcup\{\mathcal{E}(\gamma) \mid \gamma \in \Gamma\}$ of equations.

Theorem 10 *Let* $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}} \rangle$ *be a logic. The following are equivalent:*

1. L *is algebraizable.*
2. *For every* $\mathcal{L}$*-algebra* $\mathcal{A}$*,* $\Omega_{\mathcal{A}}$ *is injective and it commutes with inverse images by homomorphisms.*
3. *For every* $\mathcal{L}$*-algebra* $\mathcal{A}$*,* $\Omega_{\mathcal{A}}$ *is a lattice isomorphism of* $\mathcal{F}i_{\mathrm{L}}(\mathcal{A})$ *and* $\mathcal{C}o_{\mathbf{ALG}^*(\mathrm{L})}(\mathcal{A})$ *that commutes with inverse images by homomorphisms.*
4. *There exists a set* $E(p, q)$ *of formulae in two variables and a set* $\mathcal{E}(p) \subseteq \mathrm{Eq}_{\mathcal{L}}$ *of equations in one variable such that:*
   - *For every* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$*,* $\Gamma \vdash_{\mathrm{L}} \varphi$ *iff* $\mathcal{E}[\Gamma] \models_{\mathbf{ALG}^*(\mathrm{L})} \mathcal{E}(\varphi)$*,*
   - $p \approx q \models_{\mathbf{ALG}^*(\mathrm{L})} \mathcal{E}[E(p, q)]$ *and* $\mathcal{E}[E(p, q)] \models_{\mathbf{ALG}^*(\mathrm{L})} p \approx q$*,*
   - *For every* $\Pi \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}$*,* $\Pi \models_{\mathbf{ALG}^*(\mathrm{L})} \varphi \approx \psi$ *iff* $E[\Pi] \vdash_{\mathrm{L}} E(\varphi, \psi)$*,*
   - $p \dashv\vdash_{\mathrm{L}} E[\mathcal{E}(p)]$.

   In this case $\mathbf{ALG}^*(\mathrm{L})$ *is called* the equivalent algebraic semantics *of* L.

If L *is finitary, we can add:*

5. L *is weakly algebraizable and* $\mathbf{ALG}^*(\mathrm{L})$ *is a quasivariety.*

We say that a (possibly parameterized) set $E(p, q, \overrightarrow{r}) \subseteq \mathrm{Fm}_{\mathcal{L}}$ of formulae in two variables satisfies the *G-rule* in the logic L if $p, q \vdash_{\mathrm{L}} E(p, q, \overrightarrow{r})$. This property and the finiteness of the corresponding sets E allow the definition of some other important classes of logics. Let L be a logic. Then:

1. L is called *finitely equivalential (algebraizable)* if it is equivalential (algebraizable) with a finite equivalence set.
2. L is called *regularly weakly algebraizable* if it has a parameterized equivalence set satisfying the G-rule.
3. L is called *regularly (finitely) algebraizable* if it has a (finite) equivalence set satisfying the G-rule.

Theorem 11 *A protoalgebraic logic* L *is regularly weakly algebraizable iff* $\mathbf{MOD}^*(\mathrm{L})$ *is a unital class of matrices, i.e., filters in reduced matrices are just singletons.*

All those classes constitute the so-called *Leibniz hierarchy*. They are depicted in Figure 1 together with their subsumption order (with the largest class at the bottom, i.e. infima are shown as joins in the usual depiction of a lattice). The intersection of any two classes of the Leibniz hierarchy is exactly their infimum w.r.t. the subsumption order. Examples showing that these classes are mutually different can be found in the literature.

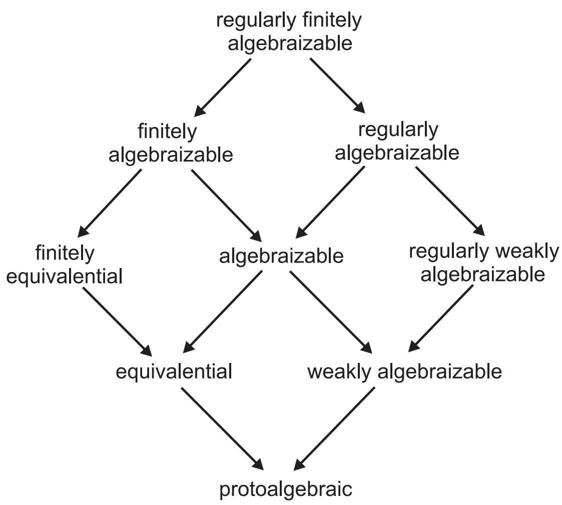

Fig. 1 Leibniz hierarchy.

3 Implications and semilinear implications

First, we introduce some useful notation. Let $\mathcal{L}$ be a propositional language and let $\Rightarrow(p, q, \overrightarrow{r}) \subseteq \mathrm{Fm}_{\mathcal{L}}$ be a set of formulae in two variables and, possibly, with parameters $\overrightarrow{r}$ (a sequence of variables). Given formulae $\varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}$ and a sequence of formulae $\overrightarrow{\alpha}$, $\Rightarrow(\varphi, \psi, \overrightarrow{\alpha})$ denotes the set obtained by substituting the formulae for the corresponding variables in $\Rightarrow(p, q, \overrightarrow{r})$, and $\varphi \Rightarrow_{\mathcal{L}} \psi$ denotes the set $\bigcup\{\Rightarrow(\varphi, \psi, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in \mathrm{Fm}_{\mathcal{L}}^{\leq\omega}\}$. Again, we omit the parameter $\mathcal{L}$ when it is clear from the context. When there are no parameters in the set $\Rightarrow(p, q)$ and it has only one element, we write $\varphi \to \psi$ instead of $\varphi \Rightarrow \psi$. Moreover, we use the following notation for the symmetrized set: $\Leftrightarrow(p, q, \overrightarrow{r}) = \Rightarrow(p, q, \overrightarrow{r}) \cup \Rightarrow(q, p, \overrightarrow{r})$ and $\varphi \Leftrightarrow \psi = (\varphi \Rightarrow \psi) \cup (\psi \Rightarrow \varphi)$.

3.1 A hierarchy of implications

In this section, we consider a collection of properties typically satisfied by implication connectives and generalize them to sets of (parameterized) formulae, yielding a hierarchy of generalized implications.

Definition 1 Let L be a logic and $\Rightarrow(p, q, \overrightarrow{r}) \subseteq \mathrm{Fm}_{\mathcal{L}}$ a parameterized set of formulae. We say that $\Rightarrow$ is a *weak p-implication* in L if[5]

(R) $\quad \vdash_{\mathrm{L}} \varphi \Rightarrow \varphi$,
(MP) $\quad \varphi, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \psi$,
(T) $\quad \varphi \Rightarrow \psi, \psi \Rightarrow \chi \vdash_{\mathrm{L}} \varphi \Rightarrow \chi$,
(sCng) $\quad \varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} c(\chi_1, \ldots, \chi_i, \varphi, \ldots, \chi_n) \Rightarrow c(\chi_1, \ldots, \chi_i, \psi, \ldots, \chi_n)$
for each $\langle c, n\rangle \in \mathcal{L}$ and each $i < n$.

We change the prefix 'weak' to 'algebraic' if there is a set $\mathcal{E}(p)$ of equations in one variable such that[6]

(Alg) $\quad p \dashv\vdash_{\mathrm{L}} \Leftrightarrow[\mathcal{E}(p)]$.

We change the prefix 'weak' to 'regular' if

(Reg) $\quad \varphi, \psi \vdash_{\mathrm{L}} \psi \Rightarrow \varphi$.

We change the prefix 'weak' to 'Rasiowa' if

(W) $\quad \varphi \vdash_{\mathrm{L}} \psi \Rightarrow \varphi$.

Finally, if $\Rightarrow$ is parameter-free we drop the prefix 'p-'.

The consecutions (R), (MP), (T), (sCng), and (W) correspond to usual properties fulfilled by implication connectives: *reflexivity*, *modus ponens*, *transitivity*, *symmetrized*[7] congruence and *weakening*. Furthermore, (Alg) (resp. (Reg)) corresponds to the class of algebraizable (resp. regularly algebraizable) logics, as will be justified later. Strictly speaking, the names of these consecutions should be parameterized by the used implication (and the set $\mathcal{E}(p)$ in the case of (Alg)).

[5] Throughout the paper we use acronyms inside parentheses to denote logical properties expressed by the satisfaction of certain consecutions, and acronyms without parentheses to denote metalogical properties.

[6] Recall that $E[\Pi]$ denotes the set $\bigcup\{E(\varphi, \psi, \overrightarrow{\alpha}) \mid \varphi \approx \psi \in \Pi, \overrightarrow{\alpha} \in \mathrm{Fm}_{\mathcal{L}}^{\leq\omega}\}$ of formulae.

[7] Notice that we could also study the following symmetrized version of *modus ponens* (sMP): $\varphi, \varphi \Rightarrow \psi, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} \psi$. Many of the theorems we are going to prove would also be valid for this more general notion, but due to the lack of any good motivating examples we are not going to do so. The reader can easily recognize where we use full (MP) and which results would hold more generally (see e.g. the next proposition). Notice that if $\Rightarrow$ satisfies (W), then (sMP) implies (MP). Indeed, from (W) we obtain $\varphi, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \psi \Rightarrow \varphi$ and hence by (sMP) we obtain $\varphi, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \psi$.

Proposition 3 *All the properties except* (sCng) *are preserved in any expansion*[8] *and* (sCng) *is preserved in those expansions in which the new connectives also have the symmetrized congruence property.*

Furthermore, if $\Rightarrow$ is parameter-free, then all the properties are preserved in fragments containing the language of $\Rightarrow$ (and that of $\mathcal{E}(p)$ in the case of (Alg)*).*

Proof The only non-trivial part is to show the first claim for parameterized sets. We present the proof for (T); the others are analogous. Let $\langle \mathcal{L}, \vdash_{\mathrm{L}} \rangle$ be a logic satisfying (T) and $\langle \mathcal{L}', \vdash_{\mathrm{L}'} \rangle$ some of its expansions. We know that $p \Rightarrow_{\mathcal{L}} q, q \Rightarrow_{\mathcal{L}} s \vdash_{\mathrm{L}} \delta(p, s, \overrightarrow{r})$ for every $\delta(p, q, \overrightarrow{r})$ from $\Rightarrow$. We can safely assume that none of p, q, s occurs in $\overrightarrow{r}$. For every $\overrightarrow{\nu} \in \mathrm{Fm}_{\mathcal{L}'}^{<\omega}$, we define an $\mathcal{L}'$-substitution σ by $\sigma(p) = \varphi$, $\sigma(q) = \psi$, $\sigma(s) = \chi$, and $\sigma(\overrightarrow{r}) = \overrightarrow{\nu}$. Thus we obtain that $\sigma[p \Rightarrow_{\mathcal{L}} q], \sigma[q \Rightarrow_{\mathcal{L}} s] \vdash_{\mathrm{L}} \delta(\varphi, \chi, \overrightarrow{\nu})$. To complete the proof, just notice that $\sigma[p \Rightarrow_{\mathcal{L}} q] \subseteq \varphi \Rightarrow_{\mathcal{L}'} \psi$ and $\sigma[q \Rightarrow_{\mathcal{L}} s] \subseteq \psi \Rightarrow_{\mathcal{L}'} \chi$.

Notice that, unlike in the definition of (parameterized) equivalence sets, we need to require the transitivity condition in the case of weak (p-)implications. Another crucial difference is the fact that the congruence property we had for (parameterized) equivalences cannot be translated into weak (p-)implications by writing the weak (p-)implication in the place of the (parameterized) equivalence, but we need to use the symmetrized set in the hypotheses. Let us now clearly state the relation between the equivalence and weak (p-)implication sets.

Proposition 4 *Let* L *be a logic and $\Rightarrow$ a weak* (p-)implication *in* L. *Then* L *is equivalential (protoalgebraic) with the (parameterized) equivalence set $\Leftrightarrow$.*

On one hand, observe that if L is an equivalential (protoalgebraic) logic then its (parameterized) equivalence set is a weak (p-)implication in the sense of the definition above. Recall that all (parameterized) equivalence sets in a given equivalential (protoalgebraic) logic are interderivable. On the other hand, there could be different implications in a given logic, take e.g. the classical logic: both the implication and the equivalence connective are weak implications in the sense of the definition above, in fact, they are regular implications; but equivalence is not a Rasiowa implication.

Proposition 5 *Each Rasiowa p-implication is a regular p-implication and each regular p-implication is an algebraic p-implication.*

Proof The first claim is trivial. To prove the second one, let α denote a formula in one variable p such that $\vdash_{\mathrm{L}} \alpha$ (such formula clearly exists: take any formula from $p \Rightarrow p$ and substitute p for all parameters). Define $\mathcal{E}(p) = \{p \approx \alpha\}$ and observe that (Reg) gives the first part of (Alg), whereas the second part follows from (MP).

Several classes of logics are defined by the existence of distinct kinds of implications.

Definition 2 Let L be a logic. We say that L is a *weakly/algebraically/regularly/Rasiowa- (p-)implicational logic* if there is a (parameterized) set of formulae $\Rightarrow$ which is a weak/algebraic/regular/Rasiowa (p-)implication in L.

We add the prefix *'finitely'* if the set $\Rightarrow$ is finite and we use the adjective *implicative* instead of *implicational* if $\Rightarrow$ is a *parameter-free singleton.*

[8] Recall that an expansion of a logic *can* contain new connectives. Extensions are those expansions where the language remains the same.

Rasiowa-implicative logics were already defined in 1974 by Rasiowa [24] and weakly implicative logics in 2006 by Cintula [6]. Notice that for the so-called *conjunctive* logics (i.e., those having a definable binary connective $\wedge$ and consecutions $\varphi \wedge \psi \vdash \varphi$, $\varphi \wedge \psi \vdash \psi$, and $\varphi, \psi \vdash \varphi \wedge \psi$, see e.g. [19]) the classes of finitely weakly implicational and weakly implicative logic coincide (analogously in the algebraic, regular, and Rasiowa case).

Now we study the relation between the classes of logics just defined and those in the Leibniz hierarchy. First notice that the first two claims of the next proposition follow directly from Proposition 4, while the remaining ones are corollaries of the characterizations of the corresponding classes in the Leibniz hierarchy presented in the Preliminaries.

Proposition 6 *The following pairs of classes of logics coincide:*

- *weakly p-implicational logics and protoalgebraic logics,*
- *(finitely) weakly implicational logics and (finitely) equivalential logics,*
- *algebraically p-implicational logics and weakly algebraizable logics,*
- *regularly p-implicational logics and regularly weakly algebraizable logics,*
- *(finitely) algebraically implicational logics and (finitely) algebraizable logics,*
- *(finitely) regularly implicational logics and (finitely) regularly algebraizable logics.*

Proposition 7 *Let* L *be a weakly implicative logic. If* L *is regularly/algebraically p-implicational, then it is regularly/algebraically implicative.*

Proof Since L is regularly p-implicational, we know that there is a parameterized equivalence set E such that $\varphi, \psi \vdash_{\mathrm{L}} E(\varphi, \psi, \overline{r})$. On the other hand, since L is weakly implicative, we know that $\{p \to q, q \to p\}$ is an equivalence set as well. Since all (parameterized) equivalence sets are interderivable, the claim follows easily. The proof for the other case is analogous.

Thus, we have obtained a new classification of logics expanding the Leibniz hierarchy as drawn in Figure 2. We call it the *hierarchy of implicational logics*.[9] However, our intention is not to replace the traditional terminology. We only have provided a new systematic way to describe it: on one axis, we are simplifying the structure of an implication and go from p-implicational, implicational, and finitely implicational to implicative; on the other axis, we strengthen the properties of the implication and use the prefixes 'weakly', 'algebraically', 'regularly', or 'Rasiowa-'. In the rest of the paper, we will use the traditional names for particular old classes, and we will use the new systematic names only for the new classes or when we need to formulate general theorems for more classes at once (see e.g. the previous or the next proposition).

Proposition 8 *Let* L *be an algebraically/regularly p-implicational logic. Then any weak p-implication is algebraic/regular.*

Obviously, the analogous statement is not true for Rasiowa p-implications.

Theorem 12 *All classes of logics in the implicational hierarchy depicted in Figure 2 are mutually different.*

[9] The class of *finitely protoalgebraic logics* (finitely weakly p-implicational logics) has not been investigated so far as part of the Leibniz hierarchy. For this reason, and because they would make the diagram 3-dimensional, we will also disregard the classes of *finitely algebraically/regularly/Rasiowa- p-implicational logics.*

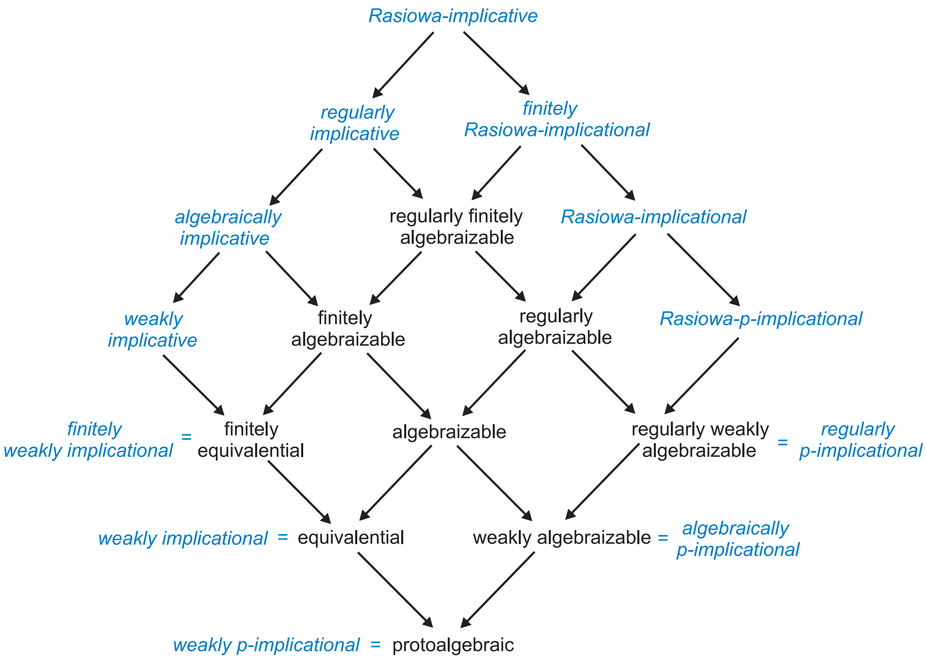

Fig. 2 The hierarchy of implicational logics

Below we give a series of examples which proves this theorem: the first three examples show the strictness of all right-to-left arrows; the next three ones show the strictness of all left-to-right arrows.

Example 1 Let us consider the following examples showing the separation of classes in the hierarchy of implicational logics:

1. Consider first the equivalence fragment of classical logic. This logic is clearly regularly implicative and we show that it is not Rasiowa-p-implicational. We know that this fragment is complete with respect to the two-valued matrix $\mathbf{M} = \langle\langle\{0,1\}, \leftrightarrow\rangle, \{1\}\rangle$, where $\leftrightarrow$ is the classical equivalence operation. Assume that there is a Rasiowa p-implication $\Rightarrow$. We know that $x, x \Rightarrow y \models_{\mathbf{M}} y$. Take an evaluation e such that $e(x) = 1$ and $e(y) = 0$. There has to be a formula $\chi(x, y, \overrightarrow{z}) \in\ \Rightarrow$ and a sequence of formulae $\overrightarrow{\psi}$ (i.e. $\varphi = \chi(x, y, \overrightarrow{\psi}) \in x \Rightarrow y$) such that $e(\varphi) = 0$. Let us define a substitution $\sigma(v) = x$ if $e(v) = 1$ and $\sigma(v) = y$ otherwise. Define the formula $\hat{\varphi} = \chi(x, y, \overrightarrow{\sigma(\psi)})$. Observe that $\hat{\varphi} = \sigma\varphi$, it has just two variables x and y, $\hat{\varphi} \in x \Rightarrow y$, and $e(\hat{\varphi}) = 0$. Let us write it as $\hat{\varphi}(1, 0) = 0$. Observe that from (R) we obtain $\hat{\varphi}(1, 1) = \hat{\varphi}(0, 0) = 1$ and from (W) we obtain that also $\hat{\varphi}(0, 1) = 1$. Thus we conclude that $\hat{\varphi}$ is the classical implication. As classical implication is not definable in the pure equivalence fragment, we have reached a contradiction and the proof is done.

2. Let UL be the Uninorm Logic studied in [22]. This logic is algebraically implicative but it is not regularly weakly algebraizable. Indeed, it has a binary primitive connective $\to$ which fulfills the properties of algebraic implication; however, it is not regularly weakly algebraizable because $\mathbf{MOD}^*(\mathrm{UL})$ is not unital (see Theorem 11). We can find many other examples with these features among well-known substructural logics: for instance, in [14], all the axiomatic extensions of FL which are not extensions of FL_w.

3. Consider the truth-degree-preserving three-valued Łukasiewicz logic $Ł_3^{\leq}$ defined in [3]: $\Gamma \models_{Ł_3^{\leq}} \varphi$ if, and only if, for every evaluation e on the three-element Łukasiewicz chain, $\min\{e(\gamma) \mid \gamma \in \Gamma\} \leq e(\varphi)$. This logic is weakly implicative with $E(p,q) = \{(p \leftrightarrow q)^2\}$ but it is not weakly algebraizable, as shown in [3]. A better known example of the same sort is the logic BCI which is clearly weakly implicative but known to be not weakly algebraizable (see [4]).

4. Let L be the logic in the language $\mathcal{L} = \{\rightarrow_1, \rightarrow_2\}$ with two binary connectives given by the matrix $\mathbf{A}$ with a three element domain $\{0, a, 1\}$, the filter $\{1\}$ and the operations

$\rightarrow_1$	0	a	1
0	1	1	1
a	0	1	1
1	1	0	1

$\rightarrow_2$	0	a	1
0	1	1	1
a	0	1	1
1	0	1	1

We show that L is finitely Rasiowa-implicational but not weakly implicative. On one hand, it is easily checked that the finite set $p \Rightarrow q = \{p \rightarrow_1 q, p \rightarrow_2 q\}$ is a Rasiowa implication. On the other hand, assume in search of a contradiction that a formula $\delta(p,q)$ defines a weak implication. Observe that the element a never appears in the truth-table of a non-atomic formula. Due to (R) and (MP) the truth-table for δ has to look like this:

δ	0	a	1
0	1	?	?
a	?	1	?
1	0	0	1

Operations definable in $\mathbf{A}$ are obtained as combinations of atoms by $\rightarrow_1$ and $\rightarrow_2$. One can prove that the truth-table of any binary operation has at most two zeros (it is routine to check the claim for definitions involving only two primitive operations, and then observe that, because of the definitions of $\rightarrow_1$ and $\rightarrow_2$, any other combination will have the same property). Therefore, δ is not a weak implication, because it fails to satisfy (sCng): $\delta(0,a) = \delta(a,0) = 1$, but also $\delta(a \rightarrow_1 a, a \rightarrow_1 0) = 0$.

5. Dellunde's logic presented in [9] is finitary and regularly algebraizable but it is not finitely equivalential. It is, as far as we know, the only example with these properties one can find in the literature. We can improve her example to show Rasiowa-implicational logics which are not finitely equivalential logic. We will do it in two different ways:

5.1. First, let L be any regularly algebraizable logic not finitely equivalential, for instance that given by Dellunde. Let $\Leftrightarrow$ be an infinite equivalence set without parameters for L, consider the class of its reduced models $\mathbf{MOD}^*(\mathrm{L})$ and define any linear order $\leq_{\mathbf{A}}$ for each $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})$ such that the unique element in the filter, say $t_{\mathbf{A}}$, is the maximum (recall that every reduced matrix is unital). We expand the language by adding a new binary connective $\wedge$ and expand the matrices by defining for each $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})$ and each $a, b \in A$: $a \wedge^{\mathbf{A}} b = a$ if $a \leq_{\mathbf{A}} b$, $a \wedge^{\mathbf{A}} b = b$ otherwise. Let $\mathrm{L}_\wedge$ be the logic of these expanded matrices which, obviously, is a conservative expansion of L. Define the set $p \Rightarrow q = p \wedge q \Leftrightarrow p$. Now, for every expanded matrix $\mathbf{A} = \langle \mathcal{A}, \{t_{\mathbf{A}}\}\rangle$ and $a, b \in A$: $a \Rightarrow^{\mathbf{A}} b = \{t_{\mathbf{A}}\}$ iff $a \wedge^{\mathbf{A}} b \Leftrightarrow a = \{t_{\mathbf{A}}\}$ iff $a \wedge^{\mathbf{A}} b = a$ iff $a \leq_{\mathbf{A}} b$. Therefore, it is clear that $\Rightarrow$ is a Rasiowa implication in $\mathrm{L}_\wedge$. To show that it is not finitely equivalential, first observe that $\Leftrightarrow$ is also an infinite equivalence set without parameters for $\mathrm{L}_\wedge$. If $\mathrm{L}_\wedge$ would have a finite equivalential set, then it would be finite subset $\Leftrightarrow_0 \subseteq \Leftrightarrow$, due to Proposition 2. But then, since $\mathrm{L}_\wedge$ is a conservative expansion of L, $\Leftrightarrow_0$ would be an equivalence set for L as well – a contradiction. Moreover, since it is a conservative expansion of L, it cannot be finitely equivalential.

5.2. Second, we will construct a particular example by generalizing the ideas in the previous example of a finitely Rasiowa-implicational but not weakly implicative logic. Let $\langle \{\to_i \mid i \in \omega\}, \vdash_{\mathcal{L}} \rangle$ be the logic, where all connectives are binary, given by the matrix $\mathbf{A}$ with domain ω^+, the filter $\{\omega\}$ and the operation $\to_0$:

$\to_0$	0	1	2	$\cdots$	$i{-}1$	i	$i{+}1$	$\cdots$	ω
0	ω	ω	ω	$\cdots$	ω	ω	ω	$\cdots$	ω
1	ω	ω	ω	$\cdots$	ω	ω	ω	$\cdots$	ω
2	ω	0	ω	$\cdots$	ω	ω	ω	$\cdots$	ω
$\vdots$									
$i{-}1$	ω	0	0	$\cdots$	ω	ω	ω	$\cdots$	ω
i	ω	0	0	$\cdots$	0	ω	ω	$\cdots$	ω
$i{+}1$	ω	0	0	$\cdots$	0	0	ω	$\cdots$	ω
$\vdots$									
ω	0	ω	ω	$\cdots$	ω	ω	ω	$\cdots$	ω

and $\to_i$ for every $i > 0$:

$\to_i$	0	1	2	$\cdots$	$i{-}1$	i	$i{+}1$	$\cdots$	ω
0	ω	ω	ω	$\cdots$	ω	ω	ω	$\cdots$	ω
1	ω	ω	ω	$\cdots$	ω	ω	ω	$\cdots$	ω
2	ω	0	ω	$\cdots$	ω	ω	ω	$\cdots$	ω
$\vdots$									
$i{-}1$	ω	0	0	$\cdots$	ω	ω	ω	$\cdots$	ω
i	0	0	0	$\cdots$	0	ω	ω	$\cdots$	ω
$i{+}1$	ω	0	0	$\cdots$	0	0	ω	$\cdots$	ω
$\vdots$									
ω	ω	ω	ω	$\cdots$	ω	0	ω	$\cdots$	ω

Notice that the truth table of $\to_0$ has to be defined separately as $0 \to_0^{\mathbf{A}} 0 = \omega$ rather than 0 (as tables for $i > 0$ could suggest). It is easily checked that the set $p \Rightarrow q = \{p \to_i q \mid i \in \omega\}$ is an infinite Rasiowa implication. Assume that L is finitely equivalential, i.e. there is a finite equivalence set E'. If we show that for each formula $\varphi(p, q)$ of two variables, there is at most one $j \in \omega$ such that $\varphi^{\mathbf{A}}(\omega, j) \neq \omega$, the proof is done. Indeed, in such a case there has to be a $j \in \omega$ such that $E'^{\mathbf{A}}(\omega, j) = \{\omega\}$ (because E' is finite) and so E' cannot define the Leibniz congruence on $\mathbf{A}$ for it identifies an element in the filter with some element outside.

We show the needed fact by induction over the complexity of the formula. First notice that for non-atomic formulae ψ we have $\psi^{\mathbf{A}}(\omega, j) \in \{0, \omega\}$. We distinguish several cases:

- $\varphi = p \to_i q$: $\varphi^{\mathbf{A}}(\omega, j) \neq \omega$ iff $j = i$.
- $\varphi = \psi \to_i p$: clearly $\varphi^{\mathbf{A}}(\omega, j) = \omega$ for each $j \in \omega$.
- $\varphi = q \to_i q$: clearly $\varphi^{\mathbf{A}}(\omega, j) = \omega$ for each $j \in \omega$.

Assume that $\psi \neq q$ and χ is not an atom:

- $\varphi = \psi \to_i q$: $\varphi^{\mathbf{A}}(\omega, j) \neq \omega$ only if $j = i$.

- $\varphi = q \to_i \chi$: $\varphi^{\mathbf{A}}(\omega, j) \neq \omega$ only if $j = i$.
- $\varphi = \psi \to_i \chi$ and $i > 0$: $\varphi^{\mathbf{A}}(\omega, j) = \omega$ for each $j \in \omega$.
- $\varphi = \psi \to_0 \chi$: $\varphi^{\mathbf{A}}(\omega, j) \neq \omega$ only if $\chi^{\mathbf{A}}(\omega, j) = 0$. From the induction assumption we know that $\chi^{\mathbf{A}}(\omega, j) \neq \omega$ for at most one $j \in \omega$.

6. The logic of ortholattices (and, in general, any orthologic which is not orthomodular) is regularly weakly algebraizable but it is not equivalential (see [8, Chapter 4.7.]). Another example of such a situation is a subtractive logic defined by Aglianò and Ursini (as proved in [8, Page 368]); it is the logic L given by a finite matrix $\mathbf{A}$ with filter $\{1\}$ and the following binary operation:

$+$	1	a	b	c	d
1	1	a	b	c	d
a	a	1	c	a	b
b	b	c	1	a	a
c	c	a	a	1	1
d	d	b	a	1	1

The set $p \Leftrightarrow q = \{(p+u)+(q+u)\}$ is a parameterized equivalence for L. Moreover one can prove that L is regularly weakly algebraizable (because it is assertional; see [8]) but not equivalential (because the class of its reduced models is not closed under submatrices; see Theorem 6). We can build from this example a Rasiowa p-implicational logic which is not equivalential. It is enough to repeat the trick we have used in Example 5.1. Consider the linear ordering $d < c < b < a < 1$ and expand $\mathbf{A}$ with a binary connective $\wedge$ such that for every $x, y \in A$, $x \wedge y = x$ if $x \leq y$, and $x \wedge y = y$ otherwise. Let $\mathrm{L}_\wedge$ be the logic of the expanded matrix. Again, $\mathrm{L}_\wedge$ is a conservative expansion of L. The parameterized set $p \Rightarrow q = p \wedge q \Leftrightarrow p$ is a Rasiowa p-implication for $\mathrm{L}_\wedge$. Moreover this logic is not equivalential because the submatrix defined on the subuniverse $\{c, d, 1\}$ is not reduced (its Leibniz congruence identifies c and d).

An interesting question is whether the relation of subsumption has the same nice intersection property that the original Leibniz hierarchy has, i.e. the intersection of any two classes is their infimum w.r.t. the subsumption order. Of course, everything works fine in the original part of the hierarchy and due to Proposition 7 also in the weakly/algebraically/regularly implicative part. The rest is an open problem:

Problem 1 What are the intersections in the hierarchy involving Rasiowa classes?

3.2 Semantics of implications

The syntactical notion of a weak p-implication that we have introduced has a natural semantical counterpart: a preorder in the models that becomes an order in the reduced models. Let us formalize this notion.

Definition 3 Let L be a logic, $\Rightarrow$ a weak p-implication, and $\mathbf{A} = \langle \mathcal{A}, D \rangle$ a matrix. We define a binary relation $\leq_{\mathbf{A}}^{\Rightarrow}$ on A by: $a \leq_{\mathbf{A}}^{\Rightarrow} b$ iff $a \Rightarrow^{\mathcal{A}} b \subseteq D$.

Proposition 9 *Let* L *be a logic,* $\Rightarrow$ *a weak p-implication, and* $\mathbf{A} = \langle \mathcal{A}, D \rangle \in \mathbf{MOD}(\mathrm{L})$. *Then:*

- $\leq_{\mathbf{A}}^{\Rightarrow}$ *is a preorder.*

- *The symmetrization of* $\leq^{\Rightarrow}_{\mathbf{A}}$ *is the Leibniz congruence of* $\mathbf{A}$*, i.e.* $\Omega_{\mathcal{A}}(D) = \leq^{\Rightarrow}_{\mathbf{A}} \cap (\leq^{\Rightarrow}_{\mathbf{A}})^{-1}$.
- $\leq^{\Rightarrow}_{\mathbf{A}}$ *is an order if, and only if,* $\mathbf{A}$ *is reduced.*
- D *is an up-set w.r.t.* $\leq^{\Rightarrow}_{\mathbf{A}}$*, i.e. if* $a \in D$ *and* $a \leq^{\Rightarrow}_{\mathbf{A}} b$*, then* $b \in D$.

Proof All the properties are easily checked.

If there is a weak p-implication $\Rightarrow$ in L, then, by virtue of Theorem 3, L is complete with respect to the class of ordered matrices. Also, for an L-matrix $\mathbf{A}$ and a weak p-implication $\Rightarrow$ we call $\leq^{\Rightarrow}_{\mathbf{A}}$ the *matrix (pre)order* of $\mathbf{A}$. The following theorem shows an interesting link between reduced models and the regularity of implication.

Theorem 13 *Let* L *be a logic and* $\Rightarrow$ *a weak p-implication. Then:*

- $\Rightarrow$ *is a regular p-implication if, and only if, for each* $\mathbf{A} = \langle \mathcal{A}, D \rangle \in \mathbf{MOD}^*(\mathrm{L})$*, there is an element* $a \in A$ *such that* $D = \{a\}$.
- $\Rightarrow$ *is a Rasiowa p-implication if, and only if, for each* $\mathbf{A} = \langle \mathcal{A}, D \rangle \in \mathbf{MOD}^*(\mathrm{L})$*, there is an element* $a \in A$ *such that* $D = \{a\}$ *and* a *is the maximum of* $\leq^{\Rightarrow}_{\mathbf{A}}$.

Proof The first claim follows from Theorem 11. For the second claim, assume that $\Rightarrow$ is a Rasiowa p-implication, i.e. $q \vdash_{\mathrm{L}} p \Rightarrow q$, and take any $\mathbf{A} = \langle \mathcal{A}, D \rangle \in \mathbf{MOD}^*(\mathrm{L})$. By the first claim, $D = \{a\}$ for some $a \in A$. Let b be an arbitrary element of A and e and $\mathcal{A}$-evaluation such that $e(q) = a$ and $e(p) = b$. Then $b \Rightarrow^{\mathcal{A}} a = \{a\}$ and hence $b \leq^{\Rightarrow}_{\mathbf{A}} a$. Conversely, if all reduced matrices have a singleton filter whose element is the maximum, then it is clear that $q \vdash_{\mathrm{L}} p \Rightarrow q$.

Given a logic L with a weak p-implication $\Rightarrow$ (i.e. a protoalgebraic logic) and a matrix $\mathbf{A} = \langle \mathcal{A}, D \rangle \in \mathbf{MOD}^*(\mathrm{L})$, we denote by $[D, A]$ the set of all filters from $\mathcal{F}i_{\mathrm{L}}(\mathcal{A})$ that contain D. Recall that $\mathcal{F}i_{\mathrm{L}}(\mathcal{A})$ is a complete lattice (hence bounded) where the meet is the set intersection, the bottom is the intersection of all filters and the top is the set A, and thus $[D, A]$ is a complete sublattice. It is easy to show that if L is weakly algebraizable, then we actually have $\mathcal{F}i_{\mathrm{L}}(\mathcal{A}) = [D, A]$. Let us denote by $\mathcal{U}p^{\Rightarrow}(\mathbf{A})$ the complete bounded lattice of $\leq^{\Rightarrow}_{\mathbf{A}}$-up-sets.

Proposition 10 *Let* L *be a logic,* $\Rightarrow$ *a weak p-implication, and* $\mathbf{A} = \langle \mathcal{A}, D \rangle$ *a reduced matrix. Then* $[D, A]$ *forms a sublattice of* $\mathcal{U}p^{\Rightarrow}(\mathbf{A})$.

Proof Take any $D' \in [D, A]$ and consider the matrix $\mathbf{A}' = \langle \mathcal{A}, D' \rangle$. Assume that $a \in D'$ and $a \leq^{\Rightarrow}_{\mathbf{A}} b$. Then $a \Rightarrow^{\mathcal{A}} b \subseteq D$ and, since $D \subseteq D'$, we obtain $a \leq^{\Rightarrow}_{\mathbf{A}'} b$. As we know, D' is an up-set w.r.t. $\leq^{\Rightarrow}_{\mathbf{A}'}$, and thus we obtain $b \in D'$, and hence $D' \in \mathcal{U}p^{\Rightarrow}(\mathbf{A})$.

Definition 4 Let L be a logic, $\Rightarrow$ a weak p-implication, and $\mathbf{A} = \langle \mathcal{A}, F \rangle \in \mathbf{MOD}(\mathrm{L})$. Then F is called $\Rightarrow$*-linear* if $\leq^{\Rightarrow}_{\mathbf{A}}$ is a total preorder, i.e. for every $a, b \in A$, $a \Rightarrow^{\mathcal{A}} b \subseteq F$ or $b \Rightarrow^{\mathcal{A}} a \subseteq F$. Furthermore, we say that $\mathbf{A}$ is a *linearly ordered model* (or just a *linear model*) with respect to $\Rightarrow$ if $\leq^{\Rightarrow}_{\mathbf{A}}$ is a linear order (equivalently: F is $\Rightarrow$-linear and $\mathbf{A}$ is reduced). We denote the class of all linear models with respect to $\Rightarrow$ as $\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$.

Observe that the class of linear models is not intrinsically defined for a given logic: it depends on the chosen implication. However, we shall see later that, in a reasonably wide class of logics, all *semilinear implications* define the same linear models. But even in the general case we can make an interesting observation about the linear models. Observe that, if $\leq^{\Rightarrow}_{\mathbf{A}}$ is a linear order, then $\mathcal{U}p^{\Rightarrow}(\mathbf{A})$ is linearly ordered by inclusion, and hence by Proposition 10 we easily obtain the following corollary:

Corollary 1 *Let* L *be a logic,* $\Rightarrow$ *a weak p-implication, and* $\mathbf{A} = \langle \mathcal{A}, D\rangle \in \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$. *Then* $[D, A]$ *is linearly ordered by inclusion.*

Theorem 14 *Let* L *be a protoalgebraic logic. Then* $\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L}) \subseteq \mathbf{MOD}^{*}(\mathrm{L})_{\mathrm{RFSI}}$ *for any weak p-implication.*

Proof Given any linear model $\langle \mathcal{A}, D\rangle$, we know that the filters in $[D, A]$ are linearly ordered and, hence, D is finitely meet-irreducible. Thus, by Proposition 1, the model is relatively finitely subdirectly irreducible.

Another interesting question is under which conditions the $\Rightarrow$-linear theories form a basis of the closure system $Th(\mathrm{L})$. We formulate this question as a kind of extension property:[10] LEP. Our characterization is based on a generalization of the so-called 'Prelinearity property' (see [6]). However, we prefer the new name 'Semilinearity Property', following the tradition of Universal Algebra to call a class of algebras 'semi$X$' whenever its subdirectly irreducible members have the property X (see Theorem 16).

Definition 5 Let $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}}\rangle$ be a logic and $\Rightarrow$ a parameterized set of formulae. We say that L has the

- *Linear Extension Property* LEP with respect to $\Rightarrow$ if for every theory $T \in Th(\mathrm{L})$ and every formula $\varphi \in \mathrm{Fm}_{\mathcal{L}} \setminus T$, there is a $\Rightarrow$-linear theory $T' \supseteq T$ such that $\varphi \notin T'$.
- *Semilinearity Property* SLP with respect to $\Rightarrow$ if the following metarule is valid:

$$\frac{\Gamma, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} \chi}{\Gamma \vdash_{\mathrm{L}} \chi}$$

Notice that LEP and SLP are properties of the pair $\langle \mathrm{L}, \Rightarrow\rangle$; however, we will often slightly abuse the terminology by saying that just a logic or a set of formulae has one of these properties when the other element of the pair is clear from the context. The proof of the next proposition relating LEP and SLP can be obtained as a generalization of [6, Lemmata 10 and 11].

Proposition 11 *Let* L *be a logic and* $\Rightarrow$ *a parameterized set of formulae. Then we have:*

- *If* L *has the* LEP*, then it has the* SLP*.*
- *If* L *is finitary, then* L *has the* LEP *iff it has the* SLP*.*

The next theorem shows that, in finitary logics, the notion of the LEP can also be meaningfully defined for other than the Lindenbaum matrices.[11]

Theorem 15 *Let* $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}}\rangle$ *be a finitary logic with the* LEP *and* $\mathcal{A} \in \mathbf{ALG}^{*}(\mathrm{L})$. *Then the linear filters form a basis of* $\mathcal{F}i_{\mathrm{L}}(\mathcal{A})$.

Proof We must show that for each $F \in \mathcal{F}i_{\mathrm{L}}(\mathcal{A})$ and $t \in A \setminus F$ there is an $\Rightarrow$-linear filter $F' \supseteq F$ such that $t \notin F'$. We distinguish two cases based on the cardinality of A.

[10] Recall the equivalent definitions of basis in the Preliminaries.

[11] This theorem can be seen as one of the so-called *transfer theorems* in the theory of g-matrices (see [13]), since it transfers a property (in this case the LEP) from the Lindenbaum basic full g-models to other basic full g-models.

1) Firstly assume that A is at most countable. We can assume that the set of propositional variables contains (or is equal to) the set $\{v_a \mid a \in A\}$. Let $\varphi(v_{a_1}, \dots v_{a_n})$ mean that a formula φ includes variables $v_{a_1}, \dots v_{a_n}$, and by $\varphi^{\mathcal{A}}(a_1, \dots a_n)$ the value of φ in $\mathbf{A}$ in the evaluation $e(v_{a_i}) = a_i$. Consider the theory T axiomatized by the following set of formulae:

$$\Gamma = \{v_a \mid a \in F\} \cup \bigcup_{\varphi(v_{a_1}, \dots v_{a_n}) \in \mathrm{Fm}_{\mathcal{L}}} \varphi(v_{a_1}, \dots v_{a_n}) \Leftrightarrow v_{\varphi^{\mathcal{A}}(a_1, \dots a_n)}.$$

Clearly, $v_t \notin T$ (because for the $\mathbf{A}$-evaluation $e(v_a) = a$ we obtain $e[\Gamma] \subseteq F$ and $e(v_t) \notin F$). Now we use the LEP to obtain a linear theory $T' \supseteq T$ such that $v_t \notin T'$. Consider a subset of A defined as $F' = \{a \mid v_a \in T'\}$. Clearly $F' \supseteq F$ and $t \notin F'$, what remains to be shown is that $F' \in \mathcal{F}i_{\mathrm{L}}(\mathcal{A})$ and it is linear.

Linearity is simpler: if we show that $v_a \Rightarrow v_b \subseteq T'$ implies $a \Rightarrow^{\mathcal{A}} b \subseteq F'$, the proof can be finished by the linearity of T'. To show $a \Rightarrow^{\mathcal{A}} b \subseteq F'$, we need to have $\chi^{\mathcal{A}}(a, b, \overrightarrow{r}) \in F'$ for any $\chi \in \Rightarrow$ and any vector $\overrightarrow{r}$ of values assigned to the parameters of χ. From our assumption we know that $\chi(v_a, v_b, \overrightarrow{v_r}) \in T'$ and from the construction of T' we have $\chi(v_a, v_b, \overrightarrow{v_r}) \Leftrightarrow v_{\chi^{\mathcal{A}}(a,b,\overrightarrow{r})} \subseteq T'$ and so by (MP) we obtain $v_{\chi^{\mathcal{A}}(a,b,\overrightarrow{r})} \in T'$ and hence $\chi^{\mathcal{A}}(a, b, \overrightarrow{r}) \in F'$.

To show that $F' \in \mathcal{F}i_{\mathrm{L}}(\mathcal{A})$, we first prove the following chain of equivalences for any $\mathbf{A}$-evaluation e and any formula $\psi(p_1, \dots p_n)$: $e(\psi) \in F'$ iff $v_{e(\psi)} \in T'$ iff $v_{\psi^{\mathcal{A}}(e(p_1), \dots, e(p_n)))} \in T'$ iff $\psi(v_{e(p_1)}, \dots, v_{e(p_n)}) \in T'$ iff $\sigma\psi \in T'$ for the substitution $\sigma p = v_{e(p)}$. The first equivalence is the definition of F', the second and the last ones are simple, and the third one follows from the fact that $\psi(v_{e(p_1)}, \dots, v_{e(p_n)}) \Leftrightarrow v_{\psi^{\mathcal{A}}(e(p_1), \dots, e(p_n))} \subseteq T'$ and (MP). Assume that $S \vdash_{\mathrm{L}} \varphi$ and for some $\mathbf{A}$-evaluation e it is the case that $e[S] \subseteq F'$. Using the chain of equivalences, we obtain $\sigma[S] \subseteq T'$ and thus also $\sigma\varphi \in T'$ (because $\sigma[S] \vdash_{\mathrm{L}} \sigma\varphi$). Thus finally $e(\varphi) \in F'$.

2) *Secondly assume that A is uncountable.* We introduce a new set of propositional variables $\mathbf{VAR} = \{v_a \mid a \in A\}$; we can safely assume that it contains the original propositional variables. We define a new logic L' in the language $\mathcal{L}'$ which has the same connectives as $\mathcal{L}$ and atoms from $\mathbf{VAR}$ (let us, in this proof, assume that the set of atoms is part of the notion of a propositional language) in the following way: $T \vdash_{\mathrm{L}'} \varphi$ iff there is finite subset $T' \subseteq T$ and $\mathcal{L}'$-substitution σ such that $\sigma[T'] \cup \{\sigma\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ and $\sigma[T'] \vdash_{\mathrm{L}} \sigma\varphi$. From [8] (remark after Exercise 0.3.3.) we know that L' is a finitary logic and it is a conservative extension of L.

Notice that $\Rightarrow$ is a weak p-implication in L'; we show that L' has the SLP: assume that $T, \varphi \Rightarrow \psi \vdash_{\mathrm{L}'} \chi$ and $T, \psi \Rightarrow \varphi \vdash_{\mathrm{L}'} \chi$. Since L' is a finitary logic, we know that there is a finite $T' \subseteq T$ such that $T', \varphi \Rightarrow \psi \vdash_{\mathrm{L}'} \chi$ and $T', \psi \Rightarrow \varphi \vdash_{\mathrm{L}'} \chi$. Obviously, there is an $\mathcal{L}'$-substitution σ such that $\sigma[T'] \cup \{\sigma\varphi, \sigma\psi\sigma\chi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$. Clearly also $\sigma[T'], \sigma\varphi \Rightarrow \sigma\psi \vdash_{\mathrm{L}'} \sigma\chi$ and $\sigma[T'], \sigma\psi \Rightarrow \sigma\varphi \vdash_{\mathrm{L}'} \sigma\chi$. Using the fact that L' expands L conservatively, we obtain $\sigma[T'], \sigma\varphi \Rightarrow \sigma\psi \vdash_{\mathrm{L}} \sigma\chi$ and $\sigma[T']\sigma\psi \Rightarrow \sigma\varphi \vdash_{\mathrm{L}} \sigma\chi$. From the SLP of L we know that $\sigma[T'] \vdash_{\mathrm{L}} \sigma\chi$. The definition of L' gives $T \vdash_{\mathrm{L}'} \chi$ and thus we obtain the SLP in L'.

By the previous proposition (notice that the cardinality of the set of atoms does not play any rôle in its proof), L' has also the LEP. Knowing this, we can repeat the constructions from the first part of this proof; we construct T (we obtain $v_t \notin T$ from the obvious observation that $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L}')$), T' (because L' has the LEP), and F'. The rest of the proof is fully analogous.

3.3 Semilinear implications

Now we introduce the central concept of a *semilinear implication* which, among others, provides another characterization of the LEP and allows us to characterize implications such that the converse inclusion of Theorem 14 holds.

Definition 6 Let L be a logic and $\Rightarrow$ a weak p-implication. We say that $\Rightarrow$ is a weak *semilinear* p-implication if $\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})}$.

Theorem 16 (A characterization of semilinear implications) *Let* L *be a logic and* $\Rightarrow$ *a weak p-implication. Then the following are equivalent:*

1. $\Rightarrow$ *is semilinear in* L,
2. L *has the* LEP *w.r.t.* $\Rightarrow$.

Furthermore, if L *is finitary, the list of equivalences can be expanded with:*

3. L *has the* SLP *w.r.t.* $\Rightarrow$,
4. $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}} \subseteq \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$.

Moreover, if $\Rightarrow$ *is finite (but possibly with parameters), we can add:*

5. $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}} \subseteq \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$.

Proof The equivalence of 1 and 2 is proved by generalizing the proof of [6, Theorem 1]; 2 and 3 are equivalent due to Proposition 11; 4 implies 1 is an easy consequence of Theorem 4; we prove that 2 implies 4. Assume that $\mathbf{A} = \langle \mathcal{A}, F\rangle \notin \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$. If $\mathbf{A} \notin \mathbf{MOD}^*(\mathrm{L})$, then trivially $\mathbf{A} \notin \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$. Assume that $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})$. By the LEP and Theorem 15 we obtain that F is the intersection of all $\Rightarrow$-linear filters containing F and, since F is not $\Rightarrow$-linear, it follows that F is meet-reducible. By Proposition 1 we obtain $\mathbf{A} \notin \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$.

Clearly 5 implies 4, thus to complete the proof we just show that 1 implies 5: from the assumptions we know that L is a finitary protoalgebraic logic complete with respect to the class $\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$, which clearly contains the trivial matrix. Thus, from Theorem 7, we obtain $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}} \subseteq \mathbf{H}_{\mathrm{S}}\mathbf{SP}_{\mathrm{U}}(\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L}))$. By [8, Theorem 0.6.1] we know that $\mathbf{SP}_{\mathrm{U}}(\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})) \subseteq \mathbf{MOD}(\mathrm{L})$. It suffices to show that the matrix preorder is linear for each matrix in $\mathbf{SP}_{\mathrm{U}}(\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L}))$ (since all models in $\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$ are reduced and $\mathbf{H}_{\mathrm{S}}$ only yields isomorphic copies when applied to reduced matrices). Consider the matrices as first-order structures with one unary predicate F. The matrix preorder can be formally defined as:

$$a \leq^{\Rightarrow}_{\mathbf{A}} b \text{ iff } \mathbf{A} \models \forall \overrightarrow{x}\Big(\bigwedge_{\chi \in \Rightarrow} F(\chi(a, b, \overrightarrow{x}))\Big).$$

Then the fact that $\leq^{\Rightarrow}_{\mathbf{A}}$ is a linear ordering is expressible in the first-order language and hence preserved under $\mathbf{P}_{\mathrm{U}}$. Finally, the preservation under $\mathbf{S}$ is obvious.

Note that the implication '1 implies 3' holds even without the additional assumption of finitarity of the logic (by Proposition 11). Also the implication '5 implies 1' holds even without the additional assumption of finiteness of the implications.

The previous theorem has several interesting and important corollaries. From Theorem 14 we obtain that, at least for finitary logics, being the class of linear models with respect to any finite semilinear implication is an intrinsic property of a logic, for it corresponds to the class of relatively finitely subdirectly irreducible matrices.

Corollary 2 *Let* L *be a finitary protoalgebraic logic. Then, for any finite weak semilinear p-implication* $\Rightarrow$*, the equality* $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}} = \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$ *holds.*

The restriction to finite implications in this corollary (and in the last part of the previous theorem) is certainly an eyesore, but fortunately can be removed in a wide class of logics (namely, those having a suitable definable disjunction, as proved in the follow-up of the present paper [7]).

The second corollary uses the trivial observation that $\varphi, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} \psi \Rightarrow \varphi$ and for regular implications also $\varphi, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \psi \Rightarrow \varphi$. Thus, if $\Rightarrow$ is semilinear, we can use the SLP to obtain that $\Rightarrow$ is a Rasiowa implication.

Corollary 3 *Each regular semilinear p-implication is a Rasiowa p-implication.*

Another interesting corollary is obtained by a simple observation that the LEP of a logic is preserved in all its axiomatic extensions.

Corollary 4 *Let* L *be a logic and* $\Rightarrow$ *a weak semilinear p-implication. Then* $\Rightarrow$ *is semilinear in every axiomatic extension of* L*.*

This corollary will be particularly useful for showing that some logic has *no* semilinear implication: all we have to do is to find an axiomatic extension with this property.

It is quite easy to show that an implication in some logic is not semilinear, consider e.g. the normal implication of the intuitionistic logic. The well-known fact that the linear Heyting algebras do not generate the variety of Heyting algebras does the job. The next proposition uses the characterization theorem (together with Corollary 1) to show much more: there is *no weak semilinear p-implication definable* in the intuitionistic logic, i.e. not only the standard nice Rasiowa implication given by a single formula is not semilinear but even if using an infinite set with parameters we could never obtain an implication whose linearly ordered Heyting algebras would generate the variety of Heyting algebras.

Proposition 12 *No weak semilinear p-implication is definable in intuitionistic logic.*

Proof We provide two alternative proofs of this fact. First a very simple *ad hoc* one, and then a more sophisticated proof using the machinery introduced in the present paper which has the advantage of providing a general method for showing the undefinability of weak semilinear p-implications in many logics.

1. Assume that $\Rightarrow$ is a weak semilinear p-implication in intuitionistic logic; we show that $p \Rightarrow q \dashv\vdash_{\mathrm{IPC}} p \to q$ (where $\to$ is the usual intuitionistic implication), which entails that $\to$ is a semilinear implication—a contradiction. One direction is simple: from $p, p \Rightarrow q \vdash_{\mathrm{IPC}} q$ we obtain (using the Deduction Theorem:) $p \Rightarrow q \vdash_{\mathrm{IPC}} p \to q$. The reverse direction: using the first direction we obtain $q \Rightarrow p, p \to q \vdash_{\mathrm{IPC}} q \to p$. Since, trivially, $q \Rightarrow p, p \to q \vdash_{\mathrm{IPC}} p \to q$ and all equivalence sets are interderivable (Proposition 2), we obtain $q \Rightarrow p, p \to q \vdash_{\mathrm{IPC}} p \Rightarrow q$. Now, using the trivial fact that $p \Rightarrow q, p \to q \vdash_{\mathrm{IPC}} p \Rightarrow q$ and the SLP, we conclude that $p \to q \vdash_{\mathrm{IPC}} p \Rightarrow q$.
2. Consider IPC in the language $\mathcal{L} = \{\wedge, \vee, \to, \bot, \top\}$. It is well known that it is a regularly algebraizable logic whose equivalent quasivariety semantics is the variety of Heyting algebras ($\mathbb{HA}$) with $E(p,q) = \{p \to q, q \to p\}$ and $\mathcal{E}(p) = \{p \approx \top\}$. On one hand, the regularity implies that the class of reduced models is unital, hence the filters of these models are just singletons of the form $\{\top^{\mathcal{A}}\}$. On the other hand, the

Leibniz operator defines a bijective correspondence between filters and congruences in any Heyting algebra and hence $\{\top^{\mathcal{A}}\}$ is meet-irreducible in $\mathcal{F}i_{\mathrm{IPC}}(\mathcal{A})$ if, and only if, the identity relation is meet-irreducible in $Co(\mathcal{A})$, i.e. $\mathcal{A}$ is subdirectly irreducible. Thus, $\mathbf{MOD}^*(\mathrm{IPC})_{\mathrm{RSI}} = \{\langle \mathcal{A}, \{\top^{\mathcal{A}}\}\rangle \mid \mathcal{A} \in \mathbb{HA}_{\mathrm{SI}}\}$. Assume now, in search of a contradiction, that $\Rightarrow(p, q, \overrightarrow{r}) \subseteq \mathrm{Fm}_{\mathcal{L}}$ is a weak p-implication in IPC. By Theorem 16, we have $\{\langle \mathcal{A}, \{\top^{\mathcal{A}}\}\rangle \mid \mathcal{A} \in \mathbb{HA}_{\mathrm{SI}}\} \subseteq \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{IPC})$. Now, it is sufficient to consider a subdirectly irreducible Heyting algebra where the natural lattice order is not linear (it is well-known that such algebras exist) and it will have two incomparable filters (IPC-filters are known to be the same as lattice filters over the Heyting algebra). Then Corollary 1 yields the contradiction.

Corollary 5 *Let* L *be the logic of any quasivariety of pointed residuated lattices*[12] *containing the variety of Heyting algebras. Then there is no weak semilinear p-implication definable in* L.

Let us give a list of some well-known logics falling under the scope of the previous proposition: the multiplicative-additive fragment of (Affine) Intuitionistic Linear logic, Full Lambek logic (possibly extended by exchange/weakening/contraction), Relevance logic R. On the other hand, observe that the second proof of Proposition 12 can be extended to many other protoalgebraic finitary logics L: all one needs to do is to find a relatively subdirectly irreducible member of $\mathbf{MOD}^*(\mathrm{L})$ with two incomparable logical filters. For instance, consider the variety V of pointed residuated lattices generated by the symmetric rotation (for this construction, see e.g. [15, 20]) of all Heyting algebras. Clearly, its corresponding logic has an involutive negation, that is, it proves $\neg\neg\varphi \to \varphi$. Exactly the same reasoning as before yields that no weak semilinear p-implication is definable in this logic and thus the same holds for the logic of any quasivariety of pointed residuated lattices containing V. In particular, this shows that no weak *semilinear* p-implication is definable in Girard's Linear logic (precisely, in its reduct in the language of pointed residuated lattices).

We close this subsection with another corollary of Theorem 16 that shows that semilinearity of implications is preserved under intersections of logics; we also discuss some of its consequences.

Corollary 6 *Let $\Rightarrow$ be a parameterized set of formulae, $\mathcal{I}$ be a family of logics in the same language and $\hat{\mathrm{L}}$ its intersection. If $\Rightarrow$ is a weak semilinear p-implication in every logic of $\mathcal{I}$, then so is it in $\hat{\mathrm{L}}$.*

Proof We show that $\hat{\mathrm{L}}$ has the LEP. Let T be an $\hat{\mathrm{L}}$-theory and $\varphi \notin T$, i.e. $T \nvdash_{\hat{\mathrm{L}}} \varphi$. Thus there has to be a logic $\mathrm{L} \in \mathcal{I}$ such that $T \nvdash_{\mathrm{L}} \varphi$, i.e. $\varphi \notin \bar{T}$ where $\bar{T}$ is the L-theory generated by T. Thus by the LEP of L there is a linear L-theory $T' \supseteq \bar{T} \supseteq T$ and $\varphi \notin T'$. Since T' is clearly an $\hat{\mathrm{L}}$-theory as well, the proof is complete.

The following theorem states that each logic with an implication can be extended to the weakest logic where that implication is semilinear. The claim is a trivial consequence of the previous corollary because any weakly p-implicational logic has at least one extension where its weak p-implication is semilinear, namely the inconsistent logic.

Theorem 17 *Let* L *be a logic and $\Rightarrow$ a weak p-implication. Then there is the weakest logic extending* L *where $\Rightarrow$ is semilinear (the intersection of all its extensions where $\Rightarrow$ is semilinear). Let us denote this logic as $\mathrm{L}^{\ell}_{\Rightarrow}$.*

[12] See e.g. monograph [14].

In the paper [7] we show how to axiomatize $\mathrm{L}^{\ell}_{\Rightarrow}$. However, determining a complete semantics is simple, as described in the following straightforward proposition.

Proposition 13 *Let* L *be a logic and* $\Rightarrow$ *a weak p-implication. Then* $\vdash_{\mathrm{L}^{\ell}_{\Rightarrow}} = \models_{\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})}$ *and* $\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L}^{\ell}_{\Rightarrow}) = \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$.

Moreover, we can show that if a logic L is finitary, than so is $\mathrm{L}^{\ell}_{\Rightarrow}$.

Proposition 14 *Let* L *be a finitary logic and* $\Rightarrow$ *a weak p-implication. Then* $\mathrm{L}^{\ell}_{\Rightarrow}$ *is finitary.*

Proof Recall that the finitary companion of a logic S, denoted as $\mathcal{FC}(\mathrm{S})$, is defined as follows: $\Gamma \vdash_{\mathcal{FC}(\mathrm{S})} \varphi$ iff there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathrm{S}} \varphi$. Observe that $\mathcal{FC}(\mathrm{S})$ is the largest finitary logic contained in S. Thus, since L is finitary, we know that $\mathrm{L} \subseteq \mathcal{FC}(\mathrm{L}^{\ell}_{\Rightarrow}) \subseteq \mathrm{L}^{\ell}_{\Rightarrow}$. If we show that $\Rightarrow$ is semilinear in $\mathcal{FC}(\mathrm{L}^{\ell}_{\Rightarrow})$ we obtain $\mathcal{FC}(\mathrm{L}^{\ell}_{\Rightarrow}) = \mathrm{L}^{\ell}_{\Rightarrow}$ and hence $\mathrm{L}^{\ell}_{\Rightarrow}$ is finitary. Actually, one can easily show in general that if $\Rightarrow$ is semilinear in a logic S, then so is it in $\mathcal{FC}(\mathrm{S})$, just by checking that satisfaction of the SLP is preserved.

Finally, observe that if L is finitary and $\Rightarrow$ a weak p-implication, then $\mathrm{L}^{\ell}_{\Rightarrow}$ is the intersection of all the *finitary* extensions of L where $\Rightarrow$ is semilinear.

4 The hierarchy of implicational semilinear logics

According to [2], *fuzzy logics are the logics of chains* in the sense that they enjoy a complete semantics *based on linearly ordered algebras*. Such a claim must be read as a methodological statement, pointing at a roughly defined class of logics, rather than a precise mathematical description of what fuzzy logics are, since there could be many different ways in which a logic might enjoy a complete semantics based on chains. Nevertheless, the framework we have developed in the present paper provides, in a natural way, a particular technical notion corresponding to this intuition. We define *implicational semilinear logics* as logics possessing some weak semilinear p-implication. Obviously, they are fuzzy logics in the sense of [2] for they happen to be complete w.r.t. the class of models where the weak semilinear p-implication induces a linear order.

Notice that we choose the term ‘semilinear’ instead of ‘fuzzy’ in spite of the fact that a first step towards the general definition we are offering here had been done by the first author in [6], where he defined the class of *weakly implicative fuzzy logics* (in our new framework: logics with a weak semilinear implication given by a single binary connective). We have realized that the attempt of [6] to use the term ‘fuzzy’ to formally define a class of logics was rather futile because the word is heavily charged with many conflicting potential meanings which are hard to be put away for many. Therefore, we have now opted for the new neutral name ‘semilinear’ (which, on the other hand, has the advantage of describing an equivalent algebraic property in the finitary case) although our intention remains the same: to capture the class of fuzzy logics among the protoalgebraic ones (originally, among the weakly implicative ones only). Nevertheless, by doing so we do not expect to capture in a mathematical definition *the whole* intuitive notion of an *arbitrary* fuzzy logic yet, for there could be (and some works in the literature suggest that this is the case; see e.g. [3] or some recent work on modal fuzzy logics) several other ways in which a logic might have a complete semantics somehow based on chains. But at least we aim to clearly define a broad class of logics which can arguably be regarded as fuzzy logics that contains almost all the prominent examples known so far. To this end, analogously as in Definition 2, we define classes of implicational semilinear logics based on the form of semilinear implication they possess.

Definition 7 Let L be a logic. We say that L is a *weakly/algebraically/Rasiowa-(p-)implicational semilinear logic* if there is a (parameterized) set of formulae $\Rightarrow$ such that it is a weak/algebraic/Rasiowa semilinear (p-)implication in L. We add the prefix *'finitely'* if the set $\Rightarrow$ is finite and we use the adjective *'implicative'* instead of *'implicational'* if $\Rightarrow$ is a parameter-free singleton.

Notice that if we would have defined the class of *regularly* (p-)implicational (implicative) semilinear logics, by the Corollary 3 we would obtain that each regularly p-implicational semilinear logic is a Rasiowa-p-implicational semilinear logic (and analogously for the other three Rasiowa classes in the hierarchy of implicational logics).

In accordance with Proposition 6 and our decision to preserve traditional terminology as much as possible, we will use 'protoalgebraic semilinear logics' instead of 'weakly p-implicational semilinear logics', '(finitely) equivalential semilinear logics' instead of '(finitely) weakly implicational semilinear logics', and '(finitely) algebraizable semilinear logics' instead of '(finitely) algebraically implicational semilinear logics'. However, in the light of the previous observation, we will have no 'regularly (finitely/weakly) algebraizable semilinear logics' and we will use '(finitely) Rasiowa-(p-)implicational semilinear logics' instead. See all the classes and their inclusions in Figure 3.

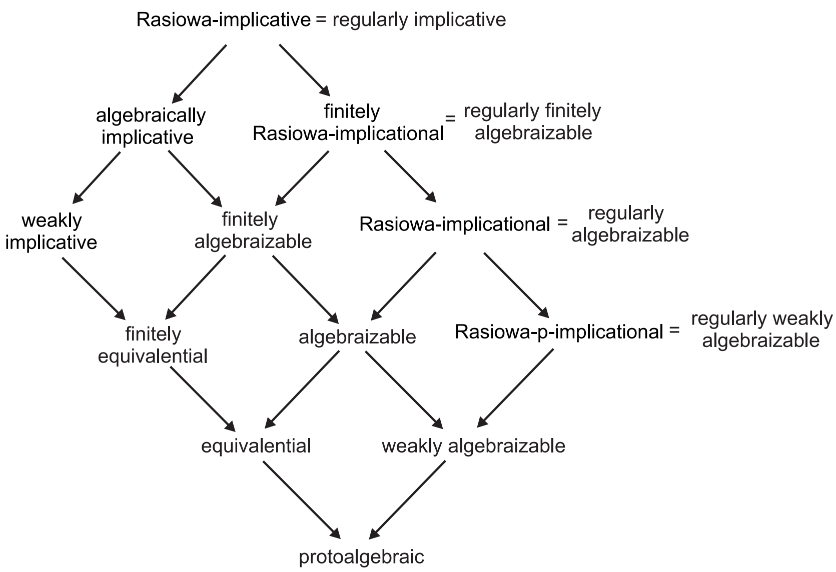

Fig. 3 The hierarchy of implicational semilinear logics

Proposition 15 *Let $\mathcal{X}$ be one of the following expressions: 'protoalgebraic', 'equivalential', 'finitely equivalential', or 'weakly implicative'. Then, a logic is algebraically/Rasiowa- $\mathcal{X}$ semilinear logic iff it is simultaneously an $\mathcal{X}$ semilinear logic and an algebraically/Rasiowa- p-implicational logic.*

Proof Let a logic L be a $\mathcal{X}$ semilinear logic, then it possesses a semilinear implication $\Rightarrow$ of a proper form (a parameterized set, a parameter-free set, a finite parameter-free set or a singleton). Let $\Leftrightarrow$ be the (parameterized) equivalence set given by symmetrization of this implication . Further assume that L is a Rasiowa-p-implicational logic. Therefore, there is a parameterized equivalence set $\Leftrightarrow'$ such that $\varphi, \psi \vdash_{\mathrm{L}} \varphi \Leftrightarrow' \psi$. Since all parameterized equivalence sets are interderivable, $\Rightarrow$ is a regular semilinear p-implication and hence by Corollary 3 it is a Rasiowa p-implication. Thus the claim follows easily. The proof for an algebraically p-implicational L is analogous.

Roughly speaking, this proposition says that in order to locate a logic in the hierarchy of implicational semilinear logics it is enough to place it in one of the *semilinear* classes on the left-down branch (i.e. those given by $\mathcal{X}$) and in one of the classes on the right-down branch of the general (not semilinear) diagram (i.e. weakly algebraizable or regularly weakly algebraizable). Notice that no semilinearity is required in the second step. This proposition has an interesting corollary and raises an open problem.

Corollary 7 *The intersection of any two classes of the hierarchy of implicational semilinear logics is their infimum w.r.t. the subsumption order.*

The proof is simple if we notice that it is sufficient to show it for the first class being one of the equivalential, finitely equivalential, or weakly implicative semilinear logics and the second being one of the weakly algebraizable or Rasiowa-implicational semilinear logics, and for these classes the claim follows from the previous proposition. The open problem is a kind of dual of the proposition, where we switch the right-down and the left-down branches.

Problem 2 Let $\mathcal{X}$ be either 'protoalgebraic', 'weakly algebraizable' or 'Rasiowa-p-implicational'. Is a logic (finitely) $\mathcal{X}$ implicational/implicative semilinear logic iff it is simultaneously an $\mathcal{X}$ semilinear logic and a (finitely) implicational/implicative logic?

If we inspect the logics in Example 1 showing the separation of the classes in the hierarchy of implicational logics, we notice that 2, 3, 4, 5.1 and 6 have, in fact, semilinear implications. Thus, they also show the separation of all the corresponding classes of implicational semilinear logics.

Theorem 18 *All classes of logics in the implicational semilinear hierarchy depicted in Figure 3 are mutually different.*

Recall that Corollary 2 together with Corollary 1 suggest how to show that a finitary weakly algebraizable logic is not semilinear: all we need to do is to find some relatively subdirectly irreducible reduced model and show that it has two incomparable filters. From Corollary 5 we conclude that many important logics (those which can be axiomatically extended to intuitionistic or affine linear logic) are not protoalgebraic semilinear logics and hence do not belong to the hierarchy of implicational semilinear logics. In particular, since the intuitionistic logic is Rasiowa-implicative we can conclude:

Proposition 16 *Let $\mathcal{X}$ be any class in the hierarchy of implicational logics. Then there is an $\mathcal{X}$ logic which is not an $\mathcal{X}$ semilinear logic.*

Finally, it will be interesting for the reader coming from the Fuzzy Logic world to locate some well known families of fuzzy logics in the hierarchy. The three main logics based on continuous t-norms (Łukasiewicz logic, Gödel-Dummett logic and Product logic) as well as the logic of all continuous t-norms BL have a primitive implication connective $\rightarrow$ which is known to be a Rasiowa semilinear implication in our terminology. Thus, they are Rasiowa-implicative semilinear logics. The same is true in general about left-continuous t-norm-based logics such as MTL and its t-norm based axiomatic extensions, and even for all *axiomatic extensions of* MTL (even those which are not complete with respect to a semantics of t-norms) because all of them are complete with respect to a subvariety of MTL-algebras generated by its linearly ordered members. Two incomparable superclasses of the class of axiomatic extensions of MTL have been considered in the literature (see Figure 4). On one hand, we have the so-called *($\triangle-$)core fuzzy logics* introduced in [17] as finitary protoalgebraic logics axiomatically expanding MTL (MTL$_\triangle$). On the other hand, we can consider the class of all *semilinear finitary extensions of MTL*. Their equivalent quasivariety semantics are the subquasivarieties of MTL-algebras generated by chains. Since such quasivarieties need not be varieties, we have that this class is strictly bigger than that of axiomatic extensions of MTL. Both of these incomparable classes are included in the class of *semilinear expansions of* MTL, and finally this class is included in the Rasiowa-implicative semilinear logics.

In the recent paper [22], the fuzzy logic UL based on uninorms instead of t-norms has been studied. It is an algebraizable logic without weakening, so it belongs to the class of algebraically implicative semilinear logics. We can consider the same structure of classes as above without weakening by replacing MTL for UL. See the resulting hierarchy of classes of semilinear logics in Figure 4.

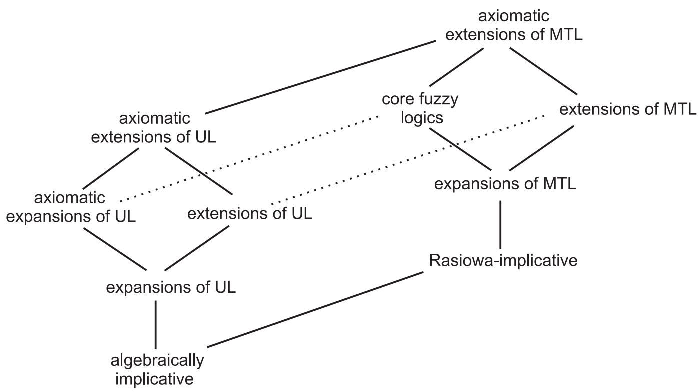

Fig. 4 Prominent classes of fuzzy logics on the top of the hierarchy of implicational semilinear logics. All of them are mutually different.

We realize that all of them lie on the top of our classification, above Rasiowa-implicative or algebraically implicative semilinear logics. But if we have succeeded in capturing an important characteristic of fuzzy logics by means of the definition of semilinear implication presented in this paper, then fuzzy logics are a much wider class than those studied so far. Thus we believe that the future research in the field will bring new significant examples of fuzzy logics throughout the whole hierarchy of implicational semilinear logics.

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