An alternative axiomatization for a fuzzy modal logic of preferences

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***Abstract*—In a recent paper, the authors have proposed an axiomatic system for a modal logic of gradual preference on fuzzy propositions that was claimed to be complete with respect to the intended semantics. Unfortunately, the completeness proof has a flaw, that leaves still open the question of whether the proposed system is actually complete. In this paper, we propose an alternative axiomatic system with a multi-modal language, where the original modal operators are definable and their semantics are preserved, and for which completeness results are proved.**

***Index Terms*—fuzzy preferences, fuzzy modal logic, completeness**

I. INTRODUCTION

Reasoning about preferences is a topic that has received a lot of attention in Artificial Intelligence since many years, see for instance [HGY12], [DHKP11], [Kac11]. Two main approaches to representing and handling preferences have been developed: the relational and the logic-based approaches.

In the classical setting, every preorder (i.e. reflexive and transitive) relation $R \subseteq W \times W$ on a set of alternatives W can be regarded as a (weak) preference relation by understanding $(a, b) \in R$ as denoting b *is not less preferred than* a. From R one can define three disjoint relations:

- the *strict preference* $P = R \cap R^d$,
- the *indifference relation* $I = R \cap R^t$, and
- the *incomparability relation* $J = R^c \cap R^d$.

where $R^d = \{(a, b) \in R : (b, a) \notin R\}$, $R^t = \{(a, b) : (b, a) \in R\}$ and $R^c = \{(a, b) \in R : (a, b) \notin R\}$. It is clear that P is a strict order (irreflexive, antisymmetric and transitive), I is an equivalence relation (reflexive, symmetric and transitive) and J is irreflexive and symmetric. The triple (P, I, J) is called a *preference structure*, where the initial weak preference relation can be recovered as $R = P \cup I$.

In the fuzzy setting, preference relations can be attached degrees (usually belonging to the unit interval $[0, 1]$) of fulfilment or strength, so they become *fuzzy relations*. A weak fuzzy preference relation on a set X will be now a fuzzy preorder $R : X \times X \to [0, 1]$, where $R(a, b)$ is interpreted as the degree in which b is at least as preferred as a. Given a t-norm $\odot$, a fuzzy $\odot$-preorder satisfies reflexivity ($R(a, a) = 1$ for each $a \in X$) and $\odot$-transitivity ($R(a, b) \odot R(b, c) \leq R(a, c)$ for each $a, b, c \in X$). The most influential reference is the book by Fodor and Roubens [FR94], that was followed by many other works like, for example [DBM07], [DBM10], [DMB04], [DBM08], [DGLM08]. In this setting, many questions have been discusssed, like e.g. the definition of the strict fuzzy order associated to a fuzzy preorder (see for example [Bod08a], [Bod08b], [BD08], [EGV18]).

The basic assumption in logical-based approaches is that preferences have structural properties that can be suitably described in a fomalized language. This is the main goal of the so-called *preference logics*, see e.g. [HGY12]. The first logical systems to reason about preferences go back to S. Halldén [Hal57] and to von Wright [vW63], [vW72], [Liu10]. Others related works are [EP06], [vBvOR05]. More recently van Benthem et al. in [vBGR09] have presented a modal logic-based formalization of representing and reasoning with preferences. In that paper the authors first define a basic modal logic with two unary modal operators $\Diamond^{\leq}$ and $\Diamond^{<}$, together with the universal and existential modalities, A and E respectively, and axiomatize them. Using these primitive modalities, they consider several (definable) binary modalities to capture different notions of preference relations on classical propositions, and show completeness with respect to the intended preference semantics. Finally they discuss their systems in relation to von Wright axioms for *ceteris paribus* preferences [vW63]. On the other hand, with the motivation of formalising a comparative notion of likelihood, Halpern studies in [Hal97] different ways to extend preorders on a set X to preorders on subsets of X and their associated strict orders. He studies their properties and relations among them, and he also provides an axiomatic system for a *logic of relative likelihood*, that is proved to be complete with respect to what he calls *preferential structures*, i.e. Kripke models with preorders as accessibility relations. All these works relate to the classical (modal) logic and crisp preference (accessibility) relations.

In the fuzzy setting, as far as the authors are aware, there are not many formal logic-based approaches to reasoning with fuzzy preference relations, see e.g. [BEFG01]. More recently, in the first part of [EGV18] we studied and characterized

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different forms to define fuzzy relations on the set $\mathcal{P}(\mathcal{W})$ of subsets of W, from a fuzzy preorder on W, in a similar way to the one followed in [Hal97], [vBGR09] for classical preorders, while in the second part we have semantically defined and axiomatized several two-tiered graded modal logics to reason about different notions of preferences on crisp propositions, see also [EGV17]. On the other hand, in [VEG17a] we considered a modal framework over a many-valued logic with the aim of generalizing Van Benthem et al.'s modal approach to the case of both fuzzy preference accessibility relations and fuzzy propositions. To do that, we first extended the many-valued modal framework for only a necessity operator $\Box$ of [BEGR11], by defining an axiomatic system with both necessity and possibility operators $\Box$ and $\Diamond$ over the same class of models. Unfortunately, in the last part of that paper, there is a mistake in the proof of Theorem 3 (particularly, equation (4)). This leaves open the question of properly axiomatizing the logic of graded preferences defined there.

In this paper we address this problem, and propose an alternative approach to provide a complete axiomatic system for a logic of fuzzy preferences. Namely, given a finite MTL-chain $\mathbf{A}$ (i.e. a finite totally ordered residuated lattice) as set of truth values, and given an A-valued preference Kripke model (W, R, e), with R a fuzzy preorder valued on A, we consider the a-cuts R_a of the relation R for every $a \in A$, and for each a-cut R_a, we consider the corresponding modal operators $\Box_a, \Diamond_a$. These operators are easier to be axiomatized, since the relations R_a are not fuzzy any longer, they are a nested set of classical (crisp) relations. The good news is that, in the our rich (multi-modal) logical framework, we can show that the original modal operators $\Box$ and $\Diamond$ are definable, and vice-versa if we expand the logic with Monteiro-Baaz's Δ operator. So we obtain a different, but equivalent, system where the original operators can be properly axiomatized in an indirect way through the graded operators.

The paper is structured as follows. After this introduction, in Section II we present the multi-modal language and the intended semantics given by *graded preference Kripke models*, which allows the formalization of different notions dealing with preferences taking values in some arbitrary MTL-chain $\mathbf{A}$. In Section III, we discuss different possibilities to formalize notions of preferences on fuzzy propositions in preference Kripke models. In Section IV we will exhibit a complete axiomatization of an alternative preference logic that is not, however, equivalent to the one from [VEG17a], since the language is intrinsically different. Nevertheless, we will see in Section V how, by the addition to the logic of the so-called Monteiro-Baaz Δ operation, we can also provide an axiomatization of the original logic of graded preference models pursued in [VEG17a]. We finish with some conclusions and open problems.

II. A MULTI-MODAL PREFERENCE LOGIC: LANGUAGE AND SEMANTICS

Let us begin by defining the formal language of our underlying many-valued propositional setting. Let $\mathbf{A} = (A, \wedge, \vee, \odot, \to, 0, 1)$ be a *finite* and linearly ordered (bounded, integral, commutative) residuated lattice (equivalently, a finite MTL-chain) [GJKO07], and consider its canonical expansion $\mathbf{A^c}$ by adding a new constant $\overline{a}$ for every element $a \in A$ (canonical in the sense that the interpretation of $\overline{a}$ in $\mathbf{A^c}$ is a itself). A negation operation $\neg$ can always be defined as $\neg x = x \to 0$.

The logic associated with $\mathbf{A^c}$ will be denoted by $\mathbf{\Lambda(A^c)}$, and its logical consequence relation $\models_{\mathbf{A^c}}$ is defined as follows: for any set $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}$ of formulas built in the usual way from a set of propositional variables $\mathcal{V}$ in the language of residuated lattices (we will use the same symbol to denote connectives and operations), including constants $\{\overline{a} : a \in A\}$,

- $\Gamma \models_{\mathbf{A^c}} \phi$ if, and only if,
  $\forall h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A}^c)$, if $h[\Gamma] \subseteq \{1\}$ then $h(\phi) = 1$,

where $\mathrm{Hom}(\mathbf{Fm}, \mathbf{A}^c)$ denotes the set of evaluations of formulas on $\mathbf{A^c}$.

Lifting to the modal level, we extend the propositional language by graded modal operators $\Box_a, \Diamond_a$, one pair for each element a of the algebra $\mathbf{A}$. We let the set $\mathbf{MFm}$ of *multi-modal formulas* defined as usual from a set $\mathcal{V}$ of propositional variables, residuated lattice operations $\{\wedge, \vee, \odot, \to\}$, truth constants $\{\overline{a} : a \in A\}$, and modal operators $\{\Box_a, \Diamond_a : a \in A\}$.

We are now ready to introduce $\mathbf{A}$-valued preference Kripke models.

Definition II.1. *An* $\mathbf{A}$*-preference model* is a triple $\mathfrak{M} = \langle W, R, e\rangle$ such that

- W is a set of worlds,
- $R\colon W \times W \to A$ is an A-valued fuzzy pre-order, i.e. a reflexive and $\odot$-transitive A-valued binary relation between worlds, and
- $e\colon W \times \mathcal{V} \to A$ is a world-wise $\mathbf{A}$-evaluation of variables. This evaluation is uniquely extended to formulas of $\mathbf{MFm}$ by using the operations in $\mathbf{A}$ for what concerns propositional connectives, and letting for each $a \in A$,

$$\begin{aligned} e(v, \Box_a\varphi) &= \bigwedge_{w : v \preceq_a w} \{e(w, \varphi)\} \\ e(v, \Diamond_a\varphi) &= \bigvee_{w : v \preceq_a w} \{e(w, \varphi)\} \end{aligned}$$

  where $v \preceq_a w$ stands for $R(v, w) \geq a$.

We will denote by $\mathbb{P}_{\mathbf{A}}$ the class of $\mathbf{A}$-preference models. Given an $\mathbf{A}$-preference model $\mathfrak{M} \in \mathbb{P}_{\mathbf{A}}$ and $\Gamma \cup \{\varphi\} \subseteq \mathbf{MFm}$, we write $\Gamma \Vdash_{\mathfrak{M}} \varphi$ whenever for any $v \in W$, if $e(v, \gamma) = 1$ for all $\gamma \in \Gamma$, then $e(v, \varphi) = 1$ too. Analogously, we write $\Gamma \Vdash_{\mathbb{P}_{\mathbf{A}}} \varphi$ whenever $\Gamma \Vdash_{\mathfrak{M}} \varphi$ for any $\mathfrak{M} \in \mathbb{P}_{\mathbf{A}}$.

We will denote by differentiated names some particular definable modal operators that enjoy a special meaning in our models. Namely:

- $\Box\varphi := \bigwedge_{a \in A} \overline{a} \to \Box_a\varphi$ and $\Diamond\varphi := \bigvee_{a \in A} \overline{a} \odot \Box_a\varphi$.
  It is easy to check that the evaluation of these operators in a preference model as defined here, coincides with the

usual one for fuzzy Kripke models, i.e.,

$$e(v, \Box\varphi) = \bigwedge_{w\in W} \{R(v,w) \to e(w,\varphi)\}$$

$$e(v, \Diamond\varphi) = \bigvee_{w\in W} \{R(v,w) \odot e(w,\varphi)\}$$

- $A\varphi := \Box_0\varphi$ and $E\varphi := \Diamond_0\varphi$.

  Again, it is easy to see that these operators coincide with the global necessity and possibility modal operators respectively, i.e.,

$$e(v, A\varphi) = \bigwedge_{w\in W} \{e(w,\varphi)\}, \quad e(v, E\varphi) = \bigvee_{w\in W} \{e(w,\varphi)\}.$$

III. Modeling fuzzy preferences on propositions

The preference models introduced above are a very natural setting to formally address and reason over graded or fuzzy preferences over non-classical contexts. They are similar to the (classical) preference models studied by van Benthem et. al in [vBGR09], but offering a lattice of values (and so, a many-valued framework) where to evaluate both the truth degrees of formulas and the accessibility (preference) relation. The latter can be naturally interpreted as a graded preference relation between possible worlds or states (assignments of truth-values to variables). The question is then how to lift a (fuzzy) preference relation $\leq$ on worlds to (fuzzy) preference relations among formulas.

In the classical case, for instance in [vBGR09], [EGV18] the following six extensions are considered, where $[\varphi]$ and $[\psi]$ denote the set of models of propositions φ and ψ respectively:

- $\varphi \leq_{\exists\exists} \psi$ iff $\exists u \in [\varphi], v \in [\psi]$ such that $u \leq v$
- $\varphi \leq_{\exists\forall} \psi$ iff $\exists u \in [\varphi]$, such that $\forall v \in [\psi]$, $u \leq v$
- $\varphi \leq_{\forall\exists} \psi$ iff $\forall u \in [\varphi]$, $\exists v \in [\psi]$ such that $u \leq v$
- $\varphi \leq_{\forall\forall} \psi$ iff $\forall u \in [\varphi]$ and $v \in [\psi]$, $u \leq v$
- $\varphi \leq_{\exists\forall 2} \psi$ iff $\exists v \in [\psi]$, such that $\forall u \in [\varphi]$, $u \leq v$
- $\varphi \leq_{\forall\exists 2} \psi$ iff $\forall v \in [\psi]$, $\exists u \in [\varphi]$ such that $u \leq v$

However, not all these extensions can be expressed in our framework. For instance, we can express the orderings $\leq_{\exists\exists}$ and $\leq_{\forall\forall}$ as follows:

- $\varphi \leq_{\exists\exists} \psi := E(\varphi \wedge \Diamond\psi)$
- $\varphi \leq_{\forall\exists} \psi := A(\varphi \to \Diamond\psi)$

Some others would need to consider the inverse order $\geq$ of $\leq$ in the models or to assume the order $\leq$ be total, and some other are not just expressible (see [vBGR09]). On the other hand, not all the extensions above are also equally reasonable, for instance some of them are not even preorders. This is not the case of $\leq_{\forall\exists}$ and $\leq_{\forall\exists 2}$, that are indeed preorders.

In the fuzzy case, the formulas

$$E(\varphi \wedge \Diamond\psi),$$
$$A(\varphi \to \Diamond\psi)$$

make full sense as a fuzzy generalizations of the $\leq_{\exists\exists}$ and $\leq_{\forall\exists}$ preference orderings respectively, and moreover, as shown in [VEG17a], the expression $A(\varphi \to \Diamond\psi)$ models a fuzzy preorder in formulas (i.e. it satisfies reflexivity and $\odot$-transitivity).

Using the graded modalities $\Diamond_a$, one could also consider other intermediate extensions like

$$E(\varphi \wedge \Diamond_a\psi),$$
$$A(\varphi \to \Diamond_a\psi)$$

which would correspond to the fuzzy extensions of the following preference orderings $\leq^a_{\exists\exists}$ $\leq^a_{\forall\exists}$ on crisp propositions defined from the a-cut of the fuzzy preorder R:

- $\varphi \leq^a_{\exists\exists} \psi$ iff $\exists u \in [\varphi]$, $\exists v \in [\psi]$ such that $R(u,v) \geq a$.
- $\varphi \leq^a_{\forall\exists} \psi$ iff $\forall u \in [\varphi]$, $\exists v \in [\psi]$ such that $R(u,v) \geq a$.

Indeed, given an $\mathbf{A}$-valued preference model $\mathfrak{M} = \langle W, R, e\rangle$, one can define the following fuzzy preference relations on formulas:

- $\varphi \preceq^a_{\exists\exists} \psi$ iff there are worlds $v, w \in W$ such that $R(v,w) \geq a$ and $e(v,\varphi) \leq e(w,\psi)$
- $\varphi \preceq^a_{\forall\exists} \psi$ iff for each world $v \in W$, there is a world $w \in W$ such that $R(v,w) \geq a$ and $e(v,\varphi) \leq e(w,\psi)$.

Then, it is not difficult to check that

$$\Vdash_{\mathfrak{M}} E(\varphi \wedge \Diamond_a\psi) \text{ iff } \varphi \preceq^a_{\exists\exists} \psi$$
$$\Vdash_{\mathfrak{M}} A(\varphi \to \Diamond_a\psi) \text{ iff } \varphi \preceq^a_{\forall\exists} \psi\ .$$

So, we think our many-valued logical framework is expressive enough to capture many notions of (fuzzy) preferences among formulas. In the next section we provide an axiomatisation for this fuzzy multi-modal preference logic.

IV. Axiomatizing fuzzy preference models

In [VEG17a], we proposed the following axiomatic system $\mathsf{P}_{\mathbf{A}}$, in the language only with $\Box$ and $\Diamond$ modal operators (i.e. without the $\Box_a$'s and $\Diamond_a$'s):

- The axioms and rules of the minimal modal logic $\mathsf{BM}_{\mathbf{A}}$ for the pairs $(\Box, \Diamond)$ and (A, E) of modal operators (see [VEG17a, Def. 2])
- T: $\Box\varphi \to \varphi, \varphi \to \Diamond\varphi, \quad A\varphi \to \varphi, \varphi \to E\varphi$
- 4: $\Box\varphi \to \Box\Box\varphi, \Diamond\Diamond\varphi \to \Diamond\varphi,\ A\varphi \to AA\varphi, EE\varphi \to E\varphi$
- B: $\varphi \to AE\varphi$
- The *inclusion* axioms: $\mathsf{A}\varphi \to \Box\varphi$, $\Diamond\varphi \to \mathsf{E}\varphi$

In [VEG17a, Th. 3], this system was claimed to be complete with respect to the class $\mathbb{P}_{\mathbf{A}}$ of preference models. Unfortunately, we have discovered there is a flaw at the end of the proof, so the claim of the theorem remains unproved. In this section we remedy this problem by considering an alternative axiomatic system, based on the use of the graded modalities $\Box_a$ and $\Diamond_a$, for $a \in A$, introduced in Section II.

To this end, we introduce next the axiomatic system $\mathsf{mM}_{\mathbf{A}}$ defined by the following axioms and rules:

1) For each $a \in A$,
   - Axioms of minimum modal logic $\mathsf{BM}_{\mathbf{A}}$ for each pair of operators $(\Box_a, \Diamond_a)$ (see [VEG17a, Def. 2])
2) For each $a \in A$, the axiom
   - C_a: $\Box_a(\overline{k} \vee \varphi) \to \overline{k} \vee \Box_a\varphi$
3) For each $a, b \in A$, axioms K, T and 4:

- $K_a\colon \Box_a(\varphi \to \psi) \to (\Box_a\varphi \to \Box_a\psi)$
- $T_a\colon \Box_a\varphi \to \varphi, \qquad \varphi \to \Diamond_a\varphi$
- $4_{a,b}\colon \Box_{a\odot b}\varphi \to \Box_a\Box_b\varphi, \qquad \Diamond_a\Diamond_b\varphi \to \Diamond_{a\odot b}\varphi$

4) For each $a \leq b$, nestedness axioms:
   - $\Box_a\varphi \to \Box_b\varphi, \quad \Diamond_b\varphi \to \Diamond_a\varphi$

5) For $a = 0$, axiom
   - $B_0\colon \varphi \to \Box_0\Diamond_0 A$

6) Rules: Modus Ponens and the necessitation for $\Box_0$:[1]
   - from φ derive $\Box_0\varphi$

Letting $\vdash_{\mathsf{mM}_{\mathbf{A}}}$ be the consequence relation of the previous axiomatic system defined as usual, we can show that it is indeed complete with respect to our intented semantics given by the class of preference structures $\Vdash_{\mathbb{P}_{\mathbf{A}}}$. Formally,

Theorem IV.1. *For any* $\Gamma, \varphi \subseteq \mathbf{MFm}$*,*

$\Gamma \vdash_{\mathsf{mM}_{\mathbf{A}}} \varphi$ *if and only if* $\Gamma \Vdash_{\mathbb{P}_{\mathbf{A}}} \varphi$*.*

Proof. Soundness (left to right direction) is easy to check. For what concerns completeness (right to left direction), we can define a canonical model

$$\mathfrak{M}^c = (W^c, \{R^c_a\}_{a\in A}, e^c)$$

with a set of crisp accessibility relations as follows, where $Th(\mathsf{mM}_{\mathbf{A}}) = \{\varphi : \vdash_{\mathsf{mM}_{\mathbf{A}}} \varphi\}$ denotes the set of theorems of $\mathsf{mM}_{\mathbf{A}}$:

- $W^c = \{v \in Hom(\mathbf{MFm}, \mathbf{A})\colon v(Th(\mathsf{mM}_{\mathbf{A}})) = \{1\}\}$,
- $R^c_a(v, w)$ if and only if $v(\Box_a\varphi) = 1 \Rightarrow w(\varphi) = 1$ for all $\varphi \in \mathbf{MFm}$,
- $e^c(v, p) = v(p)$, for any propositional variable p.

It is clear (since the only modal inference rules affects only theorems of the logic) that if $\Gamma \not\vdash_{\mathsf{mM}_{\mathbf{A}}} \varphi$, then there is $v \in W^c$ such that $v(\Gamma) \subset \{1\}$ and $v(\varphi) < 1$. It is then only necessary to prove that the evaluation in the model can be defined in that way, namely, to prove the corresponding Truth Lemma, which follows from [BEGR11] and [VEG17a], i.e., for each formula $\varphi \in \mathbf{MFm}$ and each $v \in W^c$, it holds that

$$e^c(v, \Box_a\varphi) = \bigwedge_{R^c_a(v,w)} w(\varphi) \text{ and } e^c(v, \Diamond_a\varphi) = \bigvee_{R^c_a(v,w)} w(\varphi).$$

The nestedness axioms allow us to easily prove that for any $a \leq b \in A$, it holds that $R^c_b \subseteq R^c_a$. Consider then the fuzzy relation R^c defined by

$$R^c(v, w) = \max\{a \in A\colon R^c_a(v, w)\}.$$

It is clear that $R^c(w, v) \geq a$ if and only if $R^c_a(v, w)$. Then, the truth lemma for the original model directly implies both

$$e^c(v, \Box_a\varphi) = \bigwedge_{w\in W^c, R^c(v,w)\geq a} w(\varphi),$$

$$e^c(v, \Diamond_a\varphi) = \bigvee_{w\in W^c, R^c(v,w)\geq a} w(\varphi).$$

It follows from axioms T_a that each R^c_a is reflexive, and so, R^c is a reflexive relation as well. Moreover, from axioms $4_{a,b}$, we get that R^c is $\odot$-transitive. The only remaining step is to prove is that we can obtain an equivalent model (in the sense of preserving the truth-values of formulas) in which R^c_0 is the total relation (in order to really get that $\Box_0$ and $\Diamond_0$ are global modalities). Observe that in the model defined above, thanks to axioms $T_0, 4_{0,0}$ and B_0, R^c_0 can be proven to be an equivalence relation, even though it is not necessarily the case that $R^c_0 = W^c \times W^c$. Nevertheless, since $R^c_b \subseteq R^c_0$ for all $b \in A$, for any arbitrary $v \in W^c$, we can define the model $\mathfrak{M}^c_v$ from $\mathfrak{M}^c$ by restricting the universe to $W^c_v = \{u \in W^c\colon R^c_0(v, u)\}$ and get that, for any $u \in W^c_v$ and any formula $\varphi \in \mathbf{MFm}$,

$$e^c(u, \varphi) = e^c_v(u, \varphi).$$

All the previous considerations allow us to prove that if $\Gamma \not\vdash_{\mathsf{mM}_{\mathbf{A}}} \varphi$ there is $v \in W^c_v$ such that $e^c_v(v, \Gamma) \subseteq 1$ and $e^c_v(v, \varphi) < 1$. Given that the model $\mathfrak{M}^c_v$ defined above is indeed an $\mathbf{A}$-preference model, this concludes the completeness proof. □

V. Closing the loop: from graded to fuzzy modalities

In the previous section, we have seen that we have been able to provide a complete axiomatic system $\mathsf{mM}_{\mathbf{A}}$ for the graded preference modalities $\Box_a$'s and $\Diamond_a$'s, and in Section II we have seen that the original fuzzy modalities $\Box$ and $\Diamond$ can be expressed from them. Thus, the system $\mathsf{mM}_{\mathbf{A}}$ can be considered in fact as a sort of indirect axiomatization of the modalities $\Box$ and $\Diamond$ as well. In this section, generalising an approach introduced in [BEGR09], we will see that, by enriching our language with the well-known Monteiro-Baaz Δ connective (see e.g. [Háj98]), the graded modalities $\Box_a, \Diamond_a$ can also be expressed in terms of the original modal operators $\Box$ and $\Diamond$. Surprisingly enough we can do it using only the $\Diamond$ operator, while it is not clear using only $\Box$ would suffice.

Recall that the Monteiro-Baaz Δ operation over a linearly ordered MTL-chain $\mathbf{A}$ is the operation defined as

$$\Delta(a) = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$

for all $a \in A$.

In the following, we will write $\varphi \equiv \psi$ to denote that φ and ψ are logically equivalent in the class of preferece models $\mathbb{P}_{\mathbf{A}}$. We will also denote by $\varphi \approx \overline{b}$ the formula $\Delta(\varphi \leftrightarrow \overline{b})$.

Lemma V.1.

$$\Box_a\varphi \equiv \bigwedge_{b\in A} (\Delta(\overline{a} \to \Diamond(\varphi \approx \overline{b})) \to \overline{b})$$

$$\Diamond_a\varphi \equiv \bigvee_{b\in A} (\Delta(\overline{a} \to \Diamond(\varphi \approx \overline{b}))\&\overline{b})$$

Proof. As in [BEGR09] we can check that

$$e(v, \Diamond(\varphi \approx \overline{b})) = \bigvee_{e(w,\varphi)=b} R(v, w).$$

[1]Observe that, together with the nestedness axioms, this rule implies the necessitation rule for each $\Box_a$.

Then $e(v, \Delta(\overline{a} \to \lozenge(\varphi \approx \overline{b}))) = \Delta(a \to \bigvee_{e(w,\varphi)\approx b} R(v,w))$, and thus

$$e(v, \Delta(\overline{a} \to \lozenge(\varphi \approx \overline{b}))) = \begin{cases} 1, & \text{if } a \leq \bigvee_{e(w,\varphi)=b} R(v,w) \\ 0, & \text{otherwise} \end{cases}.$$

Letting $S = \{b \in A \colon a \leq \bigvee_{e(w,\varphi)=b} R(v,w)\}$, the previous trivially implies both that

$$e(v, \Delta(\overline{a} \to \lozenge(\varphi = \overline{b})) \to \overline{b}) = \begin{cases} b, & \text{if } b \in S \\ 1, & \text{otherwise} \end{cases}$$

$$e(v, \Delta(\overline{a} \to \lozenge(\varphi = \overline{b}))\&\overline{b}) = \begin{cases} b, & \text{if } b \in S \\ 0, & \text{otherwise} \end{cases}.$$

Moreover, it is easy to see that

$$\{b \in A \colon a \leq \bigvee_{e(w,\varphi)=b} R(v,w)\} = \{e(w,\varphi) \colon a \leq Rvw\}.$$

Then, we have

$e(v, \bigwedge_{b\in A}(\Delta(\overline{a} \to \lozenge(\varphi = \overline{b})) \to \overline{b})) = \bigwedge S = \bigwedge_{a\leq R(v,w)} e(w,\varphi) = e(v, \Box_a\varphi)$

and $e(v, \bigvee_{b\in A}(\Delta(\overline{a} \to \lozenge(\varphi = \overline{b}))\&\overline{b})) = \bigvee S = \bigvee_{a\leq R(v,w)} e(w,\varphi) = e(v, \lozenge_a\varphi)$, concluding the proof. □

It is then the case that it is possible to provide an axiomatization for the fragment with only $\Box, \lozenge, A$ and E of the logic $\Vdash_{\mathbb{P}_\mathbf{A}}$ plus Δ. First, it is easy to provide an axiomatic system for the whole logic $\Vdash_{\mathbb{P}_\mathbf{A}}$ plus Δ by adding to $\mathsf{mM}_\mathbf{A}$ an axiomatization for Δ (see eg. [Háj98], [VEG17b]) and the interaction axioms

$$\Delta\Box_a\varphi \to \Box_a\Delta\varphi.$$

From here, it is clear that we can use the interdefinability of $\Box_a, \lozenge_a$ from $\lozenge$ proven above, and obtain in that way an axiomatic system complete with respect to the intended semantics.

This system is, however, quite more involved than the one presented in [VEG17a] (that did not achieve completeness with respect to its intended semantics). An open problem for future works is to study possible simplifications of this axiomatization, since the $(\Box, \lozenge, A, E)$-fragment is possibly the best suited to formalise graded preference relations while maintaining a lower level of elements in the language (and so, probably a lower complexity level).

As a side result, the previous characterization allows us to get a definition of the $\Box$ operation in terms of the $\lozenge$ very different from the usual one arising in classical modal logic. In particular, we get the following result.

Lemma V.2.

$$\Box\varphi \equiv \bigwedge_{a\in A}\bigwedge_{b<a\in A} \Delta(\overline{a} \to \lozenge(\varphi \approx \overline{b})) \to (\overline{a \to b})$$

Proof. We know by definition that

$$\Box\varphi \equiv \bigwedge_{a\in A} \overline{a} \to \Box_a\varphi.$$

Then, using the previously proven equivalences, we prove the lemma by the following chain of equalities

$$\begin{aligned} \Box\varphi &\equiv \bigwedge_{a\in A}(\overline{a} \to \bigwedge_{b\in A}(\Delta(\overline{a} \to \lozenge(\varphi \approx \overline{b})) \to \overline{b})) \\ &\equiv \bigwedge_{a\in A}\bigwedge_{b\in A} \overline{a} \to (\Delta(\overline{a} \to \lozenge(\varphi \approx \overline{b})) \to \overline{b}) \\ &\equiv \bigwedge_{a\in A}\bigwedge_{b\in A} \Delta(\overline{a} \to \lozenge(\varphi \approx \overline{b})) \to (\overline{a} \to \overline{b}) \\ &\equiv \bigwedge_{a\in A}\bigwedge_{b<a\in A} \Delta(\overline{a} \to \lozenge(\varphi \approx \overline{b})) \to (\overline{a \to b}) \end{aligned}$$

□

VI. Conclusions and Ongoing work

The aim of this work is to provide a formal framework generalising the treatment of preferences in the style of eg. [vBGR09] to a fuzzy context. We have presented an axiomatic system encompassing reflexive and transitive modalities plus global operators, that is shown to be the syntactical counterpart of many-valued Kripke models with (reflexive and transitive) graded (weak) preference relations between possible worlds or states. It is based on considering the cuts of the relations over the elements of the algebra of evaluation, solving in this way some problems arising from [VEG17a], for what concerns systems extended with the projection connective Δ. This logical framework stands towards the use of modal many-valued logics in the representation and management of graded preferences, in the same fashion that (classical) modal logic has served in the analogous Boolean preference setting.

The generalization of the previous logical system to cases when strict preferences are taken into account is part of ongoing work. The addition of those operators would allow a richer axiomatic definition of preference relations between formulas, in the sense of Section III. Moreover, further study of the introduced preference models should be pursued towards the formalisation of particular notions like indifference or incomparability, and aiming towards the incorporation of these systems in graded reasoners or recommender systems.

On the other hand, the study of the previous systems over other classes of algebras of truth-values (e.g. including infinite algebras like those defined on the real unit interval $[0, 1]$ underlying Łukasiewicz, Product or Gödel fuzzy logics) is also of great interest, both from a theoretical point of view and towards the modelization of situations needing of continuous sets of values.

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