Sure-Wins Under Coherence: A Geometrical Perspective

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Abstract. In this contribution we will present a generalization of de Finetti's betting game in which a gambler is allowed to buy and sell unknown events' betting odds from more than one bookmaker. In such a framework, the sole coherence of the books the gambler can play with is not sufficient, as in the original de Finetti's frame, to bar the gambler from a sure-win opportunity. The notion of *joint coherence* which we will introduce in this paper characterizes those coherent books on which sure-win is impossible. Our main results provide geometric characterizations of the space of all books which are jointly coherent with a fixed one. As a consequence we will also show that joint coherence is decidable.

Keywords: Coherence · Sure-win · De Finetti's betting game · Geometry of coherence · Decidability

1 Introduction

The logical foundations of subjective probability theory find in the work of de Finetti, started with [1] and culminated with [2], a solid ground which, especially in the last years has been the object of a deep study and several generalizations (see for instance [6,8,9]).

To set the scene of de Finetti's approach to probability, let us consider a bookmaker who fixes a finite number of events $e_1, \ldots, e_k$ which are represented by sentences of classical propositional logic and a book β on them, i.e., a complete assignment $\beta\colon \{e_1, \ldots, e_k\} \to [0,1]$ of betting odds $\beta(e_i) = \beta_i$. In order to bet on the events, a gambler chooses *stakes* $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$, one for each event, and pays the bookmaker the amount $\sigma_i \cdot \beta_i$ for each e_i (with $i \in \{1, \ldots, k\}$). Note that σ_i may be negative, in which case, paying $\sigma_i \cdot \beta_i$ means receiving $-\sigma_i \cdot \beta_i$, as money transfer is orientated from the gambler to the bookmaker. When a (classical propositional) valuation h determines the truth-value of each e_i, the gambler

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gains σ_i if $h(e_i) = 1$, i.e., the event e_i has actually occurred, and 0 otherwise[1]. The book β is said to be *coherent* if there is no choice of stakes $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$ which forces gambler's balance not to be strictly positive under every valuation h. In other words, coherent books are those which bar the gambler from a *sure-win* opportunity, i.e., a strictly positive gain, independently of the truth-value of the events involved.

A slightly more general, yet completely realistic, situation is the one in which a gambler is allowed to place her stakes on two or more coherent books[2]. As we are going to point out in the present contribution, in such a case the sole coherence of the books the gambler decides to play with is not sufficient to bar her from a sure-win. Consider, for instance, the following very elementary example: Two bookmakers B_1 and B_2 fix betting odds to the events "Heads" and "Tails" of the typical coin-tossing game according to the following scheme: B_1 assigns 1/2 to both "Heads" and "Tails", while B_2 assigns 1/3 to "Heads" and 2/3 to "Tails". Notwithstanding the coherence of the two assignments, buying "Tails" from B_1 for $\sigma_1 = 1$ euro and "Heads" from B_2 for $\sigma_2 = 1$ euro leads the gambler to a sure-win.

Situations of this kind have been studied by Nau and his collaborators (see [10,11]) in the context of noncooperative games. There, given a set of players, a (conjoined) strategy is said to be *jointly coherent*, if it does not expose the group to arbitrage. In other words, "players who subscribe to the standard of joint coherence, are those who do not let themselves be used *collectively* as a money pump" (see [10, p. 426]).

In this paper we deepen this research line sticking within de Finetti's original betting framework and we move the first steps towards a logico-mathematical formalization of those coherent books which avoid sure-win (i.e., arbitrage) opportunities. They will be called *jointly coherent* books. In particular, we will give an answer to the following question: given a coherent book, which other coherent books, if any, bar a malicious gambler from a sure-win opportunity? More precisely, for every coherent book β, we will provide a geometric characterization of the set of all (coherent) books which are jointly coherent with it.

This paper is organized as follows: in the next section we will recall, in a more precise way, de Finetti's coherence criterion, de Finetti's theorem and, in particular, we will focus on its geometric version. In Sect. 3, we will formally introduce the concepts of sure-win and joint coherence of a book. In Sect. 4 we will study the geometry of joint coherence and provide the main result of the paper.

2 Preliminaries

Along this paper we fix a finite set of events that we denote by $\Phi = \{e_1, \ldots, e_k\}$. As we recalled in Sect. 1, a book β on Φ is *coherent* if for each choice of stakes

[1] For details, see for instance [5].

[2] The maybe unrealistic assumption which sees the bookmakers to consider exactly the same set of events can indeed be relaxed with an inessential modification.

$\sigma_1, \ldots, \sigma_k \in \mathbb{R}$ there exists at least a possible world h such that gambler's total balance $\sum_{i=1}^{k} \sigma_i(h(e_i) - \beta(e_i)) \leq 0$.

A book β is said to be *incoherent* if it is not coherent. Obviously, incoherent books are those which allow the gambler for a sure-win opportunity.

Recall that a finitely additive *probability measure* over a Boolean algebra $\mathbf{A}$ is a map $P\colon \mathbf{A} \to [0,1]$ such that $P(1) = 1$ and $P(a \vee b) = P(a) + P(b)$, provided that $a \wedge b = 0$. De Finetti's theorem then states that a book $\beta : \Phi \to [0,1]$ is coherent iff it extends to a finitely additive probability measure over the Boolean algebra generated by the events in Φ, denoted by $\mathbf{B}_\Phi$, see [2].

This result admits an equivalent geometrical formulation (see [12]), which we are going to recall here. Any finite set of events $\Phi = \{e_1, \ldots, e_k\}$ determines a polytope in $[0,1]^k$ by the following construction. Let $h_1, \ldots, h_t$ be the homomorphisms from $\mathbf{B}_\Phi$ to the two element Boolean algebra $\mathbf{2} = \langle \{0,1\}, \wedge, \vee, \neg, 0 \rangle$. For every $j = 1, \ldots, t$ let q_j be the point of $\{0,1\}^k$

$$q_j = (h_j(e_1), \ldots, h_j(e_k)). \tag{1}$$

Finally, let $\mathscr{C}_\Phi$ be the polytope of $[0,1]^k$ generated by $q_1, \ldots, q_t$:

$$\mathscr{C}_\Phi = \mathrm{co}(\{q_j \mid j = 1, \ldots, t\}),$$

where co stands for *convex hull*.

A book $\beta : \Phi \to [0,1]$ determines a point $\beta = (\beta(e_1), \ldots, \beta(e_k)) \in [0,1]^k$. The following result, whose proof can be found in [6, Lemma 6.3] and [12, Theorem 2], provides a geometric characterization of coherent books.

Theorem 1. *For a book $\beta\colon \Phi \to [0,1]$ the following conditions are equivalent:*

1. *β is coherent;*
2. *$\beta \in \mathscr{C}_\Phi$.*

The construction illustrated above is better visualized towards an example.

Example 1. Consider $\Phi = \{e_1, e_2\}$, where $e_1 = p$ and $e_2 = p \vee q$ in a language with two propositional variables p, q. Following de Finetti [1], the above mentioned events may be thought as of referring to a horse race: the atomic event p can be interpreted as "Horse number 1 is the winner", while $p \vee q$ could stand for "An Italian horse is winning", under the assumption that only two horses are Italian (and one of them is actually the number 1).

The algebra $\mathbf{B}_\Phi$ counts of four homomorphisms to $\mathbf{2}$, namely those maps $h_1, h_2, h_3, h_4\colon \{p, q\} \to \{0,1\}$ which assign, respectively, to p and q the values $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. Therefore, we obtain the following points $q_1, \ldots, q_4 \in \mathbb{R}^2$:

$$\begin{aligned}
q_1 &= (h_1(e_1), h_1(e_2)) = (h_1(p), h_1(p \vee q)) = (0,0);\\
q_2 &= (h_2(e_1), h_2(e_2)) = (h_2(p), h_2(p \vee q)) = (0,1);\\
q_3 &= (h_3(e_1), h_3(e_2)) = (h_3(p), h_3(p \vee q)) = (1,1);\\
q_4 &= (h_4(e_1), h_4(e_2)) = (h_4(p), h_4(p \vee q)) = (1,1).
\end{aligned}$$

Since $q_3 = q_4$, we have:

$$\mathscr{C}_\Phi = \mathrm{co}(\{q_1, q_2, q_3\}) = \mathrm{co}(\{(0,0),(0,1),(1,1)\}),$$

depicted as in Fig. 1 below. Theorem 1 tells us that a book β is coherent if and only if it is a convex combination of q_1, q_2 and q_3.

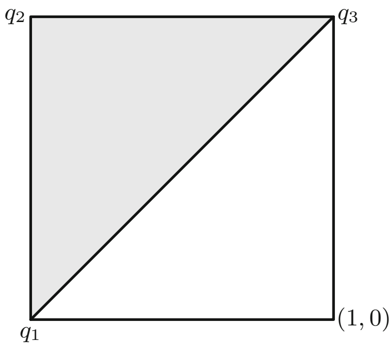

Fig. 1. The convex set $\mathscr{C}_\Phi$ (in gray) of all coherent books on events $e_1 = p$ and $e_2 = p \vee q$.

3 Sure-Wins and Jointly Coherent Books

As mentioned in the Introduction, we are interested in situation where a gambler has the opportunity of betting on two (or ideally more) books over the same set of events. The gambler's concept of *sure-win* opportunity becomes wider and it is made precise in the following.

Definition 1. *Let β_1, β_2 be coherent books on the set of events $\Phi = \{e_1, \ldots, e_k\}$. We say that a gambler has a* sure-win opportunity on β_1, β_2 *if there exists a total map $\xi : \{1, \ldots, k\} \to \{1, 2\}$ such that, the book*

$$\beta : e_i \mapsto \beta_{\xi(i)}(e_i)$$

is incoherent. If such function ξ does not exist, then β_1 and β_2 are said to be jointly coherent.

Therefore, a gambler has a sure-win opportunity on β_1, β_2 if there exists a map $\xi : \{1, \ldots, k\} \to \{1, 2\}$ and stakes $\sigma_1, \ldots, \sigma_k \in \mathbb{R}$ such that in very every possible world h, gambler's balance

$$\sum_{i=1}^{k} \sigma_i (h(e_i) - \beta_{\xi(i)})) > 0,$$

where $\beta : e_i \mapsto \beta_{\xi(i)}(e_i)$ is as in Definition 1.

Remark 1. Notice that two coherent books β_1 and β_2 are jointly coherent if any book in the set

$$\Xi(\beta_1, \beta_2) = \{\beta_1(e_1), \beta_2(e_1)\} \times \{\beta_1(e_2), \beta_2(e_2)\} \times \ldots \times \{\beta_1(e_k), \beta_2(e_k)\}$$

is coherent as well. For any pair of coherent books β_1, β_2 we will call $\Xi(\beta_1, \beta_2)$ the set of *crossed-books* of β_1 and β_2.

Also notice that, by Definition 1, a gambler is not allowed to buy (or sell) a bet on the same event from both β_1 and β_2. This restriction is imposed in order to not trivialize her opportunities of sure-win. Indeed, since β_1 and β_2 are distinct, there always exists at least an event e such that $\beta_1(e) \neq \beta(e)$. Thus, assuming that $\beta_1(e) < \beta_2(e)$ without loss of generality, buying $\beta_1(e)$ for 1 euro and $\beta_2(e)$ for -1 euro (i.e., selling $\beta_2(e)$ for 1 euro) would immediately ensure the gambler a sure-win.

Example 2. Let $\Phi = \{e_1, e_2\}$ as in Example 1 and consider the books:

1. $\beta_1(e_1) = 1/2$ and $\beta_1(e_2) = 2/3$;
2. $\beta_2(e_1) = 1/4$ and $\beta_2(e_2) = 2/3$;
3. $\beta_3(e_1) = \beta_3(e_2) = 1/3$.

Then, β_1 is jointly coherent with β_2, which is joint coherent with β_3. On the other hand β_1 is not jointly coherent with β_3. Indeed, the book $\alpha \in \Xi(\beta_1, \beta_3)$ defined by $e_1 \mapsto \beta_1(e_1) = 1/2, e_2 \mapsto \beta_3(e_2) = 1/3$ is incoherent (see Fig. 2). Therefore, a gambler who is allowed to choose, for each event, which book to bet with has a sure-win opportunity if the books into play are β_1 and β_3.

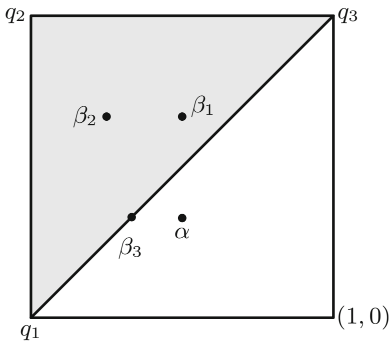

Fig. 2. The convex hull $\mathscr{C}_\Phi$ (gray); the coherent books β_1, β_2, β_3 and the incoherent book $\alpha \in \Xi(\beta_1, \beta_3)$.

We now present a first easy result. Recall that a subset $B = \{b_1, \ldots, b_r\}$ of a Boolean algebra $\mathbf{A}$ is a *partition* if $\bigvee_{i=1}^{r} b_i = \top$ and, for all $i \neq j$, $b_i \wedge b_j = \bot$.

Proposition 1. *If $\Phi = \{e_1, \ldots, e_k\}$ is a partition of $\mathbf{B}_\Phi$, then for any two coherent books β_1, β_2 on Φ the following conditions are equivalent:*

1. *$\beta_1 \neq \beta_2$;*
2. *β_1 and β_2 are not jointly coherent, i.e. the gambler has a sure-win opportunity on β_1, β_2.*

Proof. The direction (2)⇒(1) is trivial. In order to prove (1)⇒(2), observe that, since by hypothesis Φ is a partition of $\mathbf{B}_\Phi$, any book β on Φ is coherent if and only if

$$\sum_{i=1}^{k} \beta(e_i) = 1. \tag{2}$$

Now, since $\beta_1 \neq \beta_2$, there exists $e_i \in \Phi$ such that $\beta_1(e_i) \neq \beta_2(e_i)$. Let us assume, without loss of generality, that $\beta_1(e_i) < \beta_2(e_i)$ and let us consider the book $\beta \colon \Phi \to [0,1]$ defined as follows: for every $e \in \Phi$,

$$\beta(e) = \begin{cases} \beta_1(e) & \text{if } e \neq e_i, \\ \beta_2(e) & \text{otherwise.} \end{cases}$$

Notice that $\beta \in \Xi(\beta_1, \beta_2)$ and it is not coherent since $\sum_{j=1}^{k} \beta(e_j) < 1$. Therefore, β_1 and β_2 are not jointly coherent. This settles the claim. □

4 The Geometry of Joint Coherence

We are interested in providing a full characterization of all those coherent books which are jointly coherent with a fixed one. In this section, we will give geometric characterizations of the space of these books.

We set the background for proving this result. For every book $\beta \colon \Phi \to [0,1]$ and for every $i = 1, \ldots, k$, let δ_i be the pair $(d_i^+, d_i^-) \in \mathbb{R}^2$ be such that:

1. $d_i^\pm \geq 0$;
2. the books $\beta_{d_i^+} = (\beta_1, \ldots, \beta_{i-1}, \beta_i + d_i^+, \beta_{i+1}, \ldots, \beta_k)$ and $\beta_{d_i^-} = (\beta_1, \ldots, \beta_{i-1}, \beta_i - d_i^-, \beta_{i+1}, \ldots, \beta_k)$ are coherent;
3. for all $\varepsilon > 0$, $(\beta_1, \ldots, \beta_{i-1}, \beta_i + d_i^+ + \varepsilon, \beta_{i+1}, \ldots, \beta_k)$ and $(\beta_1, \ldots, \beta_{i-1}, \beta_i - d_i^- - \varepsilon, \beta_{i+1}, \ldots, \beta_k)$ are incoherent.

Let us hence define the rectangle

$$\mathscr{R}_\beta = \{\gamma \in \mathbb{R}^k \mid (\forall i = 1, \ldots, n)\ d_i^- \leq |\gamma_i - \beta_i| \leq d_i^+\},$$

and the convex set

$$\mathscr{C}_\beta = \mathscr{C}_\Phi \cap \mathscr{R}_\beta. \tag{3}$$

Obviously $\mathscr{C}_\beta$ is nonempty iff β is coherent.

Example 3. Let Φ and $\beta_1 : \Phi \to [0,1]$ be as in Example 2. Thus, $\beta_1(p) = 1/2$ and $\beta_1(p \vee q) = 2/3$. The vertices (extreme points) of the rectangle $\mathscr{R}_{\beta_1}$ are easy to compute: $v_1 = (2/3, 1/2)$; $v_2 = (0, 1/2)$; $v_3 = (0, 1)$; $v_4 = (2/3, 1)$. Thus, $\mathscr{C}_{\beta_1} = \mathscr{C}_\Phi \cap \mathscr{R}_{\beta_1}$ coincides with the gray region as in Fig. 3.

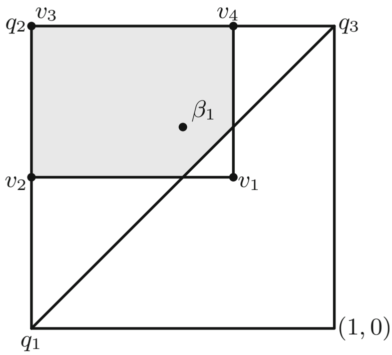

Fig. 3. The coherent book β_1 from Example 2 and the convex set $\mathscr{C}_{\beta_1}$ (the gray region) obtained by intersecting $\mathscr{C}_\Phi$ (the triangle of vertices q_1, q_2 and q_3) and the rectangle $\mathscr{R}_{\beta_1}$ (whose vertices are v_1, v_2, v_3 and v_4).

The following result shows that, for every fixed coherent book β, the convex set $\mathscr{C}_\beta$ characterizes all the coherent books which are jointly coherent with β.

Proposition 2. *Let $\beta, \beta' \colon \Phi \to [0,1]$ be two coherent books. Then the following conditions are equivalent:*

1. *β' is jointly coherent with β;*
2. *$\beta' \in \mathscr{C}_\beta$.*

Proof. (1) $\Rightarrow$ (2). Assume that β and β' are jointly coherent. Since β' is coherent then, by Theorem 1, $\beta' \in \mathscr{C}_\Phi$. We only have to show that $\beta' \in \mathscr{R}_\beta$. Suppose, by contradiction, that $\beta' \notin \mathscr{R}_\beta$, i.e. there exists $1 \le i \le k$ such that $\mid \beta_i - \beta'_i \mid > \ d_i^+$ (or $\mid \beta_i - \beta'_i \mid < \ d_i^-$, but the reasoning is analogous). The definition of $\mathscr{R}_\beta$ immediately implies that β' is not coherent, a contradiction.

(2) $\Rightarrow$ (1). Let $\beta' \in \mathscr{C}_\beta = \mathscr{C}_\Phi \cap \mathscr{R}_\beta$, i.e. β' is a coherent book which satisfies the above conditions 1.-3. Let α be any book in $\Xi(\beta, \beta')$. Since β is coherent, we have that $\beta \in \mathscr{C}_\beta$ and, by assumption, $\beta' \in \mathscr{C}_\beta$, which together imply that $\alpha \in \mathscr{C}_\beta$. Thus $\alpha \in \mathscr{C}_\Phi$, which, by Theorem 1, implies that is coherent, therefore β and β' are jointly coherent books. □

It is immediate to see that the relation of being jointly coherent is symmetric. Therefore, from Proposition 2 above β and β' are jointly coherent iff $\beta' \in \mathscr{C}_\beta$ iff $\beta \in \mathscr{C}_{\beta'}$. Therefore the following is immediate.

Corollary 1. *Let $\beta, \beta' \colon \Phi \to [0,1]$ be two coherent books. Then β and β' are jointly coherent iff $\beta, \beta' \in \mathscr{C}_\beta \cap \mathscr{C}_{\beta'}$.*

In the next we will show, for every coherent book β, another geometric characterization of $\mathscr{C}_\beta$. Recalling that every closed convex subsets of $\mathbb{R}^k$ is an intersection of halfspaces (see [4, Theorem 3.8]), for every polytope $\mathscr{P} \subseteq \mathbb{R}^k$, there are linear polynomials $f_1, \ldots, f_n$ such that $\mathscr{P} = \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid \forall\, i = 1, \ldots, n, f_i(x_1, \ldots, x_k) \geq 0\}$. In what follows, without danger of confusion, for every finite set of events $\Phi = \{e_1, \ldots, e_k\}$, we will write $f_1, \ldots, f_n$ for those polynomials such that

$$\mathscr{C}_\Phi = \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid \forall\, i = 1, \ldots, n, f_i(x_1, \ldots, x_k) \geq 0\}. \tag{4}$$

For the sake of a lighter notation, we will denote by K the index set $\{1, \ldots, k\}$.

Let us fix two points $a = (a_1, \ldots, a_k)$ and $b = (b_1, \ldots, b_k)$ of $\mathbb{R}^k$, and a subset J of K. Then, we denote by $(a_J, b_{K\setminus J})$ the tuple obtained by substituting b_j by a_j, in b, for each $j \in J$.

Theorem 2. *Let $\mathscr{C}_\Phi$, $f_1, \ldots, f_n$ be as in (4) and let β, β' be two coherent books. Then, $\beta' \in \mathscr{C}_\beta$ iff β' is solution of the following system,*

$$\mathscr{S}(\beta) = \{f_i(\beta_J, x_{K\setminus J}) \geq 0 \mid i = 1, \ldots, n,\ J \subseteq K\}.$$

In other words, $\mathscr{C}_\beta$ coincides with the set of solutions of $\mathscr{S}(\beta)$.

Proof. ($\Leftarrow$). Assume, by contraposition, that $\beta' \notin \mathscr{C}_\beta$. Thus, by Proposition 2, β and β' are not jointly coherent and hence, by Remark 1, there exists a $\hat{\beta} \in \Xi(\beta, \beta')$ which is not coherent. This means, by Theorem 1, that $\hat{\beta} \notin \mathscr{C}_\Phi$. Therefore, by (4), there exists an index $i \in \{1, \ldots, n\}$ such that $f_i(\hat{\beta}_1, \ldots, \hat{\beta}_k) < 0$ and in particular $\hat{\beta}$ is not a solution of $\mathscr{S}(\beta)$.

($\Rightarrow$). Assume, again by contraposition, that β' is not solution of $\mathscr{S}(\beta)$. Thus, there exists a $J \subseteq K$ and an index $i \in \{1, \ldots, n\}$ such that $f_i(\beta_J, \beta'_{K\setminus J}) < 0$. Therefore, the claim immediately follows by observing that $(\beta_J, \beta'_{K\setminus J}) \in \Xi(\beta, \beta')$ and $(\beta_J, \beta'_{K\setminus J}) \notin \mathscr{C}_\Phi$ proving that β and β' are not jointly coherent. □

An immediate consequence of the above theorem is the decidability of the problem determining if two rational-books are jointly coherent. For the following result to make sense, we will hence assume that the books involved take value into the rational unit interval $[0, 1] \cap \mathbb{Q}$.

Corollary 2. *Given two rational-valued books $\beta_1, \beta_2 \in \mathscr{C}_\Phi$, the problem of determining if β_1 and β_2 are jointly coherent is decidable.*

Proof. The following (sketched) procedure, which takes in input the events $e_1, \ldots, e_k$ and the rational numbers $\beta_1(e_i)$'s and $\beta_2(e_i)$'s, decides if β_1 and β_2 are jointly coherent.

Step 1: Determine the extremal points of $\mathscr{C}_\Phi$ by computing, for each truth-assignment h_j, $q_j = (h_j(e_1), \ldots, h_j(e_k))$ as in (1).

Step 2: Let $q_{t_1}, \ldots, q_{t_r}$ be, among the q_j's, the extremal points of a face $\mathscr{F}_t$ of $\mathscr{C}_\Phi$ and let $f_t(x_1, \ldots, x_k)$ the be equation of the hyperplane through $q_{t_1}, \ldots, q_{t_r}$.

Step 3: Iterate **Step 2** for all faces $\mathscr{F}_1, \ldots, \mathscr{F}_n$ of $\mathscr{C}_\Phi$ and hence determine $f_1, \ldots, f_n$ such that $\mathscr{C}_\Phi = \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid \forall\, i = 1, \ldots, n, f_i(x_1, \ldots, x_k) \geq 0\}$ as in (4).

Step 4: Introduce the system of inequalities $\mathscr{S}(\beta_1)$ as in the statement of Theorem 2.

Therefore, in the end, check if $\beta_2 = (\beta_2(e_1), \ldots, \beta_2(e_k))$ is a (rational) solution of $\mathscr{S}(\beta_1)$. □

Our last example applies the result of Theorem 2 and exemplifies also the procedure sketched in the proof of Corollary 2.

Example 4. Let $\Phi = \{p, p \vee q\}$ and $\beta_1(p) = \frac{1}{2}$, $\beta_1(p \vee q) = \frac{2}{3}$. Then

$$\mathscr{C}_\Phi = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1, x_2 \leq 1, x_1 \leq y_2\}.$$

A book β is jointly coherent with β_1 if and only if it satisfies the system given by the following inequalities

- for $J = \emptyset$, we have $x_1 \geq 0$, $1 - x_1 \geq 0$, $x_2 \geq 0$, $1 - x_2 \geq 0$ and $x_2 - x_1 \geq 0$;
- for $J = \{1\}$, we have $1/2 \geq 0$, $1 - 1/2 \geq 0$, $x_2 \geq 0$, $1 - x_2 \geq 0$ and $x_2 - 1/2 \geq 0$;
- for $J = \{2\}$, we have $x_1 \geq 0$, $1 - x_1 \geq 0$, $2/3 \geq 0$, $1 - 2/3 \geq 0$ and $2/3 - x_1 \geq 0$;
- for $J = \{1, 2\}$, we have $1/2 \geq 0$, $1 - 1/2 \geq 0$, $2/3 \geq 0$, $1 - 2/3 \geq 0$ and $2/3 - 1/2 \geq 0$.

Notice that the inequalities obtained for $J = \emptyset$ just describes $\mathscr{C}_\Phi$ and it is also immediate to see that for $J = \{1, 2\}$, we get the inequalities which assure that β_1 is a coherent book.

5 Conclusion and Future Work

The present work is motivated by the observation that when two different bookmakers assign betting quotes over the same set of events, the notion of coherence is not enough to prevent a gambler who is allowed to bet on both assignments from a sure-win opportunity. We thereby proposed the notion of *joint coherence* of two books.

Our main results consist of geometrical characterizations of the space of books which are jointly coherent with a given one. Such a space is a closed convex subset of the set of all coherent books and it identifies which books can be considered "safe" once β has been fixed.

We believe that the mathematics of joint coherence as well as its computational aspects deserve further attention. In particular, since (two) jointly coherent books are necessarily coherent, we wonder what is the effect of the property of being jointly coherent on the sets of probability measures which extend them, by de Finetti's theorem, on the Boolean algebra generated by the events. Further, still on this line, it would be interesting to extend the notion of joint coherence to more general theories of uncertainty and, in particular, to Walley's definition of coherence for imprecise probabilities where negative betting rates are forbidden [13].

We are interested also in providing different characterizations of the notion of joint coherence. In particular, following [7], where it is shown that (strict) coherence admits three characterizations (algebraic, geometrical and logical), one of the aims for the future is to extend such characterizations to joint coherence as well.

A very natural question that computer scientists may rise is whether it is possible to establish a computational bound to the problem of determining whether two books are jointly coherent. Although we proved that checking joint coherence of two books is decidable, providing a NP-bound for the same seems challenging and it will be object of further investigation.

Joint coherence arises from allowing a multiplicity of bookmakers publishing coherent books, who can be viewed as individually *rational agents*. Grounding on this consideration, it is reasonable to think that this notion may suggest an alternative way to approach *collective judgments* (see for instance [3]) and *collective rationality*. This will also be addressed in our future work.

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