A fuzzy probability logic for compound conditionals

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***Abstract*—In this paper we propose a fuzzy modal logic for conditional probability that allows to represent and reason about the probability of not only basic conditional expressions of the form "φ given ψ", written $(\varphi \mid \psi)$, but also compound conditional sentences such as "φ given ψ *and* γ given χ", written $(\varphi \mid \psi) \wedge (\gamma \mid \chi)$, and more in general, any Boolean combination of basic ones. In order to formalize compound conditional formulas we will adopt the recently defined *Logic for Boolean Conditionals* (LBC) and hence formalize conditional probability as a simple (unconditional) probability of conditional sentences. In addition to such basic fuzzy modal logic for the probability of compound conditionals, we will also present some extensions and prove that each of them is sound and complete w.r.t. to a suitable class of probabilistic models. Furthermore, we will prove how to recover the usual interpretation of conditional probability, showing that, under minimal requirements, in these logics the probability of a basic conditional $(\varphi \mid \psi)$ can be safely taken as the conditional probability of φ given ψ, i.e. as the ratio $P(\varphi \wedge \psi)/P(\psi)$.**

***Index Terms*—Conditional probability; Compound conditional; Fuzzy logic; Fuzzy modal logic.**

I. INTRODUCTION

Fuzzy sets-based models and numerical uncertainty models, although sharing the feature of evaluating sentences in a totally ordered scale, usually the real unit interval $[0, 1]$, account for radically different notions of gradualness. From a formal point of view, these differences can be easily grasped if we consider their corresponding logics: fuzzy logics and uncertainty logics (in particular probability logics), respectively. In fact, while the former are truth-functional, i.e. the truth-value of a compound formula like $\varphi \vee \psi$ only depends on the truth-values of its components φ and ψ, the latter are not, since, for instance, the probability of $\varphi \vee \psi$ cannot be computed only from the probability of φ and the probability of ψ (it is also needed to know what is the probability of $\varphi \wedge \psi$).

Despite these differences, however, probability logics can be properly handled in a fuzzy logical setting by expanding the language of a fuzzy logic with a unary modality $P(\cdot)$ and interpreting, for every classical formula φ, the modal formula $P(\varphi)$ as "φ is probable". Clearly, $P(\varphi)$ is a fuzzy proposition, whose truth-degree can be taken as the probability of φ. More precisely, the fuzzy modal logic FP(Ł), as firstly introduced in [11] and improved in [10], extends the language of Łukasiewicz logic Ł by the modal operator $P(\cdot)$ and uses the ground logic Ł to express the basic properties of a probability function. In particular, it is worth to recall that the finite additivity of P can be expressed in FP(Ł) by using the Łukasiewicz connective $\oplus$ whose standard interpretation is the truncated sum: for all $x, y \in [0, 1]$, $x \oplus y = \min\{1, x+y\}$. Very recently, in [1] the authors have studied in depth the relationship of this fuzzy logic-based approach to more traditional probability logics after Halpern et al. see e.g. [13].

In addition to simple probability, the paper [8] presents the logic FP(ŁΠ) to deal with conditional probability by considering, instead of Ł, the stronger logic ŁΠ. Such formalism can be roughly regarded as the expansion of Łukasiewicz logic by the connectives of product conjunction $\odot$ and product implication $\rightarrow_\Pi$. The standard semantics of $\odot$ and $\rightarrow_\Pi$ interprets them, respectively, by the usual product $\cdot$ and the function $x \rightarrow_\Pi y = 1$ if $x \leq y$ and $x \rightarrow_\Pi y = y/x$ otherwise. Thus, if $P(\psi)$ is not zero, the conditional probability $P(\varphi \mid \psi)$ can be written in FP(ŁΠ) as $P(\psi) \rightarrow_\Pi P(\varphi \wedge \psi)$ and hence interpreted in its semantics as $P(\varphi \wedge \psi)/P(\psi)$. A related approach can be found in [9], where Popper conditional probabilities are formalised in a similar setting.

In this paper we propose a fuzzy modal logic FP$(\mathrm{LBC}, Ł\Pi)$ for conditional probability that extends FP(ŁΠ) in the expressive power. In particular, FP$(\mathrm{LBC}, Ł\Pi)$ formalizes conditional events by the recently defined logic LBC (Logic of Boolean Conditionals) for conditional events. The latter allows to represent not only basic conditional expressions "φ given ψ", written $(\varphi \mid \psi)$, but also compound conditional sentences such as "φ given ψ *and* γ given χ", written in LBC as $(\varphi \mid \psi) \wedge (\gamma \mid \chi)$, or more in general, any Boolean combination of basic ones [5]. For each of such (basic and compound) conditional sentences, FP$(\mathrm{LBC}, Ł\Pi)$ permits to represent and reason about their probability. Thus, the conditional probability of "φ given ψ" is treated in FP$(\mathrm{LBC}, Ł\Pi)$ as the *unconditional* probability of the basic conditional formula $(\varphi \mid \psi)$.

In addition, we will present extensions of FP$(\mathrm{LBC}, Ł\Pi)$ that capture a more refined notion of probability functions. For FP$(\mathrm{LBC}, Ł\Pi)$ and each of its extensions, we prove soundness and completeness results w.r.t. suitable classes of probability models.

This paper is organized as follows. Section II gathers extensive preliminaries: on the Logic for Boolean Conditionals (LBC) in Subsection II-A; on the ground propositional logic ŁΠ in Subsection II-B; and on the fuzzy modal logic for conditional probability FP(ŁΠ) in Subsection II-C. In Section

III we will define the probability logic for compound conditionals FP(LBC, ŁΠ). In the same section we will consider the class of *separable* models and prove completeness. Moreover, we will show that, for every basic conditional $(\varphi \mid \psi)$ such that ψ has positive probability, the logic FP(LBC, ŁΠ) proves that the modal formula $P(\varphi \mid \psi)$ is equivalent to $P(\psi) \to_\Pi P(\varphi \wedge \psi)$ and hence, in every separable model, $P(\varphi \mid \psi)$ is interpreted as the ratio $\mu(\varphi \wedge \psi)/\mu(\psi)$. A first extension of FP(LBC, ŁΠ), namely the logic $\text{FP}(\text{LBC}, \text{Ł}\Pi)^+$, will be defined in Section IV and there we will show it to be complete w.r.t. to a subclass of separable models called *positive* separable models. Section V deals with the logic $\text{FP}(\text{LBC}, \text{Ł}\Pi)^+_c$, a further extension meant to capture the behavior of the so-called *canonical* extensions to $\mathcal{C}(\mathbf{A})$ of positive (unconditional) probabilities on $\mathbf{A}$ in [5]. For $\text{FP}(\text{LBC}, \text{Ł}\Pi)^+_c$ we prove soundness and completeness w.r.t. to the proper subclass of positive separable models that will be called *canonical* in that section. Conclusions and future work on this subject will be discussed in the final Section VI.

II. Preliminaries

A. The logic LBC

In this section we recall from [5] the Logic of Boolean Conditionals (LBC). The idea is to consider basic conditional formulas of the form $(\varphi \mid \psi)$ as primitive objects that can be freely combined with Boolean connectives. A difference with the so-called *measure-free conditionals* is that the combination of two basic conditionals need not be another basic conditional, only in some special cases specified in the axioms of the logic. Indeed, formulas of LBC correspond to Boolean combinations of basic conditional formulas $(\varphi \mid \psi)$, where φ, ψ are classical propositions. In more detail, let L be a propositional language built from a finite set of propositional variables $p_1, p_2, \ldots, p_k$ and classical logic connectives $\wedge, \vee, \neg, \to, \leftrightarrow$. We will denote by $\vdash_{CPL}$ derivability in Classical Propositional Logic. Based on L, we define the language LBC of conditionals by the following stipulations:

- Basic (or atomic) conditional formulas, expressions of the form $(\varphi \mid \psi)$ where $\varphi, \psi \in \mathsf{L}$ and such that $\not\vdash_{CPL} \neg\psi$, are in LBC.
- Further, if $\Phi, \Psi \in \mathsf{LBC}$, then $\neg\Phi, \Phi \wedge \Psi \in \mathsf{LBC}$.[1] Other connectives like $\vee$, $\to$ and $\leftrightarrow$ are defined as usual.

Note that we do not allow the nesting of conditionals, as usually done in the vast literature on the modal approaches to Conditional Logics Actually, purely propositional formulas from L can also be considered to be part of LBC since, as a matter of fact, any proposition φ can be identified with the conditional $(\varphi \mid \top)$, where $\top$ is an abbreviation for $\psi \vee \neg\psi$.

Definition 2.1: The *Logic of Boolean conditionals* (LBC for short) has the following axioms:

[1]We use the same symbols for connectives in L and in LBC without danger of confusion.

- (CPL) For any tautology of CPL, the formula resulting from a uniform replacement of the variables by basic conditionals.
- (A1) $(\psi \mid \psi)$
- (A2) $\neg(\varphi \mid \psi) \leftrightarrow (\neg\varphi \mid \psi)$
- (A3) $(\varphi \mid \psi) \wedge (\delta \mid \psi) \leftrightarrow (\varphi \wedge \delta \mid \psi)$
- (A4) $(\varphi \mid \psi) \leftrightarrow (\varphi \wedge \psi \mid \psi)$
- (A5) $(\varphi \mid \psi) \leftrightarrow (\varphi \mid \chi) \wedge (\chi \mid \psi)$, if $\vdash_{CPL} \varphi \to \chi$ and $\vdash_{CPL} \chi \to \psi$
- (R1) from $\vdash_{CPL} \varphi \to \psi$ derive $(\varphi \mid \chi) \to (\psi \mid \chi)$
- (R2) from $\vdash_{CPL} \chi \leftrightarrow \psi$ derive $(\varphi \mid \chi) \leftrightarrow (\varphi \mid \psi)$
- (MP) Modus Ponens: from Φ and $\Phi \to \Psi$ derive Ψ

The notion of proof in LBC, $\vdash_{LBC}$, is defined as usual from the above axioms and rules.

In [5] it is shown that the Lindenbaum algebra corresponding to LBC, that is, the algebra of LBC-formulas modulo logical equivalence, is a certain type of Boolean algebra, called Boolean algebra of conditionals, which is finite if the set propositional variables is so, that is our case. Then, the algebra is atomic and its set of atoms are conjunctions of basic conditionals of length $n-1$, where $n = 2^m$ with m being the number of propositional variables, of the following form:

$$(\alpha_1 \mid \top) \wedge (\alpha_2 \mid \neg\alpha_1) \wedge \ldots \wedge (\alpha_{n_1} \mid \wedge_{i=1,n-2}\, \alpha_i),$$

where $\alpha_1, \ldots, \alpha_{n-1}$ are propositional atoms of the Lindenbaum algebra of the underlying propositional language L.

The semantics of the LBC logic is based on sequences $\overline{w} = \langle w_1, \ldots, w_n \rangle$ of pairwise different propositional interpretations for the underlying language L, in such a way that $\overline{w}$ satisfies a conditional $(\varphi \mid \psi)$, written $\overline{w} \models_{LBC} (\varphi \mid \psi)$, if $w_i \models_{CPL} \varphi$ for the lowest index i such that $w_i \models \psi$.

B. The logic ŁΠ

The ŁΠ is a powerful fuzzy logic system that suitably combines the connectives from Łukasiewicz logic with the connectives of Product fuzzy logic [4]. The language of the ŁΠ logic is built in the usual way from a countable set of propositional variables, three binary connectives $\to_L$ (Łukasiewicz implication), $\odot$ (Product conjunction) and $\to_\Pi$ (Product implication), and the truth constant $\bar{0}$. A truth-evaluation is a mapping e that assigns to every propositional variable a real number from the unit interval [0, 1] and extends to all formulas as follows:

$$e(\bar{0}) = 0, \qquad e(\varphi \odot \psi) = e(\varphi) \cdot e(\psi)$$
$$e(\varphi \to_L \psi) = \min(1 - e(\varphi) + e(\psi), 1),$$
$$e(\varphi \to_\Pi \psi) = \begin{cases} 1, & \text{if } e(\varphi) \le e(\psi) \\ e(\psi)/e(\varphi), & \text{otherwise} \end{cases}.$$

The truth constant $\bar{1}$ is defined as $\varphi \to_L \varphi$. In this way we have $e(\bar{1}) = 1$ for any truth-evaluation e. Moreover, many other connectives can be defined from those introduced above:

$$\begin{array}{rcl} \neg_L \varphi & \text{is} & \varphi \to_L \bar{0} \\ \neg_\Pi \varphi & \text{is} & \varphi \to_\Pi \bar{0}, \\ \varphi \wedge \psi & \text{is} & \varphi \& (\varphi \to_L \psi), \\ \varphi \vee \psi & \text{is} & \neg_L(\neg_L \varphi \wedge \neg_L \psi), \end{array}$$

$$\begin{array}{rcl} \varphi \oplus \psi & \text{is} & \neg_L \varphi \to_L \psi, \\ \varphi \otimes \psi & \text{is} & \neg_L(\neg_L \varphi \oplus \neg_L \psi), \\ \varphi \ominus \psi & \text{is} & \varphi \& \neg_L \psi, \\ \varphi \leftrightarrow_L \psi & \text{is} & (\varphi \to_L \psi)\&(\psi \to_L \varphi), \\ \Delta\varphi & \text{is} & \neg_\Pi \neg_L \varphi, \\ \nabla\varphi & \text{is} & \neg_\Pi \neg_\Pi \varphi, \end{array}$$

with the following interpretations:

$$\begin{array}{rcl} e(\neg_L \varphi) & = & 1 - e(\varphi), \\ e(\neg_\Pi \varphi) & = & \begin{cases} 1, & \text{if } e(\varphi) = 0 \\ 0, & \text{otherwise} \end{cases}, \\ e(\varphi \wedge \psi) & = & \min(e(\varphi), e(\psi)), \\ e(\varphi \vee \psi) & = & \max(e(\varphi), e(\psi)), \\ e(\varphi \oplus \psi) & = & \min(1, e(\varphi) + e(\psi)), \\ e(\varphi \otimes \psi) & = & \max(0, e(\varphi) + e(\psi) - 1), \\ e(\varphi \ominus \psi) & = & \max(0, e(\varphi) - e(\psi)), \\ e(\varphi \leftrightarrow_L \psi) & = & 1 - |e(\varphi) - e(\psi)|, \\ e(\Delta\varphi) & = & \begin{cases} 1, & \text{if } e(\varphi) = 1 \\ 0, & \text{otherwise} \end{cases}, \\ e(\nabla\varphi) & = & \begin{cases} 1, & \text{if } e(\varphi) > 0 \\ 0, & \text{otherwise} \end{cases}. \end{array}$$

The logic ŁΠ is defined Hilbert-style as the logical system whose axioms and rules are the following:[2]

(i) Axioms of Łukasiewicz Logic:

(ii) Axioms of Product Logic

(iii) The following additional axioms relating Łukasiewicz and Product logic connectives:
   - $(\neg)$ $\quad \neg_\Pi \varphi \to_L \neg_L \varphi$
   - (Δ) $\quad \Delta(\varphi \to_L \psi) \equiv \Delta(\varphi \to_\Pi \psi)$
   - $(L\Pi)$ $\quad \varphi \odot (\psi \ominus \chi) \equiv (\varphi \odot \psi) \ominus (\varphi \odot \chi)$

(iv) Deduction rules of ŁΠ are modus ponens for $\to_L$ (modus ponens for $\to_\Pi$ is derivable), and necessitation for Δ: from φ derive $\Delta\varphi$.

C. The probability logic FP(ŁΠ)

This final section on preliminaries is on the probability logic FP(ŁΠ) (FP for Fuzzy Probability), introduced in [8] and defined as a sort of modal extension with a unary operator $P(\cdot)$ over the fuzzy logic ŁΠ described in the previous section. This logic allows for reasoning about the probability of classical propositions.

Actually, the propositional language L is extended by a fuzzy unary modal operator P. If φ is a proposition of L, then $P\varphi$ is a modal proposition whose intended reading is that "φ is *probable*", and whose truth-degree will be taken as the probability of φ.

The language of FP(ŁΠ) is defined as follows. *Formulas* of FP(ŁΠ) are of two types:

- Non-modal: they are exactly the (classical) formulas of L, i.e. those built from a set Var of propositional variables $\{p_1, p_2, \dots p_n, \dots\}$ using the classical binary connectives $\wedge$ and $\neg$. Other connectives like $\vee$ and $\to$ are defined from $\wedge$ and $\neg$ in the usual way. We shall denote them by lower case Greek letters φ, ψ, etc.
- Modal: they are built from elementary modal formulas of the form $P\varphi$, where φ is a non-modal formula, using the connectives of ŁΠ ($\to_L$, $\odot$, $\to_\Pi$). We shall denote them by upper case Greek letters Φ, Ψ, etc.

These are all the formulas of FP(ŁΠ). Notice that nested modalities, among other things, are not allowed.

Axioms and rules of FP(LBC, ŁΠ) are as follows:

(CPL) All axioms and rules of classical propositional logic restricted to classical, non-modal, formulas;

(ŁΠ) All axioms and rules of ŁΠ for modal formulas;

(P) The following axioms and rules for the modality P: for all propositions $\varphi, \psi \in \mathsf{L}$,
   - (P1) $P(\varphi \to \psi) \to_L (P(\varphi) \to_L P(\psi))$
   - (P2) $P(\neg\varphi) \leftrightarrow_L \neg P(\varphi)$
   - (P3) $P(\varphi \vee \psi) \leftrightarrow_L [P(\varphi) \oplus (P(\psi) \ominus P(\varphi \wedge \psi))]$
   - (Nec) if $\vdash_{CPL} \varphi$, derive $P(\varphi)$.

Models of FP(ŁΠ) are probability Kripke structures $K = \langle W, e, \mu \rangle$, where:

- W is a non-empty set of possible worlds;
- $e : V \times W \to \{0, 1\}$ provides for each world a *Boolean* (two-valued) evaluation of the proposition variables, that is, $e(p, w) \in \{0, 1\}$ for each propositional variable $p \in Var$ and each world $w \in W$; and
- $\mu : 2^W \to [0, 1]$ is a finitely additive probability measure on a Boolean algebra of subsets of W such that for each p, the set $\{w \mid e(p, w) = 1\}$ is measurable (cf. [10] 8.4.1).

A truth evaluation e is extended to non-modal formulas in the classical way, to elementary modal formulas as follows:

$$e(P\varphi, w) = \mu(\{w \in W \mid e(\varphi, w) = 1\}),$$

and to compound modal formulas by using the truth-functions of the ŁΠ logic.

Soundness and completeness of the logic FP(ŁΠ) w.r.t. to the class of probability Kripke models is proved in [8]: if $T \cup \{\Phi\}$ is a finite set of FP(ŁΠ)-formulas, then T proves Φ in FP(ŁΠ) iff for any probability Kripke model $K = (W, e, \mu)$ and any world $w \in W$, $e(\Phi, w) = 1$ whenever $e(\Psi, w) = 1$ for all $\Psi \in T$.

III. Probability logic over conditionals

In this section we define a logic to reason about the probability of basic and compound conditionals over the fuzzy logic ŁΠ. In the same line as with the logic FP(ŁΠ) described in Section II-C, we extend the language of LBC with a fuzzy (unary) modal operator P, so that, for every basic conditional $(\varphi \mid \psi)$, the intended meaning of a formula $P(\varphi \mid \psi)$ is that the conditional "φ given ψ" is *probable*, and that the truth-degree of $P(\varphi \mid \psi)$ is the probability of the conditional "$(\varphi \mid \psi)$". The relation of this probability to the usual notion of conditional probability of φ given ψ will become clear later.

The logic FP(LBC, ŁΠ) is obtained by replacing, in the definition of FP(ŁΠ), classical logic for events by the conditional logic LBC defined as in Section II-A. Formulas of FP(LBC, ŁΠ) are of two types:

[2] This definition, proposed in [3], is actually a simplified version of the original definition of $L\Pi$ given in [4].

- **Conditional formulas** are formulas of the logic LBC, that is, basic conditionals of the form $(\varphi \mid \psi)$ for all classical formulas φ and ψ such that ψ is not a classical logic contradiction (in other words, $\not\vdash_{CPL} \neg\psi$) and compound conditional formulas obtained as Boolean combinations of basic ones. Compound conditional formulas will be denoted as $\Phi, \Psi, \ldots$;

- **Modal formulas**: for every (basic or compound) conditional formula Φ, $P(\Phi)$ is an atomic modal formula. Compound modal formulas are combinations of atomic ones by means of the ŁΠ connectives.

Thus, for instance, $P(\varphi \mid \psi)$, $P((\varphi \mid \psi) \wedge (\gamma \mid \delta))$ and $P((\varphi \mid \psi) \wedge (\gamma \mid \delta)) \to_L P(\chi \mid \tau)$ are compound modal formulas for all classical formulas $\varphi, \psi, \gamma, \delta, \chi, \tau$ such that $\not\vdash \neg\psi$, $\not\vdash \neg\delta$, $\not\vdash \neg\tau$. However, neither $(\varphi \mid \psi) \to_L P(\gamma \mid \delta)$ nor $P((\varphi \mid \psi) \oplus P(\chi \mid \tau))$ are well formed formulas in this language.

Axioms and rules of FP(LBC, ŁΠ) are as follows:

(LBC) All axioms and rules of LBC restricted to conditional formulas;

(ŁΠ) All axioms and rules of ŁΠ for modal formulas;

(P) The axioms and rules for the modality P are those for FP(ŁΠ), but now for all conditional formulas $\Phi, \Psi \in$ LBC, plus a new rule (Sep):

- (P1) $P(\Phi \to \Psi) \to_L (P(\Phi) \to_L P(\Psi))$;
- (P2) $P(\neg\Phi) \leftrightarrow_L \neg P(\Phi)$;
- (P3) $P(\Phi \vee \Psi) \leftrightarrow_L [P(\Phi) \oplus (P(\Psi) \ominus P(\Phi \wedge \Psi))]$;
- (Nec) if $\vdash_{LBC} \Phi$, derive $P(\Phi)$;
- (Sep) if $\vdash_{CPL} (\varphi \to \chi) \wedge (\chi \to \psi)$, derive $P((\varphi \mid \chi) \wedge (\chi \mid \psi)) \leftrightarrow_L P(\varphi \mid \chi) \odot P(\chi \mid \psi)$.

The notion of proof according to these axioms and rules will be denoted $\vdash_{FP}$. The axioms and rules of FP(LBC, ŁΠ) are meant to capture the behavior of an unconditional, *separable* probability measure on the Lindenbaum algebra of the logic LBC. In fact, let us recall from [5] that, in particular, a probability μ on a Boolean algebra of conditionals $\mathcal{C}(\mathbf{A})$ is *separable* if for all $a, b, c \in A \setminus \{\bot\}$ such that $a \leq b \leq c$, then $\mu((a \mid b) \wedge (b \mid c)) = \mu(a \mid b) \cdot \mu(b \mid c)$. As we will show later on, separability is captured by the rule (Sep) above.

In what follows Ω will denote the set of Boolean interpretations for the variables Var, and $Seq_n(\Omega)$ will denote the set of sequences of n pairwise different interpretations from Ω.

Definition 3.1: A probability LBC-Kripke model is a structure $\mathcal{C} = \langle W, e, \mu \rangle$ where

- W is a set of worlds;
- $e : W \to Seq_n(\Omega)$ maps every world $w \in W$ to a LBC-evaluation $e(w) = \overline{w} \in Seq_n(\Omega)$;
- μ is a probability on $2^{e[W]}$, where $e[W] = \{e(w) \mid w \in W\} \subseteq 2^{Seq_n(\Omega)}$.

Each LBC-Kripke model $\mathcal{C} = \langle W, e, \mu \rangle$ induces a probability on LBC formulas in the natural way, in particular for each basic conditional $(\varphi \mid \psi)$ we define:

$$\mu(\text{“}(\varphi \mid \psi)\text{”}) = \mu(\{\overline{w} \in Seq_n(\Omega) \mid w \in W, \overline{w} \models (\varphi \mid \psi)\}, [3]$$

and similarly for every compound conditional. Notice that, when $e[W] = Seq_n(\Omega)$, the Boolean algebra $2^{e[W]}$ on which μ is defined, actually is the Lindenbaum algebra of LBC. It was proved in [5, Theorem 7.3] that such Lindenbaum algebra is isomorphic to $\mathcal{C}(\mathbf{L})$, that is, the conditional algebra generated by $\mathbf{L}$, the Lindenbaum algebra of classical propositional logic. Thus, every LBC-Kripke model determines a probability measure μ on $\mathcal{C}(\mathbf{L})$.

Definition 3.2: A probability LBC-Kripke model $\mathcal{C} = \langle W, e, \mu \rangle$ is called *separable* when

$$\mu((\varphi \mid \psi) \wedge (\psi \mid \chi)) = \mu(\varphi \mid \psi) \cdot \mu(\psi \mid \chi) \tag{1}$$

for every φ, ψ, χ such that $\vdash_{CPL} (\varphi \to \psi) \wedge (\psi \to \chi)$ and $\not\vdash_{CPL} \neg\psi$.

An immediate consequence of the definition above is that every μ of a separable model, satisfies $\mu(\varphi \mid \top) = \mu((\varphi \mid \psi) \wedge (\psi \mid \top)) = \mu(\varphi \mid \psi) \cdot \mu(\psi \mid \top)$ for all φ, ψ such that $\vdash_{CPL} \varphi \to \psi$ and $\not\vdash \neg\psi$. Note that the first equality is due to Axiom (A5) of LBC.

Given a formula F of FP(LBC, ŁΠ) and a separable model $\mathcal{S} = \langle W, e, \mu \rangle$, the evaluation of F in $\mathcal{S}$ at $w \in W$ is inductively defined by the following stipulations:

- If F is a conditional formula Φ, then $\|\Phi\|_{\mathcal{S},w} = w_1(\Phi) \in \{0, 1\}$, where $e(w) = \langle w_1, w_2, \ldots \rangle$;
- If $F = P(\Phi)$ is atomic modal, then $\|P(\Phi)\|_{\mathcal{S},w} = \mu(\Phi)$;
- If F is compound modal, then $\|F\|_{\mathcal{S},w}$ is computed by evaluating its atomic components and then by using the truth-functionality of the connectives of ŁΠ in the standard algebra $[0, 1]$.

Notice that if F is modal, then $\|F\|_{\mathcal{S},w}$ does not depend on the world w. Finally, the truth-degree of F in $\mathcal{C}$ is defined as $\|F\|_{\mathcal{S}} = \inf_{w \in W} \|F\|_{\mathcal{S},w}$.

Definition 3.3: If $T \cup \Phi$ is a set of modal formulas, Φ logically follows from T, written $T \models_{SFP} \Phi$, when for all separable probability LBC-Kripke model $\mathcal{S}$, if $\|F\|_{\mathcal{S}} = 1$ for every $F \in T$, then $\|\Phi\|_{\mathcal{S}} = 1$ as well.

Next, we prove that FP(LBC, ŁΠ) is sound and complete with respect to the class of separable models. Its proof, a main part of which will follow from a general result we will recall below, is based on the fact that the logic for events, LBC is locally finite. This means that the Lindenbaum algebra $\mathbf{CL}$ over a finite set of variables is finite. Indeed, as proved in [5, Theorem 7.3], $\mathbf{CL}$ is isomorphic to $\mathcal{C}(\mathbf{L})$ to the Boolean algebra of conditionals of the Lindenbaum algebra of classical logic, on the same set of variables.

Theorem 3.1: The logic FP(LBC, ŁΠ) is sound and complete for deductions from probabilistic modal formulas w.r.t. to the class of separable models.

Proof: Soundness of (P1), (P2), (P3) and (Nec) follows directly from [10, Lemma 8.4.5.]. Thus, let us show that (Sep) holds in every separable model. If $(\varphi \to \chi) \wedge (\chi \to \psi)$ is a theorem of classical logic and ψ is not a contradiction, then $\|P((\varphi \mid \chi) \wedge (\chi \mid \psi))\|_{\mathcal{S}} = \|P(\varphi \mid \chi)\|_{\mathcal{S}} \cdot \|P(\chi \mid \psi)\|_{\mathcal{S}}$ in any separable model $\mathcal{S} = \langle W, e, \mu \rangle$, because μ satisfies Equation (1) above, and hence $\mathcal{S}$ satisfies $P((\varphi \mid \chi) \wedge (\chi \mid \psi)) \leftrightarrow_L P(\varphi \mid \chi) \odot P(\chi \mid \psi)$.

[3] Without danger of confusion, we will write $\mu(\varphi \mid \psi)$ for $\mu(\text{“}(\varphi \mid \psi)\text{”})$.

As for completeness, we will show that $\not\vdash_{FP} F$ implies $\not\models_{SFP} F$, for any modal formula F. By either adapting the completeness proofs in [8], [11] or adapting the general result proved in [6, Theorem 20 (1)], together with the fact that LBC is locally finite and ŁΠ is finitely strong standard complete, we can show that deductions in $\mathrm{FP}(\mathrm{LBC}, Ł\Pi)$ can be translated to deductions in ŁΠ by considering atomic modal formulas $P\Phi$ as new ŁΠ propositional variables p_Φ. Indeed, it holds that, for any modal formula G, $\vdash_{FP} G$ iff $T \vdash_{Ł\Pi} G^*$, where G^* is the translation of G with the new variables, and T consists of the following three sets of formulas:

(i) $T_0 = \{H^* : H$ is an instance of axioms (P1), (P2), (P3)$\}$
(i)i $T_1 = \{p_\Psi : \psi$ is an LBC-theorem$\}$, that translates the rule (Nec), and
(iii) $T_2 = \{p_{(\varphi|\chi)\wedge(\chi|\psi)} \leftrightarrow_L p_{\varphi|\chi} \odot p_{\chi|\psi} : \vdash_{CPL} (\varphi \to \chi) \wedge (\chi \to \psi)$ and $\not\vdash_{CPL} \neg\chi\}$, that translates the rule (Sep).

Then by the finite-strong completeness of ŁΠ,[4] if F is not a theorem of $\mathrm{FP}(\mathrm{LBC}, Ł\Pi)$, there is a ŁΠ-evaluation v, model of the sets T_0, T_1, T_2 and $v(F) < 1$. Then one can define the probability Kripke model $\mathcal{S} = (W, e, \mu)$, where $W = Seq_n(\Omega)$, $e(w, p) = \overline{w}_1(p)$ for any propositional variable p, and $\mu(\Phi) = v(\Phi)$ for any conditional formula Φ, and show that $\|F\|_\mathcal{S} = v(F) < 1$, that is, $\mathcal{S}$ is a countermodel of F. Thus, it is left to prove that μ is separable. Indeed, in particular v is a model of T_2, that means $v(p_{\varphi|\chi} \odot p_{\chi|\psi}) = v(p_{\varphi|\chi}) \cdot v(p_{\chi|\psi}) = \mu(\varphi \mid \chi) \cdot \mu(\chi \mid \psi)$, for all those conditionals $(\varphi \mid \chi), (\chi \mid \psi)$ such that $\vdash_{CPL} (\varphi \to \chi) \wedge (\chi \to \psi)$ and $\not\vdash_{CPL} \neg\chi$. Thus, μ is separable and the claim is settled. ■

We end this section by noticing that in a separable LBC-Kripke model, a formula $P(\varphi \mid \psi)$ is evaluated by its corresponding conditional probability.

Corollary 3.1: For every basic conditional $(\varphi \mid \psi)$, the following deduction holds in $\mathrm{FP}(\mathrm{LBC}, Ł\Pi)$:[5]

$$\nabla P(\psi \mid \top) \vdash_{FP} P(\varphi \mid \psi) \leftrightarrow_L (P(\psi \mid \top) \to_\Pi P(\varphi \wedge \psi \mid \top))$$

IV. Positive separable models

In this section we will consider a first extension of the logic $\mathrm{FP}(\mathrm{LBC}, Ł\Pi)$ that allows to deal with positive probabilities on basic conditionals of the form $(\varphi \mid \top)$.

Definition 4.1: The logic $\mathrm{FP}(\mathrm{LBC}, Ł\Pi)^+$ is the schematic extension of $\mathrm{FP}(\mathrm{LBC}, Ł\Pi)$ obtained by adding the rule

(Pos) if $\not\vdash_{CPL} \neg\varphi$, derive $\nabla P(\varphi \mid \top)$.

The effect of axiom (Pos) is to force the probability of non-contradictory classical propositions φ (once identified as conditionals $(\varphi \mid \top)$) to be strictly positive. Therefore, it is relatively easy to see that the following holds.

Theorem 4.1: The logic $\mathrm{FP}(\mathrm{LBC}, Ł\Pi)^+$ is sound and complete w.r.t. the class of *positive* separable LBC-Kripke models, i.e. models $\mathcal{S} = \langle W, e, \mu\rangle$ in which μ is a positive probability measure, that is to say, such that $\mu(\Phi) > 0$ for all conditional formula $\Phi \neq \bot$.

Let us call *basic modal formulas* any combination of atomic modal formulas of the form $P(\varphi_i \mid \psi_i)$ with ŁΠ connectives. If we restrict ourselves to this sublanguage of $\mathrm{FP}(\mathrm{LBC}, Ł\Pi)$, we can in fact consider simpler probabilistic models.

Definition 4.2: A *positive simple model* is a pair $\mathcal{P} = \langle\Omega, \sigma\rangle$ where Ω is the set of Boolean interpretations for the base language L, and σ is a positive probability measure on 2^Ω.

Given a basic modal formula B and a positive model $\mathcal{P} = \langle\Omega, \sigma\rangle$, we interpret B in $\mathcal{P}$ as follows:

- If $B = P(\varphi \mid \psi)$, then $\|P(\varphi \mid \psi)\|_\mathcal{P} = \frac{\sigma(\varphi\wedge\psi)}{\sigma(\psi)}$;
- If B is compound use again the truth functionality of ŁΠ connectives interpreted in $[0, 1]$.

Theorem 4.2: (1) For every positive separable LBC-Kripke model $\mathcal{S}$ there exists a positive simple model $\mathcal{P}$ such that $\|B\|_\mathcal{S} = \|B\|_\mathcal{P}$ for every basic modal formula B.
(2) Vice-versa, for every positive simple model $\mathcal{P}$ there exists a positive separable LBC-Kripke model $\mathcal{S}$ such that $\|B\|_\mathcal{P} = \|B\|_\mathcal{S}$ for every basic modal formula B.

Proof: As for (1), let us prove the clam for $B = (\varphi \mid \psi)$. The case of compound conditional formulas, indeed, follows by truth-functionality of the connectives of ŁΠ. Given a positive separable LBC-Kripke model $\mathcal{S} = \langle W, e, \mu\rangle$, define $\sigma(\varphi) = \mu(\varphi \mid \top) = \mu(\{w \in W \mid \overline{w} \models (\varphi \mid \top)\}$. This is a probability on Boolean formulas that can be identified as a probability on 2^Ω. Since μ is positive and separable, we have

$$\mu(\varphi \mid \psi) = \frac{\mu(\varphi \wedge \psi \mid \top)}{\mu(\psi \mid \top)} = \frac{\sigma(\varphi \wedge \psi)}{\sigma(\psi)}.$$

On the other hand, (2) follows by adapting to our logical setting a main result in [5, Theorem 6.13] stated in algebraic terms. Indeed, in that theorem it is proved that, for any positive probability P on an algebra of events $\mathbf{A}$, there is a (plain) probability μ_P on the algebra of conditional events $\mathcal{C}(\mathbf{A})$ such that $\mu_P(a \mid b) = \frac{\mu(a\wedge b)}{\mu(b)}$ whenever $b \neq \bot$. The proof is rather involved and we refer the reader to [5] for full details. ■

V. A logic for (conditional) canonical extensions

As proved in [5] the atoms of a Boolean algebra of conditionals $\mathcal{C}(\mathbf{A})$ can be fully characterized by the atoms of the original algebra $\mathbf{A}$ and, in particular, if $\alpha_1, \ldots, \alpha_n$ are the atoms of $\mathbf{A}$, those of $\mathcal{C}(\mathbf{A})$ are conditional expressions of the form

$$\omega_i = (\alpha_{i_1} \mid \top) \wedge (\alpha_{i_2} \mid \neg\alpha_{i_1}) \wedge \ldots \wedge (\alpha_{i_{n-1}} \mid \bigwedge_{j \leq n-2} \neg\alpha_{i_j}).$$

Since the atoms of the Lindenbaum algebra of classical logic (with, say, k variables $x_1, \ldots, x_k$) are writable as minterms $\alpha_j = \bigwedge x_i^*$, for $x_i^* \in \{x_i, \neg x_i\}$, the atoms of $\mathcal{C}(\mathbf{L})$ are expressible as above. To ease the reading, we will denote them by $\omega_1, \omega_2, \ldots$. Recall form [5] that, if classical logic is defined on k propositional variables, there are $(2^k)!$ atoms of $\mathcal{C}(\mathbf{L})$.

Although not every probability measure on $\mathcal{C}(\mathbf{A})$ satisfies all the axioms of a conditional probability, every positive probability σ on the original algebra $\mathbf{A}$, has an extension to

[4] For our purposes, T can be considered to be finite, because the Lindenbaum Boolean algebra of LBC is finite, that is, there are only finitely-many non-logically equivalent basic conditionals $(\varphi \mid \psi)$, and hence only finitely-many non-logically equivalent formulas in T.

[5] Recall from Section II-B that the interpretation of the connective ∇ is $\nabla(x) = 1$ if $x > 0$, and $\nabla(x) = 0$ otherwise.

a positive probability μ_σ on $\mathcal{C}(\mathbf{A})$. These measures, called *canonical* in [5] are such that, for every atom ω_i of $\mathcal{C}(\mathbf{L})$,

$$\mu_\sigma(\omega_i) = \sigma(\alpha_{i_1} \mid \top)\cdot\sigma(\alpha_{i_2} \mid \neg\alpha_{i_1})\cdot\ldots\cdot\sigma(\alpha_{i_{n-1}} \mid \bigwedge_{j\leq n-2} \neg\alpha_{i_j}).$$

In this section we will show how to further extend the logic $\text{FP}(\text{LBC}, \text{Ł}\Pi)^+$ in order for its models to be defined by canonical extensions μ_σ of this kind. In order to do that, let us consider the following $\text{FP}(\text{LBC}, \text{Ł}\Pi)^+$-formulas:

$$(\text{Can}_i) \quad P(\omega_i) \leftrightarrow_L P(\alpha_{i_1} \mid \top)\odot\ldots\odot P(\alpha_{i_{n-1}} \mid \bigwedge_{j\leq n-2} \neg\alpha_{i_j}),$$

where $\omega_i = (\alpha_{i_1} \mid \top) \wedge \ldots \wedge (\alpha_{i_{n-1}} \mid \bigwedge_{j\leq n-2} \neg\alpha_{i_j})$, with $\alpha_{i_1}, \ldots, \alpha_{i_n}$ being minterms of the propositional language L.

Definition 5.1: Let L be a propositional language with k variables. Then, the logic $\text{FP}(\text{LBC}, \text{Ł}\Pi)^+_c$ is the schematic extension of $\text{FP}(\text{LBC}, \text{Ł}\Pi)^+$ obtained by adding the axioms (Can_i) for all $i = 1, \ldots, (2^k)!$

A separable model $\mathcal{S} = \langle W, e, \mu\rangle$ is *canonical* if there exists a positive probability σ on Ω such that $\mu = \mu_\sigma$, i.e., μ is the canonical extension of some positive σ on Ω.

Finally, we can prove that $\text{FP}(\text{LBC}, \text{Ł}\Pi)^+_c$ is sound and complete w.r.t. to canonical models.

Theorem 5.1: The logic $\text{FP}(\text{LBC}, \text{Ł}\Pi)^+_c$ is sound and complete with respect to the class of canonical models.

Proof: Following the lines we sketched in the proof of Theorem 3.1, it is enough to show that a positive separable model satisfies all the axioms (Can_i) iff the model is canonical.

(Left-to-Right). Let $\mathcal{S} = \langle W, e, \mu\rangle$ be positive separable and satisfying (Can_i) for all i. Thus, $\|P(\omega_i)\|_\mathcal{S} =$
$\|P(\alpha_{i_1} \mid \top) \odot \ldots \odot P(\alpha_{i_{n-1}} \mid \bigwedge_{j\leq i_{n-2}} \neg\alpha_j)\|_\mathcal{S} =$
$\|P(\alpha_{i_1} \mid \top)\|_\mathcal{S} \cdot \ldots \cdot \|P(\alpha_{i_{n-1}} \mid \bigwedge_{j\leq i_{n-2}} \neg\alpha_j)\|_\mathcal{S} =$
$\mu(\alpha_{i_1} \mid \top) \cdot ldots \cdot \mu(\alpha_{i_{n-1}} \mid \bigwedge_{j\leq i_{n-2}} \neg\alpha_j)$.

Since $\mathcal{S}$ is positive and separable, by Theorem 4.2 there is a positive simple model $\mathcal{P} = (\Omega, \sigma)$ such that, for every basic conditional $(\varphi \mid \psi)$, $\mu(\varphi \mid \psi) = \frac{\sigma(\varphi\wedge\psi)}{\sigma(\psi)}$. In particular, $\mu(\alpha_{i_1} \mid \top)\cdot\ldots\cdot\mu(\alpha_{i_{n-1}} \mid \bigwedge_{j\leq i_{n-2}} \neg\alpha_j) = \frac{\alpha_{i_1}}{1} \cdot \frac{\sigma(\alpha_{i_2}\wedge\neg\alpha_{i_1})}{\sigma(\neg\alpha_{i_1})} \cdot \ldots \cdot \frac{\sigma(\alpha_{i_{n-1}}\wedge\bigwedge_{j\leq i_{n-2}}\neg\alpha_j)}{\sigma(\bigwedge_{j\leq n-2}\neg\alpha_{i_j})}$. Thus, $\mu = \mu_\sigma$ and $\mathcal{S}$ is canonical.

(Right-to-Left). Conversely, if $\mathcal{S}$ is canonical, then (Can_i) holds in $\mathcal{S}$ by the very definition of canonical model and the way formulas are interpreted in separable models. ■

In the light of the above argument, we can hence slightly improve the result of Corollary 3.1 as follows.

Corollary 5.1: The following formulas are theorems of $\text{FP}(\text{LBC}, \text{Ł}\Pi)^+_c$:

1) $P(\varphi \mid \psi) \leftrightarrow_L (P(\psi \mid \top) \rightarrow_\Pi P(\varphi\wedge\psi \mid \top))$;

2) $P(\omega) \leftrightarrow_L [P(\top) \rightarrow_\Pi P(\alpha_{i_1}) \odot P(\neg\alpha_{i_1}) \rightarrow_\Pi P(\alpha_{i_2}) \odot \ldots \odot P(\bigwedge_{j\leq i_{n-2}} \alpha_j) \rightarrow_\Pi P(\alpha_{i_{n-1}})]$, for $\omega = (\alpha_{i_1} \mid \top) \wedge (\alpha_{i_2} \mid \neg\alpha_{i_1}) \wedge \ldots \wedge (\alpha_{i_{n-1}} \mid \bigwedge_{j\leq n-2} \alpha_{i_j})$, where the α_{i_j}'s are pairwise different minterms of the propositional language L.

VI. Conclusions

In this paper we have introduced fuzzy modal logics for reasoning about the probability of compound conditionals, the latter being Boolean combinations of basic conditionals $(\varphi \mid \psi)$ and formalized within the recently introduced Logic for Boolean Conditionals LBC [5]. For each of the logics we define, we have proved completeness w.r.t. suitable classes of probability models, where a formula of the kind $P(\varphi \mid \psi)$ is evaluated by a (plain) probability $\mu(\text{“}(\varphi \mid \psi)\text{”})$ of the conditional formula $(\varphi \mid \psi)$. We have shown that, if $\psi \not\vdash_{CPL} \bot$, this probability is in fact a conditional probability, and thus evaluated by the ratio $\mu(\text{“}(\varphi\wedge\psi \mid \top)\text{”})/\mu(\text{“}(\psi \mid \top)\text{”})$.

There are a number of issues on this subject left for future work. Among them, we plan to investigate the relationship of our probability logics for compound conditionals with the approach developed by Sanfilippo et al. to probabilistic inference with conjoined and iterated conditionals based on a different notion of conditional, see e.g. [14], [15]. Another topic of interest is the application of these logics to reason with (semi-) fuzzy quantifiers [2]. In addition, we plan to investigate complexity bounds for the SAT problem for $\text{FP}(\text{LBC}, \text{Ł}\Pi)$. Although it seems reasonable to conjecture that logic to be decidable, while the logics ŁΠ and FP(ŁΠ) are known to be in PSPACE [12], the complexity of the logic LBC is not known yet, and this latter non-trivial fact needs to be solved first.

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