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Maximality in finite-valued Łukasiewicz logics defined by order filters

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1 Preliminaries and first results

In this talk we consider the logics \mathbf{L}_n^i obtained from the $(n+1)$ -valued Łukasiewicz logics \mathbf{L}_{n+1} by taking the order filter generated by i/n as the set of designated elements. The $(n+1)$ -valued Łukasiewicz logic can be semantically defined as the matrix logic

$$\mathbf{L}_{n+1} = \langle \mathbf{LV}_{n+1}, \{1\} \rangle,$$

where $\mathbf{LV}_{n+1} = (LV_{n+1}, \neg, \rightarrow)$ with $LV_{n+1} = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$, and the operations are defined as follows: for every $x, y \in LV_{n+1}$, $\neg x = 1 - x$ and $x \rightarrow y = \min\{1, 1 - x + y\}$.

Observe that \mathbf{L}_2 is the usual presentation of classical propositional logic CPL as a matrix logic over the two-element Boolean algebra \mathbf{B}_2 with domain $\{0, 1\}$ and signature $\{\neg, \rightarrow\}$. The logics \mathbf{L}_n can also be presented as Hilbert calculi that are axiomatic extensions of the infinite-valued Łukasiewicz logic \mathbf{L}_∞ .

The following operations can be defined in every algebra \mathbf{LV}_{n+1} : $x \otimes y = \neg(x \rightarrow \neg y) = \max\{0, x + y - 1\}$ and $x \oplus y = \neg x \rightarrow y = \min\{1, x + y\}$. For every $n > 1$, $x^n = x \otimes \dots \otimes x$ (n -times) and $nx = x \oplus \dots \oplus x$ (n -times).

For $1 \leq i \leq n$ let $F_{i/n} = \{x \in LV_{n+1} : x \geq i/n\} = \{\frac{i}{n}, \dots, \frac{n-1}{n}, 1\}$ be the order filter generated by i/n , and let

$$\mathbf{L}_n^i = \langle \mathbf{LV}_{n+1}, F_{i/n} \rangle$$

be the corresponding matrix logic. From now on, the consequence relation of \mathbf{L}_n^i is denoted by $\models_{\mathbf{L}_n^i}$. Observe that $\mathbf{L}_{n+1} = \mathbf{L}_n^n$ for every n . In particular, CPL is \mathbf{L}_1^1 (that is, \mathbf{L}_2). If $1 \leq i, m \leq n$, we can also consider the following matrix logic: $\mathbf{L}_m^{i/n} = \langle \mathbf{LV}_{m+1}, F_{i/n} \cap LV_{m+1} \rangle$.

The logic $\mathbf{L}_2^1 = \langle LV_3, \{1, 1/2\} \rangle$ was already known as the 3-valued paraconsistent logic \mathbf{J}_3 , introduced by da Costa and D'Ottaviano see [4] in order to obtain an example of a paraconsistent logic maximal w.r.t. CPL.

Definition 1. Let L_1 and L_2 be two standard propositional logics defined over the same signature Θ such that L_1 is a proper sublogic of L_2 . Then, L_1 is *maximal* w.r.t. L_2 if, for every formula φ over Θ , if $\vdash_{L_2} \varphi$ but $\not\vdash_{L_1} \varphi$, then the logic L_1^+ obtained from L_1 by adding φ as a theorem, coincides with L_2 .

In order to study maximality among finite-valued Łukasiewicz logics defined by order filters we obtain the following sufficient condition:

Theorem 1. Let $L_1 = \langle \mathbf{A}_1, F_1 \rangle$ and $L_2 = \langle \mathbf{A}_2, F_2 \rangle$ be two distinct finite matrix logics over a same signature Θ such that \mathbf{A}_2 is a subalgebra of \mathbf{A}_1 and $F_2 = F_1 \cap A_2$. Assume the following:

1. $A_1 = \{0, 1, a_1, \dots, a_k, a_{k+1}, \dots, a_n\}$ and $A_2 = \{0, 1, a_1, \dots, a_k\}$ are finite such that $0 \notin F_1$, $1 \in F_2$ and $\{0, 1\}$ is a subalgebra of \mathbf{A}_2 .
2. There are formulas $\top(p)$ and $\perp(p)$ in $\mathcal{L}(\Theta)$ depending at most on one variable p such that $e(\top(p)) = 1$ and $e(\perp(p)) = 0$, for every evaluation e for L_1 .
3. For every $k+1 \leq i \leq n$ and $1 \leq j \leq n$ (with $i \neq j$) there exists a formula $\alpha_j^i(p)$ in $\mathcal{L}(\Theta)$ depending at most on one variable p such that, for every evaluation e , $e(\alpha_j^i(p)) = a_j$ if $e(p) = a_i$.

Then, L_1 is maximal w.r.t. L_2 .

We use this result to prove that

Theorem 2. Let $1 \leq i, m \leq n$. Then \mathbb{L}_n^i is maximal w.r.t. $\mathbb{L}_m^{i/n}$ if the following condition holds: there is some prime number p and $k \geq 1$ such that $n = p^k$, and $m = p^{k-1}$.

Corollary 1. Let $1 \leq i \leq p$. For every prime number p , \mathbb{L}_p^i is maximal w.r.t. CPL

Notice that the above corollary generalizes the well known result: L_{p+1} is maximal w.r.t. CPL for every prime number p .

Definition 2. Let L_1 and L_2 be two standard propositional logics defined over the same signature Θ such that L_1 is a proper sublogic of L_2 . Then, L_1 is *strongly maximal* w.r.t. L_2 if, for every finitary rule $\varphi_1, \dots, \varphi_n / \psi$ over Θ , if $\varphi_1, \dots, \varphi_n \vdash_{L_2} \psi$ but $\varphi_1, \dots, \varphi_n \not\vdash_{L_1} \psi$, then the logic L_1^* obtained from L_1 by adding $\varphi_1, \dots, \varphi_n / \psi$ as structural rule, coincides with L_2 .

Let i be a strictly positive integer, the i -explosion rule is the rule $(exp_i) \frac{i(\varphi \wedge \neg \varphi)}{\perp}$.

Lemma 1. For every $1 \leq i \leq n$, the rule (exp_i) is not valid in \mathbb{L}_n^i .

Corollary 2. Let $1 \leq i \leq p$. For every prime number p , \mathbb{L}_p^i is not strongly maximal w.r.t. CPL

2 Equivalent systems

Blok and Pigozzi introduce in [3] the notion of equivalent deductive systems in the following sense: Two propositional deductive systems S_1 and S_2 in the same language \mathcal{L} are equivalent iff there are two translations τ_1, τ_2 (finite subsets of \mathcal{L} -propositional formulas in one variable) such that:

- $\Gamma \vdash_{S_1} \varphi$ iff $\tau_1(\Gamma) \vdash_{S_2} \tau_1(\varphi)$,
- $\Delta \vdash_{S_2} \psi$ iff $\tau_2(\Delta) \vdash_{S_1} \tau_2(\psi)$,
- $\varphi \dashv\vdash_{S_1} \tau_2(\tau_1(\varphi))$,
- $\psi \dashv\vdash_{S_2} \tau_1(\tau_2(\psi))$.

Theorem 3. For every $n \geq 2$ and every $1 \leq i \leq n$, \mathbb{L}_n^i and \mathbb{L}^{n+1} are equivalent deductive systems.

From the equivalence among \mathbf{L}_n^i and \mathbf{L}_{n+1} , we can obtain, by translating the axiomatization of the finite valued Łukasiewicz logic \mathbf{L}_{n+1} , a calculus sound and complete with respect \mathbf{L}_n^i that we denote by \mathbf{H}_n^i .

Since \mathbf{L}_∞ is algebraizable and the class MV of all MV-algebras is its equivalent quasivariety semantics, finitary extensions of \mathbf{L}_∞ are in 1 to 1 correspondence with quasivarieties of MV-algebras. Actually, there is a dual isomorphism from the lattice of all finitary extensions of \mathbf{L}_∞ and the lattice of all quasivarieties of MV . Moreover, if we restrict this correspondence to varieties of MV we get the dual isomorphism from the lattice of all varieties of MV and the lattice of all axiomatic extensions of \mathbf{L}_∞ . Since $\mathbf{L}_{n+1} = \mathbf{L}_n^n$ is an axiomatic extension of \mathbf{L}_∞ , \mathbf{L}_{n+1} is an algebraizable logic with the class $MV_n = \mathcal{Q}(\mathbf{LV}_{n+1})$, the quasivariety generated by \mathbf{LV}_{n+1} , as its equivalent variety semantics. It follows from the previous theorem that \mathbf{L}_n^i , for every $1 \leq i \leq n$, is also algebraizable with the same class of MV_n -algebras as its equivalent variety semantics. Thus, the lattices of all finitary extensions of \mathbf{L}_n^i are isomorphic, and in fact, dually isomorphic to the lattice of all subquasivarieties of MV_n , for all $0 < i < n$.

Therefore maximality conditions in the lattice of finitary (axiomatic) extensions correspond to minimality conditions in the lattice of subquasivarieties (subvarieties). Thus, given two finitary extensions L_1 and L_2 of a given logic \mathbf{L}_n^i , where K_{L_1} and K_{L_2} are its associated MV_n -quasivarieties, L_1 is strongly maximal with respect L_2 iff K_{L_1} is a minimal subquasivariety of MV_n among those MV_n -quasivarieties properly containing K_{L_2} . Moreover, if L_1 and L_2 are axiomatic extensions of \mathbf{L}_n^i , then K_{L_1} and K_{L_2} are indeed MV_n -varieties. In that case, L_1 is maximal with respect L_2 iff K_{L_1} is a minimal subvariety of MV_n among those MV_n -varieties properly containing K_{L_2} .

The lattice of all axiomatic extensions \mathbf{L}_∞ is fully described also by Komori in [7], thus from the equivalence of Theorem 3, we can obtain the following maximality conditions for all axiomatic extensions of \mathbf{L}_n^i .

Theorem 4. *Let $0 < i, m \leq n$ be natural numbers such that $m|n$. If L is an axiomatic extension of \mathbf{L}_n^i , then L is maximal with respect to $\mathbf{L}_m^{i/n}$ iff $L = \mathbf{L}_m^{i/n} \cap \mathbf{L}_{p^{k+1}}^{i/n}$ for some prime number p with $p|n$ and a natural $k \geq 0$ such that $p^k|m$ and $p^{k+1} \nmid m$.*

As a corollary we obtain that the sufficient condition of Theorem 2 is also necessary.

Corollary 3. *Let $1 \leq i, m \leq n$. Then \mathbf{L}_n^i is maximal w.r.t. $\mathbf{L}_m^{i/n}$ if and only if there is some prime number p and $k \geq 1$ such that $n = p^k$, and $m = p^{k-1}$.*

To obtain results on strong maximality we need to study finitary extensions of \mathbf{L}_∞ . The task of fully describing the lattice of all all finitary extensions of \mathbf{L}_∞ , isomorphic to the lattice of all subquasivarieties of MV , turns to be an heroic task since the class of all MV-algebras is Q -universal [1]. For the finite valued case it is much simpler, since MV_n is a locally finite discriminator variety. Any locally finite quasivariety is generated by its critical algebras [5]. Critical MV-algebras were fully described in [6] and using this description we can obtain some results on strong maximality.

First we need to introduce the following matrix logics: For every $1 \leq i, m \leq n$,

$$\bar{\mathbf{L}}_n^i = \langle \mathbf{LV}_{n+1} \times \mathbf{LV}_2, F_{i/n} \times \{1\} \rangle \quad \bar{\mathbf{L}}_m^{i/n} = \langle \mathbf{LV}_{m+1} \times \mathbf{LV}_2, (F_{i/n} \cap \mathbf{LV}_{m+1}) \times \{1\} \rangle$$

Theorem 5. *Let $0 < i \leq n$ be natural numbers, let p be a prime number and let $r = \max\{j \in \mathbb{N} : p^j|n\}$. Then we have: For every j such that $(i-1)p < j \leq ip$, $\mathbf{L}_n^i \cap \bar{\mathbf{L}}_{p^{r+1}}^{j/n}$ is strongly maximal with respect to \mathbf{L}_n^i . Moreover, every finitary extension of some \mathbf{L}_k^j is strongly maximal with respect \mathbf{L}_n^i iff it is one of the preceding types.*

As a particular case we can also prove the following result.

Theorem 6. *Let p be a prime number. Then, for every j such that $0 < j \leq p$:*

- \bar{L}_p^j is strongly maximal with respect to CPL and it is axiomatized by H_p^j plus the j -explosion rule (exp_j) $j(\varphi \wedge \neg\varphi)/\perp$.
- L_p^j is strongly maximal w.r.t. \bar{L}_p^j .

3 Ideal paraconsistent logics

Arieli, Avron and Zamansky introduced in [2] the concept of *ideal paraconsistent logics*.

Definition 3. Let L be a propositional logic defined over a signature Θ (with consequence relation \vdash_L) containing at least a unary connective \neg and a binary connective \rightarrow such that:

- (i) L is paraconsistent w.r.t. \neg , i.e. there are formulas $\varphi, \psi \in \mathcal{L}(\Theta)$ such that $\varphi, \neg\varphi \not\vdash_L \psi$; and \rightarrow is a deductive implication, i.e. $\Gamma \cup \{\varphi\} \vdash_L \psi$ iff $\Gamma \vdash_L \varphi \rightarrow \psi$.
- (ii) There is a presentation of CPL as a matrix logic $L' = \langle \mathbf{A}, \{1\} \rangle$ over the signature Θ such that the domain of \mathbf{A} is $\{0, 1\}$, and \neg and \rightarrow are interpreted as the usual 2-valued negation and implication of CPL, respectively, such that L is a sublogic of CPL.

Then, L is said to be an *ideal paraconsistent logic* if it is maximal w.r.t. CPL, and every proper extension of L over Θ is not \neg -paraconsistent.

Lemma 2. *Let $0 < i \leq n$. L_n^i is paraconsistent w.r.t. \neg iff $\frac{i}{n} \leq \frac{1}{2}$*

Since for every $0 < i \leq n$, there is a term definable implication \Rightarrow_n^i which is deductive implication next result follows from Theorem 6

Theorem 7. *Let p be a prime number, and let $1 \leq i < p$ such that $i/p \leq 1/2$. Then, L_p^i is a $(p+1)$ -valued ideal paraconsistent logic.¹*

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¹Strictly speaking, in this claim we implicitly assume that the signature of L_p^i has been changed by adding the definable implication \Rightarrow_p^i as a primitive connective.