Interpolation of fuzzy data: Analytical approach and overview

Irina Perfilieva[a,*], Didier Dubois[b], Henri Prade[b], Francesc Esteva[c],
Lluis Godo[c], Petra Hoďáková[a]

[a] *University of Ostrava, Institute for Research and Applications of Fuzzy Modeling, 30. dubna 22, 701 03 Ostrava 1, Czech Republic*
[b] *IRIT, CNRS and Universit de Toulouse, 118 route de Narbonne, 31062 Toulouse, France*
[c] *Artificial Intelligence Research Institute (IIIA), Spanish National Research Council (CSIC), 08193 Bellaterra, Spain*

Abstract

We propose a general framework for the interpolation problem. Our framework stems from the classical elaboration of the problem. We introduce the notion of an interpolating fuzzy function and show how this function can be characterized. We examine and analyze previously published fuzzy interpolation algorithms to choose those algorithms that can be represented analytically. We also propose an analytic solution of the interpolation problem that unifies various algorithmic approaches.
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1. Introduction

As is well known, a fuzzy rule base is a characterization of a partially given mapping (fuzzy function) between sets of fuzzy sets (fuzzy spaces). Each rule characterizes a node (argument-value) of a respective fuzzy function in the sense that the antecedent of the rule characterizes the argument, whereas the consequent of the rule characterizes the dependent value. The entire finite rule base characterizes the behavior of that function at the respective finite set of nodes. For practical applications, it is desirable to interpolate that partial fuzzy function to know its values at an arbitrary node (fuzzy or crisp). Therefore, the *problem of interpolation* of a partial fuzzy function consists of its extension to a fuzzy function that is defined on an extended domain. Moreover, interpolation assumes that if we restrict the extended (interpolating) fuzzy function to its original domain, we then obtain the original partial fuzzy function.

Many papers are focused on the problem of fuzzy interpolation and give an algorithmic description of one of the possible extensions of a partial fuzzy function. Below, we list various approaches to the problem of fuzzy interpolation and indicate their principles: level cuts interpolation [22,23], analogy-based interpolation [4,5,7], interpolation by convex completion [10,28], interpolation by geometric transformations [1], interpolation in a family of interpolating

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* Corresponding author.
E-mail addresses: Irina.Perfilieva@osu.cz (I. Perfilieva), dubois@irit.fr (D. Dubois), prade@irit.fr (H. Prade), esteva@iiia.csic.es (F. Esteva), godo@iiia.csic.es (L. Godo), Petra.Hodakova@osu.cz (P. Hoďáková).

relations [2], polar cut interpolation [20], interpolation based on closeness relations [6], flank functions interpolation [18,19], and analytic fuzzy relation-based interpolation [25]. This paper will develop and extend the latter approach.

The main drawback of algorithmic approaches to the problem of fuzzy interpolation is their insufficient description of the new extended domain of an interpolating fuzzy function. As a result of this insufficient description, some of the proposed algorithms fail when applied to fuzzy sets outside of a respective domain. One "unhappy" example, proposed in [22,23], was analyzed in [29] where it has been shown that on the extended domain, the proposed interpolation does not fulfil the necessary requirement (the level cuts of the result of the interpolation are no longer nested). A precise description of the extended domain of fuzzy interpolation is especially important in the case of a sparse rule base interpolation (see Section 4.3).

We will base our research on two overview papers [5,18]. We will examine and analyze the fuzzy interpolation algorithms listed above. We will identify those algorithms that can be represented analytically as interpolating fuzzy functions. Therefore, we will concentrate on those approaches that are most suitable for this purpose. We will present an overview and analysis of the following approaches: analogy-based interpolation, interpolation by convex completion, interpolation based on closeness, and flank functions interpolation. In all cases, we will show that solutions proposed therein can be characterized by interpolating fuzzy functions and that these functions can be represented by respective fuzzy relations. They can therefore be used easily in computation. Last but not least, interpolating fuzzy functions can be used for interpolation of sparse and non-sparse (dense) fuzzy data.

Our paper is organized as follows. In Section 2, we give preliminary information about residuated lattices, fuzzy sets, and fuzzy spaces. Fuzzy functions and various approaches to this notion are discussed in Section 3. These notions are then used to formulate the problem of interpolation of fuzzy data and to present a particular case of the problem: interpolation of sparse rule bases (Sections 4). In Sections 5–8, we give a general overview of the principal approaches to this specific problem. In these sections, we analyze four particular methods: analogy-based interpolation, interpolation by convex completion, interpolation based on closeness, and flank functions interpolation. An *analytic representation* of a respective interpolating fuzzy function is presented for each of these methods.

2. Preliminaries

The notion of fuzzy function is closely related to and generalizes the notion of function. First, consider the two alternative definitions of a function. Both definitions assume that two sets, a domain X and a range Y, are given. According to the first definition, a *correspondence* $f : x \mapsto y$ is a *function* if for any $x \in X$, the value $y = f(x)$ is uniquely determined. The latter statement means that for all $x_1, x_2 \in X$ such that $x_1 = x_2$, the respective values $f(x_1), f(x_2)$ are equal as well (we will say that f respects the equality). The second definition introduces a function as a *special binary relation* $\rho \subseteq X \times Y$ such that for all $x \in X$, and all $y_1, y_2 \in Y$, $(x, y_1) \in \rho \& (x, y_2) \in \rho \Rightarrow y_1 = y_2$.

We observe immediately that the first definition (declarative) emphasizes the computational aspect of a function, whereas the second definition (descriptive) underscores the specification of a function as a binary relation. A further observation suggests that both definitions are based on the notions of set and equality. Therefore, generalized notions of a set and of an equality relation defined on that set should be used to introduce a generalized notion of function.

The following subsections introduce the notions of *L*-fuzzy space and similarity-based *L*-fuzzy space. Both notions involve classes of *L*-valued fuzzy sets enriched with fuzzy equalities.

2.1. Residuated lattice

Because the notion of fuzzy equality requires a scale of degrees (a lattice of truth values) and an algebra of operations on that scale, we will first introduce these preliminary notions. We will choose a bounded lattice for a scale of degrees, add a monoidal operation, and assume that the latter is residuated. We thus obtain a residuated lattice—our basic algebra of operations. We will use an algebraic approach for the definition of the lattice and will give a definition of a residuated lattice as an algebra.

Definition 1. A residuated lattice [1] is an algebra

$$\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, \mathbf{0}, \mathbf{1} \rangle.$$

with a support L and four binary operations and two constants such that

- $\langle L, \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$ is a lattice where the ordering $\leq$ defined using operations $\vee, \wedge$ as usual, and $\mathbf{0}, \mathbf{1}$ are the least and the greatest elements, respectively;
- $\langle L, *, \mathbf{1} \rangle$ is a commutative monoid, that is, $*$ is a commutative and associative operation with the identity $a * \mathbf{1} = a$;
- the operation $\rightarrow$ is a residuation operation with respect to $*$, i.e.,

$$a * b \leq c \quad \text{iff} \quad a \leq b \rightarrow c.$$

A residuated lattice is complete if it is complete as a lattice.

The following is the binary operation of biresiduation on $\mathcal{L}$:

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a).$$

Well known examples of residuated lattices are: Boolean algebra, Gödel, Łukasiewicz and product algebras. In the particular case of $L = [0, 1]$, the monoidal operation $*$ is in fact a left-continuous t-norm.

Hereafter and until the end of Section 3, let the complete residuated $\mathcal{L}$ be a fixed algebraic structure.

2.2. *L-valued fuzzy sets and L-fuzzy space*

Let $X \subseteq \mathbb{R}$ be a non-empty universal set, $\mathbb{R}$ the set of reals, and $\mathcal{L}$ a complete residuated lattice. An *L*-valued fuzzy set A on X (in brief, a fuzzy set) is a map $A : X \rightarrow L$. A fuzzy set is identified with its membership function. A fuzzy set A is *normal* if there exists $x_A \in X$ such that $A(x_A) = 1$. The *empty fuzzy set* is the set $\mathbf{0}$ with zero membership function. The (ordinary) set $Core(A) = \{x \in X | A(x) = 1\}$ is the *core* of the normal fuzzy set A. The (ordinary) closed set $\text{Supp}(A) = \overline{\{x \in X | A(x) > 0\}}$ is the *support* of the fuzzy set A. The class of *L*-valued fuzzy sets of X will be denoted by L^X. Similarly, we define a (*binary*) (*L-valued*) *fuzzy relation* as a fuzzy set on $X \times Y$.

A notion of *fuzzy equality* $\equiv$ between fuzzy sets $A, B \in L^X$ is defined as follows:

$$(A \equiv B) = \bigwedge_{x \in X} (A(x) \leftrightarrow B(x)).^2 \tag{1}$$

Indeed $(A \equiv B) \in L$ is a degree of coincidence between the two membership functions A and B. It is known that

$$(A \equiv B) = 1 \quad \text{iff } \forall x \in X, \quad A(x) = B(x),$$

that is, $(A \equiv B) = 1$ if and only if the membership functions A and B coincide (on X). Below, instead of $(A \equiv B) = 1$ we will write $A = B$.

Definition 2. The pair $(L^X, \equiv)$ is a fuzzy space on X.

A $*$-*fuzzy preorder* $\lesssim$ on L^X can be introduced as follows [3]:

$$(A \lesssim B) = \bigwedge_{x \in X} (A(x) \rightarrow B(x)), \tag{2}$$

where $(A \lesssim B) \in L$ is also known as a degree of "inclusion" of the fuzzy set A into the fuzzy set B [15]. Note that $(A \lesssim B) = 1$ if and only if $A(x) \leq B(x)$ for all $x \in X$. Below, instead of $(A \lesssim B) = 1$ we will write $A \leq B$. [4]

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[1] In this paper we assume a residuated lattice to be bounded, commutative and integral.

[2] For a general definition of fuzzy equality, e.g., [8,17,21].

[3] For a general definition of fuzzy preorder see e.g., [3,24].

[4] In this paper, we use $A \leq B$ for representing Zadeh fuzzy set inclusion of A in B instead of the usual notation $\subset$ because A and B denote the membership functions and $\leq$ then represent pointwise inequality between functions.

2.3. Similarity-based fuzzy space

Another definition of a fuzzy space on *X* is related to the notion of similarity (indistinguishability, fuzzy equality). See the references cited above. This space consists of similarity classes of single points of *X*. Below, we will give necessary definitions and then compare different fuzzy spaces.

A binary fuzzy relation *E* on *X* is called a *similarity* on *X* if for all $x, y, z \in X$, the following properties hold:

S.1 $E(x, x) = 1$,
S.2 $E(x, y) = E(y, x)$,
S.3 $E(x, y) * E(y, z) \leq E(x, z)$.

It is informative to observe that the operation $*$ in the definition of similarity belongs to the residuated lattice $\mathcal{L}$, of fixed structure as specified above. Given a similarity *E* on *X*, certain fuzzy sets of *X* can be characterized as similarity classes of single elements of *X*, i.e., as fuzzy sets determined by membership functions of the form E_t, with $t \in X$, where $E_t(x) = E(t, x)$ for all $x \in X$. The fuzzy set E_t is called an *E-fuzzy point* of *X*. Because $E_t(t) = 1$, E_t is a normal fuzzy set. Denote by

$$(X, E) = \{E_t | t \in X\}$$

the class of all *E*-fuzzy points of *X*, and call it a *similarity-based fuzzy space* of *X*. It is not difficult to see that for any $t, p \in X$,

$$(E_t \equiv E_p) = E(t, p).$$

According to [21], a subclass $\mathcal{S} \subseteq L^X$ of normal fuzzy sets of *X* is a similarity-based fuzzy space on *X* if and only if for any two fuzzy sets $A, B \in \mathcal{S}$, the following inequality holds:

$$\bigvee_{x \in X} (A(x) * B(x)) \leq \bigwedge_{x \in X} (A(x) \leftrightarrow B(x)). \tag{3}$$

In this case, we have $\mathcal{S} = (X, E_{\mathcal{S}})$, where the similarity $E_{\mathcal{S}}$ on *X* is given by

$$E_{\mathcal{S}}(x, y) = \bigwedge_{A \in \mathcal{S}} (A(x) \leftrightarrow A(y)).$$

Thus the subclass $\mathcal{S}$ forms a fuzzy partition of *X*. In the Boolean case, inequality (3) means that if *A* is not the same as *B*, then they are disjoint. In the fuzzy case, this inequality does not prevent elements of a fuzzy partition from having overlapping supports.

Remark 1. If *(X,E)* is a fuzzy space on *X*, and E_t is a *E*-fuzzy point for some $t \in X$, then $t \in Core(E_t)$. However, $Core(E_t)$ can contain other elements than *t*. It can be easily shown that if $t_1 \in Core(E_t)$ and $t_1 \neq t$, then $E_{t_1} = E_t$.

2.4. Relation-based fuzzy space

Let us associate a fuzzy space on *X* with a fuzzy relation that is weaker than a similarity. A binary fuzzy relation *R* on *X* is *reflexive* if for all $x \in X$, $R(x, x) = 1$. Given a reflexive fuzzy relation *R* on *X*, consider the fuzzy set R_t, for $t \in X$, defined by $R_t(x) = R(t, x)$ for all $x \in X$ and call it *R-fuzzy point* of *X*. By reflexivity of *R*, R_t is a normal fuzzy set. The class $(X, R) = \{R_t | t \in X\}$ that consists of all *R*-fuzzy points of *X* is called the *R-fuzzy space* on *X*.

3. Fuzzy functions

There exist several definitions of a fuzzy function, e.g., [8,11,16,21,25]. In [8,16,21], a fuzzy function is viewed as a special fuzzy relation, in [11] it is a fuzzy set of functions; in all cases, the authors' approach is descriptive. In [25], a fuzzy function is a mapping between two fuzzy spaces (in other words, classes of fuzzy sets), and therefore, the authors' approach is declarative. In this section, we review both approaches briefly, and we explain under what circumstances they coincide. To distinguish different notions, we will use the name "*E–F*-fuzzy function" for a fuzzy

function that is a special fuzzy relation. We will retain the simple name "fuzzy function" to denote a mapping between two fuzzy spaces.

3.1. Fuzzy function as a mapping

We will define a *fuzzy function* following [25]. Recall that for the purposes of our discussion, a fuzzy function is a mapping that respects the (ordinary) equality relation. Moreover, we will define an *extensional fuzzy function* which respects a certain fuzzy equality.

Definition 3. Let $\mathcal{L}$ be a complete residuated lattice and $(L^X, \equiv)$, $(L^Y, \equiv)$ fuzzy spaces on X and Y, respectively.

- A correspondence $f : L^X \to L^Y$ is a fuzzy function if for every $A, B \in L^X$,

$$A = B \text{ implies } f(A) = f(B). \tag{4}$$

- A fuzzy function $f : L^X \to L^Y$ is extensional if for every $A, B \in L^X$,

$$(A \equiv B) \leq (f(A) \equiv f(B)).$$

Note that a fuzzy function so defined is indeed nothing more than a conventional function from L^X to L^Y. The adjective "fuzzy" stresses a correspondence between fuzzy spaces. An example of a fuzzy function is given below.

Example 1 (*Fuzzy functions determined by a fuzzy relation*). Let $(L^X, \equiv)$, $(L^Y, \equiv)$ be fuzzy spaces on X and Y, respectively, $R \in L^{X \times Y}$ a fuzzy relation. For every $A \in L^X$, we define the $\circ$-*composition* of A and R by

$$(A \circ R)(y) = \bigvee_{x \in X} (A(x) * R(x, y)), \tag{5}$$

and the $\triangleleft$-*composition* of A and R by

$$(A \triangleleft R)(y) = \bigwedge_{x \in X} (A(x) \to R(x, y)), \tag{6}$$

Both compositions determine respective fuzzy sets $A \circ R$ and $A \triangleleft R$ of Y. The corresponding mappings $f_{\circ R} : A \mapsto A \circ R$ and $f_{\triangleleft R} : A \mapsto A \triangleleft R$ are fuzzy functions defined on the whole fuzzy space L^X. It can be proved that both $f_{\circ R}$ and $f_{\triangleleft R}$ are extensional fuzzy functions.

3.2. E–F-fuzzy function

Below, we recall the notion of fuzzy function (see [21]). [5] In discussing this notion, we will use the term *E–F-fuzzy function*.

Definition 4. Let E, F be respective similarities on X and Y. An *E–F-fuzzy function* is a binary fuzzy relation ρ on $X \times Y$ such that for all $x, x' \in X$, $y, y' \in Y$ the following axioms hold true:

- $\rho(x, y) * E(x, x') \leq \rho(x', y)$,
- $\rho(x, y) * F(y, y') \leq \rho(x, y')$,
- $\rho(x, y) * \rho(x, y') \leq F(y, y')$.

The following example [21] relates an ordinary function and an *E–F*-fuzzy function.

Example 2. Let E, F be respective similarities on X and Y, and a function $g : X \to Y$ extensional with respect to E and F so that for all $x, x' \in X$,

$$E(x', x) \leq F(g(x'), g(x)). \tag{7}$$

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[5] In [21], it has been introduced as partial fuzzy function.

Then the fuzzy relation ρ_g given by

$$\rho_g(x, y) = \bigvee_{x' \in X} (E(x', x) * F(g(x'), y)), \quad x \in X, y \in Y \tag{8}$$

is an *E–F*-fuzzy function. Moreover, for all $x \in X$ and $y \in Y$,

$$\rho_g(x, y) = F(g(x), y). \tag{9}$$

Note that an *E–F*-fuzzy function is neither a correspondence between *X* and *Y* nor a correspondence between L^X and L^Y. However, the corresponding fuzzy relation ρ determines a fuzzy function $f_{\circ\rho}$ in the sense of (5). Therefore, it is informative to investigate the relationship between a fuzzy function expressed by (5) and an *E–F*-fuzzy function when both are determined by a common fuzzy relation.

Let us recall the following statement [27]: if *E* and *F* are similarities on *X* and *Y*, respectively, and ρ_g is given by (8), then the following equality:

$$(E_{x'} \circ \rho_g)(y) = F(g(x'), y) \tag{10}$$

holds for all $x' \in X$ if and only if function *g* satisfies (7).

Accordingly, a relationship between fuzzy functions and *E–F*-fuzzy functions can be characterized as follows: if *E* and *F* are similarities on *X* and *Y*, respectively, the function $g : X \to Y$ fulfills (7), and the fuzzy relation ρ_g is given by (8), then ρ_g is an *E–F*-fuzzy function and ρ_g also determines the fuzzy function $f_{\circ\rho_g}$. Moreover, by (9) and (10), the values of ρ_g and $f_{\circ\rho_g}$ at respective "points" x' and $E_{x'}$ coincide, i.e., for all $y \in Y$,

$$\rho_g(x', y) = f_{\circ\rho_g}(E_{x'})(y).$$

3.3. R–F-fuzzy function

Note that an *E–F*-fuzzy function establishes a correspondence between two similarity-based fuzzy spaces (*X,E*) and (*Y,F*). However, in our investigation we will work also with an *R*-fuzzy space (*X,R*) where *R* is a reflexive fuzzy relation. Therefore, we need to extend an *E–F*-fuzzy function to an *R–F*-fuzzy function, the latter establishes a correspondence between (*X,R*) and (*Y,F*). Below, we give a new definition that slightly modifies Definition 4.

Definition 5. Let *R* be a reflexive fuzzy relation on *X* and *F* a similarity on *Y*. An *R–F*-fuzzy function is a binary fuzzy relation ρ on $X \times Y$ such that for all $x, x' \in X$, $y, y' \in Y$ the same axioms (where *E* should be replaced by *R*) as in Definition 4 hold true.

We will prove that fuzzy relation ρ_g expressed by formula (12) is an *R–F*-fuzzy function. Moreover, the corresponding fuzzy functions $f_{\circ\rho_g}$ establish a correspondence between the *R*-fuzzy space (*X,R*) and the similarity-based fuzzy space (*Y,F*).

Theorem 1. *Let R be a reflexive fuzzy relation on X and F a similarity on Y. Let $g : X \to Y$ be an extensional with respect to R and F function, i.e. for all $x, x' \in X$ the following holds*:

$$R(x', x) \le F(g(x'), g(x)). \tag{11}$$

Then the fuzzy relation

$$\rho_g(x, y) = \bigvee_{x' \in X} (R(x', x) * F(g(x'), y)), \quad x \in X, \ y \in Y, \tag{12}$$

fulfils the following properties:

(i) *for all $x \in X$ and $y \in Y$*,

$$\rho_g(x, y) = F(g(x), y),$$

(ii) *ρ_g is an R–F-fuzzy function*,

(iii) *for all $x' \in X$*,

$$f_{\circ \rho_g}(R'_x) = F_{g(x')}. \tag{13}$$

Proof. Assume that the assumptions of Theorem 1 are fulfilled.

(i) We will split the equality $\rho_g(x, y) = F(g(x), y)$ into two opposite inequalities and prove them. By (11) and transitivity of F,

$$\rho_g(x, y) = \bigvee_{x' \in X} (E(x', x) * F(g(x'), y))$$

$$\leq \bigvee_{x' \in X} (F(g(x'), g(x)) * F(g(x'), y)) \leq F(g(x), y).$$

On the other hand, by reflexivity of R,

$$\rho_g(x, y) = \bigvee_{x' \in X} (E(x', x) * F(g(x'), y)) \geq E(x, x) * F(g(x), y) = F(g(x), y).$$

(ii) To prove that ρ_g is an R–F-fuzzy function means to verify the axioms of Definition 4 where E is replaced by R. We will verify the first axiom (the other two are simple exercises), i.e., for all $x, x' \in X$, $y, y' \in Y$, $\rho_g(x, y) * R(x, x') \leq \rho_g(x', y)$. Indeed,

$$\rho_g(x, y) * R(x, x') = R(x, x') * F(g(x), y)$$

$$\leq \bigvee_{x \in X} (R(x, x') * F(g(x), y)) = \rho_g(x', y).$$

(iii) Finally, let us verify (13) by proving two opposite inequalities for arbitrarily chosen $x' \in X$ and $y \in Y$:

$$f_{\circ \rho_g}(R'_x)(y) = \bigvee_{x \in X} R(x', x) * \left(\bigvee_{t \in X} (R(t, x) * F(g(t), y)) \right)$$

$$\geq \bigvee_{x \in X} (R(x', x) * R(x', x) * F(g(x'), y)) = F(g(x'), y) = F_{g(x')}(y)$$

and

$$f_{\circ \rho_g}(R'_x)(y) = \bigvee_{x \in X} \left(R(x', x) * \bigvee_{t \in X} (R(t, x) * F(g(t), y)) \right)$$

$$\leq \bigvee_{x \in X} R(x', x) * \left(\bigvee_{t \in X} (F(g(t), g(x)) * F(g(t), y)) \right)$$

$$\leq \bigvee_{x \in X} (R(x', x) * F(g(x), y)) = \rho_g(x', y) = F(g(x'), y) = F_{g(x')}(y). \qquad \square$$

4. General approach to the problem of interpolation

The method developed in this paper adapts the standard framework for interpolation to interpolation of fuzzy data (fuzzy interpolation). First, we recall the standard setting of interpolation and then propose the extension to interpolation

of fuzzy data where the latter belong to a similarity-based or relation-based fuzzy space. Our principal idea is to identify interpolation (ordinary or fuzzy) with the process of choosing an interpolating function (ordinary or fuzzy) from a predetermined set of functions. Finally, we will discuss four different fuzzy interpolation methods and show interpolating fuzzy functions that correspond to them.

4.1. Interpolation in the classical sense

To begin, we will present the simplest formulation of an interpolation problem. In this formulation, all functions are real functions of real arguments. Moreover, these functions depend on one argument. Let *interpolation data* be given by a set of ordered pairs $\{(a_i, b_i), i = 1, \ldots, n\}$, where $A = \{a_1, \ldots, a_n\} \subset \mathbb{R}$ and $B = \{b_1, \ldots, b_n\} \subset \mathbb{R}$, and elements in A are pairwise different. Let $\mathcal{P} \subseteq \{\wp | \wp : D \rightarrow \mathbb{R}\}$, where $A \subset D \subseteq \mathbb{R}$ be a subset of functions that is chosen for interpolation. We say that $\wp \in \mathcal{P}$ is an *interpolating function* for data $\{(a_i, b_i), i = 1, \ldots, n\}$, if

$$\wp(a_i) = b_i, \quad i = 1, \ldots, n. \tag{14}$$

By the assumption that elements in A are pairwise different, a mapping $f : A \rightarrow B$ such that $f(a_i) = b_i, i = 1, \ldots, n$, is functional. Moreover, $\wp \in \mathcal{P}$ is an interpolating function for $\{(a_i, b_i), i = 1, \ldots, n\}$ if and only if $f = \wp|_A$.[6] In other words, we say that the interpolating function $\wp$ is an *extension* of f on the domain D.

Remark 2. Note that we can impose additional conditions on interpolating functions and by this, restrict $\mathcal{P}$. For example, $\mathcal{P}$ can be a class of polynomials, as it is traditionally assumed. Other possible restrictions for $\mathcal{P}$ are: a class of piecewise polynomials (splines), a class of smooth functions that guarantee interpolation "in between", i.e., for all $i = 1, \ldots, n - 1, \min\{b_i, b_{i+1}\} \leq \wp(a) \leq \max\{b_i, b_{i+1}\}$ if $a_i \leq a \leq a_{i+1}$, provided that $a_1 < a_2 < \cdots < a_n$, etc.

Two important conditions should be recognized in every formulation of the interpolation problem:

I₁. Interpolation data $\{(a_i, b_i), i = 1, \ldots, n\}$ determine a functional relation on $A \times B$.
I₂. A set $\mathcal{P}$ of interpolating functions contains at least one interpolating function for $\{(a_i, b_i), i = 1, \ldots, n\}$.

4.2. Interpolation of fuzzy data

This paper focuses on the interpolation approach to a computation with fuzzy data. A precise definition of *interpolation of fuzzy data* will be given below. Loosely speaking, this problem involves extension of a fuzzy function (in the sense of Definition 3) given on a restricted domain to a fuzzy function given on a wider domain (similar to the case considered above).

In the language of fuzzy functions, the problem of interpolation of fuzzy data can be formulated explicitly as follows:

Let fuzzy data be given by a set of pairs $\{(A_i, B_i) | i = 1, \ldots, n\}$, where $A_i \in L^X$, $B_i \in L^Y$, $i = 1, \ldots, n$, and $A_1, \ldots, A_n$ are pairwise different with respect to $=$. Moreover, let $f : A_i \mapsto B_i$ be a fuzzy function on the domain $\mathcal{A} = \{A_1, \ldots, A_n\}$, and $\mathcal{P} \subseteq \{\wp | \wp : D \rightarrow L^Y\}$, where $\mathcal{A} \subset D \subseteq L^X$, is a chosen subset of fuzzy functions. The problem is to find $\wp \in \mathcal{P}$ such that the following interpolation condition

$$\wp(A_i) = B_i, \quad i = 1, \ldots, n \tag{15}$$

is fulfilled. In this case, $\wp$ is called an interpolating fuzzy function for fuzzy data $\{(A_i, B_i) | i = 1, \ldots, n\}$. In other words, the interpolating fuzzy function $\wp$ is an extension of f on the domain D.

It is easy to see that both conditions **I₁**–**I₂** are respected in the formulation above.

Remark 3. Note that fuzzy interpolation, with the stress on the word "fuzzy," seeks to relax the condition (15). Let us show one possible way to achieve this result. By (4), the interpolation condition (15) can be rewritten as

$$A = A_i \text{ implies } \wp(A) = B_i, \quad i = 1, \ldots, n.$$

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[6] If $\wp : X \rightarrow Y$ is a function and $A \subseteq X$ then the restriction $\wp|_A$ is a function with the domain A such that for all $x \in A$, $\wp|_A(x) = \wp(x)$.

If both equalities above are changed to inequalities,

$$A \leq A_i \text{ implies } \wp(A) \leq B_i,$$

or even to

"The greater is the degree of inclusion of A into A_i,

the greater is the degree of inclusion of $\wp(A)$ into B_i'' (16)

where $i = 1, \ldots, n$, we arrive at a relaxed version of fuzzy interpolation. However, this relaxed version is contained in the formulation of the problem of interpolation given by (15). In fact, (16) is an additional condition that restricts the class of interpolating fuzzy functions (see Remark 2).

4.3. Interpolation of sparse rule bases

Below, we present assumptions and requirements that are common to all papers that are focused on the interpolation of sparse rule bases.

(i) A residuated lattice $\mathcal{L}$ is considered on the interval $[0, 1]$.
(ii) All fuzzy sets in a rule base are normal, convex and considered on the real line $\mathbb{R}$.
(iii) The rule base consists of two rules $A_1 \to B_1$ and $A_2 \to B_2$, where A_1, A_2 are fuzzy sets of $X \subseteq \mathbb{R}$, and B_1, B_2 are fuzzy sets of $Y \subseteq \mathbb{R}$.
(iv) It is assumed that the rule base is *sparse* in the sense that the support sets of A_1 and A_2 are disjoint, i.e., their intersection is empty.
(v) Fuzzy sets A_1, A_2 and a fuzzy set A of $\mathbb{R}$ (an "observation") are such that $A_1 \preceq A \preceq A_2$, where $\preceq$ is a *precedence* relation to be specified in each case.

The problem amounts to the estimation of a fuzzy set B of $\mathbb{R}$ (a "conclusion" of a new rule $A \to B$, where A is the given observation) such that $B_1 \preceq B \preceq B_2$ (see Remark 2).

Note that if specified, the relation $\preceq$ differs from the order relation $\leq$ defined in Section 2.2. This follows from the fact that if A_1 and A_2 are disjoint, then $(A_1 \lesssim A_2) = 0$ regardless of whether $A_1 \preceq A_2$ is true or not. Therefore, in our analytic overview we will define $\preceq$ for each approach that we analyze to obtain the extended domain of a corresponding interpolating fuzzy function.

5. Analogy-based interpolation

The analogy-based interpolation has been proposed by Bouchon-Meunier et al. [4,7]. It has been then analyzed and compared with other interpolation methods in [5]. Actually, the proposed analogy-based interpolation gives an estimate of a conclusion B. The approach is purely computational and does not seek a formal representation of a result.

In this section, we will give an algorithmic description of the analogy-based interpolation and then show how this algorithmic description can be represented analytically. The representation will be based on formula (5) where fuzzy relation R is given by (8).

5.1. Algorithmic description

In addition to the assumptions (i)–(v) considered above, this type of interpolation assumes that all fuzzy sets are L–R fuzzy intervals of $\mathbb{R}$ (see e.g., [11]). The other assumption implicitly required is that each normal fuzzy set is determined by a central point of its core and by a chosen shape. The following two steps lead to a construction of B according to the analogy-based interpolation:

• Let c, c_1, c_2 be central points of respective cores of A, A_1, A_2 such that $c_1 \leq c \leq c_2$, and similarly, d_1, d_2 are central points of respective cores of B_1, B_2. Then the central point d of B fulfills the following equation:

$$\frac{c - c_1}{c_2 - c_1} = \frac{d - d_1}{d_2 - d_1}. \tag{17}$$

- Without going into technical details, let $C_{sh}(A, A_1; A_1, A_2) \in \mathbb{R}$ (respectively $C_{sh}(A, A_2; A_1, A_2)$) be an *index of distinguishability* of A and A_1 (respectively A and A_2) in the context (A_1, A_2). It is assumed that given A_1, A_2, their central points c_1, c_2, indices $C_{sh}(A, A_1; A_1, A_2)$ and $C_{sh}(A, A_2; A_1, A_2)$, and a central point c of A, the membership function of A can be uniquely reconstructed. It is then proposed to construct B from B_1, B_2 and with the help of $C_{sh}(B, B_1; B_1, B_2)$, $C_{sh}(B, B_2; B_1, B_2)$, where

$$C_{sh}(B, B_1; B_1, B_2) = C_{sh}(A, A_1; A_1, A_2), \tag{18}$$

$$C_{sh}(B, B_2; B_1, B_2) = C_{sh}(A, A_2; A_1, A_2). \tag{19}$$

The method assumes that the membership functions A, A_1, A_2 as well as B, B_1, B_2 belong to the same group (a group can be specified by a certain shape of membership functions). Equalities (18) and (19) state two analogical proportions of the form "B is to B_1 (respectively, B_2) as A is to A_1 (respectively, A_2)", in terms of an index of distinguishability.

5.2. Interpolating fuzzy functions and the analogy-based interpolation

It is clear that the analogy-based interpolation presented above is a formal scheme that requires a specification of involved membership functions, i.e., a specification of a domain and range of a corresponding interpolating fuzzy function. After that the scheme becomes an interpolation that uniquely determines a fuzzy set B for given A and given context: A_1, A_2 and B_1, B_2. The uniqueness of B guarantees that condition (4) is fulfilled so that the analogy-based interpolation scheme thus specified actually defines an interpolating fuzzy function.

Below, we give our specification of the analogy-based interpolation scheme, and we place assumptions on interpolation data such that the interpolating fuzzy function is determined by a fuzzy relation.

(a) Let $X \subset \mathbb{R}$, $Y \subset \mathbb{R}$ be bounded domains common to fuzzy sets A, A_1, A_2 and B, B_1, B_2, respectively.
(b) Let E, F be similarities on X and Y such that
    1. if $c, c_1, c_2 \in X$ are fixed central points of the respective cores of A, A_1, A_2 then $A = E_c$, $A_1 = E_{c_1}$, $A_2 = E_{c_2}$,
    2. if $d_1, d_2 \in Y$ are fixed central points of the respective cores of B_1, B_2 then $B_1 = F_{d_1}$, $B_2 = F_{d_2}$.
(c) If $A = E_c$, $B = F_d$, where $c \in X$, $d \in Y$ then the indices of distinguishability of A and A_i, and of B and B_i, $i = 1, 2$, are defined by

$$C_{sh}(A, A_i; A_1, A_2) = \frac{c - c_i}{c_2 - c_1},$$

and

$$C_{sh}(B, B_i; B_1, B_2) = \frac{d - d_i}{d_2 - d_1}.$$

(d) Let $g : X \to Y$ be an ordinary linear function that characterizes a dependence between central points of cores $c \in X$ and $d \in Y$ according to (17), i.e., $g : c \mapsto d$, where

$$d = g(c) = d_1 + \frac{d_2 - d_1}{c_2 - c_1}(c - c_1). \tag{20}$$

Assume that g is extensional [7] with respect to E and F (cf. (7)), i.e., for all $x, x' \in X$,

$$E(x', x) \le F(g(x'), g(x)).$$

Let us show how the specification given above uniquely determines an interpolating fuzzy function for fuzzy data $\{(A_i, B_i)) | i = 1, 2\}$ on the domain $D = \{A | A = E_c, c_1 \le c \le c_2\}$. We denote this function $\wp_{BM}$ (initials of the first author of [4,7]) and define it as follows:

$$\wp_{BM}(E_c)(y) = F(g(c), y), \quad y \in Y. \tag{21}$$

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[7] Extensionality of g does not automatically follow from the specification above. This assumption puts an additional limitation on interpolation data.

It is easy to verify that D is linearly ordered by

$$E_{c'} \preceq E_{c''} \quad \text{if } c' \leq c'' \text{ where } c', c'' \in X, x \in X.$$

Thus $D = \{A | A = E_c, E_{c_1} \preceq A \preceq E_{c_2}\}$. We discussed $\preceq$ in Section 4.3.

To verify the interpolation conditions $\mathbf{I}_1$–$\mathbf{I}_2$, we note that $A_1 \in D$, $A_2 \in D$, and by the specification (b) and by (21), we have $\wp_{BM}(A_1) = B_1$ and $\wp_{BM}(A_2) = B_2$. Moreover, if $A = E_c$ then by (21), $B = \wp_{BM}(A) = F_d$, where $d = g(c)$ so that by (20),

- $C_{sh}(A, A_1; A_1, A_2) = C_{sh}(B, B_1; B_1, B_2) = (c - c_1)/(c_2 - c_1) = (d - d_1)/(d_2 - d_1)$,
- $C_{sh}(A, A_2; A_1, A_2) = C_{sh}(B, B_2; B_1, B_2) = (c - c_2)/(c_1 - c_2) = (d - d_2)/(d_1 - d_2)$.

Therefore, the fuzzy function $\wp_{BM}$ fulfills both necessary requirements, namely (17)–(19), of the analogy-based interpolation scheme. Its domain $D = \{A | A = E_c, c_1 \leq c \leq c_2\}$ is a similarity-based fuzzy space $([c_1, c_2], E)$, i.e., a class of E-fuzzy points of the interval $[c_1, c_2]$. Its range is a similarity-based fuzzy space $([d_1, d_2], F)$, i.e., the class $G = \{B | B = F_d, d_1 \leq d \leq d_2\}$ of F-fuzzy points of the interval $[d_1, d_2]$.

We can further extend the fuzzy function $\wp_{BM}$ to the domain $[0, 1]^X$, i.e., to all fuzzy subsets of X. For this purpose, we will represent $\wp_{BM}$ by a respective fuzzy relation. The following theorem gives all necessary details.

Theorem 2. *Let a specification of the analogy-based interpolation scheme be given by conditions* (a)–(d) *above. Let*

$$\rho_g(x, y) = \bigvee_{x' \in X} (E(x', x) * F(g(x'), y)), \quad x \in X, \ y \in Y$$

be a fuzzy relation on $X \times Y$, and $f_{\circ \rho_g} : A \mapsto A \circ \rho_g$ the fuzzy function defined on $[0, 1]^X$ determined by ρ_g in accordance with (5).

Then $f_{\circ \rho_g}$ is an interpolating fuzzy function for fuzzy data $\{(A_i, B_i))| i = 1, 2\}$. Moreover, $f_{\circ \rho_g}|_D = \wp_{BM}$, where $D = \{A | A = E_c, c_1 \leq c \leq c_2\}$ and $\wp_{BM}$ is given by (21).

Proof. The statement follows from (10), i.e., for all $A = E_c$, where $c_1 \leq c \leq c_2$,

$$f_{\circ \rho_g}(A) = E_c \circ \rho_g = F_{g(c)} = \wp_{BM}(A). \quad \square$$

6. Fuzzy interpolation by convex completion

Dubois and Prade et al. [10,13,28] proposed another approach to the problem of interpolation. They focused on interval interpolation using data $\{(A_i, B_i))| i = 1, \ldots, n\}$, where A_i and B_i are closed intervals on the real line. Intervals are considered as ordered pairs of real numbers with pointwise operations over them.

In this section, we will give an algorithmic description of interpolation by convex completion and then show how this algorithmic description can be represented analytically. The representation will be based on formula (5) where fuzzy relation R is given by (12).

6.1. Algorithmic description

Let us give more technical details in addition to the assumptions (i)–(v) above: the case $n = 2$ is considered so that $A_1 = [a_{11}, a_{12}]$, $A_2 = [a_{21}, a_{22}]$, $B_1 = [b_{11}, b_{12}]$, $B_2 = [b_{21}, b_{22}]$, where $a_{11} \leq a_{12} < a_{21} \leq a_{22}$ and $b_{11} \leq b_{12} < b_{21} \leq b_{22}$. In this case, $X = [a_{11}, a_{22}]$ and $Y = [b_{11}, b_{22}]$ are universal sets for antecedents and consequents, respectively. A proposed solution is an interpolating fuzzy function, say $\wp_D$, defined on the domain $\mathcal{A}$ that consists of all closed intervals $A = [\underline{a}, \overline{a}]$ such that $a_{11} \leq \underline{a} \leq \overline{a} \leq a_{22}$. In other words, $\mathcal{A}$ consists of all closed intervals that are inside X.

The following *algorithm* for computing $B = \wp_D(A)$ for given $A = [\underline{a}, \overline{a}]$ is proposed (the details are in [28]):

(i) if for some $\lambda \in [0, 1]$, $A = A_\lambda$ so that $\underline{a} = (1 - \lambda)a_{11} + \lambda a_{21}$, $\overline{a} = (1 - \lambda)a_{12} + \lambda a_{22}$ then $B = B_\lambda$, where $\underline{b} = (1 - \lambda)b_{11} + \lambda b_{21}$, $\overline{b} = (1 - \lambda)b_{12} + \lambda b_{22}$;

(ii) if for some $\lambda \in [0, 1]$, $A_\lambda \subset A$ then $A = \bigcup_{\lambda : A_\lambda \subset A} A_\lambda$ and

$$B = \bigcup_{\lambda : A_\lambda \subset A} B_\lambda;$$

(iii) if for some $\lambda \in [0, 1]$, $A \subset A_\lambda$ then $A = \bigcap_{\lambda : A \subset A_\lambda} A_\lambda$ and

$$B = \bigcap_{\lambda : A \subset A_\lambda} B_\lambda.$$

The preorder $\preceq$ can be specified on $\mathcal{A}$ as follows: if $[\underline{a}, \overline{a}] \subseteq X$ and $[\underline{a}', \overline{a}'] \subseteq X$ then

$$[\underline{a}, \overline{a}] \preceq [\underline{a}', \overline{a}'] \quad \text{if } \frac{\underline{a} + \overline{a}}{2} \leq \frac{\underline{a}' + \overline{a}'}{2}.$$

It is easy to see that $\mathcal{A} = \{A | A = [\underline{a}, \overline{a}] \subseteq X, A_1 \preceq A \preceq A_2\}$.

Let us discuss a *justification* of the proposed algorithm. In [28], the justification was based on the fact that according to classical reasoning, both formulas $(A_1 \vee A_2) \rightarrow (B_1 \vee B_2)$ and $(A_1 \wedge A_2) \rightarrow (B_1 \wedge B_2)$ are consequences of those two: $A_1 \rightarrow B_1$ and $A_2 \rightarrow B_2$. Therefore, if we accept the rule base $\{A_\lambda \rightarrow B_\lambda | \lambda \in [0, 1]\}$, we will also accept the results (deduced rules) of the algorithm. However, there exist deduced rules for which intervals in antecedents have non-empty intersections, whereas intervals in consequents do not. Those situations were characterized as potentially incoherent. Conditions that produce such situations were examined in [28].

To justify the proposed algorithm in terms of our approach, we must prove that the interpolating fuzzy function $\wp_D$ fulfills conditions $\mathbf{I}_1$–$\mathbf{I}_2$. The proof will consist of verification of (4) on the domain $\mathcal{A}$ and (15) for A_1, A_2. Recall that $\mathcal{A}$ consists of all closed intervals $A = [\underline{a}, \overline{a}]$ such that $a_{11} \leq \underline{a} \leq \overline{a} \leq a_{22}$. By [28], $\mathcal{A}$ can be divided into three subclasses $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$, where

$$\mathcal{A}_1 = \{A_\lambda | \lambda \in [0, 1]\},$$
$$\mathcal{A}_2 = \{A \in \mathcal{A} | (\exists \lambda \in [0, 1])(A_\lambda \subset A)\},$$
$$\mathcal{A}_3 = \{A \in \mathcal{A} | (\exists \lambda \in [0, 1])(A \subset A_\lambda)\}.$$

The range of $\wp_D$ can be specified by the set $\{0, 1\}^Y$ of all subsets of Y so that $\wp_D : \mathcal{A} \rightarrow \{0, 1\}^Y$ establishes a correspondence between $\mathcal{A}$ and $\{0, 1\}^Y$. The proof that $\wp_D$ is a fuzzy function, i.e., $\wp_D$ fulfills (4), follows from the fact that classes $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ are mutually disjoint (see [28, Proposition 1]). The proof that $\wp_D$ is an interpolating fuzzy function, i.e., $\wp_D$ fulfills (15) for A_1, A_2, follows from the case (i) of the algorithm. Therefore, $\wp_D$ is indeed an interpolating fuzzy function, and the proposed algorithm is justified. Moreover, a potentially incoherent situation (see above) means that for some $A \in \mathcal{A}_3$, $\wp_D(A) = \mathbf{0}$. In our general approach, this case makes no specific difficulty (we do not assign a logical meaning to values from the range of $\wp_D$), but it corresponds to a situation called "potential incoherence" by the authors of [28], i.e., a situation where the input interval A is logically inconsistent with the convex completion of the rule base.

6.2. Interpolating fuzzy functions and fuzzy interpolation by convex completion

The algorithmic description of the function $\wp_D$ given above determines it as an interpolating fuzzy function defined on $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$, where $\mathcal{A}$ is a class consisting of all closed intervals of X. In this subsection, we will show how the interpolating fuzzy function $\wp_D$ can be analytically represented on the domain $\mathcal{A}$ with the help of fuzzy relation ρ_g given by formula (8).

We will choose the Łukasiewicz algebra on $L = [0, 1]$ as an underlying residuated lattice. The following steps lead to ρ_g:

(a) compute the centers c_1, c_2 of intervals A_1, A_2, and d_1, d_2 of intervals B_1, B_2:

$$c_1 = \frac{a_{11} + a_{12}}{2}, \quad c_2 = \frac{a_{21} + a_{22}}{2}, \quad d_1 = \frac{b_{11} + b_{12}}{2}, \quad d_2 = \frac{b_{21} + b_{22}}{2},$$
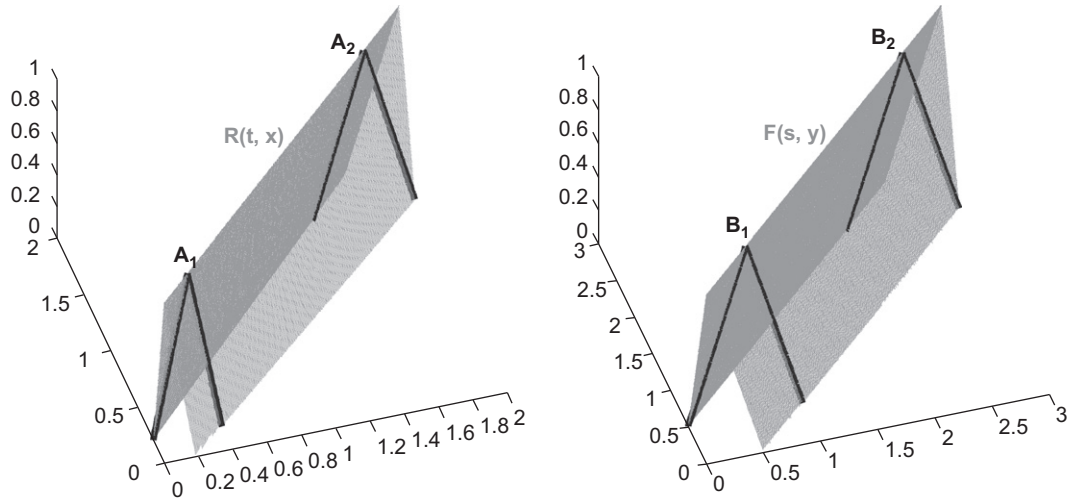
Fig. 1. Fuzzy relation R on $[c_1, c_2] \times X$ (left), and fuzzy relation F on $[d_1, d_2] \times Y$ (right).

(b) compute the lengths l_1, l_2 of intervals A_1, A_2 and m_1, m_2 of intervals B_1, B_2:

$$l_1 = a_{12} - a_{11}, \quad l_2 = a_{22} - a_{21}, \quad m_1 = b_{12} - b_{11}, \quad m_2 = b_{22} - b_{21},$$

(c) for each $\tau \in [0, 1]$, let

$$l_\tau = \tau l_2 + (1 - \tau)l_1, \quad m_\tau = \tau m_2 + (1 - \tau)m_1,$$

(d) for each $\tau \in [0, 1]$, define the similarities S^τ on $[c_1, c_2]$, and Q^τ on $[d_1, d_2]$ by

$$S^\tau(u, x) = \max\left(1 - \frac{2|u - x|}{l_\tau}, 0\right), \quad Q^\tau(v, y) = \max\left(1 - \frac{2|v - y|}{m_\tau}, 0\right),$$

(e) define the reflexive fuzzy relation R on $[c_1, c_2]$ (see Fig. 1) by

$$R(t, x) = S^{\tau_t}(t, x) \quad \text{where } \tau_t = \frac{t - c_1}{c_2 - c_1},$$

and for all $c \in [c_1, c_2]$, define the R-fuzzy point R_c as a fuzzy set on X by the following extension of the membership function $R_c : [c_1, c_2] \to [0, 1]$ to the domain $X \setminus [c_1, c_2]$:

$$R_c(x) = \begin{cases} R(c, c + |c - x|) & \text{if } x < c_1, \\ R(c, c - |c - x|) & \text{if } x > c_2. \end{cases}$$

(f) define the reflexive fuzzy relation F on $[d_1, d_2]$ (see Fig. 1) by

$$F(s, y) = Q^{\tau_s}(s, y) \quad \text{where } \tau_s = \frac{s - d_1}{d_2 - d_1},$$

and similarly to the above, for all $d \in [d_1, d_2]$, define the F-fuzzy point F_d as a fuzzy set on Y,

(g) define the ordinary (linear) function $g : [c_1, c_2] \to [d_1, d_2]$ (see Fig. 2)

$$g(x) = d_1 + \frac{x - c_1}{c_2 - c_1}(d_2 - d_1),$$

(h) finally, define the fuzzy relation ρ_g (see Fig. 3) on $X \times Y$ (see also (12)) by

$$\rho_g(x, y) = \bigvee_{x' \in [c_1, c_2]} (R'_x(x) * F_{g(x')}(y)), \quad x \in X, y \in Y,$$

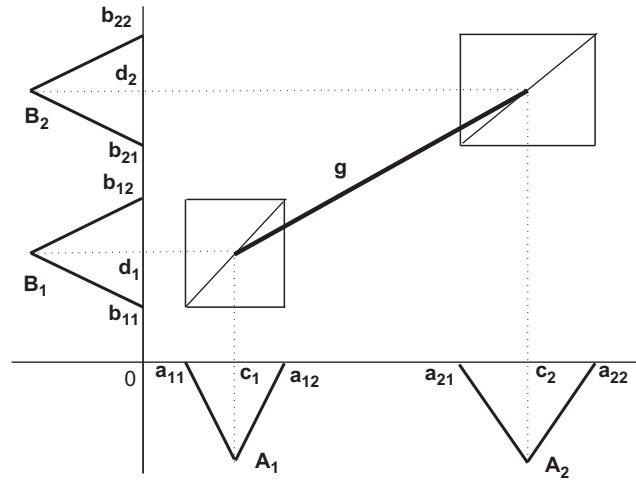where R_x and $F_{g(x')}$ are fuzzy points with extended membership functions.

Fig. 2. Function $g(x) = d_1 + ((x - c_1)/(c_2 - c_1))(d_2 - d_1)$ on the domain $[c_1, c_2]$. The slope of g is less than or equal to each of the slopes of the diagonals of rectangles $A_1 \times B_1$ and $A_2 \times B_2$.
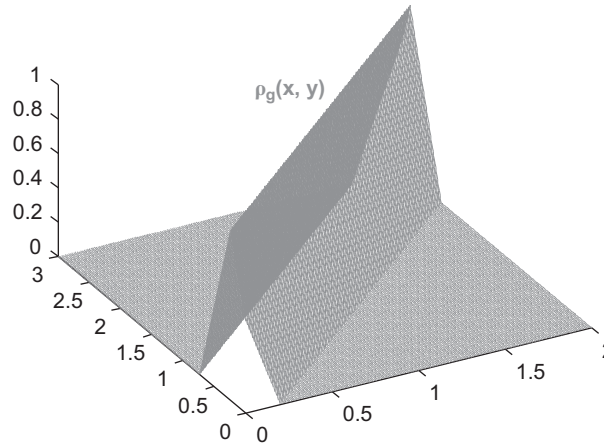

Fig. 3. Fuzzy relation ρ_g on $X \times Y$.

By the example in Section 3.1, ρ_g determines the fuzzy function $f_{\circ \rho_g}$ on the domain $[0, 1]^X$ (recall that $X = [a_{11}, a_{22}]$). However, we will consider $f_{\circ \rho_g}$ on three different subdomains $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \subset [0, 1]^X$ such that

$\mathcal{X}_1 = \{R_c | c \in [c_1, c_2]\}$ is an R-fuzzy space of X,

$\mathcal{X}_2 = \{R_{M_\cup} | M \subseteq [c_1, c_2]\}$ where $R_{M_\cup}(x) = \bigvee_{c \in M} R(c, x), \ x \in X$,

$\mathcal{X}_3 = \{R_{M_\cap} | M \subseteq [c_1, c_2]\}$ where $R_{M_\cap}(x) = \bigwedge_{c \in M} R(c, x), \ x \in X$.

We will show that restrictions $\wp_D|_{\mathcal{A}_1}, \wp_D|_{\mathcal{A}_2}, \wp_D|_{\mathcal{A}_3}$ can be obtained with the help of respective restrictions $f_{\circ \rho_g}|_{\mathcal{X}_1}$, $f_{\circ \rho_g}|_{\mathcal{X}_2}, f_{\circ \rho_g}|_{\mathcal{X}_3}$. The proof will be based on Theorem 1 which, however, requires extensionality of function g with respect to a reflexive fuzzy relation R on X and a similarity F on Y. Therefore, we will first impose limitations (Theorem 3) that guarantee that the function g (step (g) above) fulfills (11) with respect to R and F (both are reflexive) constructed on steps (e) and (f) above. Then we will show (Proposition 1) which conditions guarantee that F is a similarity on Y. Proofs of both Theorem 3 and Proposition 1 are given in the Appendix.
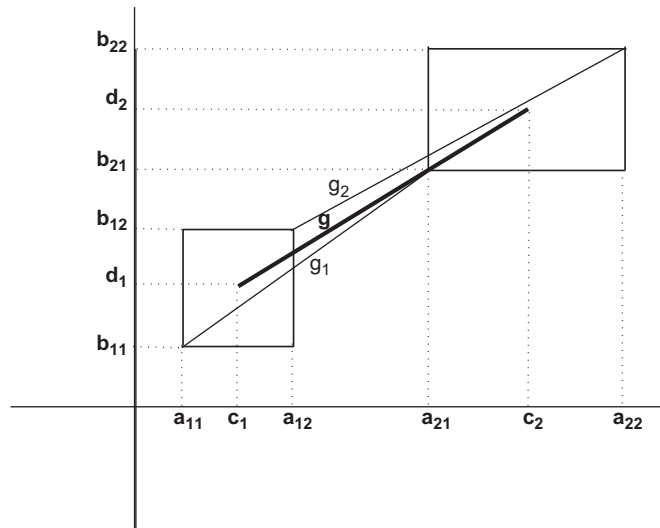
Fig. 4. The straight line g lies between g_1 and g_2.

Theorem 3. *Let us accept steps* (a)–(h) *above. Then the function g fulfills* (11), *i.e., for all* $t \in [c_1, c_2]$, $x \in X$

$$R(t, x) \leq F(g(t), g(x)),$$

if and only if for all $\tau \in [0, 1]$ *the following inequality holds*:

$$\frac{d_2 - d_1}{c_2 - c_1} \leq \min\left(\frac{m_1}{l_1}, \frac{m_2}{l_2}\right). \tag{22}$$

A geometrical meaning of the condition (22) is that the slope of the straight line g is less than or equal to each of the slopes of the diagonals of rectangles $A_1 \times B_1$ and $A_2 \times B_2$, see Fig. 2.

In [28], it is proved that the set of rules $\{A_\lambda \to B_\lambda | \lambda \in [0, 1]\}$ defines a relation $R = \bigcap_{\lambda \in [0,1]} (A_\lambda \to B_\lambda)$, which must satisfy $(\forall x \in [a_{11}, a_{22}])(\exists y \in [b_{11}, b_{22}])((x, y) \in R)$ so as to ensure the composition $A \circ R$ to be a non-empty interval. When this condition is violated, it corresponds to the potential incoherence (see the last paragraph of the preceding subsection). It is proved that potential incoherence is avoided if and only if the segment g_1 limited by points (a_{11}, b_{11}) and (a_{21}, b_{21}) lies under the segment g_2 defined by points (a_{12}, b_{12}) and (a_{22}, b_{22}) in the area limited by abscissae a_{12} and a_{21}, see Fig. 4. By construction, it is easy to see that in the latter case, the segment g lies between g_1 and g_2. Moreover, this is the case if and only if the condition on slopes given in Theorem 3 (positive slope diagonals of the rectangles in Fig. 4) is verified. In the limit (when the three slopes, i.e., of g and the diagonals, are equal the three segments are aligned).

Proposition 1. *Let us accept steps* (a)–(h) *above, and moreover, assume that the lengths* m_1, m_2 *of the intervals* B_1, B_2 *are equal. Then fuzzy relation F constructed on the step* (f) *is a similarity on* $[d_1, d_2]$.

Further on, we will accept steps (a)–(h) with the additional limitation $m_1 = m_2$ and the condition (22). They all assure that the conditions of Theorem 1 are fulfilled. Therefore, the fuzzy relation ρ_g (step (h)) determines the fuzzy function $f_{\circ \rho_g}$ such that for all $c \in [c_1, c_2]$, $f_{\circ \rho_g}(R_c) = F_d$.

Below, we will characterize restrictions $f_{\circ \rho_g}|_{\mathcal{X}_1}$, $f_{\circ \rho_g}|_{\mathcal{X}_2}$, $f_{\circ \rho_g}|_{\mathcal{X}_3}$ and show how they yield $\wp_D|_{\mathcal{A}_1}$, $\wp_D|_{\mathcal{A}_2}$, $\wp_D|_{\mathcal{A}_3}$.
The following assertions can be verified:

(1) Let $f_{\circ \rho_g}$ be restricted to the domain consisting of R-fuzzy points of $[c_1, c_2]$, i.e, to the class $\mathcal{X}_1 = \{R_c | c \in [c_1, c_2]\}$. Then by Theorem 1, the range of such a restricted $f_{\circ \rho_g}$ is equal to $\mathcal{Y}_1 = \{F_d | d \in [d_1, d_2]\}$ or to F-fuzzy points of $[d_1, d_2]$. Moreover, the restriction $f_{\circ \rho_g}|_{\mathcal{X}_1}$ behaves according to the equality $f_{\circ \rho_g}|_{\mathcal{X}_1}(R_c) = F_{g(c)}$.
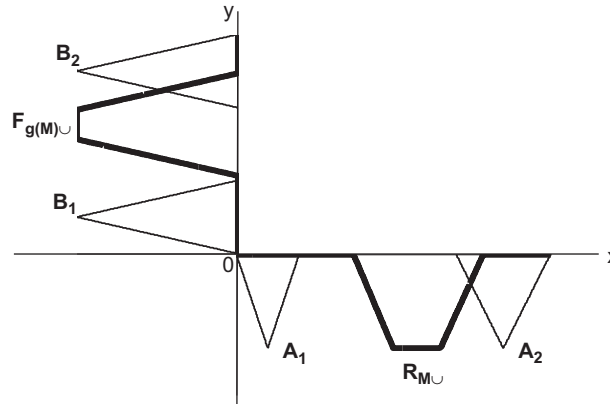
Fig. 5. Fuzzy set $R_{M_\cup}$ on X and fuzzy set $f_{\circ\rho_g}|_{\mathcal{X}_2}(R_{M_\cup}) = F_{g(M)_\cup}$ on Y.

(2) Let $\mathrm{Supp}(R_c)$ $(\mathrm{Supp}(F_d))$ be the support of the R-fuzzy point R_c (F_d). Let us consider two sets of closed intervals $\mathrm{Supp}(\mathcal{X}_1) = \{\mathrm{Supp}(R_c)|c \in [c_1, c_2]\}$ and $\mathrm{Supp}(\mathcal{Y}_1) = \{\mathrm{Supp}(F_d)|d \in [d_1, d_2]\}$, and see that $\mathrm{Supp}(\mathcal{X}_1) = \mathcal{A}_1$, where $\mathcal{A}_1$ belongs to the domain of $\wp_D$. The following function $f'_{\circ\rho_g} : \mathrm{Supp}(\mathcal{X}_1) \rightarrow \mathrm{Supp}(\mathcal{Y}_1)$, given by the equality

$$f'_{\circ\rho_g}(\mathrm{Supp}(R_c)) = \mathrm{Supp}(f_{\circ\rho_g}(R_c)) = \mathrm{Supp}(F_{g(c)}),$$

coincides with the restriction $\wp_D|_{\mathcal{A}_1}$, where $\wp_D|_{\mathcal{A}_1}$ was described in step (i) of the algorithm above. In other words, if $A = A_\lambda$, where $\lambda \in [0, 1]$, and if $c = \lambda c_2 + (1 - \lambda)c_1$, then $B = f'_{\circ\rho_g}(\mathrm{Supp}(R_c))$.

(3) Let $f_{\circ\rho_g}$ be restricted to the domain consisting of unions of R-fuzzy points of $[c_1, c_2]$, i.e., to the class $\mathcal{X}_2 = \{\bigvee_{c\in M} R(c, x)|M \subseteq [c_1, c_2]\}$. Denote $R_{M_\cup}$ a fuzzy set of X given by the membership function $\bigvee_{c\in M} R(c, x)$. By the equality below (the proof is a simple technical exercise)

$$f_{\circ\rho_g}(R_{M_\cup}) = \bigvee_{c\in M} F_{g(c)}, \tag{23}$$

we see that the range of the restriction $f_{\circ\rho_g}|_{\mathcal{X}_2}$ is equal to $\mathcal{Y}_2 = \{\bigvee_{h\in H} F(h, y)|H \subseteq [d_1, d_2]\}$. Denote $F_{H_\cup}$ a fuzzy set of Y given by the membership function $\bigvee_{h\in H} F(h, y)$. Then the restriction $f_{\circ\rho_g}|_{\mathcal{X}_2}$ behaves according to the equality $f_{\circ\rho_g}|_{\mathcal{X}_2}(R_{M_\cup}) = F_{g(M)_\cup}$ (see Fig. 5).

(4) Let $\mathrm{Supp}(R_{M_\cup})$ $(\mathrm{Supp}(F_{H_\cup}))$ be the support of $R_{M_\cup}$ $(F_{H_\cup})$. It is easy to see that if $M(H)$ is an interval, then $\mathrm{Supp}(R_{M_\cup})$ $(\mathrm{Supp}(F_{H_\cup}))$ is an interval as well.

Let us consider two sets of closed intervals $\mathrm{Supp}(\mathcal{X}_2) = \{\mathrm{Supp}(R_{M_\cup})|M = [m_1, m_2] \subseteq [c_1, c_2]\}$ and $\mathrm{Supp}(\mathcal{Y}_2) = \{\mathrm{Supp}(F_{H_\cup})|H = [h_1, h_2] \subseteq [d_1, d_2]\}$, and see that $\mathrm{Supp}(\mathcal{X}_2) = \mathcal{A}_2$, where $\mathcal{A}_2$ belongs to the domain of $\wp_D$. The following function $f''_{\circ\rho_g} : \mathcal{A}_2 \rightarrow \mathrm{Supp}(\mathcal{Y}_2)$, given by the equality

$$f''_{\circ\rho_g}(\mathrm{Supp}(R_{M_\cup})) = \mathrm{Supp}(f_{\circ\rho_g}(R_{M_\cup})) = \mathrm{Supp}(F_{g(M)_\cup}),$$

coincides with the restriction $\wp_D|_{\mathcal{A}_2}$, where $\wp_D|_{\mathcal{A}_2}$ was described in the step (ii) of the algorithm above. In other words, step (ii) considers arguments A such that for some $\lambda \in [0, 1]$, $A \supset A_\lambda$. By [28], the inclusion $A \supset A_\lambda$ is valid for $\lambda \in [\underline{\lambda}, \overline{\lambda}]$, where the latter is a certain interval. Let $\underline{c} = \underline{\lambda}c_2 + (1 - \underline{\lambda})c_1$, $\overline{c} = \overline{\lambda}c_2 + (1 - \overline{\lambda})c_1$ and $M = [\underline{c}, \overline{c}]$. Then it can be verified that $B = f''_{\circ\rho_g}(\mathrm{Supp}(R_M))$.

(5) Let $f_{\circ\rho_g}$ be restricted to the domain consisting of intersections of R-fuzzy points of $[c_1, c_2]$, i.e., to the class $\mathcal{X}_3 = \{\bigwedge_{c\in M} R(c, x)|M \subseteq [c_1, c_2]\}$. Denote by $R_{M_\cap}$ the fuzzy set of X given by the membership function $\bigwedge_{c\in M} R(c, x)$. It can be proved that (the proof is a simple technical exercise)

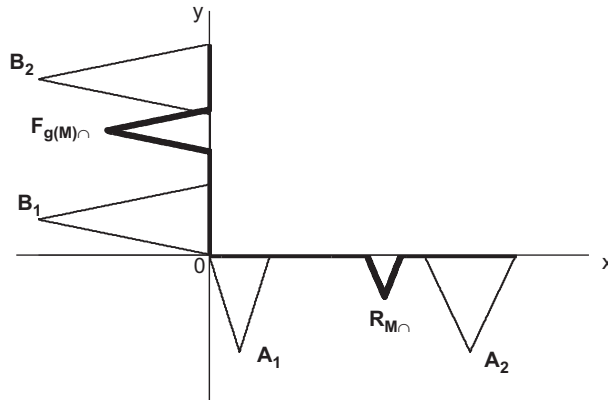$$R_{M_\cap} \circ \rho_g \leq \bigwedge_{c\in M} F_{g(c)}. \tag{24}$$

Fig. 6. Fuzzy set $R_{M_\cap}$ on X and fuzzy set $F_{g(M)_\cap}$ on Y.

Denote by $F_{H_\cap}$ the fuzzy set of Y given by the membership function $\bigwedge_{h \in H} F(h, y)$. Then the restriction $f_{\circ\rho_g}|_{\mathcal{X}_3}$ behaves according to the inequality $f_{\circ\rho_g}|_{\mathcal{X}_3}(R_{M_\cap}) \leq F_{g(M)_\cap}$ (the graph of $F_{g(M)_\cap}$ can be seen from Fig. 6).

(6) In a manner similar to that employed in assertion (4) above, we consider two sets of closed intervals $\mathrm{Supp}(\mathcal{X}_3) = \{\mathrm{Supp}(R_{M_\cap})|M = [m_1, m_2] \subseteq [c_1, c_2]\}$ and $\mathrm{Supp}(\mathcal{Y}_3) = \{\mathrm{Supp}(F_{H_\cap})|H = [h_1, h_2] \subseteq [d_1, d_2]\}$, and see that $\mathrm{Supp}(\mathcal{X}_3) = \mathcal{A}_3$ where $\mathcal{A}_3$ is a part of the domain of $\wp_D$. Let us define two functions: $f'''_{\circ\rho_g} : \mathcal{A}_3 \to \mathrm{Supp}(\mathcal{Y}_2)$ and $f''' : \mathcal{A}_3 \to \mathrm{Supp}(\mathcal{Y}_2)$ so that

$$f'''_{\circ\rho_g}(\mathrm{Supp}(R_{M_\cap})) = \mathrm{Supp}(f_{\circ\rho_g}(R_{M_\cap}))$$

and

$$f'''(\mathrm{Supp}(R_{M_\cap})) = \mathrm{Supp}(F_{g(M)_\cap}).$$

It is easy to see that f''' coincides with the restriction $\wp_D|_{\mathcal{A}_3}$, whereas for all $\mathrm{Supp}(R_{M_\cap}) \in \mathrm{Supp}(\mathcal{X}_3)$, the following inequality holds:

$$f'''_{\circ\rho_g}(\mathrm{Supp}(R_{M_\cap})) \leq f'''(\mathrm{Supp}(R_{M_\cap})).$$

Summarizing assertions (2), (4) and (6) above, we conclude that both functions $f'_{\circ\rho_g} \cup f''_{\circ\rho_g} \cup f'''$ and $f'_{\circ\rho_g} \cup f''_{\circ\rho_g} \cup f'''_{\circ\rho_g}$ are interpolating fuzzy functions defined on the extended domain $\mathcal{A}$. These functions coincide on $\mathcal{A}_1 \cup \mathcal{A}_2$. The interpolating fuzzy function $\wp_D$ coincides with $f'_{\circ\rho_g} \cup f''_{\circ\rho_g} \cup f'''$ and is greater than or equal to $f'_{\circ\rho_g} \cup f''_{\circ\rho_g} \cup f'''_{\circ\rho_g}$. The behavior of the latter fuzzy function is determined by the fuzzy relation ρ_g. Therefore, the function can be further extended to the domain $[0, 1]^X$.

6.3. Analogy-based interpolation and fuzzy interpolation by convex completion

In this subsection, we will compare the two interpolating fuzzy functions $\wp_{BM}$ and $\wp_D$ previously considered in Sections 5.2 and 6.2. Both functions are formally represented with the help of fuzzy relations that we denoted by the same symbol ρ_g. In general, however, those fuzzy relations differ because their constituents have different properties. Therefore, fuzzy functions $\wp_{BM}$ and $\wp_D$ differ as well. We give additional details below.

The domain $D = \{A|A = E_c, c_1 \leq c \leq c_2\}$ of $\wp_{BM}$ is a similarity-based fuzzy space $([c_1, c_2], E)$, i.e., a class of E-fuzzy points of the interval $[c_1, c_2]$. The domain $\mathcal{A}$ of $\wp_D$ is a union of three subdomains $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$, where the first $\mathcal{A}_1$ consists of closed supports of fuzzy sets from $\mathcal{X}_1$. The latter is an R-fuzzy space on X, i.e., a class of R-fuzzy points of the respective interval $[c_1, c_2]$. Thus, domains D and $\mathcal{X}_1$ have similar structures if and only if both fuzzy relations, denoted by E and R, are similarities.

Another difference between $\wp_{BM}$ and $\wp_D$ is the further extension of $\wp_D$ to closed supports of fuzzy sets in $\mathcal{X}_2 \cup \mathcal{X}_3$, i.e., to $\mathcal{A}_2 \cup \mathcal{A}_3$.

7. Fuzzy interpolation based on closeness relations

Another approach to fuzzy interpolation is that proposed in [6]. In the processing of this method, we will use formula (6) for analytical representation of the respective interpolating fuzzy function, where fuzzy relation R is given by (12).

7.1. Algorithmic description

Fuzzy data $\{A_i, B_i)|i = 1, \ldots, n\}$ are characterized as a set of fuzzy if–then rules $K = \{$"If $X = A_i$ then $Y = B_i$"$|i = 1, \ldots, n\}$. The basic idea is to interpret each fuzzy rule "If $X = A_i$ then $Y = B_i$" as a kind of meta-inference rule stating

"The *closer* the input A is to A_i, the *closer* the output B is to B_i".

To use this interpretation for fuzzy interpolation, we need to have a pair of L-valued fuzzy relations, $close_X : L^X \times L^X \to L$ and $close_Y : L^Y \times L^Y \to L$, on the input and output spaces, respectively, modeling a graded notion of *closeness*. For the moment, we will only assume that they are reflexive relations, i.e., $close_X(A, A) = 1$ for all $A \in L^X$ and $close_Y(B, B) = 1$ for all $B \in L^Y$. Then the above interpretation of the fuzzy rule "If $X = A_i$ then $Y = B_i$" comes down to the condition

$$close_X(A, A_i) \leq close_Y(B, B_i). \tag{25}$$

Now, if K is the set of fuzzy if–then rules given above, then the *suitability* of the $close_X$ and $close_Y$ relations for interpolation is understood as follows. We require that condition (25) be satisfied by all rules in K. That is, we require that the inequalities

$$\{close_X(A_j, A_i) \leq close_Y(B_j, B_i)|i, j = 1, \ldots, n\} \tag{26}$$

be satisfied. In such a case, we will call the pair $(close_X, close_Y)$ *interpolative* with respect to the rule set K. If so, then given an input $X = A$, a compatible output $Y = B$ according to K and $(close_X, close_Y)$ would be among those B' that solve the following set of inequalities:

$$\alpha_i \leq \{close_Y(B', B_i)|i = 1, \ldots, n\}, \tag{27}$$

where $\alpha_i = close_X(A, A_i)$. For instance, it is meaningful to choose B as the least specific (i.e., the pointwise biggest) B' satisfying (27).

This approach can be refined when the input and output spaces provide not only a pair of interpolative closeness relations but also an entire parametric family $\mathcal{M} = \{(close_\lambda^X, close_\lambda^Y)\}_{\lambda \in \Lambda}$, with $\Lambda \subset \mathbb{R}^+$, of decreasingly stricter interpolative closeness relations, i.e., such that

(1) for all λ, $close_\lambda^X(A_j, A_i) \leq close_\lambda^Y(B_j, B_i)$ for all $i, j = 1, \ldots, n$
(2) if $\lambda \leq \lambda'$ then $close_\lambda^X \leq close_{\lambda'}^X$ and $close_\lambda^Y \leq close_{\lambda'}^Y$
(3) $close_X^0$ and $close_Y^0$, restricted to the (classical) singletons of X and Y, respectively, are the crisp identity relations.
(4) for all $A, A' \in L^X$ and $B, B' \in L^Y$, $\sup_{\lambda \in \Lambda} close_X^\lambda(A, A') = 1$ and $\sup_{\lambda \in \Lambda} close_Y^\lambda(B, B') = 1$.

In such conditions, for a given input $X = A$ and according to K and a family of closeness relations $\{(close_\lambda^X, close_\lambda^Y)\}_{\lambda \in \Lambda}$, a compatible output for Y will be any fuzzy set $B \in L^Y$ that satisfies the following inequalities:

$$\{close_\lambda^X(A, A_i) \leq close_\lambda^Y(B, B_i)|i = 1, \ldots, n, \lambda \in \Lambda\}. \tag{28}$$

However, it is meaningful to apply the *minimal specificity principle* here (e.g., [9]) and to take as the solution the least specific B satisfying (28).

For example, we describe an instance (in fact, two instances) of the general approach. This example uses a family of fuzzy similarity relations over the reals to define the closeness relations $close_\lambda^X$ and $close_\lambda^Y$. Suppose that the input and output spaces are the real line $\mathbb{R}$ for simplicity, and suppose further that we have a parametric nested family of fuzzy

similarity relations on $\mathbb{R}$, $\mathscr{S} = \{S_\lambda : \mathbb{R}^2 \to [0, 1] | 0 \leq \lambda < +\infty\}$, such that S_0 is the crisp equality and $S_{+\infty} = 1$. An example of such a parametric family of relations on $\mathbb{R}^2$ is the family defined by

$$S_\lambda(x, y) = \max\left(1 - \frac{|x - y|}{\lambda}, 0\right) \tag{29}$$

for any real $\lambda > 0$, and $S_0(x, y)$ being the classical equality.

One can then consider two families of closeness relations on $[0, 1]^{\mathbb{R}}$, $\{H_\lambda^1\}_{\lambda \in \Lambda}$ and $\{H_\lambda^2\}_{\lambda \in \Lambda}$, induced by the family of similarities $\mathscr{S}$. The first family is defined by

$$H_\lambda^1(E, D) = (E \lesssim S_\lambda \circ D) \tag{30}$$

i.e., according to (2), $H_\lambda^1(E, D)$ is nothing but the degree of inclusion of E into the extended fuzzy set D with the similarity S_λ.[8] The second family is defined by symmetrization:

$$H_\lambda^2(E, D) = \min((E \lesssim S_\lambda \circ D), (D \lesssim S_\lambda \circ E)).$$

Obviously, H_λ^1 is reflexive and H_λ^2 is both reflexive and symmetric. Moreover, they are $*$-transitive whenever S_λ is. The proof is in Lemma 1 in the Appendix.

In the following we will write H_λ when we do not distinguish between H_λ^1 and H_λ^2.

Actually, given a fuzzy rule base $K = \{ R_i : \text{"If } X = A_i \text{ then } Y = B_i \text{" } | i = 1, \ldots, n\}$, it is not customary to take the full K into account to perform interpolation given an input $X = A$. Rather, it is usual to consider only a subset of rules $K(A) \subseteq K$ that are most closely related to A. According to our assumptions in Section 4.3, $K(A)$ will consist of only two rules.

Finally let us define the family of interpolative closeness relations $\mathscr{M} = \{(close_\lambda^X, close_\lambda^Y)\}_{\lambda \in [0, +\infty)}$ based on the H_λ measures as follows: for each λ,

$$close_\lambda^X = H_\lambda, \quad close_\lambda^Y = H_{f(\lambda)},$$

where

$$f(\lambda) = \inf\{\mu \in \mathbb{R}^+ | \forall R_i, R_j \in K(A), H_\lambda(A_i, A_j) \leq H_\mu(B_i, B_j)\}. \tag{31}$$

Note that if this infimum is actually a minimum, then the family of closeness relations $\mathscr{M}$ is indeed interpolative with respect to the rule set $K(A)$. Moreover, if the A_i's are totally pairwise disjoint, then $H_0(A_i, A_j) = 0$ for $i \neq j$, and thus $f(0) = 0$.

In this setting, the proposed approach consists of taking, as the interpolated solution for an input $X = A$, the output $Y = B$ where

$$B = \bigcap\{F_\lambda(A, R_i) | R_i \in K(A), S_\lambda \in \mathscr{S}\} \tag{32}$$

with the fuzzy subset $F_\lambda(A, R_i)$ of Y being the least specific B' such that $close_\lambda^X(A, A_i) \leq close_\lambda^Y(B', B_i)$. In [6] it is shown that $F_\lambda(A, R_i)$ is then defined by

$$F_\lambda(A, R_i)(y) = H_\lambda(A, A_i) \to (S_{f(\lambda)} \circ B_i)(y) \quad \text{for all } y \in Y. \tag{33}$$

Note that even though the formal expression for the output (32) is the same for both types of closeness relations defined by H_λ^1 and H_λ^2, the output fuzzy set is not generally the same.

It is informative to note that the method described in [14] can be formulated, in part, in terms of this approach. In fact, Esteva et al. use $close_\lambda^X, close_\lambda^Y$ defined by Ruspini's implication measures H_λ^1 induced by some parametric families of similarity relations on the input and output spaces, and the output B is required to satisfy (28) with equalities. Sometimes this stronger condition is too strong to yield a fuzzy set as solution. As noted in [14], this method applied to triangular fuzzy sets gives the same results for pairs of sparse rules as does the analogy-based interpolation explained in Section 6.2.

---

[8] Recall that the fuzzy set $S_\lambda \circ D$ is defined as $(S_\lambda \circ D)(x) = \bigvee_{y \in \mathbb{R}} S_\lambda(x, y) * D(y)$.

7.2. Interpolating fuzzy functions and fuzzy interpolation based on closeness

Let us analyze the above described instantiation of the approach to fuzzy interpolation that results in formula (32). The latter gives an interpolated solution B for an input A in the sense described above and is different from the one proposed in Section 4.2. Because (32) uniquely assigns B to A, it can be viewed as a formal representation of a fuzzy function. We denote this fuzzy function by $\wp_G : L^X \to L^Y$. Moreover, if we substitute $F_\lambda(A, R_i)$ in (32) for the right-hand side of (33), we then obtain a formal representation of $\wp_G : L^X \to L^Y$ in the form

$$\wp_G(A) = \bigwedge_{\lambda \in \Lambda} \bigwedge_{i=1}^{2} (H_\lambda(A, A_i) \to (S_{f(\lambda)} \circ B_i)), \tag{34}$$

where $\Lambda = \{\lambda | 0 \le \lambda \le +\infty\}$, $f(\lambda)$ is given by (31), and for each $\lambda \ge 0$, S_λ is a similarity relation from the nested family $\mathscr{S} = \{S_\lambda : 0 \le \lambda \le +\infty\}$ of similarity relations on $\mathbb{R}$. In the following, we will assume that if $0 \le \lambda_1 \le \lambda_2$, then $S_{\lambda_1} \le S_{\lambda_2}$.

In this subsection, we will show that under certain restrictions on the fuzzy data and on the domain of $\wp_G$, the latter satisfies the interpolation condition (15). Moreover, we will show that such a restricted fuzzy function $\wp_G$ can be represented formally with the help of a certain fuzzy relation and the $\lhd$-composition given by (6).

Let us list the assumptions made about the fuzzy data A_1, A_2:

(a) A_1, A_2 are normal fuzzy sets with $x_{A_i} \in Core(A_i)$, $i = 1, 2$,

(b) there exists $\lambda^* \ge 0$ such that for every $i = 1, 2$, $A_i \le S_{x_{A_i}}^{\lambda^*}$,

(c) $S_{x_{A_1}}^{\lambda^*} \wedge S_{x_{A_2}}^{\lambda^*} = 0$.

The set $\Lambda = \{\lambda | 0 \le \lambda \le +\infty\}$ will be partitioned into two subsets $\Lambda = \Lambda_1 \cup \Lambda_2$ so that

$$\Lambda_1 = \{\lambda | 0 \le \lambda \le \lambda^*\}, \quad \Lambda_2 = \{\lambda | \lambda^* < \lambda \le +\infty\}. \tag{35}$$

For every $\lambda \in \Lambda_2$, we consider

$$S_{x_{A_i}}^{\lambda} \lhd R = S_{f(\lambda)} \circ B_i, \quad i = 1, 2 \tag{36}$$

as an auxiliary system of fuzzy relation equations (where R is unknown). In [27], we proved that the inequality (31) guarantees the solvability of (36) and moreover, the following fuzzy relation

$$\hat{R}^\lambda(x, y) = \bigwedge_{i=1}^{2} (S_{x_{A_i}}^{\lambda}(x) \to (S_{f(\lambda)} \circ B_i)(y)) \tag{37}$$

is a solution of (36).

Our results will be based on the propositions given below. We retain the notation of Section 7 (see [26] for proofs).

Theorem 4. *Let $A_1, A_2 \in L^X$, $B_1, B_2 \in L^Y$ be normal fuzzy sets with $x_{A_i} \in Core(A_i)$ and $x_{B_i} \in Core(B_i)$, $i = 1, 2$. Let the family of interpolative closeness relations $\mathscr{M}$ be based on a H_λ measure, let $f : \Lambda \to \Lambda$ be the function defined by (31), and let λ^* determine the partition (35). Then for every $j = 1, 2$, the interpolated solution (32) for A_j is equal to B_j, and the fuzzy function $\wp_G$ given by (34) satisfies the equality*

$$\wp_G(A_j) = B_j.$$

Corollary 1. *Let all the assumptions of Theorem 4 be satisfied, and for every $j = 1, 2$, let the interpolated solution for A_j be given by*

$$\tilde{\wp}_G(A) = \bigwedge_{\lambda \ge \lambda^*} \bigwedge_{i=1}^{2} (H_\lambda(A, A_i) \to (S_{f(\lambda)} \circ B_i)), \tag{38}$$

where, contrary to (34), *the external* inf *is taken over a subset of* Λ. *Then for every* $j = 1, 2$,

$$\tilde{\wp}_G(A_j) = B_j.$$

Theorem 4 and its corollary confirm that fuzzy functions (34) and (38) are interpolating in both senses: the classical one based on the interpolation condition (15) and the one based on closeness (25). Note that both functions are defined on the domain L^X.

We will now show that the interpolating fuzzy function (38) on the restricted domain D (see below) can be represented with the aid of the fuzzy relation $\hat{R}^\lambda$ given by (37), and the $\lhd$-composition given by (6). The domain D consists of those normal fuzzy sets that are between A_1 and A_2, i.e.,

$$D = \{A \in L^X | x_A \in Core(A), x_A \in [x_{A_1}, x_{A_2}], A \le S_{x_A}^{\lambda^*}\},$$

where the core points x_{A_1}, x_{A_2} of the fuzzy sets $A_1, A_2 \in L^X$ fulfill the inequality: $x_{A_1} < x_{A_2}$, and λ^* determines the partition (35).

Let fuzzy relation $\hat{R}^\lambda$ be given by (37) and for all $A \in D$, and all $\lambda \ge \lambda^*$, let fuzzy function $\sigma_\lambda : D \to L^Y$ be defined by

$$\sigma_\lambda(A)(y) = f_{\lhd \hat{R}^\lambda}(S_{x_A}^\lambda) = \bigwedge_{x \in X}(S_\lambda(x_A, x) \to \hat{R}^\lambda(x, y)), \quad y \in Y. \tag{39}$$

Moreover, for all $A \in D$, let

$$\sigma(A) = \bigwedge_{\lambda \ge \lambda^*} \sigma_\lambda(A),$$

so that $\sigma : A \to L^Y$ is a fuzzy function on D as well. The next theorem shows that for all $A \in D$, $\tilde{\wp}_G(A)$ is equal to $\sigma(A)$. In other words, the respective fuzzy functions coincide.

Theorem 5. *Let all the assumptions of Theorem 4 be fulfilled and* $\tilde{\wp}_G|_D$ *be a restriction of* $\tilde{\wp}_G$ *on* D. *Then the fuzzy functions* σ *and* $\tilde{\wp}_G|_D$ *coincide.*

8. An axiomatic approach to fuzzy interpolation

An axiomatic approach that considered both approaches proposed in [4,28] appeared in [18]. Seven necessary conditions (axioms) were formulated in order to state a "logical characterization of the existing rule interpolation methods." In addition to two axioms that we formulated above as conditions $\mathbf{I}_1$–$\mathbf{I}_2$, other axioms demand that an interpolating fuzzy function should have the following properties: monotonicity, continuity, closeness with respect to union and intersection, linearity, and preserving "in between." Moreover, a method (in algorithmic form) for obtaining an interpolating fuzzy function that fulfills all required axioms was proposed in [18] as well. Implicitly, the proposed method justifies the logical flavor of this axiomatic approach (in the same sense as a model justifies consistency of a logical theory).

In this section, we will consider the axiomatic approach to fuzzy interpolation in more (but not in all) technical details to compare it with approaches previously discussed.

8.1. Algorithmic description

Let assumptions (i)–(iv) above be satisfied and let fuzzy sets A_1, A_2 be represented by the following triplets: $(p_{A_1}, f_{A_1}, g_{A_1})$ and $(p_{A_2}, f_{A_2}, g_{A_2})$ where for each $i = 1, 2$, p_{A_i} is a *reference point* (e.g., a center of a core), and $f_{A_i}, g_{A_i} : [0, 1] \to [0, M_i]$ are *flank functions* (non-decreasing, left-continuous mappings from [0, 1] to $(-\infty, 0]$) such that for all $x \in X$,

$$A_i(x) = \begin{cases} f_{A_i}^{(-1)}(x - p_{A_i}) & \text{if } x \le p_{A_i}, \\ g_{A_i}^{(-1)}(p_{A_i} - x) & \text{if } x \ge p_{A_i}. \end{cases}$$

Similarly, we suppose fuzzy sets B_1, B_2 to be represented by triplets: $(p_{B_i}, f_{B_i}, g_{B_i})$, $i = 1, 2$. Let $A = (p_A, f_A, g_A)$ be an observation, i.e., a fuzzy subset of X with a triplet representation. We again propose to represent the respective value B of an interpolating fuzzy function by the triplet (p_B, f_B, g_B). The unknown components are computed in three steps:

Step 1: Compute $r = (p_{A_2} - p_A)/(p_{A_2} - p_{A_1})$.

Step 2: Compute $A' = (p_{A'}, f_{A'}, g_{A'})$, $B' = (p_{B'}, f_{B'}, g_{B'})$ where

$$
\begin{aligned}
p_{A'} &= p_A, & p_{B'} &= r p_{B_1} + (1 - r) p_{B_2}, \\
f_{A'} &= r f_{A_1} + (1 - r) f_{A_2}, & f_{B'} &= r f_{B_1} + (1 - r) f_{B_2}, \\
g_{A'} &= r g_{A_1} + (1 - r) g_{A_2}, & g_{B'} &= r g_{B_1} + (1 - r) g_{B_2}.
\end{aligned}
$$

Step 3: Transform A' into A so that

$$
f_A = t_{A,f} \circ f_{A'},
$$
$$
g_A = t_{A,g} \circ g_{A'}.
$$

Rescale $t_{A,f}$ and $t_{A,g}$ into $t_{B,f}$ and $t_{B,g}$ and apply to $f_{B'}$ and $g_{B'}$ to obtain f_B and g_B so that

$$
f_B = t_{B,f} \circ f_{B'},
$$
$$
g_B = t_{B,g} \circ g_{B'}.
$$

We have shown that the approach proposed above fulfills all axioms if and only if the following conditions hold:

(a) $A_1, A_2 : X \to [0, 1]$ are continuous functions,
(b) the reference points of A, A_1, A_2, B_1, B_2 are centers of their cores,
(c) the observation $A : X \to [0, 1]$ is a unimodal function (i.e., it has one extremum-maximum),
(d) functions A, A_1, A_2, B_1, B_2 are balanced (see [18]).

At the start of [18], the domain of the proposed extension was formally characterized as a set of *valid* fuzzy subsets of X. Subsequently, this characterization was specified as a set of fuzzy subsets of X with triplet representation. However, the claim given above shrinks the domain by imposing additional requirements. (In our subsequent discussion, we will characterize the domain so that all requirements are met).

8.2. Interpolating fuzzy functions and axiomatic approach to fuzzy interpolation

It is not difficult to see that any model of the axiomatic approach to fuzzy interpolation is a combination of the analogy-based interpolation and the interpolation by convex completion discussed in Sections 5 and 6. From that point of view, the former is an interpolating fuzzy function, say $\wp_J$. Conditions (a)–(d) above characterize a class of interpolating fuzzy functions that are models of the axiomatic approach to fuzzy interpolation. It follows from these conditions that the domain of $\wp_J$ is a class of all fuzzy sets A (observations) on X such that their respective membership functions are continuous, normal, convex, unimodal and have reference points at the centers of their cores.

Finally, we postulate conditions in addition to (a)–(d) that guarantee that $\wp_J$ has a formal representation. To do so, we will invoke the fuzzy relation ρ_g given by (8):

- The Łukasiewicz algebra on $L = [0, 1]$ is chosen as an underlying residuated lattice.
- Let $h_{A_1}, h_{A_2}, h_{B_1}, h_{B_1} : [0, 1] \to [0, 1]$ be decreasing functions such that for all $x \in [0, 1]$,

$$
h_{A_1}(x) \le 1 - x, \quad h_{A_2}(x) \le 1 - x, \quad h_{B_1}(x) \ge 1 - x, \quad h_{B_2}(x) \ge 1 - x.
$$

- For all $x \in X$, let

$$
\begin{aligned}
A_1(x) &= h_{A_1}^{(-1)}(|x - p_{A_1}|), & A_2(x) &= h_{A_2}^{(-1)}(|x - p_{A_2}|), \\
B_1(x) &= h_{B_1}^{(-1)}(|x - p_{B_1}|), & B_2(x) &= h_{B_2}^{(-1)}(|x - p_{B_2}|),
\end{aligned}
$$

- Let us define the ordinary function $g : [p_{A_1}, p_{A_2}] \rightarrow [p_{B_1}, p_{B_2}]$

$$g(x) = p_{B_1} + \frac{x - p_{A_1}}{p_{A_2} - p_{A_1}}(p_{B_2} - p_{B_1})$$

A proof of this claim will appear in a forthcoming paper on interpolating fuzzy functions.

9. Conclusion

We have characterized the problem of interpolation of fuzzy data. Our approach was to treat the problem in terms of extension of a fuzzy function (in the sense of Definition 3) given on a restricted domain to a fuzzy function given on a wider domain. Accordingly, we introduced and subsequently discussed all notions involved: fuzzy space, fuzzy functions, fuzzy equality, and similarity. These formalized notions were used to formulate the problem of interpolation of fuzzy data. We gave a general overview of principal approaches to this problem. We then analyzed the following four methods: analogy-based interpolation, interpolation by convex completion, interpolation based on closeness, and flank functions interpolation.

Our main purpose was to develop fuzzy interpolation methods centered on relation-based analytic representation of a resulting interpolating fuzzy function. For all four methods considered above, we showed how to characterize the proposed solutions by interpolating fuzzy functions. We also showed how to represent these functions by respective fuzzy relations and compositions. Our analytic representations reveal the common nature of these previously proposed methods. Still other approaches have been envisaged [12], where fuzzy data are viewed as ill-known points (rather than points equipped with similarity or weaker relations), which always leads to a fuzzy output for the result of an interpolation of a given type (e.g., linear) between such fuzzy points, even when applied to a precise input. Furthermore, our work offers better understanding of these specific methods. By doing so, it contributes to better general understanding of the problem of interpolation of fuzzy data.

Appendix

Theorems 3 and 1, proved below, confirm that computation on the basis of convex completion (Section 6) can be analytically represented by the fuzzy function $f_{\circ \rho_g}$ on the domain $\mathcal{X}_1$.

Theorem 2. *Let us accept steps* (a)–(h) *in Section* 6.2. *Then function g fulfills* (11), *i.e., for all* $t \in [c_1, c_2]$, $x \in X$,

$$R(t, x) \leq F(g(t), g(x)),$$

if and only if inequality (22) *holds, i.e.*

$$\frac{d_2 - d_1}{c_2 - c_1} \leq \min\left(\frac{m_1}{l_1}, \frac{m_2}{l_2}\right).$$

Proof. At first, we will prove that under the assumptions of the theorem, g fulfills (11) if and only if for all $\tau \in [0, 1]$ the following inequality holds:

$$\frac{d_2 - d_1}{c_2 - c_1} \leq \frac{\tau m_2 + (1 - \tau)m_1}{\tau l_2 + (1 - \tau)l_1}. \tag{40}$$

Let $t \in [c_1, c_2]$, $x \in X$ be chosen and let $\tau_t = (t - c_1)/(c_2 - c_1)$, $\tau_{g(t)} = (g(t) - d_1)/(d_2 - d_1)$ be computed. We show that $\tau_t = \tau_{g(t)}$. Indeed, using the expression for g (step (g)), we obtain

$$\tau_{g(t)} = \frac{g(t) - d_1}{d_2 - d_1} = \frac{1}{d_2 - d_1} \frac{(d_2 - d_1)(t - c_1)}{c_2 - c_1} = \frac{t - c_1}{c_2 - c_1} = \tau_t.$$

Then

$$R(t, x) \leq F(g(t), g(x)) \Leftrightarrow S^{\tau_t}(t, x) \leq Q^{\tau_{g(t)}}(g(t), g(x)) \Leftrightarrow \frac{|t - x|}{l_{\tau_t}} \geq \frac{|g(t) - g(x)|}{m_{\tau_{g(t)}}}.$$

where $l_{\tau_t} = \tau_t l_2 + (1 - \tau_t)l_1$, $m_{\tau_t} = \tau_t m_2 + (1 - \tau_t)m_1$. Because

$$|g(t) - g(x)| = \frac{(d_2 - d_1)|t - x|}{c_2 - c_1},$$

then

$$R(t, x) \leq F(g(t), g(x)) \Leftrightarrow \frac{m_{\tau_{g(t)}}}{l_{\tau_t}} \geq \frac{d_2 - d_1}{c_2 - c_1} \Leftrightarrow \frac{\tau_t m_2 + (1 - \tau_t)m_1}{\tau_t l_2 + (1 - \tau_t)l_1} \geq \frac{d_2 - d_1}{c_2 - c_1}.$$

Because $t \in [c_1, c_2]$ is an arbitrary point, τ_t runs over $[0, 1]$.

We now prove that inequalities (40) and (22) are equivalent. If we assume (40) and let $\tau = 0$ and 1 then

$$\frac{d_2 - d_1}{c_2 - c_1} \leq \frac{m_1}{l_1} \quad \text{and} \quad \frac{d_2 - d_1}{c_2 - c_1} \leq \frac{m_2}{l_2},$$

and thus

$$\frac{d_2 - d_1}{c_2 - c_1} \leq \min\left(\frac{m_1}{l_1}, \frac{m_2}{l_2}\right).$$

If on the other hand we assume (22) then it is sufficient to prove (40) for $\tau \in (0, 1)$. Let $m_2/l_2 \leq m_1/l_1$ and $r = m_2/l_2$ wherefrom $m_2 = rl_2$ and $m_1 \geq rl_1$. Then

$$\frac{\tau m_2 + (1 - \tau)m_1}{\tau l_2 + (1 - \tau)l_1} \geq \frac{\tau rl_2 + (1 - \tau)rl_1}{\tau l_2 + (1 - \tau)l_1} = r = \min\left(\frac{m_1}{l_1}, \frac{m_2}{l_2}\right).$$

Therefore,

$$\frac{d_2 - d_1}{c_2 - c_1} \leq \min\left(\frac{m_1}{l_1}, \frac{m_2}{l_2}\right) \leq \frac{\tau m_2 + (1 - \tau)m_1}{\tau l_2 + (1 - \tau)l_1}. \quad \square$$

Theorem 3. *Let us accept steps* (a)–(h) *in Section* 6.2, *and moreover, assume that the lengths* m_1, m_2 *of the intervals* B_1, B_2 *are equal. Then fuzzy relation* F *constructed on the step* (f) *is a similarity on* $[d_1, d_2]$.

Proof. It is easy to see that by the assumption $m_1 = m_2$, for each $\tau \in [0, 1]$, $m_\tau = m_1 = m_2$. Therefore, all similarities Q^τ on $[d_1, d_2]$ are equal and they are equal to the fuzzy relation F on $[d_1, d_2]$. Thus F is a similarity on $[d_1, d_2]$ too.

In Lemma 1, the $*$-transitivity of the closeness relation H^1_λ is proved. $\square$

Lemma 1. *If S is reflexive and $*$-transitive, then $(E \lesssim S \circ D) * (D \lesssim S \circ F) \leq (E \lesssim S \circ F)$.*

Proof. The proof is based on the following properties, all of which can be checked easily for any fuzzy sets E, D, F and any $\alpha \in [0, 1]$:

(1) $(E \lesssim \alpha * D) \geq \alpha * (E \lesssim D)$.
(2) $S \circ (\alpha * D) = \alpha * (S \circ D)$.
(3) $S \circ (S \circ D) = S \circ D$.
(4) $(E \lesssim D) * (D \lesssim F) \leq (E \lesssim F)$.
 Proof: let $\alpha = E \lesssim D = \bigwedge_x E(x) \to D(x)$. Then it holds that $D(x) \geq \alpha * E(x)$ for every x, noted $D \geq \alpha * E$. Therefore, $(S \circ E \lesssim S \circ D) \geq (S \circ E \lesssim S \circ (\alpha * E)) \geq \alpha * (S \circ E \lesssim S \circ E) = \alpha * 1 = \alpha = E \lesssim D$.

Then we have the following chain of inequalities: $(E \lesssim S \circ D) * (D \lesssim S \circ F) \leq (E \lesssim S \circ D) * (S \circ D \lesssim S \circ (S \circ F)) = (E \lesssim S \circ D) * (S \circ D \lesssim S \circ F) \leq (E \lesssim S \circ F)$. $\square$

Acknowledgments

References

[1] P. Baranyi, T.D. Gedeon, Rule interpolation by spatial geometric representation, in: Proceedings of IPMU'96 Conference, Granada, 1996, pp. 483–488.

[2] P. Baranyi, L.T. Kóczy, T.D. Gedeon, A generalized concept for fuzzy rule interpolation, IEEE Transactions on Fuzzy Systems 12 (2004) 820–837.

[3] U. Bodenhofer, M. De Cock, E.E. Kerre, Openings and closures of fuzzy pre- orderings: theoretical basis and applications to fuzzy rule-based systems, International Journal of General Systems 32 (2003) 343–360.

[4] B. Bouchon-Meunier, J. Delechamp, C. Marsala, N. Mellouli, M. Rifqi, L. Zerrouki, Analogy and fuzzy interpolation in case of sparse rules, in: Proceedings of the EUROFUSE-SIC Joint Conference, 1999, pp. 132–136.

[5] B. Bouchon-Meunier, D. Dubois, C. Marsala, H. Prade, L. Ughetto, A comparative view of interpolation methods between sparse fuzzy rule, in: Proceedings of the Joint 9th IFSA World Congress and 20th NAFIPS International Conference, 2001, pp. 2499–2504.

[6] B. Bouchon-Meunier, F. Esteva, L. Godo, M. Rifqi, S. Sandri, A principled approach to fuzzy rule base interpolation using similarity relations, in: Proceedings of the EUSFLAT-LFA Joint Conference, 2005, pp. 757–763.

[7] B. Bouchon-Meunier, C. Marsala, M. Rifqi, Interpolative reasoning based on graduality, in: Proceedings of FUZZ-IEEE'2000 International Conference, San Antonio, 2000, pp. 483–487.

[8] M. Demirci, Fundamentals of m-vague algebra and m-vague arithmetic operations, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 10 (2002) 25–75.

[9] D. Dubois, F. Esteva, L. Godo, H. Prade, Fuzzy-set based logics – a history-oriented presentation of their main developments, in: D.M. Gabbay, J. Woods (Eds.), Handbook of the History of Logic, vol. 8, The Many Valued and Non-monotonic Turn in Logic, Elsevier, 2007, pp. 325–449.

[10] D. Dubois, R. Martin-Clouaire, H. Prade, Practical computing in fuzzy logic, in: M.M. Gupta, T. Yamakawa (Eds.), Fuzzy Computing, North-Holland, 1988.

[11] D. Dubois, H. Prade, Fuzzy Sets and Systems. Theory and Applications, Academic Press, New York, 1980.

[12] D. Dubois, H. Prade, On fuzzy interpolation, International Journal of General Systems 28 (1999) 103–114.

[13] D. Dubois, H. Prade, L. Ughetto, A new perspective on reasoning with fuzzy rules, International Journal of Intelligent Systems 18 (5) (2003) 541–567.

[14] F. Esteva, B. Bouchon-Meunier, M. Rifqi, M. Detyniecki, Similarity-based fuzzy interpolation method, in: Proceedings of IPMU'04, 2004, pp. 1443–1449.

[15] S. Gottwald, Fuzzy Sets and Fuzzy Logic. The Foundations of Application—from a Mathematical Point of View, Vieweg, Braunschweig/Wiesbaden and Teknea, Toulouse, 1993.

[16] P. Hájek, Metamathematics of Fuzzy Logic, Kluwer, Dordrecht, 1998.

[17] U. Höhle, L.N. Stout, Foundations of fuzzy sets, Fuzzy Sets and Systems 40 (1991) 257–296.

[18] S. Jenei, Interpolation and extrapolation of fuzzy quantities revisited—an axiomatic approach, Soft Computing 5 (2001) 179–193.

[19] S. Jenei, E.P. Klement, R. Konzel, Interpolation and extrapolation of fuzzy quantities revisited—the multiple-dimensional case, Soft Computing 6 (2002) 258–270.

[20] Z.C. Johanyák, S. Kovács, Fuzzy rule interpolation based on polar cuts, in: B. Reusch (Ed.), Computational Intelligence, Theory and Applications, Springer, 2006.

[21] F. Klawonn, Fuzzy points, fuzzy relations and fuzzy functions, in: V. Novák, I. Perfilieva (Eds.), Discovering the World with Fuzzy Logic, Springer, Berlin, 2000, pp. 431–453.

[22] L.T. Kóczy, K. Hirota, Approximate reasoning by linear rule interpolation and general approximation, International Journal of Approximate Reasoning 9 (1993) 197–225.

[23] L.T. Kóczy, K. Hirota, Interpolative reasoning with insufficient evidence in sparse fuzzy rule bases, Information Sciences 71 (1993) 169–201.

[24] H. Lai, D. Zhang, Fuzzy preorder and fuzzy topology, Fuzzy Sets and Systems 157 (2006) 1865–1885.

[25] I. Perfilieva, Fuzzy function as an approximate solution to a system of fuzzy relation equations, Fuzzy Sets and Systems 147 (2004) 363–383.

[26] I. Perfilieva, P. Hoďáková, M. Wrublová, Fuzzy rule base interpolation and its graphical representation, in: Proceedings of 17th Zittau Fuzzy Colloquium, Zittau, 2010, pp. 132–137.

[27] I. Perfilieva, V. Novák, System of fuzzy relation equations as a continuous model of if–then rules, Information Sciences 177 (2007) 3218–3227.

[28] L. Ughetto, D. Dubois, H. Prade, Fuzzy interpolation by convex completion of sparse rule bases, in: Proceedings of FUZZ-IEEE'2000 International Conference, San Antonio, 2000, pp. 465–470.

[29] S. Yan, M. Mizumoto, W. Qiao, Reasoning conditions on Koczy's interpolative reasoning method in sparse fuzzy rule bases, Fuzzy Sets and Systems 75 (1995) 63–71.