Some remarks about standard first order tautologies

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Abstract

In this short note we gather some known results about tautologies of first order fuzzy logics with the standard semantics, mainly for logics of a continuous t-norms, and specially for Łukasiewicz, Product and Gödel logics, and we end up with an open problem. We summarize some completeness and satisfiability results but we will not deal with decidability and complexity issues. For general notions and results on first-order fuzzy logics we refer to [1, 6, 5].

The following facts are well-known (see e.g. Hájek's book [4]):

- Gödel first order logic G∀ is the only one that is strong standard complete and thus standard semantics coincides with general semantics. This also implies that this logic is recursively axiomatizable, and the classical deduction-detachment theorem is valid in G∀.
- Łukasiewicz and Product first order logics, Ł∀ and $\Pi\forall$, are not standard complete and while the set of tautologies of these logics with the general semantics is recursively axiomatizable, those with the standard semantics are not even recursively enumerable.

On the other hand, Hájek also proved that a formula is a standard tautology of Ł∀ iff, for each $n \geq 2$, it is a tautology of $Ł_n\forall$, where $Ł_n\forall$ denotes the $(n+1)$-valued Łukasiewicz first order logic. But a similar result cannot be generalized to other first order fuzzy logics (see e.g. [3]), namely:

- If $[0,1]_*$ is the standard chain defined by a continuous t-norm $*$ different from the Łukasiewicz t-norm, then the formula (witness axiom $(C\forall)$ for the universal quantifier),
$$(\exists x)(P(x) \to (\forall y)P(y))$$
is a tautology over any finite L_*-chain but it is not a tautology over $[0,1]_*$. In particular, it does not hold in general that a formula is a standard tautology of G∀ (i.e. a tautology over $[0,1]_G$) iff, for each $n \geq 2$, is a $G_n\forall$-tautology (i.e. a tautology over $(n+1)$ element Gödel chain).
- Let $*$ be a left-continuous (non-continuous) t-norm. Then the formula,
$$(\forall x)(\chi \& \psi) \to (\chi \& (\forall x)\psi) \text{ where } x \text{ is not free in } \chi,$$

is a tautology over any finite L_*-chain, but it is not a tautology over the standard chain $[0,1]_*$.

Nevertheless, for $\Pi\forall$ the following result holds [2]:

– A formula is a standard tautology (i.e. over $[0,1]_\Pi$) iff it is a tautology over a one-element generated subchain of $[0,1]_\Pi$, i.e. iff it is a tautology over a (quasi-discrete) product algebra over a set $\{1,0\} \cup \{a^n \mid n \in \mathbb{N}\}$, for some $0 < a < 1$, and hence where the positive elements form a descending chain with limit 0.

Taking into account the above behaviour of Ł$\forall$ and $\Pi\forall$ with respect to one-element generated subchains of the standard one, one could ask whether this behaviour generalizes to logics $L_*\forall$, where $*$ is an ordinal sum of Łukasiewicz and product components, in the following sense: does the set of standard tautologies $L_*\forall$ coincide with the common tautologies of the family of logics $L_{*'}\forall$ where $*'$ is obtained from $*$ by replacing

- each Łukasiewicz component by a Ł$_n$ component, and
- each product component by a one-element generated product chain ?

However, an easy argument shows that the answer to this question is negative. The reason is that each $L_{*'}\forall$ obviously satisfies the witness axiom for the existential quantifier $(C\exists)$, while $L_*\forall$ does not. Indeed, let $a \in [0,1]_*$ be an idempotent element different from 0 and 1 and take the $[0,1]_*$-model $\mathcal{M} = (\mathbb{N}, P_M)$, where $P_M(n) = a_n$ with $\langle a_n \rangle_{n\in\mathbb{N}}$ forming an increasing sequence with limit a and such that $a_n < a$ for all $n \in \mathbb{N}$. Then the value of the formula

$$(\exists x)((\exists y)P(y) \to P(x))$$

in the model $\mathcal{M}$ is

$$\sup_n(\sup_m(a_m) \to a_n) = \sup_n(a \to a_n) = a \neq 1.$$

We finish with some short remarks about the relationships between the 1-satisfiability (SAT) and the positive-satisfiability (SAT$_{pos}$) problems for the three main logics G$\forall$, Ł$\forall$ and $\Pi\forall$ with the standard semantics. For the first two logics the following holds:

– For G$\forall$, SAT is equivalent to SAT$_{pos}$.
  Indeed, taking into account the completeness result and the deduction-detachment theorem for G$\forall$, we have: φ is not 1-satisfiable iff $\varphi \models_{G\forall} \bar{0}$ iff $\models_{G\forall} \varphi \to \bar{0}$, i.e. φ is not postively satisfiable.
– For Ł$\forall$, the previous equivalence is false since it is already false for propositional logic (for example, the formula $p \wedge \neg p$ is positively satisfiable but not 1-satisfiable).

Open problem. For the case of product logic, although SAT is equivalent to SAT$_{pos}$ in the propositional case, whether this equivalence holds for $\Pi\forall$ is currently an open problem. In fact, to solve this problem is equivalent to solve the question of whether the validity of the following instance of the deduction-detachment theorem

$$\varphi \models_{\Pi\forall} \bar{0} \quad \text{iff} \quad \models_{\Pi\forall} \varphi \to \bar{0}.$$

holds within the standard semantics.

Acknowledgments The authors acknowledge partial support the Spanish projects ARINF (TIN2009-14704- C03-03) and TASSAT(TIN2010-20967-C04-01), as well as the FP7-PEOPLE-2009-IRSES project MaToMUVI (PIRSES-GA-2009-247584).

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