Probabilistic Epistemic Spaces, Lockean Beliefs, Stable Beliefs and Change

Tommaso Flaminio[1,*,†], Lluis Godo[1,†], Ramón Pino Pérez[2,†], Ricardo O. Rodriguez[3,†] and Lluis Subirana[4,†]

[1]*Artificial Intelligence Research Institute (IIIA - CSIC), Campus de la UAB, Bellaterra, Spain*
[2]*Departamento de Matemáticas, Facultad de Ciencias, Universidad de Los Andes, Mérida, Venezuela*
[3]*Departamento de Computación, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Argentina*
[4]*Department of Mathematics, University of Barcelona (UB), Barcelona, Spain*

Abstract

The present contribution is framed in the context of probabilistic beliefs and analyzes two main approaches to the subject: the Lockean thesis and Leitgeb's notion of P-stable sets. More precisely, it investigates the connections between the deductive closure of Lockean belief sets, conformity to a characterization obtained in terms of step probabilities, and the revision of belief sets defined via P-stability. Regarding the latter approach, in addition to the original ideas and results of Leitgeb (2013), we also follow a path recently explored by Delgrande, Lakemeyer, Pagnucco, and Sack. We uncover tight links between these two approaches in the setting of probabilistic epistemic change.

Keywords

Lockean beliefs, Stable beliefs, Belief revision, Iterated belief revision, Epistemic spaces

1. Introduction

The so-called *Lockean thesis* argues that it is rational for an agent to believe a statement φ provided that the agent's subjective probability $P(\varphi)$ of φ overcomes a certain threshold λ that is strictly greater than $1/2$, see [12]. Given a threshold λ and a probability function P, one can then consider the corresponding *Lockean belief set*:

$$\mathscr{B}_{\lambda,P} = \{\varphi \in \mathcal{L} : P(\varphi) \geq \lambda\}. \tag{1}$$

The recent paper [11] is concerned with the study and the characterization of those Lockean belief sets that are closed under logical deduction. In the same paper a notion of minimal revision for Lockean belief sets has been introduced. Such a minimal revision is based on a slight modification of Jeffrey's rule for updating probabilities.

Dealing with belief sets via epistemic states that employ probabilities is not a new idea; indeed, several other alternatives appeared in the literature. Let us recall [22] by Shear and Fitelson; [14, 15] both by Hannson; [19] by Leitgeb and [7] by Cantwell and Rott. Among these, Leitgeb's formalization of P*-stable*$^\lambda$ sets particularly captured our attention because, already at a first sight, these sets seem to share a common inspiring principle with the analysis we put forward in [11] based on Lockean belief sets.

Besides Leitgeb's original formulation, we have been indeed motivated by a recent paper by Delgrande et al. [9], which builds on Leitgeb's notion of P-stability$^\lambda$. They assume $\lambda = 1/2$ and show that the set of P-stable$^{1/2}$ subsets of the set of possible worlds forms an increasing chain. Thus, they define a *belief set* to be the least P-stable$^{1/2}$ subset of possible worlds, in such inclusion order.

NMR 2026: 24th International Workshop on Nonmonotonic Reasoning July 17–19, 2026, Lisbon, Portugal
*Corresponding author.
†These authors contributed equally.
tommaso@iiia.csic.es (T. Flaminio); godo@iiia.csic.es (L. Godo); pinoperez@gmail.com (R. Pino Pérez); ricardo@dc.uba.ar (R. O. Rodriguez); lsubirana@yahoo.com (L. Subirana)

This chain of P-stable$^{1/2}$-sets determines a total pre-order on worlds, and hence it induces an AGM revision operator. The authors of [9] also study when this static revision is compatible with a dynamic setting where probabilities (understood as epistemic states) are updated by Jeffrey's rule.

An interesting fact emerging from these probability-based settings is that the distinction between epistemic states and what we might call the *doxastic state* becomes evident. The former captures *knowledge* via a probabilistic model; the latter encodes *beliefs* as a set of formulas. In the Lockean view, for example, an epistemic state is a pair (P, λ) and the induced doxastic state is $\mathscr{B}_{\lambda,P} = \{\varphi \mid P(\varphi) \geq \lambda\}$.

While the classical approaches of AGM [1] and KM [16] tend to identify the epistemic and the doxastic states, it is worth noticing that Darwiche and Pearl [8] start discussing this distinction. Consider the following question: when an agent receives new information ψ, should they revise their epistemic state $\mathcal{E}$ or their doxastic state $\mathcal{D}$? As observed, while this question makes no sense in the AGM and KM setting, Darwiche and Pearl propose to revise the epistemic state. Similarly, in the Lockean setting, revision typically updates $\mathcal{E}$ (e.g., via Bayesian conditionalization).

This situation can be summarised in the following diagram where the double downarrows labelled by $\mathcal{B}$ indicates that $\mathcal{D}$ and $\mathcal{D}^*_\psi$ are the doxastic states induced by $\mathcal{E}$ and $\mathcal{E}^*_\psi$ respectively, and where $\mathbb{er}$ indicate a generic *epistemic revision* (for instance Bayesian conditionalization in case of probabilistic epistemic states) while $\mathbb{dr}$ stands for a *doxastic revision* acting on sets of formulas (as an AGM operator for example).

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\mathbb{er}"] \arrow[d, Rightarrow, "\mathcal{B}"'] & \mathcal{E}^*_\psi \arrow[d, Rightarrow, "\mathcal{B}"] \\ \mathcal{D} = \mathcal{B}(\mathcal{E}) \arrow[r, "\mathbb{dr}"'] & \mathcal{D}^*_\psi \stackrel{?}{=} \mathcal{B}(\mathcal{E}^*_\psi) \end{tikzcd}$$

This naturally leads to the question of whether, or under which conditions of $\mathbb{er}$ and $\mathbb{dr}$, given an epistemic state $\mathcal{E}$, the doxastic revision of the doxastic state induced by $\mathcal{E}$ coincides with the doxastic state induced by the epistemic revision of $\mathcal{E}$. In other words, whether

$$\mathbb{dr}(\mathcal{B}(\mathcal{E})) = \mathcal{B}(\mathbb{er}(\mathcal{E})).$$

The paper by Delgrande et al. [9] addresses this interesting question studying an AGM-style revision of doxastic states defined by their epistemic states, leveraging the probabilistic information of the latter. Specifically, they use the probabilistic information of the epistemic state to define a total pre-order on possible worlds, which is then used in the AGM-like revision. In doing so, they show that in some cases the above diagram commutes: the $\mathbb{dr}$ AGM-revision of the doxastic state $\mathcal{D}$ induced by the initial $\mathcal{E}$ coincides with the doxastic state induced by the revised epistemic state obtained via $\mathbb{er}$.

However, the method developed in [9] does not work in general and there are cases in which the above diagram does not commute. In this paper, we therefore propose a slight generalization of the AGM-revision considered by Delgrande et al. and show that the limitation of the earlier approach can be overcome.

In the next Section 2 we will recall the basic formal setting needed for the present paper and in Section 3 we will provide some insightful description of P-stable$^{1/2}$ sets, in the sense of Leitgeb, in terms of deductively closed Lockean belief sets and particular probability distributions called *step-probabilities*. The basic construction of [9] and the setting for minimal revision investigated in [11] will be briefly recalled in Section 4 and our main result will be presented in Section 5. In the last Section 6 we conclude and briefly discusse our future work on the subject.

2. Basic Notions

The language for expressing beliefs is classical propositional logic (CPL). The set of formulas $\mathcal{L}$ is built from a finite set of propositional variables, say $x_1, \ldots, x_n$, connectives $\wedge, \vee, \neg$ for *conjunction*, *disjunction*, and *negation* respectively, and constants $\top, \bot$ for *true* and *false*. Formulas in that language will be denoted by lower case Greek letters $\varphi, \psi, \ldots$ with possible subscripts. The connective of *implication* is defined as $\varphi \to \psi = \neg\varphi \vee \psi$, and *double implication* is $\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$. The set of consistent formulas will be denoted $\mathcal{L}^\star$. The consequence relation of CPL is denoted by $\vdash$. If Γ is a set of formulas, the consequences of Γ denoted $Cn(\Gamma)$ is defined by putting $Cn(\Gamma) = \{\psi \in \mathcal{L} \mid \Gamma \vdash \psi\}$. By simplicity, we denote by $Cn(\varphi)$ the set $Cn(\{\varphi\})$. By Ω we denote the finite set of logical valuations of $\mathcal{L}$. From the semantic point of view, every formula φ can be identified with the set $[\![\varphi]\!]$ of its models, *i.e.*, those logical valuations $\omega : \mathcal{L} \to \{0, 1\}$ such that $\omega(\varphi) = 1$, also written $\omega \models \varphi$. For all $\varphi \in \mathcal{L}$, $\uparrow[\![\varphi]\!]$ denotes the set of subsets of Ω that contains $[\![\varphi]\!]$, that is to say, $\uparrow[\![\varphi]\!]$ is the *filter* of 2^Ω, the algebra of parts of Ω, principally generated by $[\![\varphi]\!]$. Sometimes we identify $\uparrow[\![\varphi]\!]$ with $Cn(\varphi)$. A set of formulas Γ is said to be a theory if it is closed by logical deduction, *i.e.*, if $Cn(\Gamma) = \Gamma$. The set of all the theories is denoted by $\mathcal{T}$ and the set of consistent theories is denoted $\mathcal{T}^\star$. Given a subset X of Ω, we define the theory of X, denoted $Th(X)$ by putting $Th(X) = \{\varphi \in \mathcal{L} \mid X \subseteq [\![\varphi]\!]\}$. This set is clearly a theory. Let us note that in the context of belief change the theories are called *beliefs sets*.

Probability distributions are defined on Ω as usual and hence the corresponding probability measures, that are denoted by the same symbol P, will be defined on 2^Ω. For every formula $\varphi \in \mathcal{L}$ we adopt the convention to write $P(\varphi)$, instead of $P([\![\varphi]\!])$, to denote the probability of the set of models of φ. By a positive probability distribution (measure, respectively), we mean a distribution (measure) P such that $P(\omega) > 0$ for all $\omega \in \Omega$ (i.e. $P(\varphi) > 0$ for all φ such that $\varphi \not\vdash \bot$). The set of positive probabilities will be denoted $\mathscr{P}$.

AGM revision operators [1, 13] are functions that take as input a pair consisting of a theory representing the current beliefs and a formula representing the new information, and the output is a new theory. More precisely, an AGM revision operator $*$ is a function $* : \mathcal{T} \times \mathcal{L} \longrightarrow \mathcal{T}$ satisfying a set of eight postulates. As usual, the infix notation $K * \varphi$ is used to denote $*(K, \varphi)$.

Darwiche and Pearl [8] showed that the revision operators of the AGM framework are not well adapted to deal with iterated revisions. Then they proposed to extend the framework AGM to define *iterated revision operators*. Intuitively, in this setting the epistemic state of an agent to be revised by a new information, is a more complex structure that contains the logical beliefs and additional information as well, for instance some conditional information about the possible evolution of the state. In order to define in a clear and formal way this class of operators, the notion of *epistemic space* has been introduced [21] as part of the domain where these operators evolve.

Definition 1. *An* epistemic space *is an ordered pair $\mathcal{E} = \langle U, \mathcal{B} \rangle$ where U is a set whose elements are called* epistemic states *and $\mathcal{B}$ is a* projection *function, $\mathcal{B} : U \longrightarrow \mathcal{T}^\star$, that maps each $\Psi \in U$ to a consistent theory $\mathcal{B}(\Psi)$ that represents the belief set associated with the epistemic state Ψ.*

The two more typical examples of epistemic spaces are the TPO epistemic space, $\mathcal{E}_{tpo}$ (TPO is the abbreviation of Total Pre-Order) and the OCF epistemic space, $\mathcal{E}_{ocf}$ (OCF is the abbreviation Ordinal Conditional Function).[1] More precisely, $\mathcal{E}_{tpo} = \langle U_{tpo}, B_{tpo} \rangle$ where U_{tpo} is the set of all total pre-orders over Ω and B_{tpo} maps each total pre-order $\preceq$ in U_{tpo} to a consistent theory $K \in \mathcal{T}^\star$ such that $K = Th(\min(\Omega, \preceq))$. And $\mathcal{E}_{ocf} = \langle U_{ocf}, B_{ocf} \rangle$ where U_{ocf} is the set of all OCFs over Ω and B_{ocf} maps an OCF κ to a consistent theory K such that $K = Th(\{\omega : \kappa(\omega) = 0\})$. More examples can be found in [4, 5] where there is a study of belief change in the epistemic space of rational rankings. Below, we will give more examples in particular epistemic spaces defined with probabilities.

Given an epistemic space $\mathcal{E} = \langle U, \mathcal{B} \rangle$, an *iterated change operator* $\circ$ on $\mathcal{E}$ is a function of the form $\circ : U \times \mathcal{L}^\star \longrightarrow U$. General epistemic states are noted by uppercase Greek letters like Ψ. The output of an operator is noted in infix notation. Thus, $\Psi \circ \varphi$ is the epistemic state obtained from Ψ with the change produced by the information φ.

Definition 2. *Let $\mathcal{E} = \langle U, \mathcal{B} \rangle$ be an epistemic space and let $\circ$ be an iterated change operator on $\mathcal{E}$. We say that $\circ$ is an iterated revision operator on $\mathcal{E}$ if it satisfies the following postulates:*

[1]An OCF [23] κ is a function associating each world with a non-negative integer such that there is a world ω with $\kappa(\omega) = 0$. In some works, OCFs are called *rankings* as well and they can take values on several ordered sets like natural or real numbers [24, 17] or even truly transfinite ordinals as originally introduced by Spohn [23, 18].

(K*1)	$\mathcal{B}(\Psi \circ \varphi)$ *is a belief set.*	**(Closure)**
(K*2)	$\varphi \in \mathcal{B}(\Psi \circ \varphi)$.	**(Success)**
(K*3)	$\mathcal{B}(\Psi \circ \varphi) \subseteq Cn(\mathcal{B}(\Psi) \cup \{\varphi\})$.	**(Inclusion)**
(K*4)	$Cn(\mathcal{B}(\Psi) \cup \{\varphi\}) \subseteq \mathcal{B}(\Psi \circ \varphi)$ *when* $\neg\varphi \notin \mathcal{B}(\Psi)$.	**(Preservation)**
(K*5)	$\mathcal{B}(\Psi \circ \varphi)$ *is consistent when* φ *is consistent.*	**(Consistency)**
(K*6)	$\varphi_1 \equiv \varphi_2 \;\Rightarrow\; \mathcal{B}(\Psi \circ \varphi_1) \equiv \mathcal{B}(\Psi \circ \varphi_2)$.	**(Equivalence)**
(K*7)	$\mathcal{B}(\Psi \circ (\varphi_1 \wedge \varphi_2)) \subseteq Cn(\mathcal{B}(\Psi \circ \varphi_1) \cup \{\varphi_2\})$.	**(Superexpansion)**
(K*8)	$Cn(\mathcal{B}(\Psi \circ \varphi_1) \cup \{\varphi_2\}) \subseteq \mathcal{B}(\Psi \circ (\varphi_1 \wedge \varphi_2))$ *when* $\neg\varphi_2 \notin \mathcal{B}(\Psi \circ \varphi_1)$.	**(Subexpansion)**
(DP1)	$\gamma \vdash \psi \;\Rightarrow\; \mathcal{B}((\Psi \circ \psi) \circ \gamma) = \mathcal{B}(\Psi \circ \gamma)$.	**(Rigidity 1)**
(DP2)	$\gamma \vdash \neg\psi \;\Rightarrow\; \mathcal{B}((\Psi \circ \psi) \circ \gamma) = \mathcal{B}(\Psi \circ \gamma)$.	**(Rigidity 2)**
(DP3)	$\psi \in \mathcal{B}(\Psi \circ \gamma) \;\Rightarrow\; \psi \in \mathcal{B}((\Psi \circ \psi) \circ \gamma)$.	**(Non worsening 1)**
(DP4)	$\neg\psi \notin \mathcal{B}(\Psi \circ \gamma) \;\Rightarrow\; \neg\psi \notin \mathcal{B}((\Psi \circ \psi) \circ \gamma)$.	**(Non worsening 2)**

Iterated revision operators are also called DP revision operators. The most well-known DP revision operators are the natural revision of Boutilier [6], the lexicographical revision of Nayak [20], and the restrained revision of Booth and Meyer [3], all defined in the epistemic space $\mathcal{E}_{tpo}$. In [21], there are DP revision operators defined on $\mathcal{E}_{ocf}$ that cannot be "represented" on $\mathcal{E}_{tpo}$.

A family of probabilistic epistemic spaces which we are concerned with in this work has been considered in [10] and they are defined as follows.

Definition 3. *For every* $\lambda \in (\frac{1}{2}, 1)$ *we define* $\mathcal{E}_\lambda = \langle \mathscr{P}_\lambda, \mathcal{B}_\lambda \rangle$ *where*

- $\mathscr{P}_\lambda = \{P \in \mathscr{P} \mid \mathscr{B}_{\lambda,P}$ *is a theory*$\}$, *and*
- *for every* $P \in \mathscr{P}_\lambda$ *we put* $\mathcal{B}_\lambda(P) = \mathscr{B}_{\lambda,P}$.

Another important probabilistic epistemic space which we are concerned with in this work has been considered in [9] and it is based on the notion of *stability* as introduced by Leitgeb in [19]. These notions will be recalled and introduced in the next section.

3. Static Case: Relations Between Leitgeb's P-Stability and Lockean Belief Closure

In this section we will recall and discuss notions and results concerning Leitgeb's notions of P-stable$^\lambda$ sets and Lockean belief sets. To start with let us fix once and forall a finite set Ω of possible worlds, a probability function $P : 2^\Omega \to [0, 1]$, and a $\lambda \geq 1/2$. To ease and uniform the presentation, we will henceforth assume that we have fixed also a propositional language $\mathcal{L}$ so that Ω represents the set of the logical valuations of $\mathcal{L}$. Thus each subset X of Ω can be seen as a formula φ (up to logical equivalence) such that $[\![\varphi]\!] = X$ and reciprocally formulas can be understood as subsets of Ω.

Leitgeb's notion of a P-stable$^\lambda$ set is the following.

Definition 4 ([19])**.** *Let* X *be a subset of* Ω *and let* $\lambda \in [\frac{1}{2}, 1)$. X *is* P*-stable*$^\lambda$ *if for any* $Y \subseteq \Omega$ *such that* $X \cap Y \neq \emptyset$ *and* $P(Y) > 0$, *it holds that* $P(X \mid Y) > \lambda$.

The next result is proved in [19] and it provides a useful characterization of P-stable$^\lambda$ sets that will be employed in the rest of the present paper.

Lemma 1. $X \subseteq \Omega$ *is* P*-stable*$^\lambda$ *iff* $P(\omega) > \frac{\lambda}{1-\lambda} P(\Omega \setminus X)$ *for all* $\omega \in X$.

In particular, $X \subseteq \Omega$ *is* P*-stable*$^{1/2}$ *if* $P(\omega) > P(\Omega \setminus X)$ *for all* $\omega \in X$.

The other notion that we will be concerned with in this paper is that of Lockean belief set which we recall again here.

Definition 5. *Given the threshold* $\lambda > 1/2$ *the set of Lockean beliefs for the probability* P *is the set* $\mathscr{B}_{\lambda,P} = \{\varphi \in \mathcal{L} \mid P(\varphi) \geq \lambda\}$.

One of the key problems that makes difficult the employment of Lockean belief sets in belief change theory is that they are not closed under logical deduction in general. However, deductively closed Lockean belief sets has been recently characterized in [11]. We now recall that result and we invite the reader to notice how closely it resembles the second claim of the above Lemma 1.[2]

Theorem 1. *$\mathscr{B}_{\lambda,P}$ is deductively closed iff there is a formula $\Phi \in \mathcal{L}$ such that $P(\Phi) = \lambda$ and $P(\omega) > 1-\lambda$ for all $\omega \in [\![\Phi]\!]$. In such a case $\mathscr{B}_{\lambda,P} = \uparrow[\![\Phi]\!]$.*

Now we can show the following relationships.

Proposition 1. *If $\mathscr{B}_{\lambda,P}$ is deductively closed, and Φ is such that $\mathscr{B}_{\lambda,P}$ is the filter generated by Φ, then $[\![\Phi]\!]$ is P-stable$^{1/2}$.*

Proof. If $\mathscr{B}_{\lambda,P}$ is the filter generated by Φ, then by Theorem 1, $P(\Phi) = \lambda_M$ and $P(\omega) > 1 - \lambda_M = P(\Omega \setminus [\![\Phi]\!])$ for all $\omega \in [\![\Phi]\!]$, hence $[\![\varphi]\!]$ is P-stable$^{1/2}$. □

The converse also holds.

Proposition 2. *If Φ is such that $[\![\Phi]\!]$ is P-stable$^{1/2}$, then $\mathscr{B}_{\lambda,P}$ is deductively closed, where $\lambda = P(\Phi)$ and $\mathscr{B}_{\lambda,P}$ is the filter generated by Φ.*

Proof. If $[\![\Phi]\!]$ is P-stable$^{1/2}$, then, for all $\omega \in [\![\Phi]\!]$, $P(\omega) > P(\Omega \setminus [\![\Phi]\!]) = 1 - P([\![\Phi]\!])$. Therefore, by Theorem 1, if $\lambda = P([\![\Phi]\!])$, then $\mathscr{B}_{\lambda,P}$ is the filter generated by Φ, and hence it is deductively closed. □

As shown in [11], besides the above recalled characterization, deductively closed Lockean belief sets can be described in terms of *step probabilities*. We recall that a probability P has an *ω-step* (or has *a step at* ω), if it satisfies $P(\omega) > \sum_{\omega':P(\omega')<P(\omega)} P(\omega') > 0$. We denote by Φ_ω a formula such that $[\![\Phi_\omega]\!] = \{\omega' \in \Omega \mid P(\omega') \geq P(\omega)\}$.

Theorem 2 ([11]). *For every positive probability P, there exists $1/2 < \lambda < 1$ such that $\mathscr{B}_{\lambda,P}$ is deductively closed non-trivial[3] iff P has an ω-step.*

As a corollary of this result and Propositions 1 and 2 we have the following

Corollary 1. *Let P be a probability function. Then P has a step at some atom ω iff $[\![\Phi_\omega]\!]$ is $P^{1/2}$-stable.*

Now, let P be a probability, and let $\omega_0, \ldots, \omega_k$ be a sequence of steps of maximum length such that

$$P(\omega_0) > P(\omega_1) > \ldots > P(\omega_k) \tag{2}$$

Note that $[\![\Phi_{\omega_i}]\!] \subsetneq [\![\Phi_{\omega_j}]\!]$ if $i < j$. Thus, as a direct consequence of the above results, we have the following corollary.

Corollary 2. *For a probability function P, the set of its $P^{1/2}$-stable sets is totally ordered, and it coincides with the sequence of sets $([\![\Phi_{\omega_0}]\!], \ldots, [\![\Phi_{\omega_k}]\!], \Omega)$ where $\omega_0, \ldots, \omega_k$ is a maximal sequence of steps such that (2) holds.*

Example 1. *Let P be the probability defined on $\Omega = \{\omega_0, \omega_1, \omega_2, \omega_3\}$ by putting $P(\omega_0) = P(\omega_1) = 0.05$, $P(\omega_2) = 0.3$ and $P(\omega_3) = 0.6$. Note that the only steps of P are in ω_3 and ω_2. Moreover, $P(\omega_3) > P(\omega_2)$. Thus, by Corollary 2, we have that the increasing chain of stable set is $\{\omega_3\} \subset \{\omega_2, \omega_3\} \subset \Omega$.*

We conclude this section by introducing the epistemic space considered by Delgrande et al. [9].

Definition 6. *$\mathcal{E}_{st} = \langle \mathscr{P}, \mathcal{B}_{st} \rangle$ where $\mathscr{P}$ is the set of positive probabilities and for every $P \in \mathscr{P}$, $\mathcal{B}_{st}(P) = Th(W_0^P)$ with W_0^P the minimal stable set for P.*

Note that for every $\lambda \in (\frac{1}{2}, 1)$, $\mathcal{E}_\lambda \neq \mathcal{E}_{st}$.

[2]In the original formulation of the next result we have to consider a value λ_M that, although it is close to it, might not coincide with the parameter λ that defines the Lockean belief set. Discuss this point would create an unnecessary confusion and thus we prefer to present the next result in that way and we invite the reader to consult [11] for more details.

[3]A set Γ deductively closed is trivial if $Cn(\Gamma) = Cn(\top)$.

4. Dynamic Case: Minimal Change Operator vs Delgrande et al.'s AGM Revision and Epistemic State Revision

We first recall the minimal change operator on Lockean belief sets from [11], where R^{λ}_{φ} denotes the Jeffrey-like update of P by φ, with parameter λ. More precisely,

$$R^{\lambda}_{\varphi}(\omega) = \begin{cases} P(\omega) \cdot \max\left\{1, \frac{\lambda}{P(\varphi)}\right\} & \text{if } \omega \in [\![\varphi]\!] \\ P(\omega) \cdot \min\left\{1, \frac{1-\lambda}{P(\neg\varphi)}\right\} & \text{if } \omega \notin [\![\varphi]\!] \end{cases} \tag{3}$$

It is proved in [11] that R^{λ}_{φ} is Jeffrey conditionalization of P when $\varphi \notin \mathscr{B}_{\lambda,P}$, while $R^{\lambda}_{\varphi} = P$ otherwise. Next, we give the definition of the minimal revision operator $*_{ml}$ defined on Lockean beliefs.

Definition 7. $\mathscr{B}_{\lambda,P} *_{ml} \varphi = \{\psi \mid R^{\lambda}_{\varphi}(\psi) \geq \lambda\} = B_{\lambda,R^{\lambda}_{\varphi}}$

Next result characterizes when the minimal revision $\mathscr{B}_{\lambda,P} *_{ml} \varphi$ is again deductively closed. This yields a very drastic condition in case φ was not a previous belief, i.e. when $\varphi \notin \mathscr{B}_{\lambda,P}$.

Theorem 3 ([11]). *Let $\varphi \notin \mathscr{B}_{\lambda,P}$, i.e. $P(\varphi) < \lambda$. Then $\mathscr{B}_{\lambda,P} *_{ml} \varphi$ is closed iff $P(\omega) > \frac{1-\lambda}{\lambda}P(\varphi)$, for all $\omega \in [\![\varphi]\!]$. In that case, $\mathscr{B}_{\lambda,P} *_{ml} \varphi = \uparrow[\![\varphi]\!]$, that is, in the revision no information from $\mathscr{B}_{\lambda,P}$ is retained.*

This allows us to give an "iterated version" of the operator $*_{ml}$ agreeing with the framework of epistemic spaces introduced in Section 2. Namely, we define $\circ_{ml}$ on $\mathcal{E}_{\lambda} = \langle \mathscr{P}_{\lambda}, \mathcal{B}_{\lambda} \rangle$, that is, $\circ_{ml} : \mathscr{P}_{\lambda} \times \mathcal{L}^{\star} \longrightarrow \mathscr{P}_{\lambda}$, by putting

$$P \circ_{ml} \varphi = \begin{cases} R^{\lambda}_{\varphi} & \text{if } \varphi \in \mathscr{C}_{\lambda,P} \\ P & \text{if } \varphi \notin \mathscr{C}_{\lambda,P} \end{cases}$$

where $\mathscr{C}_{\lambda,P}$, the set of credible formulas for P given the threshold λ is defined by

$$\mathscr{C}_{\lambda,P} = \mathscr{B}_{\lambda,P} \cup \left\{ \varphi \in \mathcal{L}^{\star} \mid \forall \omega \in [\![\varphi]\!], P(\omega) > \frac{1-\lambda}{\lambda}P(\varphi) \right\}.$$

Actually, by Theorem 3, this operator is well defined as an iterated change operator on $\mathcal{E}_{\lambda}$ and it satisfies most iterated credibility-limited revision postulates (see [10, 2]). Moreover, if $\mathscr{B}_{\lambda,P}$ is a Lockean belief in the epistemic space $\mathcal{E}_{\lambda}$, *i.e.* there exists $P \in \mathscr{P}_{\lambda}$ such that $\mathcal{B}_{\lambda}(P) = \mathscr{B}_{\lambda,P}$, we have

$$\mathscr{B}_{\lambda,P} *_{ml} \varphi = \mathcal{B}_{\lambda}(P \circ_{ml} \varphi).$$

4.1. AGM-Style Revision

The notion of P-stability gives rise to a natural definition of revision in the AGM style. This is first introduced by Leitgeb in [19] and then reworked in Delgrande et al.'s [9].

Let P a probability function that is an epistemic state belonging to an epistemic space $\mathcal{E}$. The notion of P-stability$^{1/2}$, induces a total pre-order. Let $W_0, W_1, ..., W_k$ all the proper stable sets of Ω given by $W_i = [\![\Phi_{\omega_i}]\!]$ for $i = 1, \ldots, k$, as in Corollary 2. Put $W_{k+1} = \Omega$. Then this tower of sets induces a total pre-order on Ω whose levels are $L_0, L_1, ..., L_k, L_{k+1}$, defined by putting $L_0 = W_0$ and for $i \in \{1, \ldots, k+1\}$, $L_i = W_i \setminus \cup_{j=0}^{i-1} L_j$. The pre-order $\preceq_P$ is then defined by

$$\omega \preceq_P \omega' \quad \Leftrightarrow \quad \omega \in L_i,\, \omega \in L_j \text{ and } i \leq j$$

This total pre-order $\preceq_P$ on Ω is used to define a AGM-style revision in the natural way.

Definition 8. *Given P, the (beliefs of the) revision of P by a formula ϕ, denoted $P \otimes \varphi$ is defined by putting $P \otimes \varphi = Th(\min([\![\varphi]\!], \preceq_P))$.*

Note that the operator $\otimes$ has domain $\mathscr{P} \times \mathcal{L}$ and its codomain is the set of theories. Thus, properly speaking, the operator $\otimes$ is neither an AGM revision operator nor an operator for iterated revision. Note also that if we fix the theory $K = Bel(P)$, by Katsuno and Mendelzon's representation theorem [16], if the output of the revision of K by φ is exactly $P \otimes \varphi$ this function satisfies the Postulates AGM for the revision.

Note that the set $\mathcal{B}_{st}(P)$, the beliefs of the epistemic state P in the epistemic space $\mathcal{E}_{st}$, is exactly $Th(W_0)$ (see Def. 6), and hence, by Proposition 2, $\mathcal{B}_{st}(P) = \mathscr{B}_{\lambda,P}$ with $\lambda = P(W_0)$. Thus, one can see the one shot Delgrande et al.'s revision $\otimes$ as another way of revising this particular Lockean belief: $\mathscr{B}_{\lambda,P} \otimes \varphi = P \otimes \varphi$. However, note that in general $\mathscr{B}_{\lambda,P} \otimes \varphi \neq \mathscr{B}_{\lambda,P} *_{ml} \varphi$.

4.2. Delgrande et al.'s Connection of AGM-Revision With Epistemic State Update

Delgrande et al. consider iterated change in the space $\mathcal{E}_{st} = \langle \mathscr{P}, \mathcal{B}_{st} \rangle$ using Jeffrey's conditionalization.

Definition 9. *Let $P \in \mathscr{P}$ be an epistemic state in the space $\mathcal{E}_{st}$ and let $s \in (1/2, 1)$. Then the epistemic state resulting from revising P by φ according to the parameter s is:*

$$P *_s \varphi = P^s_\varphi$$

where P^s_φ is the Jeffrey conditionalization of P given φ with probability s, i.e.,

$$P^s_\varphi(\psi) = s \cdot P(\psi \mid \varphi) + (1 - s) \cdot P(\psi \mid \neg\varphi).$$

Then, Delgrande et al. consider the following problem of finding values for s such that $\mathcal{B}_{st}(P^s_\varphi)$, the beliefs of P^s_φ, *i.e.*, the theory of the least P^s_φ-stable$^{1/2}$ set, coincides with the above AGM-like revision of P by φ. More precisely

Problem: find s such that $\mathcal{B}_{st}(P^s_\varphi) = Th(\min(\llbracket\varphi\rrbracket, \preceq_P))$.

They find two constraints on s equivalent to the solution of the problem. These are the following inequalities (4) and (5):

$$s > \frac{P(\varphi)}{P(\omega) + P(\min(\llbracket\varphi\rrbracket, \preceq_P))}, \quad \text{for } \omega \in \min(\llbracket\varphi\rrbracket, \preceq_P), \qquad (4)$$

and for all $X' \subsetneq \min(\llbracket\varphi\rrbracket, \preceq_P)$ there must exist $\omega \in X'$ such that

$$s \leq \frac{P(\varphi)}{P(X') + P(\omega)} \qquad (5)$$

Condition (4) guarantees that $\min(\llbracket\varphi\rrbracket, \preceq_P)$ is P^s_φ-stable, while the additional fulfillment of (5) guarantees that $\min(\llbracket\varphi\rrbracket, \preceq_P)$ is the minimal P^s_φ-stable set.

Proposition 3 (Delgrande et al, Prop 14). *For any P and any $\varphi \not\equiv \top$, there exists a minimal $s_1 < 1$ such that $\min(\llbracket\varphi\rrbracket, \preceq_P)$ is P^s_φ-stable for every $s \in (s_1, 1)$.*

Thus, while one can always find solutions for (4) as the above proposition guarantees, the problem may arise since (4) and (5) can be incompatible, as the following example shows.

Example 2. *(Slightly modified from Delgrande et al.) Consider $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and a probability P such that $P(\omega_1) = 0.5, P(\omega_2) = 0.25, P(\omega_3) = 0.20, P(\omega_4) = 0.05$. Let φ such that $\llbracket\varphi\rrbracket = \{\omega_1, \omega_2\}$. In this case $W_0 = \{\omega_1, \omega_2, \omega_3\}$ is the only P-stable set. Moreover $\min(\llbracket\varphi\rrbracket, \preceq_P) = \{\omega_1, \omega_2\} = \llbracket\varphi\rrbracket$. Condition (4) for s then becomes*

$$s > \frac{P(\varphi)}{\min_{\omega \in \min(\llbracket\varphi\rrbracket, \preceq_P)} P(\omega) + P(\min(\llbracket\varphi\rrbracket, \preceq_P))} = \frac{0.75}{0.25 + 0.75} = 0.75.$$

So, for $s > 0.75$, $\min(\llbracket\varphi\rrbracket, \preceq_P)$ is P^s_φ-stable, but it is not minimal, since (5) becomes

$$s \leq \frac{P(\varphi)}{P(\omega') + P(X')} = \frac{0.75}{0.5 + 0.5} = 0.75,$$

for $X' = \{\omega_1\}$ and $\omega' = \omega_1$. Thus, (4) and (5) are incompatible in this case and anything can be claimed about the beliefs of the revision of P by φ.

Therefore, there is no s such that $\mathcal{B}_{st}(P^s_\varphi) = Th(\min(\llbracket\varphi\rrbracket, \preceq_P))$.

On the other hand, Delgrande et al.'s approach also faces some counterintuitive behavior when updating an epistemic state with a formula which is already a belief. Next example shows a case where we have a probability P and a formula φ such that $\mathcal{B}_{st}(P) = Cn(\varphi)$ but the updated probability is different from P, *i.e.*, $P^s_\varphi \neq P$.

Example 3. *Consider $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and a probability P such that $P(\omega_1) = 0.4, P(\omega_2) = 0.3, P(\omega_3) = 0.2, P(\omega_4) = 0.1$. Let φ such that $\|\varphi\| = \{\omega_1, \omega_2, \omega_3\}$. $X = \{\omega_1, \omega_2, \omega_3\}$ is the minimal P-stable set. Note that e.g. $\{\omega_1, \omega_2\}$ is not P-stable, since $0.3 = P(\omega_2) \not> P(\omega \setminus \{\omega_1, \omega_2\}) = 0.3$. Moreover $\min(\|\varphi\|, \preceq_K) = \{\omega_1, \omega_2, \omega_3\} = \|\varphi\|$.*

Condition (4) for s becomes

$$s > \frac{P(\varphi)}{\min_{\omega \in \min(\|\varphi\|, \preceq_K)} P(\omega) + P(\min(\|\varphi\|, \preceq_K))} = \frac{0.9}{0.2 + 0.9} = 9/11$$

So, for $s > 9/11$, $\min(\|\varphi\|, \preceq_K)$ is P^s_φ-stable, and minimal as well for $s \leq 0.9$ since (5) becomes

$$s \leq \frac{P(\varphi)}{P(\omega') + P(X')} = \frac{0.9}{0.3 + 0.7} = 9/10$$

for $X' = \{\omega_1, \omega_2\}$ and $\omega' = \omega_2$. Thus, (4) and (5) are indeed compatible in this case.

Note that for $s = 0.9$, which satisfies (4), we have that $P^s_\varphi(\varphi) = P(\varphi) = 0.9$. In fact, $P^s_\varphi(\omega_i) = sP(\omega_i)/P(\varphi) = P(\omega_i)$ for $i = 1, 2, 3$ and $P^s_\varphi(\omega_4) = (1 - s)P(\omega_4/P(\neg\varphi) = P(\omega_4)$, so $P^s_\varphi = P$. On the other hand, for $s \in (9/11, 9/10)$, $P^s_\varphi \neq P$, although $\mathcal{B}_{st}(P^s_\phi) = \mathcal{B}_{st}(P) = Cn(\varphi)$.

The definition below proposes a convenient way to regard a revision operator which is inspired by what is proved and stated in Theorem 18 of [9].

Definition 10. *The operator $*_{st}$ is defined on $\mathcal{E}_{st}$ as a partial function,[4] $*_{st} : \mathscr{P} \times \mathcal{L}^\star \longrightarrow \mathscr{P}$ defined by*

$$P *_{st} \varphi = \begin{cases} P^s_\varphi, & \textit{if } s \textit{ satisfies conditions (4) and (5)} \\ \text{undefined}, & \textit{otherwise}\,. \end{cases}$$

5. Towards a Generalization of Delgrande et al.'s Approach

Given the above mentioned problem, here we propose a way to update an epistemic state P, given φ, that is alternative to the one proposed by Delgrande et al. Precisely, we look for suitable subsets $X \subseteq \min(\llbracket\varphi\rrbracket, \preceq_P)$ and some parameter $s \in (1/2, 1)$ such that X is the minimal P^s_φ-stable set, that is, such that $\mathcal{B}_{st}(P^s_\varphi) = Th(X)$.

We start by showing how the above conditions (4) and (5) can be equivalently reformulated in a way that is more convenient to us.

Proposition 4. *X is P^s_φ-stable iff s satisfies the following condition:*

$$s > \frac{P(\varphi)}{\min_{\omega \in X} P(\omega) + P(X)}\,. \qquad (4')$$

[4]This is not a notation in Delgrande et al. [9] but we introduce it to get a better understanding of their approach.

Proof. By definition, X is P_φ^s-stable iff $P_\varphi^s(\omega) > P_\varphi^s(\Omega \setminus X) = P_\varphi^s(\neg X)$ for all $\omega \in X$, i.e.

$$sP(\omega)/P(\varphi) > sP(\neg X \cap [\![\varphi]\!])/P(\varphi) + (1-s)P(\neg X \cap [\![\neg\varphi]\!])/P(\neg\varphi).$$

Since $X \subseteq [\![\varphi]\!]$, we have $[\![\neg\varphi]\!] \subseteq \neg X$, hence $P(\neg X \cap [\![\neg\varphi]\!])/P(\neg\varphi) = 1$, and the above inequality reduces to the following ones for any $\omega \in X$:

$$\begin{aligned}
&sP(\omega)/P(\varphi) > sP(\neg X \cap [\![\neg\varphi]\!])/P(\varphi) + (1-s),\\
&s(P(\varphi) + P(\omega) - P(\neg X \cap [\![\neg\varphi]\!])) > P(\varphi),\\
&s(P(X \cap [\![\neg\varphi]\!]) + P(\omega)) > P(\varphi),\\
&s(P(X) + P(\omega)) > P(\varphi),\\
&s > P(\varphi)/(P(\omega) + P(X)).
\end{aligned}$$

Hence it must be $s > P(\varphi)/(\min_{x\in X} P(\omega) + P(X))$. □

In addition, if we want $\mathcal{B}_{st}(P_\phi^s) = X$ to be the minimal P_ϕ^s-stable, we have to require s to further satisfy a condition similar to condition (5).

Proposition 5. *Let X be P_ϕ^s-stable set. Then it is also minimal iff*

$$s \leq \frac{P(\phi)}{\max_{X' \subsetneq X}\{(\min_{w\in X'} P(w)) + P(X')\}}\,. \qquad (5')$$

Proof. The inequality (5') directly follows from requiring that all proper subsets $X' \subsetneq X$ are not P_φ^s-stable, hence the inequalities $s \leq \frac{P(\phi)}{\min_{w\in X'} P(w) + P(X')}$ must hold for all such X'. □

The next example, that is obtained by slightly modifying Example 17 in [9], has inspired our generalization of Delgrande at al.'s approach.

Example 4. *Consider $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and a probability P such that $P(\omega_1) = 0.48$, $P(\omega_2) = P(\omega_3) = 0.24$, $P(\omega_4) = 0.04$. Let φ be such that $[\![\varphi]\!] = \{\omega_1, \omega_2\}$. It is quite clear that $W_0 = \{\omega_1, \omega_2, \omega_3\}$ is the P-stable minimal. Moreover $\min([\![\varphi]\!], \preceq_P) = \{\omega_1, \omega_2\} = [\![\varphi]\!]$. Condition (4) for s becomes*

$$s > \frac{P(\varphi)}{P(\omega_2) + P(\min([\![\varphi]\!], \preceq_P))} = \frac{0.72}{0.24 + 0.72} = 0.75.$$

So, for $s > 0.75$, $\min(\phi, \preceq_K)$ is P_φ^s-stable, but it is not minimal, since (5) becomes

$$s \leq \frac{P(\varphi)}{P(\omega') + P(X')} = \frac{0.72}{0.48 + 0.48} = 0.75,$$

for $X' = \{\omega_1\}$ and $\omega' = \omega_1$. Therefore, again, there is no s such that $\mathcal{B}_{st}(P_\varphi^s) = Th(\min([\![\varphi]\!], \preceq_P))$. However, following our proposal, let us choose $X = \{\omega_1\} \subseteq \min([\![\varphi]\!], \preceq_P)$. Then we get the following condition:

$$s > \frac{P(\varphi)}{P(\omega_1) + P(X)} = \frac{0.72}{0.48 + 0.48} = 0.75.$$

So, for $s > 0.75$, $X = \{\omega_1\}$ is P_ϕ^s-stable, and obviously minimal. Hence, for $s > 0.75$ we can claim that $\mathcal{B}_{st}(P_\phi^s) = \{\omega_1\}$.

The compatibility of the conditions (4') and (5') is not always guaranteed for any $X \subseteq \min([\![\varphi]\!], \preceq_P)$. In other words, not for every X both (4') and (5') hold. Thus, it makes sense to see for which X they are compatible, and hence, when there are minimal P_φ^s-stable sets and for which values of the parameter s meet these constraints. The next result characterizes these subsets of Ω where (4') and (5') are both satisfied.

Proposition 6. *Let $X = \{\omega_1, \ldots, \omega_n\} \subseteq \Omega$ such that $P(\omega_1) \geq \ldots \geq P(\omega_n)$. Then conditions (4') and (5') for X are compatible iff, for any $1 \leq k \leq n-1$, the following condition is satisfied:*

$$P(\omega_k) - P(\omega_n) < \sum_{i=k+1}^{n} P(\omega_i).$$

Proof. The compatibility condition of (4') and (5') reads as follows:

$$\frac{P(\varphi)}{\min_{\omega \in X} P(\omega) + P(X)} < \frac{P(\varphi)}{\max_{X' \subsetneq X}\{(\min_{\omega \in X'} P(\omega)) + P(X')\}}$$

hence, iff for all $X' \subsetneq X$ we have

$$\min_{\omega \in X'} P(\omega) + P(X') < \min_{\omega \in X} P(\omega) + P(X). \qquad (*)$$

Suppose $X_1 = \{\omega_1, \ldots, \omega_k\} \subsetneq X$ satisfies (*), and let $X_2 = \{\omega_{i_1}, \ldots, \omega_{i_k}\}$ be another subset of X of the same cardinality as X_1. Then it is easy to check that X_2 satisfies (*) as well. Indeed, $\min_{\omega \in X_2} P(\omega) + P(X_2) \leq P(\omega_k) + P(X_1)$. Hence, to check that (*) holds for any $X' \subsetneq X$ is enough to check what happens with all the subsets $X_k = \{\omega_1, \ldots, \omega_k\}$ of the k worlds with highest probability values, for $k = 1, \ldots, n-1$. Then the condition (*) for each set X_k reads as

$$\min_{\omega \in X_k} P(\omega) + P(X_k) < \min_{\omega \in X} P(\omega) + P(X)$$

that is,

$$P(\omega_k) + P(X_k) < P(\omega_n) + P(X)$$

or equivalently,

$$P(\omega_k) - P(\omega_n) < P(X) - P(X_k) = P(\omega_{k+1}) + \ldots + P(\omega_n).$$

□

In the following example, we explore the possibilities of making a good choice for $X \subseteq [\![\varphi]\!]$.

Example 5. *Consider $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ and a probability P such that $P(\omega_1) = 0.4$, $P(\omega_2) = 0.3$, $P(\omega_3) = 0.2$, $P(\omega_4) = 0.05$, $P(\omega_5) = 0.05$. And let φ such that $[\![\varphi]\!] = \{\omega_1, \omega_2, \omega_3, \omega_5\}$.*

The minimal P-stable set is $X = \{\omega_1, \omega_2, \omega_3\}$. Note that e.g. $\{\omega_1, \omega_2\}$ is not P-stable, since $0.3 = P(\omega_2) \not> P(\Omega \setminus \{\omega_1, \omega_2\}) = 0.3$. Moreover, $\min([\![\varphi]\!], \preceq_P) = \{\omega_1, \omega_2, \omega_3\} \subset [\![\varphi]\!]$.

Condition (4) for s becomes

$$s > \frac{P(\varphi)}{\min_{\omega \in X} P(\omega) + P(\min([\![\varphi]\!], \preceq_P))} = \frac{0.95}{0.2 + 0.9} = 95/110$$

So, for $s > 95/110$, $\min([\![\varphi]\!], \preceq_K)$ is P^s_φ-stable, but it is not minimal, since (5) becomes

$$s < \frac{P(\varphi)}{P(\omega') + P(X')} = \frac{0.95}{0.4 + 0.7} = 95/110$$

for $X' = \{\omega_1, \omega_2\}$ and $\omega' = \omega_1$. Thus, (4) and (5) are incompatible in this case, and the approach by Delgrande et al. cannot claim anything about the beliefs of the revision of P by φ.

Following our proposal, let us first consider $X = \{\omega_1, \omega_2\}$. Then we have

$$s > \frac{P(\varphi)}{\min_{\omega \in X} P(\omega) + P(X)} = \frac{0.95}{0.3 + 0.7} = 0.95 \qquad \text{(4')}$$

and

$$s \leq \min_{X' \subsetneq X, \omega' \in X'} \frac{P(\varphi)}{P(X') + P(\omega')} = \min\left\{\frac{0.95}{0.4 + 0.4}, \frac{0.95}{0.3 + 0.3}\right\} > 1 \qquad \text{(5')}$$

Therefore, $X = \{\omega_1, \omega_2\}$ is minimal P^s_φ-stable for $s \in (0.95, 1]$. Hence, we would choose $Bel(P^s) = Th(\{\omega_1, \omega_2\})$ as the revised beliefs by φ.

Next, let us consider $X = \{\omega_1\} \subseteq [\![\varphi]\!]$, which is the world with highest probability. Condition (4') now reads:

$$s > \frac{P(\varphi)}{P(\omega_1) + P(X)} = \frac{0.95}{0.4 + 0.4} > 1$$

so $X = \{\omega_1\}$ is not stable for any s.

Note that, even for a fixed φ and P, there might exist different disjoint intervals I_1 and I_2 for s such that the corresponding minimal P^s_φ-stable set X_1 for $s \in I_1$ is different from the P^s-stable set X_2 for $s \in I_2$. Next example shows this situation.

Example 6. *Consider $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ and a probability P such that $P(\omega_1) = 0.4, P(\omega_2) = 0.3, P(\omega_3) = 0.2, P(\omega_4) = 0.05, P(\omega_5) = 0.05$. And let φ such that $[\![\varphi]\!] = \{\omega_1, \omega_2, \omega_5\}$.*

The minimal P-stable set is $W_0 = \{\omega_1, \omega_2, \omega_3\}$. Note that e.g. $\{\omega_1, \omega_2\}$ is not P-stable, since $0.3 = P(\omega_2) \not> P(\Omega \setminus \{\omega_1, \omega_2\}) = 0.3$. In this case we have $\min([\![\varphi]\!], \preceq_P) = [\![\varphi]\!] \cap W_0 = \{\omega_1, \omega_2\} \subset [\![\varphi]\!]$.

We check for which s, the set $X = \min([\![\varphi]\!], \preceq_P)$ is P^s_φ-stable. Condition (4) for s becomes

$$s > \frac{P(\varphi)}{\min_{\omega \in X} P(\omega) + P(X)} = \frac{0.75}{0.3 + 0.7} = 0.75.$$

So, for $s > 0.75$, $\min(\varphi, \preceq_P)$ is P^s_φ-stable. Then (5) amounts to check that for every $X' \subsetneq X$ there exists $\omega' \in X'$,

$$s \leq \frac{P(\varphi)}{P(\omega') + P(X')}, \text{ that is, } \quad s \leq \min(\frac{0.75}{0.4 + 0.4}, \frac{0.75}{0.3 + 0.3}) = 75/80.$$

Thus, (4) and (5) are compatible, hence for $s \in (0.75, 0.75/0.8]$, $X = \{\omega_1, \omega_2\}$ is a minimal P^s_φ-stable set.

Let us now consider $X = \{\omega_1\} \subseteq [\![\varphi]\!]$, which is the world with highest probability. Hence:

$$s > \frac{P(\varphi)}{\min_{\omega \in X} P(\omega) + P(X)} = \frac{0.75}{0.4 + 0.4} = 0.75/0.8. \quad (4')$$

So, $X = \{\omega_1\}$ is also a minimal P^s_φ-stable set for $s \in (0.75/0.8, 1)$.

Summarising: (i) for $s \in (0.75, 0.75/0.8]$, $X = \{\omega_1, \omega_2\}$ is a minimal P^s_φ-stable set, and (ii) for $s \in (0.75/0.8, 1)$, $X = \{\omega_1\}$ is minimal P^s_φ-stable.

The previous example is indeed an illustration of a more general fact stated in the following result.

Proposition 7. *For any φ and any P, there exists an interval I, $I = (s_1, s_2]$ with $s_2 < 1$ or $I = (s_1, 1)$ such that for all $s \in I$ there exists a unique $X \subseteq \min([\![\varphi]\!], \leq_P)$ such that X is P^s_φ-stable and minimal. In fact, there exists a finite sequence of intervals $I_1 = (s_1, s_2], I_2 = (s_2, s_3], \ldots I_k = (s_k, s_{k+1} = 1)$ such that for each I_i there is X_i that is P^s_φ-stable for each $s > s_1$ and is minimal for $s \in I_i$.*

Before giving the proof of this proposition, we establish two useful lemmas.

Lemma 2. *Let X be minimal stable for $P^{s_1}_\varphi$ and $P^{s_2}_\varphi$ with $s_1 < s_2$. Then X is P^s_φ minimal stable for every $s \in [s_1, s_2]$.*

Proof. Put $m = \min_{\omega \in X} P(\omega)$. Since X is $P^{s_1}_\varphi$-stable, we have $s_1 > \frac{P(\varphi)}{m + P(X)}$. Then for all $s \in [s_1, s_2]$, we have $s > \frac{P(\varphi)}{m + P(X)}$ $(*)$. Since X is $P^{s_2}_\varphi$-stable and minimal, we have that for every $X' \subsetneq X$ there exists $\omega' \in X'$ such that $s_2 \leq \frac{P(\varphi)}{P(\omega') + P(X')}$, therefore, for all $s \in [s_1, s_2]$, $s \leq \frac{P(\varphi)}{P(\omega') + P(X')}$ $(**)$. Then, by $(*)$ and $(**)$, we have that X is stable and minimal for P^s_φ for all $s \in [s_1, s_2]$. □

Lemma 3. *Let X be a subset of Ω and $s' < 1$ such that for all $s > s'$, X is P^s_φ-stable but not minimal. Then there exists $s'' > s'$ and a unique $X' \subset X$ such that for every $s \in (s', s'']$, the set X' is minimal stable for P^s_φ.*

Proof. Let $\{s_i\}_{i\geq 0}$ a decreasing sequence converging to s' such that $s_0 < 1$. By hypothesis, for every s_i, there exists X_{s_i} which is $P_\varphi^{s_i}$-stable and minimal. Since the set of subsets of X is finite, by the pigeonhole principle, there exists a subsequence $\{s_{i_k}\}_{k\geq 0}$ of $\{s_i\}_{i\geq 0}$ and a set $X' \subsetneq X$ which is $P_\varphi^{s_{i_k}}$-stable and minimal. Put $s'' = s_{i_0}$. Then, by Lemma 2, we have that X' is P_φ^s-stable minimal for all $s \in (s', s'']$. The uniqueness of X' follows straightforward of the uniqueness of a minimal stable in P_φ^s. □

Now we are ready to prove Proposition 7.

Proof. By Proposition 3 there is a a smallest s_1 such that $\min([\![\varphi]\!], \preceq_P)$ is P_φ^s-stable for $s > s_1$. If $\min([\![\varphi]\!], \leq_P)$ is also minimal, then let s_2 the upper bound given by condition (5) and we are done. Suppose now $\min([\![\varphi]\!], \leq_P)$ is not minimal. Then, use Lemma 3 to find an s' and a unique $X \subsetneq \min([\![\varphi]\!], \leq_P)$ which is P_φ^s-stable and minimal for $s \in (s_1, s']$. In that case let s_2 be the upper bound for s which follows from condition (5'). If $s_2 < 1$ we put $I = (s_1, s_2]$. If $s_2 \geq 1$, we put $I = (s_1, 1)$.

For $s > s_2$, X is not minimal any longer, yet it remains being P_φ^s-stable. Therefore, by the same argument, there is proper subset $X' \subsetneq X$ which is P_φ^s-stable and minimal, up to a value s_3 given by condition (5'). And so forth ...

Therefore, we eventually find a finite sequence of intervals $I_1 = (s_1, s_2], I_2 = (s_2, s_3], \ldots I_k = (s_k, s_{k+1} = 1)$ such that for each I_i there is X_i is P_φ^s-stable for each $s > s_1$ and is minimal for $s \in I_i$. □

Now we can use the Proposition 7 to define a new iterated change operator as follows. In the following, recall from Definition 6 that in the epistemic space $\mathcal{E}_{st} = \langle \mathscr{P}, \mathcal{B}_{st} \rangle$, the belief projection operator is defined, for any $P \in \mathscr{P}$, by taking $\mathcal{B}_{st}(P)$ to be (the theory of) the minimal P-stable set. Moreover, for any φ and any P, let X_φ^s be the unique subset of $\min([\![\varphi]\!], \leq_P)$ such that X_φ^s is the minimal P_φ^s-stable for s.

Definition 11. *In the space $\mathcal{E}_{st} = \langle \mathscr{P}, \mathcal{B}_{st} \rangle$ we define $*_{gst} : \mathscr{P} \times \mathcal{L}^\star \longrightarrow \mathscr{P}$ by putting*

$$P *_{gst} \varphi = \begin{cases} P_\varphi^s, & \text{if } \varphi \notin \mathcal{B}_{st}(P) \text{ and } s = \frac{s_1+s_2}{2} \\ P, & \text{if } \varphi \in \mathcal{B}_{st}(P) \end{cases}$$

where the first interval associated to P and φ defined in Proposition 7 is $I_1 = (s_1, s_2]$ if $s_2 < 1$ or $I_1 = (s_1, s_2)$ if $s_2 = 1$. Then, regarding beliefs, we also have:

$$\mathcal{B}_{st}(P *_{gst} \varphi) = \begin{cases} Th(X_\varphi^s), & \text{if } \varphi \notin \mathcal{B}_{st}(P) \\ \mathcal{B}_{st}(P), & \text{if } \varphi \in \mathcal{B}_{st}(P). \end{cases} \tag{4}$$

We note that in the above definition the choice of $s = \frac{s_1+s_2}{2}$ can be seen as a kind of trade-off. If we would choose to maximise the probability of the beliefs then we would take $s = s_2$ except in the case $I_1 = (s_1, 1)$, where we could take $s = 1 - \epsilon$ for an arbitrary small $\epsilon > 0$. On the contrary, if we would like to look for a minimal change, we would probably choose $s = s_1 + \epsilon$.

We conclude with some observations about the operator $*_{gst}$ whose analysis we plan to deepen in our future work on this subject:

- The operator $*_{gst}$ generalizes the operator $*_{st}$ inspired by Delgrande et al. (see Definition 10). More precisely, when conditions (4) and (5) are fulfilled, we have $\mathcal{B}_{st}(P *_{gst} \varphi) = \mathcal{B}_{st}(P *_{st} \varphi)$.
- The operator $*_{gst}$ is more in agreement with the principle of minimal change when the new information is in the old beliefs; in such a case $P *_{gst} \varphi = P$ but $P *_{st} \varphi$ can be different of P (see Example 3).
- The operator $*_{gst}$ is deterministic in the sense that the choice of the parameter s in the output of $P *_{gst} \varphi$ is completely determined: $s = \frac{s_1+s_2}{2}$. This is in contrast with the operator $*_{st}$ where the choice of s, when $P *_{st} \varphi$ is defined, is not determined (simply an $s \in I_1$).

- The operator $*_{gst}$ is a total function while the reformulation of operator $*_{st}$ in Definition 10 is a partial function.
- The price to pay for having a total operator lies in the non-satisfaction, in general, of some postulates AGM, namely Inclusion (K*3) and Superexpansion (K*7).

6. Conclusion and Perspectives

We have established tight links between the notion of stability and the notion of Lockean beliefs. In particular, for the notion of P-stability$^{\frac{1}{2}}$, there is a one-to-one correspondence between P-stable$^{\frac{1}{2}}$ sets and closed Lockean belief sets (see Corollaries 1 and 2). This correspondence makes it easy to see that stable sets are linearly ordered by inclusion.

We have formally presented the framework of epistemic , in which iterated change operators can be defined. In particular, we have presented some probabilistic epistemic spaces and some operators defined on them.

We have studied the iterated revision operator proposed by Delgrande et al. and have identified a problematic issue in their approach, namely that their operator is a partial function. Consequently, we explored ways to extend this operator to all the consistent formulas. The main tool enabling the definition of our new operator $*_{gst}$ is Proposition 7. Our operator $*_{gst}$ is total and obeys a basic principle of minimality: do not change the epistemic state if the new piece of information is already a belief.

It remains to do a thorough study of which postulates AGM and DP are satisfied by our operator. We believe that the satisfaction of DP postulates in the probabilistic setting is a real challenge that requires the introduction of more flexible and rich epistemic spaces.

Acknowledgments

The authors are indebted to the anonymous reviewers for their insightful comments. Flaminio, Subirana and Pino Pérez acknowledge partial support by the Spanish project SHORE (PID2022-141529NB- C22) funded by MCIN/AEI/10.13039/501100011033 . Godo acknowledges support from the Spanish project LINEXSYS (PID2022-139835NB-C21) funded by MCIU/AEI/10.13039/501100011033. Rodriguez would like to remark that this work was completed despite the lack of support from the scientific funding agencies of Argentina. Flaminio, Godo and Rodriguez, also acknowledge the European MSCA-SE SemPER project No. 101299559.

Declaration on Generative AI

The authors have not employed any Generative AI tools.

References

[1] C. E. Alchourrón, P. Gärdenfors, and D. Makinson. On the logic of theory change: partial meet contraction and revision functions. *Journal of Symbolic Logic*, 50:510–530, 1985.

[2] R. Booth, E. Fermé, S. Konieczny, and R. Pino Pérez. Credibility-limited revision operators in propositional logic. In *Proceedings of the 13th International Conference on Principles of Knowledge Representation and Reasoning*, pages 116–125, 2012.

[3] R. Booth and T. A. Meyer. Admissible and restrained revision. *Journal of Artificial Intelligence Research*, 26:127–151, 2006.

[4] N. Borges, S. Konieczny, R. Pino Pérez, and N. Schwind. Belief change on rational rankings. In *Proceedings of the 21st International Conference on Principles of Knowledge Representation and Reasoning (KR'24)*, 2024.

[5] N. Borges, S. Konieczny, R. Pino Pérez, and N. Schwind. Can we improve without worsening? belief change in rational rankings. *The European Journal on Artificial Intelligence*, pages 1–25, 2026.

[6] C. Boutilier. Iterated revision and minimal change of conditional beliefs. *Journal of Philosophical Logic*, 25(3):263–305, 1996.

[7] J. Cantwell and H. Rott. Probability, coherent belief and coherent belief change. *Annals of Mathematics and Artificial Intelligence*, 87:259–291, 2019.

[8] A. Darwiche and J. Pearl. On the logic of iterated belief revision. *Artificial Intelligence*, 89(1-2):1–29, 1997.

[9] J.P. Delgrande, G. Lakemeyer, M. Pagnucco, and J. Sack. Belief revision in a probabilistic setting. In *Proceedings of the 22th International Conference on Principles of Knowledge Representation and Reasoning (KR'25)*, 2025.

[10] T. Flaminio, Ll. Godo, R. Pino Pérez, and Ll. Subirana. Minimal and credibility-limited iterated revision of deductively closed lockean belief sets. In Kai Sauerwald Maria Vanina Martinez, Nina Pardal, editor, *Proceedings of the Workshop on Foundations and Future of Change in Artificial Intelligence (FCAI 2025)*, volume 4069 of *CEUR Worshop proceedings*, pages 2–7. CEUR, 2025.

[11] T. Flaminio, Ll. Godo, R. Pino Pérez, and Ll. Subirana. On lockean beliefs that are deductively closed and minimal change. In G. Casini, B. Dundua, and T. Kutsia, editors, *Logics in Artificial Intelligence - 19th European Conference, JELIA 2025, Kutaisi, Georgia, September 1-4, 2025, Proceedings, Part II*, volume 16094 of *Lecture Notes in Computer Science*, pages 28–42. Springer, 2025.

[12] R. Foley. *Working without a net.* Oxford University Press, Oxford, 1992.

[13] P. Gärdenfors. *Knowledge in flux.* MIT Press, 1988.

[14] S. O. Hansson. Revising probabilities and full beliefs. *The Journal of Philosophical Logic*, 49:1005–1039, 2020.

[15] S. O. Hansson. A basis for agm in bayesian probability revision. *The Journal of Philosophical Logic*, 52:1535–1559, 2023.

[16] H. Katsuno and A. O. Mendelzon. Propositional knowledge base revision and minimal change. *Artificial Intelligence*, 52:263–294, 1991.

[17] G. Kern-Isberner, N. Skovgaard-Olsen, and W. Spohn. Ranking theory. In M. Knauff and W. Spohn, editors, *The handbook of rationality*, pages 337–345. MIT Press, 2021.

[18] S. Konieczny. Using transfinite ordinal conditional functions. In *Proceedings of the Tenth European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty, ECSQARU 2009*, pages 396–407, 2009.

[19] H. Leitgeb. Reducing belief simpliciter to degrees of belief. *Annals of Pure and Applied Logic*, 164:1338–1389, 2013.

[20] A. Nayak. Iterated belief change based on epistemic entrenchment. *Erkenntnis*, 41:353–390, 1994.

[21] N. Schwind, S. Konieczny, and R. Pino Pérez. On the representation of darwiche and pearl's epistemic states for iterated belief revision. In *Proceedings of the 19th International Conference on Principles of Knowledge Representation and Reasoning (KR'22)*, 2022.

[22] T. Shear and B. Fitelson. Two approaches to belief revision. *Erkenntnis*, 84:487–518, 2019.

[23] W. Spohn. Ordinal conditional functions: A dynamic theory of epistemic states. In W. L. Harper and B. skyrms, editors, *Causation in Decision: Belief Change and Statistics*, pages 105–134. Kluwer, 1988.

[24] W. Spohn. A survey of ranking theory. In F. Huber and C. Schmidt-Petri, editors, *Degrees of Belief.*, pages 185–228. Springer, Dordrecht., 2009.