# Degree-preserving companion of Nelson logic expanded with a consistency operator

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***Abstract.*** *The main aim of this paper is defining a Logic of Formal Inconsistency over the degree-preserving companion of Nelson logic with a consistency operator. In this sense, we present a quasivariety of Nelson lattices enriched with a suitable consistency operator and axiomatise the corresponding logic. As main results we present necessary and sufficient conditions to prove a categorial equivalence for the category for Nelson lattices with a consistency operator.*

***Resumo.*** *O principal objetivo desse artigo é definir uma Lógica da Inconsistência Formal sobre a lógica que preserva gaus de verdade que acompanha á lógica de Nelson com um conectivo de consistência. Nesse sentido, apresentamos uma quase-variedade de reticulados de Nelson enriquecidos com um operador de consistência adequado e axiomatizamos a lógica correspondente. Como resultado principal, apresentamos as condições necessárias e suficientes para demonstrar uma equivalência categorial para os reticulados de Nelson com um operador de consistência.*

## 1. Introduction

In the 1950's, Constructive logic with strong negation (CLSN) was formulated independently by Nelson and Markov as a result of certain philosophical objections to intuitionistic negation. The criticism of intuitionistic negation concerns its disadvantageous non-constructive property, namely, that the derivability of the formula $\neg(\alpha \wedge \beta)$ in an intuitionistic propositional calculus does not imply that at least one of the formulas $\neg\alpha$ or $\neg\beta$ is derivable.

More recently, in 2008, Spinks and Veroff have shown CLSN, also known in the literature as Nelson logic, can be considered as a substructural logic. More precisely, they have shown that the algebraic models of CLSN, *Nelson algebras*, are termwise equivalent to certain involutive, bounded, commutative and integral residuated lattices, called *Nelson (residuated) lattices*, see [6].

The main aim of this paper is defining a Logic of Formal Inconsistency (LFI) based on Nelson logic, by considering the degree-preserving companion of Nelson logic expanded with a consistency operator, in the style of [4]. To do so, we will first algebraically study the quasivariety of Nelson lattices with a consistency operator, then we look at a categorial equivalence between Heyting algebras (with extra structure) and Nelson lattices with consistency operators, and finally we define and axiomatise the logic.

## 2. Preliminaries

Nelson algebras were introduced by Rasiowa (see, for instance, [7]), under the name of N-lattices, as the algebraic counterpart of the constructive logic with strong negation considered by Nelson and Markov. A Nelson algebra is an algebra $\mathbf{N} = (N, \vee, \wedge, \Rightarrow, \neg, \top, \bot)$ of type $(2, 2, 2, 1, 0, 0)$ such that $(N, \vee, \wedge, \neg, \top, \bot)$ is a Kleene algebra, and the implication $\Rightarrow$ satisfies the following equations:

$$\begin{array}{ll} x \Rightarrow x = \top & x \Rightarrow (y \wedge z) = (x \Rightarrow y) \wedge (x \Rightarrow z) \\ x \wedge (x \Rightarrow y) = x \wedge (\neg x \vee y) & x \Rightarrow (y \Rightarrow z) = (x \wedge y) \Rightarrow z \end{array}$$

This definition of Nelson algebras is due to Brignole and Monteiro, and provides an equational characterization of the N-lattices introduced by Rasiowa.

On the other hand, as mentioned above, in [6], the authors have shown that Nelson algebras are termwise equivalent to a certain class of *involutive residuated lattices*, called Nelson residuated lattices or Nelson lattices. Recall that a *commutative, integral, bounded residuated lattice*, that we will simply call *residuated lattice*, is an algebra $\mathcal{A} = \langle A, \wedge, \vee, *, \rightarrow, \bot, \top\rangle$ of type $(2, 2, 2, 2, 0, 0)$ such that $\langle A, *, \top\rangle$ is a commutative monoid, $\mathcal{L}(A) = \langle A, \wedge, \vee, \bot, \top\rangle$ is a bounded lattice with least element $\top$ and greatest element $\top$, and such that the following condition holds: $x * y \leq z$ iff $x \leq y \rightarrow z$, where $x, y, z$ denote arbitrary elements of $A$ and $\leq$ is the order given by the lattice structure. Since we assume the neutral element of the monoid reduct coincides with the greatest element of $\mathcal{L}(A)$ we have that: $x \leq y$ iff $x \rightarrow y = \top$.

It is well-known that the class $\mathsf{RL}$ of residuated lattices is a variety, which is related to different and well-known varieties studied in substructural and fuzzy logics literature. In fact $\mathsf{RL}$ coincides with the variety of $\mathsf{FL}_{ew}$-algebras of [5]. We use the denotational conventions of [5]: $\mathsf{FL}$ refers to the "Full Lambek calculus", which is the base system and associated algebras, and subindices indicate several axiomatic extensions with properties such as exchange ($e$) or weakening ($w$).

A residuated lattice is called *involutive* if it satisfies the double negation equation: $\neg\neg x = x$, where $\neg x$ is $x \rightarrow \bot$. Besides, it is possible to prove that: $x * y = \neg(x \rightarrow \neg y)$ and $x \rightarrow y = \neg(x * \neg y)$.

A *Nelson residuated lattice*, or simply *Nelson lattice*, is an involutive residuated lattice that satisfies the following identity:

$$(((x * x) \rightarrow y) \wedge ((\neg y * \neg y) \rightarrow \neg x)) \rightarrow (x \rightarrow y) = \top.$$

Nelson lattices will be also called in the paper $\mathrm{NL}$-algebras.

In [2] the authors show that, given a Nelson algebra $\mathbf{N} = (N, \vee, \wedge, \Rightarrow, \neg, \top, \bot)$, then the algebra $\mathcal{R}(\mathbf{N}) = (N, \wedge, \vee, *, \rightarrow, \bot, \top)$ is a Nelson residuated lattice, where the operations $*$ and $\rightarrow$ are defined as follows:

$$x * y := \neg(x \Rightarrow \neg y) \vee \neg(y \Rightarrow \neg x), \qquad x \rightarrow y := (x \Rightarrow y) \wedge (\neg y \Rightarrow \neg x),$$

and moreover it holds $\neg x = x \rightarrow \bot$ for each $x \in N$. Conversely, consider now a Nelson lattice $\mathbf{A} = \langle A, \wedge, \vee, *, \rightarrow, \bot, \top\rangle$ and define a binary operation $\rightarrow_N$ by specifying $x \rightarrow_N y := (x * x) \rightarrow y$. Then, $\mathcal{N}(A) = \langle A, \wedge, \vee, \rightarrow_N, \neg, \bot, \top\rangle$ is a Nelson algebra.

For a given a Heyting algebra $\mathbf{H}$ and a Boolean filter $F$ of $\mathbf{H}$, let us define $N(H, F) := \{(x, y) \in H \times H : x \wedge y = \bot,\ x \vee y \in F\}$. Then we have that

$\mathbf{N}(\mathbf{H}, F) = (N(H, F), \vee, \wedge, \Rightarrow, \neg, *\bot, \top)$ is a Nelson lattice, where the operations are defined as follows:

$$(x, y) \vee (s, t) = (x \vee s, y \wedge t),$$
$$(x, y) \wedge (s, t) = (x \wedge s, y \vee t),$$
$$(x, y) * (s, t) = (x \wedge s, (x \to t) \wedge (s \to y)),$$
$$(x, y) \Rightarrow (s, t) = ((x \to s) \wedge (t \to y), x \wedge t),$$
$$\neg(x, y) = (y, x),$$
$$\top = (\top, \bot),$$
$$\bot = (\bot, \top).$$

Finally, we briefly recall the notion a *Logic of Formal Inconsistency*, LFI for short, see e.g. [3]. Let $\Sigma'$ be a propositional signature and assume a denumerable set $\mathcal{V} = \{p_1, p_2, \ldots\}$ of propositional variables. The propositional language generated by $\Sigma'$ from $\mathcal{V}$ will be denoted by $\mathcal{L}_{\Sigma'}$. On the other hand, let $\mathbf{L} = \langle \Sigma', \vdash \rangle$ be a Tarskian, finitary and structural logic defined over the propositional signature $\Sigma'$, which contains a negation $\neg$, and let $\circ$ be a primitive or defined unary connective. Then, $\mathbf{L}$ is said to be a *Logic of Formal Inconsistency with respect to* $\neg$ *and* $\circ$ if the following holds:

(i) $\varphi, \neg\varphi \nvdash \psi$, for some $\varphi$ and $\psi$;
(ii) there are two formulas $\alpha$ and $\beta$ such that $\circ\alpha, \alpha \nvdash \beta$ and $\circ\alpha, \neg\alpha \nvdash \beta$;
(iii) $\circ\varphi, \varphi, \neg\varphi \vdash \psi$, for every $\varphi$ and $\psi$.

## 3. Nelson residuated lattices with a consistency operator $\circ$

We start by formally defining the class of Nelson lattices with a consistency operator.

**Definition 3.1** $\mathrm{NL}_\circ$*-algebras are expansions of Nelson lattices with a new unary operation* $\circ$ *satisfying the following equation and quasi-equation:*

($\circ 1$) $x \wedge \neg x \wedge \circ(x) = 0$
($\circ 2$) *if* $x \wedge \neg x \wedge y = 0$ *then* $y \leq \circ(x)$

In the following, we will use the expression $(\mathbf{A}, \circ)$ to denote the $\mathrm{NL}_\circ$-algebra whose Nelson lattice reduct is $\mathbf{A}$.

From this definition, it is clear that the class of $\mathrm{NL}_\circ$-algebras is a quasivariety. Now, we have the following easy properties of $\mathrm{NL}_\circ$-algebras are displayed in the next lemma.

**Lemma 3.2** *The following properties hold in a* $\mathrm{NL}_\circ$*-algebra* $(\mathbf{A}, \circ)$*:*

*(i)* $\circ x = \max\{z \in A \mid x \wedge \neg x \wedge z = 0\}$
*(ii)* $\circ(x) = \circ(\neg x) = \circ(x \wedge \neg x) = \circ(x \vee \neg x)$
*(iii)* $\circ(1) = \circ(0) = 1$

It follows from (i) of Lemma 3.2 that if a consistency operator is definable in a Nelson lattice it is uniquely determined. Moreover, it also tells us that a consistency operator is always definable in every finite Nelson lattice. However it is not always the case in infinite Nelson lattices as we can see in the next example.

**Example 3.3** *Let* $\mathbf{A}$ *be the Nelson lattice over* $[0,1] \setminus \{\frac{1}{2}\}$ *with the Nilpotent Minimum operations:* $x * y = \min(x,y)$ *if* $x + y > 1$ *and* $x * y = 0$ *otherwise; and* $\neg x = 1 - x$*. Let* $\mathbf{B}$ *be the subalgebra of* $\mathbf{A} \times \mathbf{A}$ *defined on the sublattice consisting of the elements of the form* $(x,y)$ *with* $x, y$ *being positive elements of* $\mathbf{A}$ *(i.e. such that* $1 \geq x, y > \frac{1}{2}$*) and their negations* $(1-x, 1-y)$*. Take an element* $(x,1) \in B$ *such that* $1 > x > \frac{1}{2}$*. An easy computation shows that* $\circ((x,1))$ *does not exist.*

As is well-known, an implicative filter of a bounded residuated lattice $A$ is a subset $F \subseteq A$ such that $\top \in F$ and it is closed under modus pones: $x \in F$ and $x \to y \in F$ imply $y \in F$. For each implicative filter $F$, the binary relation $\Theta(F)$ defined by $(x,y) \in \Theta(F)$ if and only if $x \to y, y \to x \in F$ is a congruence of the residuated lattice $A$, and $F = \{z \in A : (z, \top) \in \Theta(F)\}$. This is actually a one-one correspondence between the lattice of congruences and the lattice of implicative filters for the variety of bounded residuated lattices. However, since the class of expanded Nelson lattices with a consistency operator involves not only equations but also a quasi-equation, it is a quasivariety. In a quasivariety, congruences that allow for the decomposition of an algebra as a subdirect product of subdirectly irreducible components are required to satisfy an additional condition: the quotient of an algebra by a congruence has to belong to the quasivariety. This condition is automatically satisfied in varieties but not in quasivarieties. Congruences satisfying this condition are usually called Q-congruences. Similarly, filters that are in a one-one correspondence between Q-congruences are implicative filters 'closed' by the quasiequations of the quasivariety, and are called Q-filters.

Therefore, a filter $F$ of a $\mathrm{NL}_\circ$-algebra, besides being implicative, has to additionally satisfy the following two conditions:

(F1) if $x \to y, y \to x \in F$ then $\circ x \to \circ y, \circ y \to \circ x \in F$
(F2) if $x \vee \neg x \vee \neg y \in F$ then $y \to \circ x \in F$.

We shall call such a filter a $\circ$-*filter*. Note that from (F1) it follows in particular that $\circ$-filters are closed by $\circ$: if $x \in F$ then $\circ x \in F$ as well.

**Lemma 3.4** *If* $(\mathbf{A}, \circ)$ *is* $\mathrm{NL}_\circ$*-algebra such that* $\mathbf{A}$ *is a subdirectly irreducible Nelson lattice, then* $\circ$ *is such that* $\circ(1) = \circ(0) = 1$ *and* $\circ(x) = 0$ *otherwise, and hence* $(\mathbf{A}, \circ)$ *is a simple* $\mathrm{NL}_\circ$*-algebra.*

In the following, if $\mathbf{A}$ is a Nelson lattice, we will denote by $B(\mathbf{A})$ the set of its Boolean elements, i.e. $B(\mathbf{A}) = \{x \in A \mid x \wedge \neg x = 0\}$.

**Theorem 3.5** *Let* $(\mathbf{A}, \circ)$ *be a subdirectly irreducible* $\mathrm{NL}_\circ$*-algebra. Then* $B(\mathbf{A}) = \{0, 1\}$.

When the algebra is finite, then we can prove more.

**Theorem 3.6** *Let* $(\mathbf{A}, \circ)$ *be a finite* $\mathrm{NL}_\circ$*-algebra. Then the following conditions are equivalent:*

(i) $(\mathbf{A}, \circ)$ *is a s.i.* $\mathrm{NL}_\circ$*-algebra,*
(ii) $B(\mathbf{A}) = \{0, 1\}$,
(iii) $(\mathbf{A}, \circ)$ *is a simple* $\mathrm{NL}_\circ$*-algebra.*

**Theorem 3.7** *Let* $(\mathbf{A}, \circ)$ *be a* $\mathrm{NL}_\circ$*-algebra, then the following conditions are equivalent:*

(i) $(\mathbf{A}, \circ)$ *is subdirectly irreducible but not simple,*
(ii) $B(\mathbf{A}) = \{0, 1\}$ *and there is a non-trivial* $\circ$*-filter* $F$ *such that* $(\circ(x))^2 \neq 0$ *for every* $x \in F$.

## 4. A categorial equivalence

In this section we show an equivalence between a category that involves Heyting algebras (with extra structure) and the algebraic category of Nelson lattices with consistency operators.

For convenience, let us represent $\mathbf{A}$ as $\mathbf{N}(\mathbf{H}, F)$ for $\mathbf{H}$ being a Heyting algebra and $F$ being a Boolean filter of $\mathbf{H}$. So, every $x \in A$ is of the form $(a, b)$ for $a, b \in H$ such that $a \wedge b = \bot$ and $(a \vee b) \in F$. Thus, condition $(i)$ of Lemma 3.2 which indeed characterizes the $\circ$ in a Nelson lattice, can be reformulated in the following way: for all $(a, b) \in A$,

$$o(a, b) = \max\{(z, z') \in A \mid (\bot, a \vee b \vee z') = (\bot, \top)\}$$

where the equality holds because $a \wedge b = \bot$ (in $H$) for all $(a, b) \in A$. The order on $A$ is defined, with respect to the order of $H$, as $(a, b) \leq (c, d)$ iff $a \leq c$ and $b \geq d$.

**Lemma 4.1** *For every Heyting algebra* $\mathbf{H}$*, for every Boolean filter* $F$ *of* $\mathbf{H}$ *and for every* $a \in H$*,* $a \vee \neg a \in F$*.*

Let us recall from that a unary operation $\urcorner$ on a Heyting algebra $\mathbf{H}$ is called a *dual pseudocomplement*, provided that the following equations are satisfied:

(D1) $x \vee \urcorner(x \vee y) = x \vee \urcorner y$,
(D2) $x \vee \urcorner 1 = x$,
(D3) $\urcorner\urcorner 1 = 1$.

In every Heyting algebra $\mathbf{H}$ and for every $x \in H$, if is the dual pseudocomplement of $x$ exists, then it is defined as

$$\urcorner x = \min\{z \in H \mid x \vee z = 1\}. \tag{1}$$

**Proposition 4.2** *Let* $\mathbf{H}$ *be a Heyting algebra,* $F$ *a Boolean filter of* $\mathbf{H}$ *and* $\mathbf{A}$ *the Nelson lattice* $\mathbf{N}(\mathbf{H}, F)$*. If* $\urcorner(a \vee b)$ *exists in* $H$*, then* $\circ(a, b)$ *exists and* $\circ(a, b) = (\neg d, d)$*, where* $d = \urcorner(a \vee b)$*. Besides, for any* $(a, b) \in A$*,* $\circ(a, b) \in A$*.*

**Corollary 4.3** *Let* $\mathbf{H}$ *be a Heyting algebra,* $F$ *a Boolean filter of* $\mathbf{H}$ *and* $\mathbf{A}$ *the Nelson lattice* $\mathbf{N}(\mathbf{H}, F)$*. Then the operation* $\circ$ *exists in* $\mathbf{A}$ *if and only if the dual pseudo-complement exists for all the elements of* $F$*. In particular, if* $\mathbf{A} = \mathbf{N}(\mathbf{H}, H)$*, then* $\circ$ *exists in* $\mathbf{A}$ *if and only if the dual pseudo-complementation is definable in the whole* $\mathbf{H}$*.*

### 4.1. A categorial equivalence for Nelson lattices with a consistency operator

To ease the notation, let us denote by $BPF(\mathbf{H})$ the set of the Boolean filters $F$ of a Heyting algebra $\mathbf{H}$ that further satisfy the following property:

(DP) For every $x \in F$, $\urcorner x$ exists in $\mathbf{H}$.

Thus, $BPF(\mathbf{H})$ is the set of Boolean filters $F$ of $\mathbf{H}$ such that every element $x \in F$ has a dual pseudocomplement in $\mathbf{H}$.

**Definition 4.4** *Consider the set* $\mathbf{HBP}$ *containing pairs* $(\mathbf{H}, F)$ *such that* $\mathbf{H}$ *is a Heyting algebra,* $F \in BPF(\mathbf{H})$ *and maps defined as follows: given two pairs* $(\mathbf{H}, F)$ *and* $(\mathbf{H}', F')$ *a map* $h$ *between them is such that:*

*(m1)* $h$ *is a Heyting homomorphism between* $\mathbf{H}$ *and* $\mathbf{H}'$*,*

*(m2)* $h(F) \subseteq F'$,
*(m3) for all* $x \in F$, $h(\neg x) = \neg' h(x)$.

It is not difficult to see that $\mathbf{HBP}$ is a category and hence we will respectively call *objects* and *morphisms* the pairs $(\mathbf{H}, F)$ and the maps $h : (\mathbf{H}, F) \to (\mathbf{H}', F')$ of $\mathbf{HBP}$ defined as above.

Moreover, let $\mathbf{NC}$ be the algebraic category of Nelson lattices with a consistency operator as in Definition 3.1 and let us consider the map $\mathcal{NC} : \mathbf{HBP} \to \mathbf{NC}$ defined in the following way:

- For every object $(\mathbf{H}, F) \in \mathbf{HBP}$, $\mathcal{NC}(\mathbf{H}, F) = (\mathbf{N}(\mathbf{H}, F), \circ)$, where $\mathbf{N}(\mathbf{H}, F)$ is the Nelson algebra of pairs $(a, b) \in H \times H$ such that $a \wedge b = \bot$ and $a \vee b \in F$ as in the above section, and $\circ(a, b) = (\neg\neg(a \vee b), \neg(a \vee b))$ as in Proposition 4.2.
- For every morphism $h : (\mathbf{H}, F) \to (\mathbf{H}', F')$ of $\mathbf{HBP}$, $\mathcal{NC}(h) : \mathcal{NC}(\mathbf{H}, F) \to \mathcal{NC}(\mathbf{H}', F')$ is so defined: for all $(a, b) \in \mathcal{NC}(\mathbf{H}, F)$,

$$\mathcal{NC}(h)(a, b) = (h(a), h(b)).$$

For every object $(\mathbf{H}, F)$, $\mathcal{NC}(\mathbf{H}, F)$ is an object in $\mathbf{NC}$. Analogously, if $h$ is a morphism of $\mathbf{HBP}$, $\mathcal{NC}(h)$ is a morphism of $\mathbf{NC}$. The we can prove the following

**Proposition 4.5** *The map* $\mathcal{NC} : \mathbf{HBP} \to \mathbf{NC}$ *is a functor.*

Recall that a functor between two categories yields a categorial equivalence iff it is full, faithful and essentially surjective.

**Theorem 4.6** *The functor* $\mathcal{NC}$ *establishes a categorial equivalence between* $\mathbf{HBP}$ *and* $\mathbf{NC}$.

## 5. Adding a consistency operator to Nelson logic and its paraconsistent companion

Let L be the logic of a variety of residuated lattices, that is, an axiomatic extension of $\mathrm{FL}_{ew}$. For each such a logic L, in [1] the authors introduce a companion logic denoted $\mathrm{L}^{\leq}$, whose associated consequence relation has the following semantics: for every set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash_{\mathrm{L}^{\leq}} \varphi \quad \text{iff} \quad \begin{array}{l}\text{for every L-algebra } \boldsymbol{A}, \text{ every } a \in A, \text{ and every } \boldsymbol{A}\text{-evaluation } e,\\ \text{if } a \leq e(\psi) \text{ for every } \psi \in \Gamma, \text{ then } a \leq e(\varphi).\end{array}$$

For this reason $\mathrm{L}^{\leq}$ is known as the *degree-preserving companion* of L.

Now, let us consider the logic NL corresponding to the variety of Nelson lattices and its degree-preserving companion $\mathrm{NL}^{\leq}$. Since the negation in NL is involutive, and hence it does not satisfy the pseudo-complementation axiom $\varphi \wedge \neg\varphi \to \overline{0}$, the logic $\mathrm{NL}^{\leq}$ is paraconsitent, i.e. in general

$$\varphi, \neg\varphi \not\vdash_{\mathrm{NL}^{\leq}} \overline{0}$$

However, $\mathrm{NL}^{\leq}$ is not a *logic of formal inconstency* (LFI) since we cannot define in the language of NL a consistency operator $\circ$ satisfying $\{\circ\varphi, \varphi, \neg\varphi\} \vdash \psi$ for every $\varphi$ and $\psi$. Therefore, we consider next the expansion of NL with a suitable consistency operator $\circ$, that we will call $\mathrm{NL}_{\circ}$, so that its degree-preserving companion $\mathrm{NL}_{\circ}^{\leq}$ is a LFI.

**Definition 5.1** *We define the logic* $\mathrm{NL}_\circ$ *as the expansion of* NL *in a language which incorporates a new unary connective* $\circ$ *with the following additional axioms and rules:*

(A1) $\circ\bar{1}$
(A2) $\circ\bar{0}$
(A3) $\neg(\varphi \wedge \neg\varphi \wedge \circ\varphi)$

*and the following inference rules:*

$$\text{(CNG)}\ \frac{\varphi \leftrightarrow \psi}{\circ\varphi \leftrightarrow \circ\psi} \qquad \text{(Max)}\ \frac{\varphi \vee \neg\varphi \vee \neg\psi}{\psi \to \circ\varphi}$$

**Lemma 5.2** *The following derivabilities hold in* $\mathrm{NL}_\circ$*:*
*i)* $\varphi \vee \neg\varphi \vdash_{\mathrm{L}_\circ} \circ\varphi$*, (ii)* $\mathrm{L}_\circ \vdash \circ\varphi \to \neg\varphi \vee \varphi$*, (iii)* $\varphi \vee \neg\varphi \dashv\vdash_{\mathrm{L}_\circ} \circ\varphi$*.*

It is easy to check that, due to the (Cong) rule for $\circ$, $\mathrm{NL}_\circ$ is a Rasiowa implicative logic and hence it is algebraizable. The equivalent algebraic semantics is given by the quasi-variety of $\mathrm{NL}_\circ$-algebras. As a direct consequence we have the following general completeness result.

**Proposition 5.3** $\mathrm{NL}_\circ$ *is strongly complete w.r.t the class of* $\mathrm{NL}_\circ$*-algebras.*

**Definition 5.4** *The degree-preserving companion of the logic NL*$_\circ$ *is the logic* $\mathrm{NL}_\circ^\leq$ *defined by the following axioms and rules:*

- *Axioms of* $\mathrm{NL}_\circ^\leq$ *are those of NL*$_\circ$
- *Rules of* $\mathrm{NL}_\circ^\leq$ *are:*
  **(Adj-**$\wedge$**)** *from* $\varphi$ *and* $\psi$ *derive* $\varphi \wedge \psi$
  **(MP-**$r$**)** *if* $\vdash_{\mathrm{NL}_\circ} \varphi \to \psi$*, then from* $\varphi$ *and* $\varphi \to \psi$*, derive* $\psi$
  **(CNG-**$r$**)** *if* $\vdash_{\mathrm{NL}_\circ} \varphi \leftrightarrow \psi$*, then from* $\varphi \leftrightarrow \psi$ *derive* $\circ\varphi \leftrightarrow \circ\psi$
  **(Max-**$r$**)** *if* $\vdash_{\mathrm{NL}_\circ} \neg(\varphi \wedge \neg\varphi \wedge \psi)$*, then from* $\neg(\varphi \wedge \neg\varphi \wedge \psi)$ *derive* $\psi \to \circ\varphi$

It is clear that the degree-preserving companion $\mathrm{NL}_\circ^\leq$ is a LFI.

Let L be a paraconsistent logic with a consistency operator $\circ$. Then we say that $\circ$ satisfies the *propagation property* in L with respect to a subset $X$ of connectives of the language of L if $\{\circ\varphi_1, \ldots, \circ\varphi_n\} \vdash_{\mathrm{L}} \circ\#(\varphi_1, \ldots, \varphi_n)$, for every $n$-ary connective $\# \in X$ and formulas $\varphi_1, \ldots, \varphi_n$ built with connectives from $X$.

In particular, checking whether $\mathrm{NL}_\circ^\leq$ satisfies the propagation property for the connectives $X = \{\bar{0}, \wedge, \&, \to\}$ amounts to check the following conditions: $\vdash_{\mathrm{NL}_\circ} \circ\bar{0}$ and $\vdash_{\mathrm{NL}_\circ} (\circ\varphi \wedge \circ\psi) \to \circ(\varphi\#\psi)$ for each binary $\# \in X$ (Prop*).

The first condition is obviously satisfied since $\circ\bar{0}$ is an axiom of the logic. Next proposition shows that, in fact, the other conditions are also satisfied.

**Proposition 1** $\mathrm{NL}_\circ$ *satisfies (Prop*).*

In the context of LFIs, it is a desirable property to recover the classical reasoning by means of the consistency connective $\circ$ (see [3]). Specifically, let **CPL** be classical propositional logic. If L is a given LFI such that its reduct to the language of **CPL** is a sublogic of **CPL**, then a DAT (Derivability Adjustment Theorem) for L with respect to

**CPL** is as follows: for every finite set of formulas $\Gamma \cup \{\varphi\}$ in the language of **CPL**, there exists a finite set of formulas $\Theta$ in the language of L, whose variables occur in formulas of $\Gamma \cup \{\varphi\}$, such that:

(DAT) $\Gamma \vdash_{\mathbf{CPL}} \varphi$ iff $\circ(\Theta) \cup \Gamma \vdash_{\mathrm{L}} \varphi$.

When the operator $\circ$ enjoys the propagation property in the logic L with respect to the classical connectives then the DAT takes the following, simplified form: for every finite set of formulas $\Gamma \cup \{\varphi\}$ in the language of **CPL**,

(PDAT) $\Gamma \vdash_{\mathbf{CPL}} \varphi$ iff $\{\circ p_1, \ldots, \circ p_m\} \cup \Gamma \vdash_{\mathrm{L}} \varphi$

where $\{p_1, \ldots, p_m\}$ is the set of propositional variables occurring in $\Gamma \cup \{\varphi\}$.

The fact that $\circ$ has the propagation property it allows us to prove the following PDAT theorem for $\mathrm{NL}_\circ^\leq$.

**Theorem 5.5 (PDAT for $\mathrm{NL}_\circ^\leq$)** *Let $\Gamma \cup \{\varphi\}$ be a finite set of formulas in the language of* **CPL** *and let $\{p_1, \ldots, p_m\}$ the set of propositional variables appearing in $\Gamma \cup \{\varphi\}$. Then*

$$\Gamma \vdash_{\mathbf{CPL}} \varphi \text{ iff } \Gamma \vdash_{\mathrm{NL}_\circ^\leq} ((\bigwedge_{i=1}^{n} \circ(p_i)) * (\bigwedge_{i=1}^{n} \circ(p_i))) \to \varphi.$$

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## References

[1] F. Bou, F. Esteva, J. M. Font, A. Gil, L. Godo, A. Torrens and V. Verdú (2009) Logics preserving degrees of truth from varieties of residuated lattices, J. Log. Comput., 19(6):1031–1069

[2] M. Busaniche and R. Cignoli (2010) Constructive Logic with Strong Negation as a Substructural Logic, J. Log. Comput. 20(4): 761–793

[3] W. Carnielli, M. Coniglio, and J. Marcos (2007) Logics of Formal Inconsistency, In Dov Gabbay and Franz Guenthner, editors, Handbook of Philosophical Logic (2nd. edition), volume 14, pages 1–93. Springer.

[4] M.E. Coniglio, F. Esteva and L. Godo (2014) Logics of formal inconsistency arising from systems of fuzzy logic, Logic Journal of the IGPL 22, n. 6: 880–904, 2014.

[5] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono (2007) Residuated Lattices: an Algebraic Glimpseat Substructural Logics, Vol. 151 of Studies in Logic and the Foundations of Mathematics. Elsevier.

[6] M. Spinks and R. Veroff (2008) Constructive logic with strong negation is a substructural logic over $\mathrm{FL}_{ew}$ I, Studia Logica, 88, 325–348, 2008.

[7] H. Rasiowa (1974) An algebraic approach to non-clasical logics, Studies in logic and the foundations of mathematics, vol. 78. North-Holland Publishing Company, Amsterdam and London, and American Elsevier Publishing Company, Inc., New York.