An approach to inconsistency-tolerant reasoning about probability based on Łukasiewicz logic

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Abstract

In this paper we consider the probability logic over Łukasiewicz logic with rational truth-constants, denoted FP(RPL), and we explore two possible approaches to reason from inconsistent FP(RPL) theories on classical events in a non-trivial way. The first one amounts to replace the logic RPL, that is explosive, by its paraconsistent companion $\text{RPL}^{\leq}$. The second one consists of suitably weakening the formulas in an inconsistent theory T, depending on the degree of inconsistency of T. We also discuss the possibility of applying a similar approach to reason about probability over the paraconsistent logic **RCi** along the lines of Bueno-Soler and Carnielli's approach to paraconsistent probability.

1 Introduction

Nowadays, with the explosion of available data and information, it is not uncommon to encounter inconsistencies among different pieces of information. Thus, finding a suitable way of handling inconsistent information has become a challenge for both logicians and computer scientists working on knowledge representation techniques and reasoning models, see e.g. [2, 7, 31] among many others.

From a logical point of view, inconsistency is ubiquitous in many contexts in which, regardless of the given information being contradictory, one is still expected to extract inferences in a sensible way. Classical logic, and in general any logic validating the so-called *explosion principle*, does not allow to reason in any interesting way in the presence of contradictions, since they trivialize deductions by allowing to extract any conclusion from an inconsistent theory. In this sense, such logics are called *explosive*. On the other hand, *paraconsistent* logics have been introduced, among other approaches (see e.g. [3]), as non-explosive deductive systems able to cope with contradictions. Indeed, paraconsistency is devoted to the study of logical systems with a negation operator, say $\neg$, such that not every contradictory set of premises $\{\Phi, \neg\Phi\}$ trivializes the system [7].

In this work we investigate a fuzzy logic-based approach for handling conflicts when the information is of a probabilistic nature. Reasoning with inconsistent probabilistic information is indeed a research topic that has received growing attention in the last years; in particular, with respect to inconsistency measurement of probabilistic knowledge bases, and how inconsistency measures can be used to devise paraconsistent inference methods, see e.g. the survey [18]. Alternatively, from a more logically-oriented perspective, Bueno-Soler and Carnielli have approached the problem in a different way. Namely, by allowing events to be formulas of a given paraconsistent logic, and then formalising a suitable notion of probability over such logic [5].

The approach we follow here is based on a logical formalization of probabilistic reasoning on classical propositions, given by a modal theory over Łukasiewicz fuzzy logic, called FP(Ł), as developed by Hájek et al. [32, 33]. The idea is to understand the probability of a classical proposition φ as the truth-degree of a fuzzy modal proposition $P\varphi$, standing for the statement "φ is probable", in such a way that the higher (resp. lower) is the probability of φ, the more (resp. less) true is $P\varphi$. Then, the $[0,1]$-based semantics of Łukasiewicz connectives, heavily relying on the usual addition and subtraction operations, make it possible to capture the postulates of probability measures (in particular the finite additivity property) with formulas in the language of FP(Ł). We observe that the logic FP(Ł) can surely be seen as a *qualitative* probability logic. For instance, one can easily express that the probability of φ is less or equal than the probability of ψ, by means of the FP(Ł) formula $P\varphi \to P\psi$. On the other hand, the expressive power of Łukasiewicz logic makes it possible to encode more *quantitative* probabilistic relations. Even more so, by expanding Łukasiewicz logic with rational truth-constants, yielding the logic called Rational Pavelka logic (RPL), it is possible to encode purely quantitative expressions like "the probability of φ is at least 0.4" as the modal formula $\overline{0.4} \to P\varphi$ in the language of FP(RPL), the probability logic obtained by replacing Ł by RPL in FP(Ł).

In this paper, we explore two possible approaches to reason from inconsistent FP(RPL) theories in a non-trivial way. The first one amounts to replace the external logic RPL, that is explosive, by its paraconsistent degree-preserving companion $\mathrm{RPL}^{\leq}$. The second one amounts to suitably weaken formulas of an inconsistent theory T depending on the degree of inconsistency of T.

In order for this chapter be self-contained, we are forced to have extensive preliminaries. The paper is then structured as follows. In the next section, we discuss the basis of a general fuzzy logic-based approach to define modal theories for reasoning about uncertainty. In the case of probability, the suitable fuzzy logic to be used is Łukasiewicz logic Ł, whose main definitions and properties, as well some of its expansions, are recalled in Section 3. In Section 4, we overview its paraconsistent degree-preserving companion $\text{Ł}^{\leq}$. The following two sections are devoted to the introduction of the probability logics based on Ł and on RPL, respectively. Section 7 is devoted to our two proposals to deal with inconsistent probability theories over FP(RPL). In Section 8, we relate our approach on measuring inconsistency of theories to other proposals in the literature, based on the use of distance-based and violation-based inconsistency measures. Finally, in Section 9 we comment on a possible formalisation of Bueno-Soler and Carnielli's approach in our fuzzy logic-based probabilistic setting.

This chapter is a revised and fully expanded version of a preliminary work reported in [26].

2 Fuzzy logics versus Uncertainty logics

In [32], a new approach to axiomatize logics of uncertainty was proposed, further elaborated later on in the celebrated Hájek's monograph [33]. This approach clarifies the different roles played on the one side by fuzzy logics and vagueness, and on the other side by probability and other types of quantified belief. We briefly recall here the main underlying ideas, instantiated in the case of probability:

(i) *Truth degrees $\neq$ uncertainty degrees.*

Fuzzy logic is a logic of gradual, imprecise notions and propositions that may be more or less true. Fuzzy logic is then a logic of partial degrees of truth. On the contrary, probability (or any other uncertainty theory) deals with crisp notions and propositions that, at least in its classical formulation, are either true or false; the probability of a proposition is then the degree of belief on the truth of that proposition. Then, clearly, fuzzy logic does not deal with uncertainty (as belief) at all. From a semantical point of view, the main difference lies in the fact that degrees of belief are not extensional (truth-functional). E.g., the probability of $p \wedge q$ is not a function of the probability of p and the probability of q, whereas degrees of truth of vague notions admit truth-functional approaches (although they are not bound to them). Therefore, formally speaking, fuzzy logics behave as many-valued logics, whereas, as we shall discuss in the next point, uncertainty or belief theories can be related to some kinds of (two-valued) modal logics.

(ii) *Interpreting probability degrees of crisp propositions as truth-degrees of fuzzy (modal) propositions.*

This is the key idea of the approach. Probability, or any uncertainty measure in general, preserves classical logical equivalence and therefore "understands" formulas as classical propositions. However, uncertainty measures are just variables (like pressure, temperature, etc.) and we can make fuzzy assertions on them. For instance, if φ is any formula we may say "φ is *probable*" (or "*probability_of_*φ is *high*"), and these are typical fuzzy propositions. Thus, in particular, there is nothing wrong in taking as truth-degree of the fuzzy proposition "φ is probable" exactly the probability degree of the crisp proposition φ being true. For what we said above on compositionality, we clearly have to distinguish between propositions like "(φ is probable) and (ψ is probable)" on the one hand, and "($\varphi \wedge \psi$) is probable" in the other.

(iii) *Probability measures as models of fuzzy theories.*

Once one builds a fuzzy proposition $P\varphi$ (with P being a modality standing for *probable*) for each classical proposition φ, one can write theories about the $P\varphi$'s over a particular fuzzy logic including, as axioms, the formulas corresponding to basic postulates of a particular uncertainty theory, probability theory in this case. In this way, models (in the sense of many-valued logic) of the theories about the $P\varphi$'s become probability measures over the crisp φ's. For instance, in [32] a propositional probability logic was defined as a theory over Rational Pavelka logic, an extension of Łukasiewicz's infinite-valued logic with rational truth constants.

Now, the question is *which kind* of fuzzy logic we can use to formalize probability theory. The very reason of using Łukasiewicz and Rational Pavelka logics in [32, 33] is that the truth-functions for the formulas of Łukasiewicz logic are based on the arithmetic operations of addition and subtraction in the unit interval $[0, 1]$, which is obviously what is needed to deal with additive measures such as probabilities. In the next section we overview Łukasiewicz logic Ł and its expansion with rational truth-constants RPL. Extensions and generalizations of this approach are clearly possible, and we invite the interested reader to consult [25] for a survey.

3 Łukasiewicz logic and some of its extensions

Łukasiewicz infinite-valued logic [36] is one of the most prominent systems falling under the umbrella of Mathematical Fuzzy Logic, see the handbooks [10, 11, 12]. In fact, together with Gödel infinite-valued logic [30], it was defined much before fuzzy logic as a discipline was born. In particular, it has received much attention since the fifties, when completeness results were proved by Rose and Rosser [45], and by Chang [14, 15] who developed, for his completeness result, the theory of MV-algebras; the latter is now largely studied in the literature. For more details and results about Łukasiewicz logic and MV-algebras the reader is referred to the monographs [8, 39].

The language of Łukasiewicz logic is built in the usual way from a set of propositional variables, one binary connective $\rightarrow$ (that is, Łukasiewicz implication) and the truth constant $\bar{0}$, that we will also denote as $\perp$. An *evaluation* e maps every propositional variable to a real number from the unit interval $[0, 1]$ and extends to all formulas in the following way:

$$\begin{array}{lcl} e(\bar{0}) & = & 0, \\ e(\varphi \rightarrow \psi) & = & \min(1 - e(\varphi) + e(\psi), 1) \end{array}$$

Other interesting connectives can be defined from them,

$$
\begin{array}{lll}
\bar{1} & \text{is} & \varphi \to \varphi, \\
\neg\varphi & \text{is} & \varphi \to \bar{0}, \\
\varphi \oplus \psi & \text{is} & \neg\varphi \to \psi, \\
\varphi \& \psi & \text{is} & \neg(\neg\varphi \oplus \neg\psi), \\
\varphi \ominus \psi & \text{is} & \varphi \& \neg\psi, \\
\varphi \equiv \psi & \text{is} & (\varphi \to \psi)\&(\psi \to \varphi), \\
\varphi \wedge \psi & \text{is} & \varphi \& (\varphi \to \psi), \\
\varphi \vee \psi & \text{is} & \neg(\neg\varphi \wedge \neg\psi),
\end{array}
$$

and they have the following interpretations:

$$
\begin{array}{lll}
e(\neg_L\varphi) & = & 1 - e(\varphi), \\
e(\varphi \oplus \psi) & = & \min(1, e(\varphi) + e(\psi)) \\
e(\varphi \& \psi) & = & \max(0, e(\varphi) + e(\psi) - 1) \\
e(\varphi \ominus \psi) & = & \max(0, e(\varphi) - e(\psi)) \\
e(\varphi \equiv \psi) & = & 1 - |e(\varphi) - e(\psi)| \\
e(\varphi \wedge \psi) & = & \min(e(\varphi), e(\psi)) \\
e(\varphi \vee \psi) & = & \max(e(\varphi), e(\psi)).
\end{array}
$$

The interpretations of Łukasiewicz logic connectives over the real unit interval $[0,1]$ define an algebra which is referred to as the *standard MV-algebra* and denoted with $[0,1]_{MV}$, which generates in universal algebraic terms the equivalent algebraic semantics of Łukasiewicz logic in the sense of Blok and Pigozzi, that is, the variety of MV-algebras [8, 39].

An evaluation e is called a *model* of a set of formulas T whenever $e(\varphi) = 1$ for each formula $\varphi \in T$.

Axioms and rules of Łukasiewicz logic are the following [8, 33]:

(Ł1)	$\varphi \to (\psi \to \varphi)$
(Ł2)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(Ł3)	$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$
(Ł4)	$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$
(MP)	Modus ponens: from φ and $\varphi \to \psi$ derive ψ

From this axiomatic system, the notion of proof from a theory (a set of formulas), denoted $\vdash_{\text{Ł}}$, is defined as usual.

The above axioms are tautologies or valid (i.e., they are evaluated to 1 by any evaluation), and the rule of modus ponens preserves validity. Moreover, the following completeness result holds.

Theorem 1. *The logic Ł is complete for deductions from finite theories. That is, if T is a finite theory, then $T \vdash_{L} \varphi$ iff $e(\varphi) = 1$ for each Łukasiewicz evaluation e model of T.*

This completeness result with respect to the standard semantics on $[0,1]_{MV}$ is not valid for deductions from general (non-finite) theories. There are two main ways to enforce such stronger completeness result. The first one is to extend Łukasiewicz logic by the following infinitary rule of inference [37]:

$$(IR)\ \frac{\varphi \to \psi^n, \text{ for each } n \in \mathbb{N}}{\neg\varphi \vee \psi}$$

where ψ^n is a shorthand for $\psi \& \overset{n}{\ldots} \& \psi$. The second is to replace the standard real chain $[0,1]_{MV}$ by an MV-chain on a hyperreal unit interval $[0,1]^*$ as the domain of truth-values and hence allowing for infinitesimal and co-infinitesimal values [24, 9].

Łukasiewicz logic does not satisfy the deduction theorem in full generality, as it only satisfies the following *local* form: $T, \varphi \vdash_{Ł} \psi$ iff there exists $n \in \mathbb{N}$ such that $T \vdash_{Ł} \varphi^n \to \psi$, where n depends on the formula φ. In this sense we refer to that form of the deduction theorem as "local".

It is also worth noticing that, as it happens with most of the many-valued or fuzzy logics, classical propositional logic (CP) can be recovered extending Łukasiewicz logic Ł with the axiom of the excluded middle principle:

(EM) $\quad \varphi \vee \neg\varphi$

In the rest of this section we briefly recall two interesting expansions of Łukasiewicz logic that will be used later on.

3.1 Ł$_\Delta$: extending Łukasiewicz with the projection Baaz-Monteiro connective Δ

A very helpful connective for our purposes is the so-called *Baaz-Monteiro projection operator* Δ, whose standard semantics is as follows: for a Łukasiewicz evaluation e to $[0,1]$,

$$e(\Delta\varphi) = \begin{cases} 1, & \text{if } e(\varphi) = 1 \\ 0, & \text{otherwise} \end{cases}$$

In a sense, even if φ takes intermediate truth-values, $\Delta\varphi$ is two-valued: it is true when φ is 1-true, and false otherwise.

Ł$_\Delta$ is axiomatized by adding to the Hilbert-style system of Ł the deduction rule of necessitation for Δ:

(NΔ) $\quad$ from φ infer $\Delta\varphi$

and the following axiom schemata:

(Δ1) $\quad \Delta\varphi \vee \neg\Delta\varphi$
(Δ2) $\quad \Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$
(Δ3) $\quad \Delta\varphi \to \varphi$
(Δ4) $\quad \Delta\varphi \to \Delta\Delta\varphi$
(Δ5) $\quad \Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$

So defined, the completeness result for Ł extends to Ł$_\Delta$ with the above semantics for Δ, see [33]. Moreover, Ł$_\Delta$ satisfies a *global deduction theorem* in the following sense: for each set of formulas $\Sigma \cup \{\varphi, \psi\}$ one has

$$\Sigma, \varphi \vdash_{\text{Ł}_\Delta} \psi \ \text{ iff } \ \Sigma \vdash_{\text{Ł}_\Delta} \Delta\varphi \to \psi.$$

This is a relevant property, considering that Łukasiewicz logic itself only satisfies the above mentioned *local* form of deduction theorem.

3.2 Rational Pavelka logic: extending Łukasiewicz logic with rational truth constants

Following Hájek [33], the language of Rational Pavelka logic, denoted RPL, is the language of Łukasiewicz logic expanded with countably-many truth-constants $\overline{r}$, one for each rational $r \in [0, 1]$.

The evaluation of RPL formulas is as in Łukasiewicz logic, with the proviso that evaluations map truth-constants to their intended value: for any rational $r \in [0, 1]$ and any evaluation e, $e(\overline{r}) = r$.

Note that, for any evaluation e, $e(\overline{r} \to \varphi) = 1$ iff $e(\varphi) \geq r$, and $e(\overline{r} \equiv \varphi) = 1$ iff $e(\varphi) = r$.

Axioms and rules of RPL are those of Ł plus the following countable set of bookkeeping axioms for truth-constants:

(BK) $\quad \overline{r} \to \overline{s} \equiv \overline{\min(1, 1 - r + s)}$, for any rationals $r, s \in [0, 1]$.

The notion of proof is defined as in Łukasiewicz logic, and the deducibility relation will be denoted by $\vdash_{RPL}$. Moreover, completeness of Łukasiewicz logic smoothly extends to RPL as follows: if T is a finite theory over RPL, then $T \vdash_{RPL} \varphi$ iff $e(\varphi) = 1$ for any RPL-evaluation e model of T.

RPL also enjoys a sort of infinitary completeness result, known as *Pavelka-style completeness*. Namely, for any set of RPL formulas $T \cup \{\varphi\}$, define:

- the *truth degree* of φ in T: $\|\varphi\|_T = \inf\{e(\varphi) : e$ RPL-evaluation model of $T\}$,
- the *provability degree* of φ from T: $|\ \varphi\ |_T = \sup\{r \in [0, 1]_{\mathbb{Q}} \mid T \vdash_{RPL} \overline{r} \to \varphi\}$.

Then the following result holds for deductions from any arbitrary (non necessarily finite) theory [33].

Theorem 2. *For any set of RPL formulas $T \cup \{\varphi\}$, we have:*

$$|\ \varphi\ |_T = \|\varphi\|_T.$$

We remark that from $\|\varphi\|_T = s$, with s being rational (in particular, in the case $s = 1$) it does not follow in general that $T \vdash_{RPL} \overline{s} \to \varphi$, only that $T \vdash_{RPL} \overline{r} \to \varphi$ for all rational $r < s$. However, the situation improves when T is a finite theory. Indeed, if T is finite, we can restrict ourselves to rational-valued Łukasiewicz evaluations and get strong completeness. The following results are proved in [33].

Proposition 1. *If T is a finite theory over RPL, then the following conditions hold:*

- $\|\varphi\|_T = 1$ *iff* $T \vdash_{RPL} \varphi$.
- $\|\varphi\|_T$ *is rational, hence* $\|\varphi\|_T = r$ *iff* $T \vdash_{RPL} \overline{r} \to \varphi$.

Remark 1 (Implicit definition of rational numbers)**.** It is interesting to notice that rational numbers can already be implicitly defined in Łukasiewicz logic, as observed in [33, Lemma 3.3.11]. Let us consider for instance the constant $\overline{0.5}$. It can be directly checked by the definition of the connectives in the standard MV-algebra that a model of the one-variable formula $\phi(p) := p \oplus p$ is an evaluation e that maps p to a real number in the interval $[0.5, 1]$. Similarly, a model of the formula $\psi(p) := \neg(p\&p)$ is an evaluation that maps p to some value the interval $[0, 0.5]$. Thus, a model of $\phi \wedge \psi$ is an evaluation which maps the variable p to exactly 0.5.[1] This means that, given any formula of Łukasiewicz logic γ, if we consider the theory $T = \{\gamma \equiv p, \phi \wedge \psi\}$, a model of T is an evaluation e such that $e(\gamma) = 0.5$.

While here we picked a simple example for the sake of clarity, the same holds for all rational numbers. That is, given any rational $r \in [0, 1]$, there are Łukasiewicz formulas $\phi_r(p), \psi_r(p)$ whose models are evaluations that map p, respectively, to a value in the intervals $[r, 1]$ and $[0, r]$. Therefore, a model of $\phi_r \wedge \psi_r$ is an evaluation that maps p exactly to r. A consequence of such implicit definition is that, as proved in [33], deductions in RPL from finite theories can be reduced to proofs in Ł.

Remark 2. One can also combine the logics $Ł_\Delta$ and RPL. Indeed, one can extend $Ł_\Delta$ with rational truth-constants (or viceversa, extend RPL with Δ) and define the logic RPL_Δ by expanding the language of $Ł_\Delta$ with rational truth-constants $\overline{r}$, and by adding to $Ł_\Delta$ the bookkeeping axioms (BK). By doing so, one can easily check that the bookkeeping axioms for the Δ connective are derivable, that is, RPL_Δ proves $\neg\Delta\overline{r}$, for every rational $r < 1$.

Also, one has to notice that RPL_Δ keeps enjoying completeness for deductions from finite theories, but the Pavelka-style completeness is lost, due to the non-continuity of the truth function for the Δ connective. As an example, if we consider the theory $T = \{\overline{r} \to \varphi \mid r < 1\}$, then $\|\Delta\varphi\|_T = 1$, but $\mid \Delta\varphi \mid_T = 0$.

4 $Ł^{\leq}$: the paraconsistent degree-preserving companion of Ł

Deductive systems in the scope of Mathematical Fuzzy Logic (MFL) have been usually studied under the paradigm of (full) *truth-preservation.* The latter, generalizing the classical notion of consequence, postulates that a formula follows from a set of premises if every algebraic evaluation that interprets the premises

[1] Alternatively, one could consider the formula $\eta(p) := p \leftrightarrow \neg p$ that again gets value 1 under an evaluation e iff $e(p) = 0.5$.

as fully true also interprets the conclusion as fully true. An alternative approach that has recently received some attention is based on the *degree-preservation* paradigm [28, 4], in which a conclusion follows from a set of premises if, for all evaluations, the truth degree of the conclusion is not lower than that of the premises. It has been argued that this approach is more coherent with the commitment of many-valued logics to truth-degree semantics because all values play an equally important rôle in the corresponding notion of consequence [27].

Moreover, while the truth-preserving fuzzy logics are explosive, i.e. from any theory containing a formula φ and its negation $\neg\varphi$ everything follows, in two recent papers [20, 16] some degree-preserving fuzzy logics have been shown to exhibit some well behaved paraconsistency properties. In particular, this is the case of Łukasiewicz logic Ł, whose degree-preserving companion $Ł^{\leq}$ is not explosive, i.e. it is *paraconsistent*. Moreover, the degree-preserving companions of finitely-valued Łukasiewicz logics $Ł_n$ are not only paraconsistent but they also belong to the family of the so-called *logics of formal inconsistency* (**LFI**s) [7].

The logical consequence relation for $Ł^{\leq}$ is defined as follows [28]. For every set of formulas $\Gamma \cup \{\varphi\}$:

> $\Gamma \models_{Ł^{\leq}} \varphi$ iff for every evaluation v over the standard MV-algebra $[0,1]_{\mathrm{MV}}$ and every $a \in [0,1]$, if $a \leq v(\gamma)$ for every $\gamma \in \Gamma$, then $a \leq v(\varphi)$.

Because of its very definition, $\models_{Ł^{\leq}}$ is known as the Łukasiewicz logic *preserving degrees of truth*, or the *degree-preserving companion* of Ł.

In fact, Ł and $Ł^{\leq}$ have the same tautologies, and for every finite set of formulas $\Gamma \cup \{\varphi\}$ we have:

$$\Gamma \models_{Ł^{\leq}} \varphi \text{ iff } \models_{Ł} \Gamma^{\wedge} \to \varphi,$$

where, if $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$, $\Gamma^{\wedge}$ denotes the conjunction $\gamma_1 \wedge \ldots \wedge \gamma_k$ (when Γ is empty then we take $\Gamma^{\wedge}$ as $\top$). It is worth noticing that the usual rule of modus ponens is not sound for $Ł^{\leq}$. However, the logic $Ł^{\leq}$ admits a Hilbert-style axiomatization with a weaker form of modus ponens.

Definition 1. Axioms of $Ł^{\leq}$ are those of Ł, and the rules of $Ł^{\leq}$ are the following:

$$\textbf{(Adj-}\wedge\textbf{)} \quad \frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad\qquad \textbf{(MP-}r\textbf{)} \quad \frac{\varphi \quad \vdash_{Ł} \varphi \to \psi}{\psi}$$

The corresponding notion of proof defined from these axioms and rules will be denoted with $\vdash_{Ł^{\leq}}$.

Note that to apply the modified form of the modus ponens rule (MP-r) it is required that the implication $\varphi \to \psi$ must be a theorem. Clearly, both the (Adj-$\wedge$) and (MP-r) rules are derivable in Łukasiewicz logic. Therefore, since the axioms are the same, for all set of formulas $T \cup \{\varphi\}$, if $T \vdash_{Ł^{\leq}} \varphi$, then $T \vdash_{Ł} \varphi$, but not vice-versa as, for instance, the Modus Ponens rule is not derivable in $Ł^{\leq}$. Thus, $Ł^{\leq}$ is a weaker logic than Ł.

This axiomatic system provides a sound and complete axiomatisation of $\models_{Ł^{\leq}}$ for deductions from a finite set of formulas [4], that is, for every finite set of formulas $\Gamma \cup \{\varphi\}$, one gets:

$$T \vdash_{Ł^{\leq}} \varphi \quad \text{iff} \quad T \models_{Ł^{\leq}} \varphi.$$

It is clear that $Ł^{\leq}$ is a paraconsistent logic, since for instance, $p, \neg p \not\vdash_{Ł^{\leq}} \perp$. However, $Ł^{\leq}$ is not a logic of formal inconsistency (LFI), as shown in [20]. Nonetheless, if we expand $Ł^{\leq}$ with the Δ operator, then we obtain an LFI. Indeed, $Ł^{\leq}_{\Delta}$ is the logic axiomatised by adding to $Ł^{\leq}$ the axioms $(\Delta 1) - (\Delta 5)$ and the following restricted necessitation rule for Δ (it only applies to theorems of Ł):

(rNΔ) if $\vdash_{Ł} \varphi$ infer $\Delta\varphi$

Then, the definable connective $\circ\varphi := \Delta(\varphi \vee \neg\varphi)$ in $Ł^{\leq}_{\Delta}$ fulfills the required properties of a consistency operator in the logics of formal inconsistency, and thus $Ł^{\leq}_{\Delta}$ is an LFI [20].

Similarly, one can define the degree-preserving companion of RPL considering the logical consequence $\models^{\leq}_{RPL}$ according with the proviso that every evaluation e over the standard MV-algebra $[0,1]_{\mathrm{MV}}$ additionally satisfies $e(\overline{r}) = r$ for every rational $r \in [0,1]$. Equivalently, one gets a sound and finite strong complete axiomatisation for $\models^{\leq}_{RPL}$ just adding to the axiomatic system for $Ł^{\leq}$ the usual bookkeeping axioms for truth-constants. Moreover, an analogous Pavelka-style completeness result for $\mathrm{RPL}^{\leq}$ can also be obtained.

For any set of RPL formulas $T \cup \{\varphi\}$, define:

- *truth degree* of φ in T: $\|\varphi\|^{\leq}_{T} = \inf\{e(T) \Rightarrow e(\varphi) : e \text{ RPL-evaluation}\}$,
- *provability degree* of φ from T: $\mid \varphi \mid^{\leq}_{T} = \sup\{r \mid T \vdash^{\leq}_{RPL} \overline{r} \to \varphi\}$,

where $e(T) = \inf\{e(\psi) : \psi \in T\}$ and $\Rightarrow$ is the truth-function of Łukasiewicz implication, i.e. $x \Rightarrow y = \min(1, 1 - x + y)$. Then, to prove a Pavelka-style completeness for $\mathrm{RPL}^{\leq}$, we need two previous lemmas.

Lemma 1. *If $T \cup \{\varphi\}$ is a finite set of RPL-formulas, we have:*

$$\mid \varphi \mid^{\leq}_{T} = \|\varphi\|^{\leq}_{T}.$$

Proof. We have the following equalities:

$$\begin{aligned} \mid \varphi \mid^{\leq}_{T} &= \sup\{r \in [0,1] \text{ rational} \mid T \vdash^{\leq}_{RPL} \overline{r} \to \varphi\} \\ &= \sup\{r \in [0,1] \text{ rational} \mid \vdash_{\mathrm{RPL}} \overline{r} \to (T^{\wedge} \to \varphi)\} \text{ (by RPL complet.)} \\ &= \inf\{e(T^{\wedge} \to \varphi) : e \text{ is an Ł-evaluation}\} = \|\varphi\|^{\leq}_{T}, \end{aligned}$$

the latter equality being due to the fact that $e(T) = e(T^{\wedge})$. □

Lemma 2. *For any theory T, $\|\varphi\|^{\leq}_{T} = \sup_{T_0 \subseteq^f T} \|\varphi\|^{\leq}_{T_0}$, where $T_0 \subseteq^f T$ stands for $T_0 \subseteq T$ with T_0 finite.*

Proof. We prove the two following inequalities:

(i) $\sup\{\|\varphi\|^{\leq}_{T_0} \mid T_0 \subseteq^f T\} \leq \|\varphi\|^{\leq}_{T}$. Indeed:

$\sup\{\|\varphi\|^{\leq}_{T_0} \mid T_0 \subseteq^f T\} = \sup\{\inf_e\{e(T_0) \to e(\varphi)\} \mid T_0 \subseteq^f T\} \leq$
$\leq \sup\{\inf_e\{e(T) \to e(\varphi)\} \mid T_0 \subseteq^f T\} = \inf_e\{e(T) \to e(\varphi)\} = \|\varphi\|^{\leq}_{T}$.

(ii) $\sup\{\|\varphi\|^{\leq}_{T_0} \mid T_0 \subseteq^f T\} \geq \|\varphi\|^{\leq}_{T}$.

Assume $\sup\{\|\varphi\|^{\leq}_{T_0} \mid T_0 \subseteq^f T\} < \|\varphi\|^{\leq}_{T}$. Then, there is a rational r such that $\sup\{\|\varphi\|^{\leq}_{T_0} \mid T_0 \subseteq^f T\} < r < \|\varphi\|^{\leq}_{T}$. Then we have:

- $\sup\{\|\varphi\|^{\leq}_{T_0} \mid T_0 \subseteq^f T\} < r$ implies that, for all finite $T_0 \subseteq T$, $\|\varphi\|^{\leq}_{T_0} < r$, that is, $\inf_e e(T_0) \to e(\varphi) < r$. Therefore, there is an evaluation e^* such that $e^*(T_0) \to e^*(\varphi) < r$, i.e. $e^*(T_0) > r \to e^*(\varphi)$. Hence, we have $\inf_{T_0} e^*(T_0) \geq r \to e^*(\varphi)$, that is, $e^*(T) \geq r \to e^*(\varphi)$.

- $r < \|\varphi\|^{\leq}_{T}$ implies $r < \inf_e e(T) \to e(\varphi) \leq e^*(T) \to e^*(\varphi) \leq (r \to e^*(\varphi)) \to e^*(\varphi) = r \vee e^*(\varphi)$. Therefore, it follows that $e^*(\varphi) > r$, that is in contradiction with the fact that $e(T_0) \to e(\varphi) < r$.

□

Then we can finally prove the following completeness result.

Theorem 3. *For any set of RPL formulas $T \cup \{\varphi\}$, we have:*

$$\mid \varphi \mid^{\leq}_{T} = \|\varphi\|^{\leq}_{T}.$$

Proof. The following is a proof based on the previous two lemmas:

$$\begin{aligned}
\mid \varphi \mid^{\leq}_{T} \quad &= \sup\{r \in [0,1] \text{ rational} \mid T \vdash^{\leq}_{RPL} \overline{r} \to \varphi\} \\
&= \sup\{r \in [0,1] \text{ rational} \mid \exists \text{ a finite } T_0 \subseteq T \text{ s.t. } T_0 \vdash^{\leq}_{RPL} \overline{r} \to \varphi\} \\
&= \sup_{T_0 \subseteq^f T} \sup\{r \in [0,1] \text{ rational} \mid T_0 \vdash^{\leq}_{RPL} \overline{r} \to \varphi\} \\
&= \sup_{T_0 \subseteq^f T} \mid \varphi \mid^{\leq}_{T_0} \text{ (by Lemma 1)} \\
&= \sup_{T_0 \subseteq^f T} \|\varphi\|^{\leq}_{T_0} \text{ (by Lemma 2)} \\
&= \|\varphi\|^{\leq}_{T}.
\end{aligned}$$

□

This completeness result shows that RPL$^{\leq}$ is well-behaved in a purely logical sense. However, we have mentioned above that the usual rule of modus ponens is not sound in Ł$^{\leq}$, and hence neither is in RPL$^{\leq}$. Actually, in RPL$^{\leq}$, one can show that only the following weakened form of modus ponens deduction holds:

$$\{\varphi, \varphi \to \psi\} \models^{\leq}_{RPL} \overline{0.5} \to \psi.$$

That is, in RPL$^{\leq}$ we are forced to lower the truth-degree of the conclusion in order to have a sound, but weaker, modus ponens rule. This is a price to pay to enjoy a paraconsistent behavior.

Let us finally remark that we can also expand RPL$^{\leq}$ with the Δ operator, denoted RPL$^{\leq}_{\Delta}$ and defined analogously to the case of Ł$^{\leq}_{\Delta}$, to come up with an LFI with rational truth-constants.

5 FP(Ł): a qualitative logic to reason about probability as a modal theory over Ł

In this section we describe the fuzzy modal logic FP(Ł) to reason qualitatively about probability, built upon Łukasiewicz logic Ł. The language of FP(Ł) is defined in two steps:

Non-modal formulas: these are built from a set V of propositional variables $\{p_1, p_2, ... p_n, \dots\}$ using the classical binary connectives $\wedge$ and $\neg$. Other connectives like $\vee$, $\rightarrow$ and $\leftrightarrow$ are defined from $\wedge$ and $\neg$ in the usual way.[2] Non-modal formulas (we will also refer to them as Boolean propositions) will be denoted by lower case Greek letters φ, ψ, etc. The set of non-modal formulas will be denoted by $\mathcal{L}$.

Modal formulas: these are built from elementary modal formulas of the form $P\varphi$, where φ is a non-modal formula and using the connectives and constants of Łukasiewicz logic Ł. We shall denote them by upper case Greek letters Φ, Ψ, etc. Notice that we do not allow nested modalities of the form $P(P(\psi) \oplus P(\varphi))$, nor mixed formulas of the kind $\psi \rightarrow P\varphi$.

Definition 2. The axioms of the logic FP(Ł) are the following:

(i) Axioms of classical propositional logic for non-modal formulas

(ii) Axioms of Ł for modal formulas

(iii) Probabilistic modal axioms:

(FP0)	$P\varphi$, for φ being a theorem of CPL
(FP1)	$P(\varphi \rightarrow \psi) \rightarrow (P\varphi \rightarrow P\psi)$
(FP2)	$P(\neg\varphi) \equiv \neg P\varphi$
(FP3)	$P(\varphi \vee \psi) \equiv (P\varphi \rightarrow P(\varphi \wedge \psi)) \rightarrow P\psi$

The only deduction rule of FP(Ł) is that of Ł (i.e. *modus ponens*)

The notion of proof for modal formulas is defined as usual from the above axioms and rule. We will denote that in FP(Ł) a modal formula Φ follows from a theory (set of modal formulas) T by $T \vdash_{FP} \Phi$. Note that FP(Ł) preserves classical equivalence. Indeed, due to axioms (FP0) and (FP1), FP(Ł) proves the formula $P\varphi \equiv P\psi$ whenever φ and ψ are classically logically equivalent, and as we will see below, this will mean that in such a case they are bound to have the same probability.

The semantics for FP(Ł) is given by *probability Kripke structures* $K = \langle W, \mathcal{U}, e, \mu \rangle$, where:

- W is a non-empty set of possible worlds.

[2] Although we are using the same symbols $\wedge, \neg, \vee, \rightarrow$ as in Łukasiewicz logic to denote the conjunction, negation, disjunction and implication, the context will help in avoiding any confusion. In particular classical logic connectives will appear only under the scope of the operator P, see below.

- $e : V \times W \to \{0, 1\}$ provides for each world a *Boolean* (two-valued) evaluation of the propositional variables, that is, $e(p, w) \in \{0, 1\}$ for each propositional variable $p \in V$ and each world $w \in W$. A truth-evaluation $e(\cdot, w)$ is extended to Boolean propositions as usual. For a Boolean formula φ, we will write $[\varphi]_W = \{w \in W \mid e(\varphi, w) = 1\}$.
- $\mu : \mathcal{U} \to [0, 1]$ is a probability over a Boolean algebra $\mathcal{U}$ of subsets of W such that $[\varphi]_W$ is μ-measurable for any non-modal φ.
- $e(\cdot, w)$ is extended to elementary modal formulas by defining

  $$e(P\varphi, w) = \mu([\varphi]_W),$$

  and to arbitrary modal formulas according to Ł semantics, that is:

  $$\begin{array}{rcl} e(\overline{0}, w) & = & 0, \\ e(\Phi \to_L \Psi, w) & = & \min(1 - e(\Phi, w) + e(\Psi, w), 1). \end{array}$$

Notice that if Φ is a modal formula the truth-evaluations $e(\Phi, w)$ depend only on the probability measure μ and not on the particular world w.

The truth-degree of a formula Φ in a probability Kripke structure $K = \langle W, \mathcal{U}, e, \mu \rangle$, written $\|\Phi\|^K$, is defined as

$$\|\Phi\|^K = \inf_{w \in W} e(\Phi, w).$$

When $\|\Phi\|^K = 1$ we say that Φ is valid in K or that K is a model for Φ, and we will also writte $K \models \Phi$. Let T be a set of modal formulas. Then we say that K is a model of T if $K \models \Phi$ for all $\Phi \in T$. Now, let $\mathcal{M}$ be a class of probability Kripke structures. Then we define the truth-degree $\|\Phi\|_T^{\mathcal{M}}$ of a formula in a theory T relative to the class $\mathcal{M}$ as

$$\|\Phi\|_T^{\mathcal{M}} = \inf\{\|\Phi\|^K \mid K \in \mathcal{M},\ K \text{ being a model of } T\} \ .$$

The notion of logical entailment relative to the class $\mathcal{M}$, written $\models_{\mathcal{M}}$, is then defined as follows:

$$T \models_{\mathcal{M}} \Phi \text{ if } \|\Phi\|_T^{\mathcal{M}} = 1 \ .$$

That is, Φ logically follows from a set of formulas T if every structure in $\mathcal{M}$ which is a model of T is also a model of Φ. If $\mathcal{M}$ denotes the whole class of probability Kripke structures we shall write $T \models_{FP} \Phi$ and $\|\Phi\|_T^{FP}$.

It is easy to check that axioms FP0-FP3 are valid formulas in the class of all probability Kripke structures. Moreover, the inference rule of substitution of equivalents preserves truth in a model, while the necessitation rule for P preserves validity in a model. Therefore we have the following soundness result.

Lemma 3. **(Soundness)** *The logic FP(L) is sound with respect to the class of probability Kripke structures.*

For any φ, ψ in the set of non-modal formulas $\mathcal{L}$, define $\varphi \sim \psi$ iff $\vdash \varphi \leftrightarrow \psi$ in classical logic. The relation $\sim$ is an equivalence relation in the crisp language $\mathcal{L}$ and $[\varphi]$ will denote the equivalence class of φ, containing the propositions provably equivalent to φ. Obviously, the quotient set $\mathcal{L}/_\sim$ forms a Boolean algebra which is isomorphic to a subalgebra $\mathbf{B}(\Omega)$ of the power set of the set Ω of Boolean interpretations of the crisp language $\mathcal{L}$.[3] For each $\varphi \in \mathcal{L}$, we can identify the equivalence class $[\varphi]$ with the set $\{\omega \in \Omega \mid \omega(\varphi) = 1\} \in \mathbf{B}(\Omega)$ of interpretations that make φ true. We denote by $\mathcal{P}(\mathcal{L})$ the set of probabilities over $\mathcal{L}/_{\sim_{FP}}$ or equivalently on $\mathbf{B}(\Omega)$.

Notice that each probability $\mu \in \mathcal{P}(\mathcal{L})$ induces a probability Kripke structure $\langle \Omega, \mathbf{B}(\Omega), e_\mu, \mu \rangle$ where $e_\mu(p, \omega) = \omega(p) \in \{0, 1\}$ for each $\omega \in \Omega$ and each propositional variable p. We shall denote by $\mathcal{PS}$ the class of Kripke structures induced by probabilities $\mu \in \mathcal{P}(\mathcal{L})$, i.e. $\mathcal{PS} = \{\langle \Omega, \mathbf{B}(\Omega), e_\mu, \mu \rangle \mid \mu \in \mathcal{P}(\mathcal{L})\}$. Abusing the language, we say that a probability $\mu \in \mathcal{P}(\mathcal{L})$ is a *model* of a modal theory T whenever the induced Kripke structure $\Omega_\mu = \langle \Omega, \mathbf{B}(\Omega), e_\mu, \mu \rangle$ is a model of T. Besides, we will often write $\mu(\varphi)$ actually meaning $\mu([\varphi])$. In fact, for many purposes we can restrict ourselves to the class of probability Kripke structures $\mathcal{PS}$.

Lemma 4. *For each probability Kripke structure $K = \langle W, \mathcal{U}, e, \mu \rangle$ there is a probability $\mu^* : \mathbf{B}(\Omega) \to [0, 1]$ such that $\|P\varphi\|^K = \mu^*(\varphi)$ for all $\varphi \in \mathcal{L}$. Therefore, it also holds that $\|\Phi\|_T^{FP} = \|\Phi\|_T^{\mathcal{PS}}$ for any modal formula Φ and any modal theory T.*

In the following we sketch a completeness proof for FP(Ł) as presented in [32, 33] and then further generalised in [23, 25], since we think it may be appealing for the interested reader.

The usual strategy to prove completeness of probabilistic modal logics like FP(Ł) w.r.t. probabilistic Kripke models consists in the following steps (see [25, 13] for more details):

(S1) First of all we define a syntactic translation $^\bullet$ from modal to propositional formulas of Łukasiewicz logic by interpreting every atomic modal formula $P(\varphi)$ in a new propositional variable p_φ and extending $^\bullet$ to compound modal formulas by truth functionality.

(S2) The translation of all instances of the axioms (FP1)-(FP3), together with the set $\{p_\varphi \mid \vdash \varphi\}$ which encodes the propositional translation of (FP0), gives rise to a propositional Łukasiewicz theory $\mathbf{FP}^\bullet$ such that, for every (finite) set of modal formulas $T \cup \{\Phi\}$, $T \vdash_{FP} \Phi$ iff $T^\bullet \cup \mathbf{FP}^\bullet \vdash \Phi^\bullet$ (see for instance [23, 25] and [33]).

Now, assume that $T \not\vdash_{FP} \Phi$ and hence $T^\bullet \cup \mathbf{FP}^\bullet \not\vdash \Phi^\bullet$. As we recalled in Section 3, Łukasiewicz logic does not enjoy the strong standard completeness, i.e. standard completeness with respect to deductions from infinite theories.

[3] Actually, $\mathbf{B}(\Omega) = \{\{\omega \in \Omega \mid \omega(\varphi) = 1\} \mid \varphi \in \mathcal{L}\}$. Needless to say, if the language has only finitely many propositional variables then the algebra $\mathbf{B}(\Omega)$ is just the whole power set of Ω, otherwise it is a strict subalgebra.

Therefore, if $T^\bullet \cup \mathbf{FP}^\bullet$ turns out to be infinite, countermodels cannot always be found in a chain over $[0,1]$. As already mentioned above, we should hence either extend Ł by the infinite rule (IR), or extend the scope of countermodels of $\Phi^\bullet$ to include also hyperreal MV-chains of the form $[0,1]^*$ [24]. However, if T is finite, $\mathbf{FP}^\bullet$ can be obtained translating all the instantiation of the axioms (P1)-(P3) using, for events, classical formulas up to logical equivalence. Those are finitely many, once we restrict to the variables occurring in the events of $T \cup \{\Phi\}$. Thus, if T is finite, $T^\bullet \cup \mathbf{FP}^\bullet$ is finite as well and, by Theorem 1, if $T^\bullet \cup \mathbf{FP}^\bullet \not\vdash \Phi^\bullet$ there exists a $[0,1]_{MV}$-valuation e which is a model of $T^\bullet \cup \mathbf{FP}^\bullet$ and maps $\Phi^\bullet$ to some value $\alpha < 1$. Finally, since e is a model of all (the translation of the instances of) (FP0)-(FP3), e is indeed a probability function that does not satisfy Φ. Thus, the following holds.

Theorem 4. (Probabilistic completeness of FP(Ł)) *Let T be a finite modal theory over FP(Ł) and Φ a modal formula. Then $T \vdash_{FP} \Phi$ iff $e_\mu(\Phi) = 1$ for each probability μ model of T.*

We end this section with the following remark that presents FP(Ł) and its extensions as logic for qualitative probabilistic reasoning.

Remark 3. Notice that FP(Ł) can be used to reason in a purely qualitative way about comparative probability statements by exploiting the fact a FP(Ł)-formula of the form $P\psi \to P\varphi$ is 1-true in a model defined by a probability μ iff $\mu(\psi) \leq \mu(\varphi)$. Therefore, if we represent the statement "the event φ is at least as probable than the event ψ" as $\psi \lhd \varphi$, then an inference of the form

$$\text{from } \psi_1 \lhd \varphi_1, \ldots, \psi_n \lhd \varphi_n \text{ infer } \chi \lhd \nu$$

can be faithfully captured by the following proof in FP(Ł)

$$P\psi_1 \to P\varphi_1, \ldots, P\psi_n \to P\varphi_n \vdash_{FP} P\chi \to P\nu.$$

Moreover, FP(Ł) also allows to reason about statements with a more quantitative flavour like "the event φ is as twice as probable as the event ψ", represented by the formula

$$P\psi \oplus P\psi \to P\varphi.$$

Indeed, such a formula is 1-true in a model defined by a probability μ iff $\mu(\varphi) \geq \min(2\mu(\psi), 1)$.

6 Introducing truth-constants in FP(Ł): the logic FP(RPL)

As we have just seen above, the logic FP(Ł) already allows to express several kinds of qualitative statements about probability, from purely comparative statements to more quantitative statements, but always without an explicit numerical representation of probability values. However, if one wants to explicitly reason about numerical statements, like "the probability of φ is 0.8", "the probability of φ is at least 0.8", or "the probability of φ is at most 0.8", a simple solution is to replace in FP(Ł) Łukasiewicz logic Ł by Rational Pavelka logic RPL, that is, its expansion of Ł with rational truth-constants. Then, one will be easily able to express:

- "the probability of φ is 0.8" as $P\varphi \equiv \overline{0.8}$,
- "the probability of φ is at least 0.8" as $\overline{0.8} \to P\varphi$, and
- "the probability of φ is at most 0.8" as $P\varphi \to \overline{0.8}$.

To define the logic FP(RPL) we just expand the language of modal formulas with the truth-constants $\overline{r}$ for every rational $r \in [0,1]$, and in the axiomatic definition of FP(Ł) we add the bookkeeping axioms of RPL. The corresponding probability Kripke structures are as in the case of FP(Ł) with the obvious minor modifications.

Theorem 5 (Probabilistic completeness of FP(RPL))**.** *Let T be a finite modal theory over FP(RPL) and Φ a modal formula. Then $T \vdash_{FP(RPL)} \Phi$ iff $e_\mu(\Phi) = 1$ for each probability μ model of T.*

As in the case of RPL, for finite theories we can still get completeness when we restrict ourselves to rational-valued probabilities.

Corollary 1. *Let T be a finite modal theory over FP(RPL) and Φ a modal formula. Then $T \vdash_{FP(RPL)} \Phi$ iff $e_\mu(\Phi) = 1$ for each rational-valued probability μ model of T.*

Moreover, also Pavelka-style completeness holds for FP(RPL): for an arbitrary set of modal formulas $T \cup \{\Phi\}$, one defines the *provability degree of* Φ *in* T as $|\Phi|_T = \sup\{r \mid T \vdash_{FP(RPL)} \overline{r} \to \Phi\}$ and the *truth degree of* Φ *in* T as $\|\Phi\|_T = \inf\{\|\Phi\|^K \mid K$ probability Kripke structure model of $T\}$. Then the following holds.

Theorem 6 (Pavelka-style completeness for FP(RPL))**.** *For an arbitrary set of modal formulas $T \cup \{\Phi\}$ of $FP(RPL)$, one has $|\Phi|_T = \|\Phi\|_T$.*

Since deductions in FP(RPL) from a finite theory can be encoded as deductions from a (larger) finite theory in RPL, as a direct corollary of Proposition 1, we get the following.

Corollary 2. *If T is finite, for any FP(RPL)-formula Φ, $\|\Phi\|_T$ is rational.*

Adding the rational values in $FP(RPL)$ is indeed quite interesting. In that manner, we can express in $FP(RPL)$ partial assignments of probability as well, and thus capture, by a logical consistency of a modal theory, de Finetti's notion of *coherence* for rational-valued probability assignments [19]. Before showing this result, let us recall that, if $\varphi_1, \ldots, \varphi_k$ are classical formulas, an assignment $\beta : \varphi_i \mapsto r_i$ of the φ_i's to $[0,1]$ is said to be *coherent* iff β can be extended to a (at least one) probability function on the algebra spanned by the events $\varphi_1, \ldots, \varphi_k$.[4] Then, we can prove the following (see [22, Theorem 8] for further details).

Theorem 7. *Let $\varphi_1, \ldots, \varphi_k$ be (finitely many) classical formulas and let $\beta : \varphi_i \mapsto r_i$ be a rational-valued assignment. Then the following conditions are equivalent:*

1. *β is coherent;*
2. *The modal theory $T_\beta = \{P(\varphi) \equiv \overline{r_i} \mid i = 1, \ldots, k\}$ is consistent, i.e., $T_\beta \not\vdash_{FP(RPL)} \bot$.*

Proof. By Theorem 5, T_β is consistent iff there exists a probability μ model of T_β. In order to prove the claim is hence sufficient to show that this is the case iff β is coherent.

Suppose that β is coherent and let $\mathbf{B}$ be the finite Boolean algebra of all events modulo provable equivalence where we identify every event with its equivalence class. Then, by de Finetti's theorem (see [19]), there exists a probability measure $\mu : \mathbf{B} \to [0, 1]$ such that, for all $i = 1, \ldots, k$, $\mu(\varphi_i) = \beta(\varphi_i) = r_i$. Thus, if W is the set of atoms of $\mathbf{B}$ and, for all $w \in W$, $e(p, w)$ is the logical valuation of p associated to w, $\langle W, \mathbf{B}, e, \mu\rangle$ is a probability Kripke model that satisfies T_β. Thus the latter is consistent.

Conversely, if T_β is consistent, then it is immediate to see that if $\langle W, \mathcal{U}, e, \mu\rangle$ is a probability Kripke model that satisfies all $P(\varphi_i) \equiv \overline{r}_i$, μ extends β and hence β is coherent again by de Finetti's theorem. □

Besides the logical characterization of coherent assignments provided by Theorem 7 above, the same notion can be also described in geometrical terms, see for instance [40]. Such geometrical representation will play a role in the next sections and we now recall how it is obtained. First of all, let us fix a finite set of events (i.e., classical formulas) $\mathcal{E} = \{\varphi_1, \ldots, \varphi_k\}$ and let $\mathbf{B}$ be the finite Boolean algebra generated by them. Let $h_1, \ldots, h_t$ the (also finitely many) homomorphisms of $\mathbf{B}$ to the two-element Boolean chain $\{0, 1\}$. For each $i = 1, \ldots, t$, let us define the string, with $\{0, 1\}$-coordinates

$$q_i = \langle h_i(\varphi_1), \ldots, h_i(\varphi_k)\rangle.$$

These q_i's can hence be regarded as point of $[0, 1]^k$ and indeed as vertices of the same cube. Thus, call $\mathscr{C}_\mathcal{E}$ the convex hull generated by $q_1, \ldots, q_t$, i.e.,

$$\mathscr{C}_\mathcal{E} = \mathrm{co}(\{q_1, \ldots, q_t\}). \tag{1}$$

Also notice that any assignment $\beta : \mathcal{E} \to [0, 1]$ determines a unique point of $[0, 1]^k$, namely, $\beta = \langle \beta(\varphi_1), \ldots, \beta(\varphi_k)\rangle$. Then, the following holds.

[4] It is worth to point out that the one reported here is the statement of de Finetti's theorem that characterizes coherent books as those that can be extended by a probability measure. The actual definition of coherence is in terms of a precisely defined zero-sum betting game. We invite the interest reader to consult [19] for further details.

Proposition 2. *Let $\mathcal{E} = \{\varphi_1, \ldots, \varphi_k\}$ be a finite set of classical formulas and let $\beta : \mathcal{E} \to [0,1]$ be an assignment. Then the following conditions are equivalent:*

1. *β is coherent;*
2. *$\beta \in \mathscr{C}_{\mathcal{E}}$.*

7 Reasoning with inconsistent probabilistic information in FP(RPL)

If we want to reason in a non-trivial way from inconsistent probabilistic theories over FP(RPL), we need to devise possible ways to define paraconsistent reasoning inference relations in a meaningful form.

In this section we will consider two possible approaches. The first approach is to replace in FP(RPL) the outer logic RPL by its degree-preserving companion RPL$^{\leq}$, or even by RPL$^{\leq}_{\Delta}$, its expansion with Δ, that are both paraconsistent logics as we have seen in Section 4. The second approach consists of, given an inconsistent theory over FP(RPL), computing a degree of inconsistency of T and using this degree to minimally weaken the formulas in the theory so that the weakened theory becomes consistent.

7.1 The degree-preserving companion of FP(RPL)

As it is clear from the definition of $FP(\text{Ł})$ and $FP(RPL)$ given in Sections 5 and 6 respectively, Łukasiewicz logic and RPL, together with (P0)-(P4) allow to reason about probability statements using Łukasiewicz calculus with possibly rational truth-constants. Thus, it makes sense, from what we recalled in Section 4, to replace those logics by their degree-preserving companion Ł$^{\leq}$ and $RPL^{\leq}$. We will mainly focus on the probability logic FP($RPL^{\leq}$), the degree-preserving companion of $FP(RPL)$ and, so as to emphasize that degree-preservation applies at the level of the probability formulas, it will be henceforth denoted by FP$^{\leq}$(RPL).

Conforming to the usual way of defining deductions in degree-preserving logics, given two probabilistic modal formulas Φ and Ψ, $\Phi \vdash_{FP^{\leq}} \Psi$ iff for every probabilistic Kripke model $\mathcal{M} = (W, e, \mu)$ of FP$^{\leq}$(RPL), $\|\Phi\|_{\mathcal{M}} \leq \|\Psi\|_{\mathcal{M}}$. More precisely, if $\Phi = t_1[P(\varphi_1), \ldots, P(\varphi_k)]$ and $\Psi = t_2[P(\varphi_1), \ldots, P(\varphi_k)]$, the above means that, for all finitely additive probability measure μ,

$$t_1^{[0,1]}[\mu(\varphi_1), \ldots, \mu(\varphi_k)] \leq t_2^{[0,1]}[\mu(\varphi_1), \ldots, \mu(\varphi_k)],$$

where, for $i = 1, 2$, t_i is a Łukasiewicz formula and $t_i^{[0,1]}$ stands for its interpretation in the standard MV-algebra $[0,1]_{MV}$.

Obviously the above definition applies to the more general case in which $T = \{\Phi_1, \ldots, \Phi_n\}$ is any finite set of modal formulas. In such a case, in fact, $T \vdash_{FP^{\leq}} \Psi$ iff for all probabilistic Kripke model $\mathcal{M}$,

$$\|\Phi_1 \wedge \ldots \wedge \Phi_n\|_{\mathcal{M}} \leq \|\Psi\|_{\mathcal{M}}.$$

Let us start noticing that the logic $\mathrm{FP}^{\leq}(RPL)$ is not explosive. Indeed, for each classical formula φ that is neither a classical theorem nor a contradiction, $P(\varphi), \neg P(\varphi) \not\vdash_{FP^{\leq}} \bot$. To make an example, notice that, in such case, $P(\varphi), \neg P(\varphi) \not\vdash_{FP^{\leq}} P(\varphi) \& P(\varphi)$ because, semantically, one can find a probability μ that assigns $\mu(\varphi) = 1/2$ and this gives

$$\min\{\mu(\varphi), \mu(\neg\varphi)\} = 1/2 > \mu(\varphi) \odot \mu(\varphi) = 0.$$

Moreover, $\mathrm{FP}^{\leq}(RPL)$ is weaker than $FP(RPL)$ in the following sense: recalling the translation map $^{\bullet}$ defined in Section 5, for all pair of modal formulas Φ and Ψ,

$$\begin{array}{ll} \Phi \vdash_{FP^{\leq}} \Psi & \Leftrightarrow \\ FP^{\bullet} \vdash_{RPL} \Phi^{\bullet} \to \Psi^{\bullet} & \Rightarrow \\ FP^{\bullet}, \Phi^{\bullet} \vdash_{RPL} \Psi^{\bullet} & \Leftrightarrow \\ \Phi \vdash_{FP} \Psi. & \end{array}$$

Therefore, if $\Phi \vdash_{FP^{\leq}} \Psi$, then $\Phi \vdash_{FP} \Psi$ but the other way around does not hold in general. In other words, we have the following.

Proposition 3. *Let $T \cup \{\Psi\}$ be a finite set of modal formulas. If $T \vdash_{FP^{\leq}} \Psi$, then $T \vdash_{FP} \Psi$. Therefore, in particular, if $T \not\vdash_{FP} \bot$, then $T \not\vdash_{FP^{\leq}} \bot$.*

The fact that $\mathrm{FP}^{\leq}(RPL)$ is not explosive is particularly interesting in the light of Theorem 7 above, where we showed how the (full) consistency of a modal theory of FP(Ł) can be used to characterize the coherence of partial probabilistic assignments. Indeed, being able to handle inconsistency in $\mathrm{FP}^{\leq}(RPL)$ allows us to handle partial incoherence as well and hence to derive some reasonable conclusions from an initial incoherent evaluation. Let us start from the following definition that uses the characterization result proved in the above Theorem 7.

Definition 3. Let $\varphi_1, \ldots, \varphi_k$ be finitely many classical formulas and let $\beta : \varphi_i \mapsto r_i$ be a rational-valued assignment. Then we say that β is *weakly coherent* if $T_\beta \not\vdash_{FP^{\leq}} \bot$.

The following is an immediate consequence of the above definition and the semantic definition of $\vdash_{FP^{\leq}}$.

Proposition 4. *Let $\varphi_1, \ldots, \varphi_k$ be finitely many classical formulas and $\beta : \varphi_i \mapsto r_i$ be a rational-valued assignment. Then the following conditions are equivalent:*

1. *β is weakly coherent;*
2. *There exists a probabilistic Kripke model $\mathcal{M}$ such that* $\min\{\|P(\varphi_i) \equiv \overline{r_i}\|_{\mathcal{M}} \mid i = 1, \ldots, k\} > 0$.

From Proposition 4, every coherent assignment β is weakly coherent, but the converse is generally false. So as to provide an example, let φ and ψ be such that $\varphi \wedge \psi \leftrightarrow \bot$ and take the incoherent assignment $\beta(\varphi) = 0.4$ and $\beta(\varphi \vee \psi) = 0.2$. Then β, although incoherent, is weakly coherent. Indeed, consider the probability Kripke model $\mathcal{M} = (W, e, \mu)$ where, in particular, μ gives to $\mu([\varphi]_W) = 0.3$ and $\mu([\psi]_W) = 0$. It is then easy to see that

$$\min\{\|P(\varphi) \equiv \overline{0.4}\|_{\mathcal{M}}, \|P(\varphi \vee \psi) \equiv \overline{0.3}\|_{\mathcal{M}}\} = 0.1 > 0.$$

However, we can see that in $\mathrm{FP}^{\leq}(RPL)$ the following deduction is valid:

$$P(\varphi) \equiv \overline{0.4}, P(\varphi \vee \psi) \equiv \overline{0.2} \vdash_{FP^{\leq}} \overline{0.1} \to P(\psi). \tag{2}$$

Semantically, the above states that, for all probability measures μ (defined on the algebra of events), one has that $\min\{1 - |\mu(\varphi) - 0.4|, 1 - |\mu(\varphi \vee \psi) - 0.2|\} \leq (1 - 0.1) \oplus \mu(\psi)$, i.e., since $\varphi \wedge \psi \leftrightarrow \perp$,

$$\min\{1 - |\mu(\varphi) - 0.4|, 1 - |\mu(\varphi) + \mu(\psi) - 0.2|\} \leq 0.9 \oplus \mu(\psi). \tag{3}$$

Notice that, for all values of $\mu(\psi) \geq 0.1$, the above is trivially satisfied. Also, for $0 \leq \mu(\psi) < 0.1$ it is valid too. Indeed, taking for instance $\mu(\psi) = 0$ (which would give the lower value for $0.9 \oplus \mu(\psi)$) and $\mu(\varphi) = 1$ (that is compatible with the fact that $\varphi \wedge \psi \leftrightarrow \perp$ and it would give the highest value to the left-hand-side of (3)), one has that $\min\{0.4, 0.2\} \leq 0.9$. Also, the function $\min\{1 - |x - 0.4|, 1 - |x + y - 0.2|\}$ gets a maximum when $x = 0.3$ and $y = 0$, where it takes the value 0.9 and, in that case, (3) yields $0.9 \leq 0.9 + 0$. Thus, (2) is always valid.

Adopting the same argument as above, we can also notice that

$$P(\varphi) \equiv \overline{0.4}, P(\varphi \vee \psi) \equiv \overline{0.3} \not\vdash_{FP^{\leq}} \overline{0.1} \equiv P(\psi).$$

Indeed, $\min\{1 - |\mu(\varphi) - 0.4|, 1 - |\mu(\varphi) + \mu(\psi) - 0.3|\} > 1 - |P(\psi) - 0.1|$ for the probability μ such that $\mu(\varphi) = 0$ and $\mu(\psi) = 1$.

A possibly more evident example that shows what can be proved in $\mathrm{FP}^{\leq}(RPL)$ from a more intuitive set of incoherent premises consists in taking the incoherent assignment $P(\varphi) = 0.8$ and $P(\neg\varphi) = 0.6$. By using a semantical argument again, we can see that the following hold:

1. $P(\varphi) \equiv \overline{0.8}, P(\neg\varphi) \equiv \overline{0.6} \vdash_{FP^{\leq}} P(\perp) \equiv \overline{r}$ for all $r \leq 0.2$
2. $P(\varphi) \equiv \overline{0.8}, P(\neg\varphi) \equiv \overline{0.6} \not\vdash_{FP^{\leq}} P(\perp) \equiv \overline{r}$ for all $r > 0.2$.

Remark 4. The above examples show that most incoherent partial assignments can be handled in the logic $\mathrm{FP}^{\leq}(RPL)$ by means of the notion of weak coherence. However, not all incoherent assignment are necessarily weakly coherent. For instance, consider the assignment $\beta(\perp) = 1$ (or, dually, $\beta(\top) = 0$). It is not difficult to see that every probability measure μ, on any algebra, gives $e_\mu(P(\perp) \equiv \overline{1}) = 0$, and hence, by Proposition 4 the above assignment is not even weakly coherent. This reasoning path clearly extends to any assignment β that contains the identity $\beta(\perp) = 1$ (or $\beta(\top) = 0$). Also notice that hence, whenever an assignment β contains such equations, its associated theory T_β is explosive also in $\mathrm{FP}^{\leq}(RPL)$.

7.2 An inconsistency-tolerant probabilistic logic

In the previous section we have seen that a possible way to reason paraconsistently from an inconsistent probabilistic theory in the language of FP(RPL) is to resort to the degree preserving companion $\mathrm{FP}^{\leq}(RPL)$. However, we have also seen that the inferential power of $\mathrm{FP}^{\leq}(RPL)$ is somehow limited. In this section, we consider other possible ways of defining inconsistency-tolerant probabilistic consequence relations that can provide, in some cases, more intuitive results.

Let us recall from Section 5 that, from a semantical point of view, the logic FP(RPL) is defined as follows: for any set of FP(RPL)-formulas $T \cup \{\Phi\}$,

$$\begin{aligned} T \models_{FP} \Phi \text{ if, for every probability } \mu \text{ on Boolean formulas,} \\ \text{if } \mu \text{ is a model of } T \text{ then } e_\mu(\Phi) = 1, \end{aligned}$$

where by μ being a model of T we mean that $e_\mu(\Psi) = 1$ for every $\Psi \in T$, or put it short, $e_\mu(T) = 1$. We will denote by $[\![T]\!]$ the set probability measures on formulas that are models of T. In other words,

$$[\![T]\!] = \{\text{probability } \mu \mid \text{for all } \Psi \in T, e_\mu(\Psi) = 1\}.$$

Of course, the above definition trivializes in the case T is inconsistent, i.e., when $[\![T]\!] = \emptyset$. However, in FP(RPL) one can take advantage of its many-valued/fuzzy setting and consider the notion of (in)consistency as being many-valued as well.

Indeed, if a probabilistic theory T has no models, it makes sense to distinguish, e.g., cases where: (1) for every probability μ there is a formula $\Phi \in T$ such that $e_\mu(\Phi) = 0$; and (2) there exists a probability μ such that, for all $\Phi \in T$, $e_\mu(\Phi)$ is close to 1. In the former case T is clearly inconsistent, while in the latter case one could say that T is close to being consistent.

This observation justifies to introduce, for each threshold α, the set of α-*generalised models* (or just α-models) of T defined as follows:

$$[\![T]\!]_\alpha = \{\text{probability } \mu \mid \text{for all } \Psi \in T, e_\mu(\Psi) \geq \alpha\}.$$

Note that the set $[\![T]\!]_1$ coincides with the set $[\![T]\!]$ of usual models of T. Moreover $[\![T]\!]_\alpha$ is a convex set of probabilities.

This in turn allows to define the degree of consistency of a theory as the highest value α for which T has at least one α-generalised model.

Definition 4. Let T be a theory of FP(RPL). The consistency degree of T is defined as

$$Con(T) = \sup\{\beta \in [0,1] \mid [\![T]\!]_\beta \neq \emptyset\}.$$

Dually, the inconsistency degree of T is defined as

$$Incon(T) = 1 - Con(T) = \inf\{1 - \beta \in [0,1] \mid [\![T]\!]_\beta \neq \emptyset\}.$$

By completeness of $FP(\text{Ł})$ with respect to probability models (Theorem 4), we can also express, for every finite modal theory T, $Con(T)$ as follows:

$$Con(T) = \sup\{\beta \in [0,1] \mid T_\beta \not\vdash \bot\}$$

and hence $Incon(T) = \inf\{1 - \beta \in [0,1] \mid T_\beta \not\vdash \bot\}$.

Remark 5. Notice that the notion of consistency degree for a theory T in FP(RPL) can be seen as a stronger and quantitative version of the notion of weak coherence as defined in the above Subsection 7.1. Indeed, an immediate computation shows that if T is a theory representing a partial assignment β on some classical formulas, i.e., $T = T_\beta$ as in the statement of Theorem 7, then we have:

1. $Con(T) = 1$ implies, by Proposition 2, that β is coherent.
2. On the other hand, if $Con(T) > 0$, then β is weakly coherent by Proposition 4.
3. Finally, if $Con(T) = 0$, then β is not even weakly coherent. Indeed, this case coincides with the one explored in Remark 4 above.

A somewhat different formulation of the degrees of consistency and inconsistency is the following.

Lemma 5. *$Con(T) = \sup\{\bigwedge_{\Phi\in T} e_\mu(\Phi) \mid \mu$ probability$\}$.*
$Incon(T) = \inf\{\bigvee_{\Phi\in T} e_\mu(\neg\Phi) \mid \mu$ probability$\}$.

Proof. Let $A = \{\beta \mid [\![T]\!]_\beta \neq \emptyset\}$ and $B = \{\bigwedge_{\Phi\in T} e_\mu(\Phi) \mid \mu \text{ probability }\}$. We have to prove that $\sup A = \sup B$.

- Let μ be a probability and let $\alpha = \bigwedge_{\Phi\in T} e_\mu(\Phi)$. Then $e_\mu(\Phi) \geq \alpha$ for all $\Phi \in T$. Therefore, $\mu \in [\![T]\!]_\alpha$, that is, $\alpha \in A$, hence $\alpha \leq \sup A$ and thus $\sup B \leq \sup A$.
- Let $\beta \in A$, that is, let β be such that $[\![T]\!]_\beta \neq \emptyset$. Then there exists μ probability such that $e_\mu(\Phi) \geq \beta$ for all $\Phi \in T$. Thus, $\bigwedge_{\Phi\in T} e_\mu(\Phi) \geq \beta$, and hence $\sup B \geq \beta$, that is, $\sup B \geq \sup A$.

Finally, we have $Incon(T) = 1 - \sup\{\bigwedge_{\Phi\in T} e_\mu(\Phi) \mid \mu \text{ probability}\} =$ $\inf\{1 - \bigwedge_{\Phi\in T} e_\mu(\Phi) \mid \mu \text{ probability}\} = \inf\{\bigvee_{\Phi\in T} 1 - e_\mu(\Phi) \mid \mu \text{ probability}\} =$ $\inf\{\bigvee_{\Phi\in T} e_\mu(\neg\Phi) \mid \mu \text{ probability}\}$. □

If T is finite, the suprema and the infima in Definition 4 and in the expressions $Con(T)$ and $Incon(T)$ as in Lemma 5, are in fact maxima and minima.

Lemma 6. *Let T be a finite theory of FP(RPL). Then:*

$$Con(T) = \max\{\bigwedge_{\Phi\in T} e_\mu(\Phi) \mid \mu \textit{ probability}\} = \max\{\beta \in [0,1] \mid T_\beta \nvdash \bot\},$$

$$Incon(T) = \min\{\bigvee_{\Phi\in T} e_\mu(\neg\Phi) \mid \mu \textit{ probability}\} = \min\{1-\beta \in [0,1] \mid T_\beta \nvdash \bot\}.$$

Proof. First, notice that if T is a finite theory, we can write the formula $T^\wedge = \bigwedge_{\Phi\in T}\Phi$, which is such that $[\![T]\!]_\alpha = [\![T^\wedge]\!]_\alpha$. In fact, $T^\wedge$ as a formula of FP(RPL) can be seen as a formula of RPL over variables of the kind $P\varphi_1, \ldots, P\varphi_k$.

Accordingly, let $T^\wedge = t[P\varphi_1, \dots, P\varphi_k]$ for some RPL term t. Then for each probability map μ, $e_\mu(T^\wedge) = t^{[0,1]^k}(e_\mu(P\varphi_1), \dots, e_\mu(P\varphi_k))$. The set of values $(p_1, \dots, p_k) \in [0,1]^k$ associated to probability values of the classical formulas in $\mathcal{E} = \{\varphi_1, \dots, \varphi_k\}$ is the set $\mathscr{C}_\mathcal{E}$ as defined in (1), which is a compact convex subset of $[0,1]^k$. Moreover, the map $t^{[0,1]^k}$ is a continuous function (since Łukasiewicz truth-functions and rational constants are continuous). Hence, the set of possible values of $e_\mu(T^\wedge)$, for any μ, is the image in $[0,1]$ of a closed set over $[0,1]^k$, therefore it is a closed subset of $[0,1]$, which has a maximum value β. Now, $\beta = Con(T) = Con(T^\wedge)$. Indeed, there exists at least a probability map μ such that $e_\mu(T^\wedge) \geq \beta$, and β is the largest element with such property. $\square$

Moreover, it can be shown that, for any finite theory T, $Con(T)$ (and thus $Incon(T)$ as well) is always a rational value.

Lemma 7. *If T is a finite theory over FP(RPL), then $Con(T)$ and $Incon(T)$ are rational.*

Proof. By Corollary 2, if T is a finite theory over RPL, then the truth-degree of any formula $\varphi \in T$ is rational, i.e. $\|\varphi\|_T$ is rational. This easily extends to FP(RPL), since deductions in FP(RPL) from a finite theory T can be encoded as deductions in RPL from the larger, but still finite, theory $T^\bullet \cup FP^\bullet$ (recall the translation map $\bullet$ defined in Section 5).

Now, one can check that $Incon(T)$ is nothing but the truth-degree of the formula $T^\# = \bigvee_{\Phi \in T} \neg\Phi$ in the theory $FP^\bullet$, that is, $Incon(T) = [\![T^\#]\!]_{FP^\bullet}$. Therefore, from the above, it follows that $Incon(T)$ is rational. $\square$

In particular, from the previous lemma it follows that for a finite theory T, if $Con(T) = \alpha$, then $[\![T]\!]_\alpha \neq \emptyset$. Moreover:

(i) If $Con(T) = 1$ then T has a model, while if $Con(T) = 0$ then, for any probability μ there is a formula $\Psi \in T$ such that $e_\mu(\Psi) = 0$.

(ii) If $Con(T) > 0$ then $\|T\|_{Con(T)} \neq \emptyset$.

(iii) If $T' \subseteq T$ then $Con(T') \geq Con(T)$.

Let us clarify what we showed so far by some very simple examples.

Example 1. Let us consider the following theory of precise probability assignments $T = \{P\varphi_i \equiv \overline{r_i}\}_{i=1,\dots,n}$. If μ is a probability, then $e_\mu(P\varphi_i \equiv \overline{r_i}) = 1 - |\mu(\varphi_i) - r_i|$. Further let

$$\beta_\mu = \bigwedge_{i=1,n} 1 - |\mu(\varphi_i) - r_i|.$$

Then $\mu \in [\![T]\!]_{\beta_\mu}$, and

$$Con(T) = \sup\{\beta_\mu \mid \mu \text{ probability}\} = \sup_\mu \bigwedge_{i=1,n} 1 - |\mu(\varphi_i) - r_i|,$$

and thus,

$$Incon(T) = 1 - Con(T) = 1 - \sup_{\mu} \bigwedge_{i=1,n} 1 - |\mu(\varphi_i) - r_i| = \inf_{\mu} \bigvee_{i=1,n} |\mu(\varphi_i) - r_i|.$$

That is to say, $Incon(T)$ is nothing but the Chebyshev distance of the point $(r_1, \ldots, r_n) \in [0,1]^n$ to the convex set of coherent assignments $\mathscr{C}_{\mathcal{E}}$ on the events $\mathcal{E} = \{\varphi_1, \ldots, \varphi_n\}$.

Example 2. Consider the theory $T_\beta = \{P(p) \equiv \overline{1/2}, P(\neg p) \equiv \overline{1/3}\}$ given by the incoherent assignment $\beta : p \mapsto 1/2; \neg p \mapsto 1/3$. Notice that, since every probability μ satisfies $\mu(\neg p) = 1 - \mu(p)$, the consistency of T_β is equivalent to the consistency of the (formally different) theory

$$T'_\beta = \{(P(p) \equiv \overline{1/2}) \wedge (P(p) \equiv \overline{2/3})\},$$

which we will now consider instead of T_β for the sake of this example.

Now, the coherent set of the unique event p that we consider in the above formulation, is the whole real unit interval $[0,1]$. Indeed, in fact, for every $\alpha \in [0,1]$, there exists a probability μ such that $\mu(p) = \alpha$. Then, we can consider the function $f_\Psi : [0,1] \to [0,1]$ defined as

$$f_\Psi(y) = \min\{1 - |y - 1/2|, 1 - |y - 2/3|\},$$

that is, the continuous and piecewise linear function with rational coefficients that represents the RPL-formula $\Psi = (P(p) \equiv \overline{1/2}) \wedge (P(p) \equiv \overline{2/3})$ once $P(p)$ is regarded as a (complex) propositional variable (left-hand side of Figure 1).

Now, the maximum for f_Ψ on the convex and compact set $\mathscr{C}_{\{p\}}$ of all coherent assignments on p, that in this particular case is the whole real unit interval $[0,1]$, i.e. $\mathscr{C}_{\{p\}} = [0,1]$, is attained at $7/12$ and $f_\Psi(7/12) = 11/12$, see the left-hand side of Figure 1. Thus,

$$Con(T'_\beta) = 11/12.$$

Another, yet equivalent, way to graphically compute the consistency degree of our theory, is to use the geometric description of coherence we briefly recalled at the end of Section 6. Indeed, if we take into account the original assignment β and hence the theory $T_\beta = \{P(p) \equiv \overline{1/2}, P(\neg p) \equiv \overline{1/3}\}$, the set of all coherent assignments on events p and $\neg p$ is the set $\mathscr{C}_{\{p,\neg p\}} = \{(x, 1-x) \mid x \in [0,1]\}$, i.e. the segment in $[0,1]^2$ with endpoints $(1,0)$ and $(0,1)$ (right-hand side of Figure 1), and the incoherent assignment β is displayed as the point $(1/2, 1/3) \notin \mathscr{C}_{\{p,\neg p\}}$. As we mentioned in the above Example 1, $Con(T_\beta)$ can be computed as 1 minus the Chebyshev distance between $(1/2, 1/3)$ and $\mathscr{C}_{\{p,\neg p\}}$. This value is attained at the point of coordinates $(7/12, 5/12)$ and then we have:

$$1 - |\beta(p) - 7/12| = 1 - |\beta(\neg p) - 5/12| = 1 - 1/12 = 11/12 = Con(T_\beta).$$

Example 3. Let us consider a theory representing an imprecise probability assignment to a set of events:

$$T = \{(\overline{r_i - \epsilon_i} \to P\varphi_i) \wedge (P\varphi_i \to \overline{r_i + \epsilon_i})\}_{i=1,\ldots,n}$$

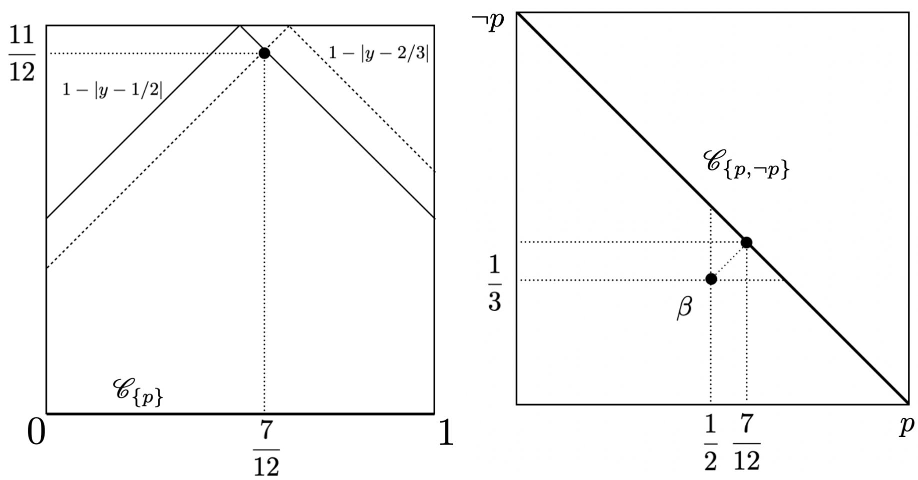

Figure 1: Two equivalent ways to compute $Con(T)$: on the left-hand side, $Con(T)$ is the maximum value attained by the function $f_\Psi(y) = \min\{1 - |y - 1/2|, 1 - |y - 2/3|\}$ computed on the compact convex set of coherent assignment. On the right-hand side $Con(T)$ is computed as 1 minus the Chebyshev distance between the point β that represents the partial assignment on p and $\neg p$, and the set of coherent assignments on p and $\neg p$.

where, for each i, $r_i - \epsilon_i \geq 0$ and $r_i + \epsilon_i \leq 1$, that is $\epsilon_i \leq r_i \leq 1 - \epsilon_i$. Then we can compute the degree of inconsistency of T as follows:

$$Incon(T) = 1 - Con(T) = 1 - \sup_\mu \bigwedge_{i=1,n} e_\mu((\overline{r_i - \epsilon_i} \to P\varphi_i) \wedge (P\varphi_i \to \overline{r_i + \epsilon_i}))$$

$$= 1 - \sup_\mu \bigwedge_{i=1,n} ((r_i - \epsilon_i) \to \mu(\varphi_i)) \wedge (\mu(\varphi_i) \to (r_i + \epsilon_i))$$

$$=^5 1 - \sup_\mu \bigwedge_{i=1,n} ((1 - \epsilon_i) \to (r_i \equiv \mu(\varphi_i)) = 1 - \sup_\mu \bigwedge_{i=1,n} (1 - \epsilon_i) \to 1 - |r_i - \mu(\varphi_i)|$$

$$= \inf_\mu 1 - \bigwedge_{i=1,n} (1 - \epsilon_i) \to 1 - |r_i - \mu(\varphi_i)| = \inf_\mu \bigvee_{i=1,n} 1 - ((1 - \epsilon_i) \to 1 - |r_i - \mu(\varphi_i)|)$$

$$= \inf_\mu \bigvee_{i=1,n} (1 - \epsilon_i) \otimes |r_i - \mu(\varphi_i)|.$$

As for paraconsistently reasoning from an inconsistent theory in FP(RPL), the idea we explore here is to use α-generalised models instead of usual models to define a context-dependent inconsistent-tolerant notion of probabilistic entailment.

[5] Here we use the property $\min((x - y) \to z, z \to (x + y)) = y \to (x \equiv z)$, if $y \leq x \leq 1 - y$.

Definition 5. Let T be a theory such that $Con(T) = \alpha > 0$. We define:

$$T \approx\!\!\!\!\models^* \Phi \text{ if } e_\mu(\Phi) = 1 \text{ for all probabilities } \mu \in [\![T]\!]_\alpha.$$

Note that for a finite theory T, if $Con(T) > 0$, then $T \not\approx\!\!\!\!\models^* \bot$, hence $\approx\!\!\!\!\models^*$ does not trivialize even if T is inconsistent ($Con(T) < 1$).

The following are some interesting properties of the consequence relation $\approx\!\!\!\!\models^*$:

- Clearly, $\approx\!\!\!\!\models^*$ is nonmonotonic. For instance, let $T' = \{P\varphi \equiv \overline{0.4}, P\varphi \to P\psi\}$ and $T = T' \cup \{\overline{0.3} \equiv P\varphi\}$. Then we have $con(T') = 1$ and trivially $T' \approx\!\!\!\!\models^* \overline{0.4} \equiv P\varphi$, while $Con(T) = 0.95$ and $T \not\approx\!\!\!\!\models^* \overline{0.4} \equiv P\varphi$.
- $\approx\!\!\!\!\models^*$ is idempotent, that is, if $S \approx\!\!\!\!\models^* \varphi$ and $T \approx\!\!\!\!\models^* \psi$ for all $\psi \in S$, then $T \approx\!\!\!\!\models^* \varphi$

Next proposition shows that paraconsistent reasoning from an inconsistent theory T by means of the inference relation $\approx\!\!\!\!\models^*$ can be reduced to usual reasoning in FP(RPL) by suitably weakening the formulas in the initial theory T.

Proposition 5. *Given a theory T, with $Con(T) = \alpha$, then the following condition holds:*

$$T \approx\!\!\!\!\models^* \Phi \textit{ iff } T_\alpha \vdash_{FP} \Phi,$$

where $T_\alpha = \{\overline{\alpha} \to \Psi \mid \Psi \in T\}$.

Proof. Indeed, if μ is a probability such that $e_\mu(\overline{\alpha} \to \Psi) = 1$, i.e. such that $e_\mu(\Psi) \geq \alpha$, for all $\Psi \in T$, this means that $\mu \in [\![T]\!]_\alpha$. But if we assume $T \approx\!\!\!\!\models^* \Phi$, then it follows that $e_\mu(\Phi) = 1$. Hence $T_\alpha \vdash_{FP} \Phi$.

Conversely, assume $T_\alpha \vdash_{FP} \Phi$ with $Con(T) = \alpha$ and that $\mu \in \|T\|_\alpha$. The latter means that $e_\mu(\Psi) \geq \alpha$ for all $\Psi \in T$, i.e. $e_\mu(\alpha \to \Psi) = 1$ for all $\Psi \in T$. But then, since $T_\alpha \vdash_{FP} \Phi$, it follows that $e_\mu(\Phi) = 1$, that is, $T \approx\!\!\!\!\models^* \Phi$. □

The weakened theory T_α can be seen as a *repair* of T. It is worth considering how the repaired theories T_α look like for two particular kinds of theories T:

- In the case the theory represents a precise probability assignment of the form $T = \{\overline{r_i} \equiv P\varphi_i\}_{i=1,n}$, then $T_\alpha = \{(\overline{\alpha \otimes r_i} \to P\varphi_i) \wedge (P\varphi_i \to \overline{\alpha \Rightarrow r_i})\}_{i=1,n}$.
- In the case the theory represents an imprecise probability assignment of the form $T = \{(\overline{r_i} \to P\varphi_i) \wedge (P\varphi_i \to \overline{s_i})\}_{i=1,n}$, then $T_\alpha = \{(\overline{\alpha \otimes r_i} \to P\varphi_i) \wedge (P\varphi_i \to \overline{\alpha \Rightarrow s_i})\}_{i=1,n}$.

Note that in both cases, the constraints represented in T_α are weaker, more imprecise than those in T: in the first case, a constraint $\mu(\varphi_i) = r_i$ is modified into the constraint $\mu(\varphi_i) \in [\alpha \otimes r_i, \alpha \Rightarrow r_i]$, while in the second case, a constraint $\mu(\varphi_i) \in [r_i, s_i]$ is modified into the constraint $\mu(\varphi_i) \in [\alpha \otimes r_i, \alpha \Rightarrow s_i]$.

7.3 A refinement of $\approx\!\!\!\!\mid^*$

The consequence relation $\mid\!\approx^*$ introduced above has some nice features, but it may also have a counter-intuitive behaviour in some cases. For instance, let $T = \{\overline{0.3} \equiv P\varphi, \overline{0.4} \equiv P\varphi, \overline{0.6} \equiv P\psi\}$, where φ and ψ are assumed to be propositional variables, then $Con(T) = 0.95$, and hence $T \mid\!\approx^* \overline{0.35} \equiv P\varphi$, but strangely enough, $T \not\mid\!\approx^* \overline{0.6} \equiv P\psi$, since we can only derive $T \mid\!\approx^* \overline{0.95} \to (\overline{0.6} \equiv P\psi)$, even though the formula $\overline{0.6} \equiv P\psi$ is not involved in the conflict in T. The reason is that $Con(T)$ is a global measure that does not take into account individual formulas. Actually, if $T' = T \cup \{\overline{0.7} \equiv P\psi\}$, we still have $Con(T) = Con(T') = 0.95$.

The above example motivates the following iterative procedure to come up with more suitable generalised models of an inconsistent theory T. The intuitive idea is to weaken the formulas of a theory, but not more than necessary. To do so, if $Con(T) = \alpha$, the following procedure starts by identifying minimal subsets of formulas that yield the same consistency degree α than T. Call the complement of that set $T^>$. Then the process starts over again replacing T by $T^>$, but we restrict ourselves to the α-generalised models of T to compute $Con(T^>)$. The procedure iteratively continues until one cannot find a non-empty subset of formulas with a consistency degree higher than the one found in the previous step.

Step 1: Let $Con(T) = \alpha_1$. Then we know that the set of probabilities $[\![T]\!]_{\alpha_1}$ is non-empty. Hence, we can partition T in the following two disjoint subtheories:

$T^= = \cup\{S \subseteq T \mid S \text{ minimal}, Con(S) = \alpha_1\}$

$T^> = T \setminus T^=$

Note that $T^= \neq \emptyset$ and if $T^> \neq \emptyset$ then $Con(T^>) > \alpha_1$. By definition $T^= \cap T^> = \emptyset$ and $T = T^= \cup T^>$.

Then we proceed to weaken only those formulas in $T^=$, so we define:

$T^{(1)} = \{\overline{\alpha_1} \to \Phi \mid \Phi \in T^=\}$.

If $T^> = \emptyset$, then we stop and we define the repaired theory as $T^w = T^{(1)}$. Otherwise we follow to the next step to repair $T^>$.

Step 2: Restrict the set of possible models to those of $[\![T]\!]_{\alpha_1}$ to compute the consistency degree of $T^>$.

Let $Con_T(T^>) = \max\{\beta \mid \text{there exists } \mu \in [\![T]\!]_{\alpha_1}, e_\mu(\Phi) \geq \beta \text{ for all } \Phi \in T^>\} = \alpha_2$.

By definition, $\alpha_2 > \alpha_1$. And we proceed similarly as above, but restricting the set of models to those in $[\![T]\!]_{\alpha_1}$, and we partition $T^>$ into the following two subtheories:

$(T^>)^= = \cup\{S \subseteq T^> \mid \text{ minimal}, Con_T(S) = \alpha_2\}$

$(T^>)^> = T^> \setminus (T^>)^=$

Again note that $(T^>)^= \neq \emptyset$, and if $(T^>)^> \neq \emptyset$ then $Con((T^>)^>) > \alpha_2$. We proceed to the weakening of the subtheory $(T^>)^=$ and define:

$T^{(2)} = \{\overline{\alpha_2} \to \Phi \mid \Phi \in (T^>)^=\}$

If $(T^>)^> = \emptyset$, then we stop and we define the repaired theory as $T^w = T^{(1)} \cup T^{(2)}$. Otherwise we follow to the next step to repair $(T^>)^>$.

Step 3: Restrict the set of possible models to those of $[\![T]\!]_{\alpha_1} \cap [\![T^>]\!]_{\alpha_2}$ to compute the consistency degree of $(T^>)^>$:

Let $Con_{T,T^>}((T^>)^>) = \max\{\beta \mid \text{there exists } \mu \in [\![T]\!]_{\alpha_1} \cap [\![T^>]\!]_{\alpha_2}, e_\mu(\Phi) \geq \beta \text{ for all } \Phi \in (T^>)^>\} = \alpha_3$.

By definition, $\alpha_3 > \alpha_2 > \alpha_1$. we then follow the same procedure as above, but restricting the set of models to those in $[\![T]\!]_{\alpha_1} \cap [\![T^>]\!]_{\alpha_2}$, and we partition $(T^>)^>$ into the following two subtheories:

$((T^>)^>)^= = \cup\{S \subseteq (T^>)^> \mid S \text{ minimal}, Con_{T,T^>}(S) = \alpha_3\}$

$((T^>)^>)^> = (T^>)^> \setminus ((T^>)^>)^=$

Now we proceed to weaken the subtheory $((T^>)^>)^=$ and define:

$T^{(3)} = \{\overline{\alpha_3} \to \Phi \mid \Phi \in ((T^>)^>)^=\}$.

If $((T^>)^>)^> = \emptyset$, then we stop and we define the repaired theory as $T^w = T^{(1)} \cup T^{(2)} \cup T^{(3)}$. Otherwise we follow to the next step to repair $((T^>)^>)^>$.

... ...

This procedure goes on until, for a first m, $(...(T^>) \overset{m}{\cdots})^>) = \emptyset$. Then the procedure stops and as a result we get a (finite) sequence of subtheories $T^{(1)}, T^{(2)}, \ldots, T^{(m)}$, with associated consistency values $\alpha_1 < \ldots < \alpha_m$.

Lemma 8. *The theory $T^w = T^{(1)} \cup \ldots \cup T^{(m)}$ is consistent.*

Proof. Indeed, by construction, $[\![T^{(1)}]\!] \cap [\![T^{(2)}]\!] \cap \ldots \cap [\![T^{(m)}]\!] \neq \emptyset$. □

This allows us to define a refined variant of the $\approx^*$ consequence relation.

Definition 6. Let T be a theory over FP(RPL). Then we define a refinement $\approx^\circ$ of the consequence relation $\approx^*$ as follows:

$$T \approx^\circ \Phi \text{ if } T^w \vdash_{FP} \Phi.$$

Compare this definition with the characterisation of $\approx^*$ in Prop. 5. It is clear that $\approx^\circ$ is stronger than $\approx^*$ while still paraconsistent.

Example 4. Let $T = \{\overline{0.3} \equiv P\varphi, \overline{0.4} \equiv P\varphi, \overline{0.6} \equiv P\psi, \overline{0.8} \equiv P\psi, \overline{0.7} \equiv P\chi\}$, where φ, ψ, χ are propositional variables. Since $Con(T) = 0.9$, we have

$$T_{0.9} = \{\overline{0.9} \to (\overline{0.3} \equiv P\varphi), \overline{0.9} \to (\overline{0.4} \equiv P\varphi), \overline{0.9} \to (\overline{0.6} \equiv P\psi), \\ \overline{0.9} \to (\overline{0.8} \equiv P\psi), \overline{0.9} \to (\overline{0.7} \equiv P\chi)\}$$

Models of $T_{0.9}$ are probabilities μ such that $\mu(\varphi) \in [0.2, 0.4] \cap [0.3, 0.5] = [0.3, 0.4]$, $\mu(\psi) \in [0.5, 0.7] \cap [0.7, 0.9] = \{0.7\}$ and $\mu(\chi) \in [0.6, 0.8]$.

Let us see what the refinement procedure yields:

Step 1: $Con(T) = 0.9$

$T^{=} = \{\overline{0.6} \equiv P\psi, \overline{0.8} \equiv P\psi\}$

$T^{>} = \{\overline{0.3} \equiv P\varphi, \overline{0.4} \equiv P\varphi, \overline{0.7} \equiv P\chi\}$

$T^{(1)} = \{\overline{0.9} \to (\overline{0.6} \equiv P\psi), \overline{0.9} \to (\overline{0.8} \equiv P\psi)\}$

Step 2: $Con_T(T^{>}) = 0.95$

$(T^{>})^{=} = \{\overline{0.3} \equiv P\varphi, \overline{0.4} \equiv P\varphi\}$

$(T^{>})^{>} = \{\overline{0.7} \equiv P\chi\}$

$T^{(2)} = \{\overline{0.95} \to (\overline{0.3} \equiv P\varphi), \overline{0.95} \to (\overline{0.4} \equiv P\varphi)\}$

Step 3: $Con_T((T^{>})^{>}) = 1$

$(T^{>})^{=} = \{\overline{0.7} \equiv P\chi\}$

$(T^{>})^{>} = \emptyset$

$T^{(3)} = \{\overline{0.7} \equiv P\chi\}$

Therefore,

$$T^w = \{\overline{0.9} \to (\overline{0.6} \equiv P\psi), \overline{0.9} \to (\overline{0.8} \equiv P\psi), \overline{0.95} \to (\overline{0.3} \equiv P\varphi),$$
$$\overline{0.95} \to (\overline{0.4} \equiv P\varphi), \overline{0.7} \equiv P\chi\},$$

that is equivalent to the theory

$$T'^w = \{\overline{0.5} \to P\psi, P\psi \to \overline{0.7}, \overline{0.7} \to P\psi, P\psi \to \overline{0.9}, \overline{0.25} \to P\varphi,$$
$$P\varphi \to \overline{0.35}, \overline{0.35} \to P\varphi, P\varphi \to \overline{0.45}, \overline{0.7} \equiv P\chi\}.$$

In this case, models of T^w are probabilities μ such that $\mu(\varphi) = 0.35$, $\mu(\psi) = 0.7$ and $\mu(\chi) = 0.7$, and hence the refined consequence relation $\approx^\circ$ is such that:

$$T \mathrel{|\!\!\approx^\circ} \overline{0.7} \equiv P\psi, \overline{0.35} \equiv P\varphi, \overline{0.7} \equiv P\chi.$$

8 Related approaches

Several papers and monographs about how to measure the inconsistency of probabilistic knowledge bases have recently appeared in the literature, see for instance [46, 38, 47, 41, 42, 17, 43, 44]. In particular, there is a nice overview by de Bona, Finger, Potyka and Thimm in [18] on which we will base the comparison with our approach.

First of all, by a *probabilistic knowledge base* one usually understands a finite set of (conditional) probability constraints on classical propositional formulas (from a given finitely generated language $\mathcal{L}$), of the form $KB = \{(\varphi_i \mid \psi_i)[\underline{q}_i, \overline{q}_i] \mid i = 1, \ldots n\}$, where $\underline{q}_i$ and $\overline{q}_i$ are rational values from the unit interval $[0, 1]$.

Such an expression $(\varphi_i \mid \psi_i)[\underline{q}_i, \overline{q}_i]$ intuitively expresses the constraint (or belief) that the conditional probability of φ_i given ψ_i lies in the interval $[\underline{q}_i, \overline{q}_i]$.

Then, a probability μ on the formulas satisfies a conditional expression $(\varphi_i \mid \psi_i)[\underline{q}_i, \overline{q}_i]$, written $\mu \models (\varphi_i \mid \psi_i)[\underline{q}_i, \overline{q}_i]$, whenever $\mu(\varphi_i \wedge \psi_i) \geq \underline{q}_i \cdot \mu(\psi_i)$, and $\mu(\varphi_i \wedge \psi_i) \leq \overline{q}_i \cdot \mu(\psi_i)$. We call such a probability μ a *model* of the formula $(\varphi_i \mid \psi_i)[\underline{q}_i, \overline{q}_i]$. Of course, if $\mu(\psi_i) > 0$, these conditions amount to state that $\mu \models (\varphi_i \mid \psi_i)[\underline{q}_i, \overline{q}_i]$ when the conditional probability $\mu(\varphi_i \mid \psi_i)$ belongs to the interval $[\underline{q}_i, \overline{q}_i]$.

For the case where a probabilistic knowledge base KB is inconsistent, i.e. when there is no probability map satisfying all the expressions in it, a number of *inconsistency measures* have been proposed in the literature. In particular, different proposals measure how inconsistent a KB can be, some of them generalising to the probabilistic case inconsistency measures already proposed for the propositional case, and some of them specifically tailored to deal with probabilistic expressions. Among them, one finds the so-called *distance-based measures* and *violation-based measures*. Roughly speaking, the former look for consistent knowledge bases that *minimize the distance* (for some suitable notion of distance) to the original inconsistent KB, while the latter look for probabilities that *minimize the violation* (for some suitable notion of violation) of the knowledge base [47, 41].

According to [18], when it comes to reasoning with an inconsistent probabilistic KB, there are two sensible ways to proceed: either repair the inconsistent knowledge base and then apply classical probabilistic reasoning, or apply paraconsistent reasoning models that can deal with inconsistent knowledge bases. For the first approach, distance-based measures are well-suited, while for the second approach violation-based measures (together with so-called fuzzy-based measures) seem to be the most appropriate ones.

We show here that our approach to reason with inconsistent probabilistic theories over FP(RPL) described in the previous section, when restricted to theories of the form $T = \{P\varphi_i \equiv \overline{r_i}\}_{i=1,\ldots,n}$, can be seen both as a distance-based approach and as violation-based approach. Note that we do not deal with conditional probability expressions, thus our case is in this respect simpler.

In the distance-based approach, given a distance d on $\mathbb{R}^n$, and two theories $T = \{P\varphi_i \equiv \overline{r_i}\}_{i=1,\ldots,n}$ and $T' = \{P\varphi_i \equiv \overline{r'_i}\}_{i=1,\ldots,n}$, one can define the distance between T and T' as the distance between their corresponding vectors of truth-constants:

$$d(T, T') = d((r_1, \ldots, r_n), (r'_1, \ldots, r'_n)).$$

Then, if $T = \{\overline{r_i} \equiv P\varphi_i\}_{i=1,\ldots,n}$ is an inconsistent theory, the aim is to look for a consistent theory (a *repair*), by modifying the truth-constants r_i's, so to be at a minimum distance from T. Note that all possible repairs of T that are precise-assignments theories are of the form

$$T_\mu = \{\overline{\mu(\varphi_i)} \equiv P\varphi_i\}_{i=1,\ldots,n},$$

for μ being a rational-valued probability on formulas. In our approach, the degree of inconsistency of T can be seen as the minimum distance from T to the set of all its repairs, indeed we have:

$$Incon(T) = \inf_{\mu} \bigvee_{i=1,n} |\mu(\varphi_i) - r_i| =$$

$$= \inf_{\mu} d_c((\mu(\varphi_1), \ldots, \mu(\varphi_n)), (r_1, \ldots, r_n)) = \inf_{\mu} d_c(T, T_\mu),$$

where d_c is the well-known Chebyshev distance in $\mathbb{R}^n$. That is to say, $Incon(T)$ is nothing but the Chebyshev distance of the point in $[0,1]^n$ given by the (inconsistent) assignment $(r_1, \ldots, r_n)$ to the convex set of coherent probability assignments to the events $\varphi_1, \ldots, \varphi_n$. The set of precise repairs at minimal distance from T is then

$$Repairs(T) = \{T_\mu \mid \mu \text{ probability}, d_c(T, T_\mu) = Incon(T)\}.$$

We observe that this set may contain more than one theory.

Suppose now that T represents an imprecise probability assignment

$$T = \{(\overline{r_i - \epsilon_i} \to P\varphi_i) \wedge (P\varphi_i \to \overline{r_i + \epsilon_i})\}_{i=1,\ldots,n}$$

where, for each i, $r_i - \epsilon_i \geq 0$ and $r_i + \epsilon_i \leq 1$, that is $\epsilon_i \leq r_i \leq 1 - \epsilon_i$. Then, as shown in Example 3, the degree of inconsistency of T is:

$$Incon(T) = \inf_{\mu} \bigvee_{i=1,n} (1 - \epsilon_i) \otimes |r_i - \mu(\varphi_i)|.$$

Therefore, by defining $d_c^*(T, T_\mu) = \bigvee_{i=1,n} (1 - \epsilon_i) \otimes |r_i - \mu(\varphi_i)|$, we can write

$$Incon(T) = \inf_{\mu} d_c^*(T, T_\mu).$$

Note that the definition of $d_c^*(T, T_\mu)$ is similar to the one of $d_c(T, T_\mu)$ but takes into account the width of the probability intervals assigned to the events in T. However, d_c^* is not symmetric in its arguments since T is in general an imprecise assignment theory, while T_μ is a precise assignment theory. The question is then whether d_c^* can still be considered as a kind of distance. What we can say in this respect is that: (i) in the particular case T is a precise assignment theory, then all the ϵ_i's are zero, and thus $d^*(T, T_\mu) = d_c(T, T_\mu)$; and (ii) a restricted form of the triangle inequality holds, as the next lemma shows.

Lemma 9. *For any imprecise assignment theory T and any probabilities μ and σ, the following condition holds:*

$$d^*(T, T_\mu) \leq d^*(T, T_\sigma) + d^*(T_\sigma, T_\mu).$$

Proof. We first show that

$$(1 - \epsilon_i) \otimes |r_i - \mu(\varphi_i)| \leq ((1 - \epsilon_i) \otimes |r_i - \sigma(\varphi_i)|) + |\sigma(\varphi_i) - \mu(\varphi_i)|. \tag{4}$$

Clearly, $|r_i - \mu(\varphi_i)| \leq |r_i - \sigma(\varphi_i)| + |\sigma(\varphi_i) - \mu(\varphi_i)|$, and since $|r_i - \mu(\varphi_i)| \leq 1$, it also holds that $|r_i - \mu(\varphi_i)| \leq |r_i - \sigma(\varphi_i)| \oplus |\sigma(\varphi_i) - \mu(\varphi_i)|$. Therefore, to show (4), it is enough to show the following inequality for every $x, y, z \in [0, 1]$:

$$x \otimes (y \oplus z) \leq (x \otimes y) \oplus z.$$

But we know that $y \leq (x \to (x \otimes y))$, and hence $y \oplus z \leq (x \to (x \otimes y)) \oplus z$, and moreover by a simple computation, it can be checked that $(x \to (x \otimes y)) \oplus z = x \to ((x \otimes y) \oplus z)$. Therefore, $y \oplus z \leq x \to ((x \otimes y) \oplus z)$, and this holds iff $x \otimes (y \oplus z) \leq (x \otimes y) \oplus z$. So we have shown (4). Finally, we have

$$d^*(T, T_\mu) = \bigvee_{i=1,n} (1-\epsilon_i) \otimes |r_i - \mu(\varphi_i)| \leq \bigvee_{i=1,n} ((1-\epsilon_i) \otimes |r_i - \sigma(\varphi_i)|) + |\sigma(\varphi_i) - \mu(\varphi_i)|$$

$$\leq \bigvee_{i=1,n} (1 - \epsilon_i) \otimes |r_i - \sigma(\varphi_i)| + \bigvee_{i=1,n} |\sigma(\varphi_i) - \mu(\varphi_i)| = d^*(T, T_\sigma) + d^c(T_\sigma, T_\mu).$$

□

From all the above, it is clear that $Incon(\cdot)$ belongs to the family of distance-based inconsistency measures.

On the other hand, in our setting, for a given inconsistent theory T over FP(RPL), a violation-based inconsistency measure aims at: first, estimating how far every interpretation (i.e. every probability) is from satisfying every formula in T (violation degrees); then, minimising a suitable aggregation of those degrees. We can show that $Incon(\cdot)$ is in fact a violation-based measure as well. Indeed, given a probability μ, we define the violation degree of a formula $\Phi \in T$ by μ as the satisfaction degree of its negation, i.e.

$$vd_\mu(\Phi) = e_\mu(\neg\Phi) = 1 - e_\mu(\Phi),$$

and then we define the global violation degree of T as $vd_\mu(T) = \max_{\Phi \in T} vd_\mu(\Phi)$. Finally, according to Lemma 5, it is straightforward to check that

$$Incon(T) = \inf_\mu dv_\mu(T),$$

that is, $Incon(T)$ is nothing but the infimum of the violation degrees of T by all possible probabilities, and the set of *generalised models* of T are those probabilities yielding a minimum violation degree:

$$GMod(T) = \{\mu \text{ probability } \mid dv_\mu(T) = Incon(T)\} = \|T\|_{Con(T)}.$$

Moreover, we can show that, in our particular case, the set of consequences entailed by the set of generalised models in fact coincides with the common consequences of all theories in $Repairs(T)$.

Proposition 6. *For a precise-assignment theory T, we have:*

$$GMod(T) \subseteq \|\Phi\| \text{ iff for all } T_\mu \in Repairs(T),\ T_\mu \vdash_{RPL} \Phi.$$

Proof. Let $T = \{\overline{r_i} \equiv P\varphi_i\}_{i=1,\ldots,n}$. Assume that $GMod(T) \subseteq \|\Phi\|$, that is, for every $\mu \in GMod(T) = \|T\|_{Con(T)}$, $e_\mu(\Phi) = 1$. Now, let μ be a probability such that $T_\mu \in Repairs(T)$. This means that $d_c(T, T_\mu) = Incon(T)$, hence $|r_i - \mu(\varphi_i)| \geq \alpha$ for all i, that is equivalent to $e_\mu(\overline{r_i} \equiv \varphi_i) \geq \alpha$ for all i, i.e. $\mu \in GMod(T)$. Let σ be a probability such that $\sigma \in \|T_\mu\|$. This implies that $\sigma(\varphi_i) = \mu(\varphi_i)$ for all i, hence $\sigma \in GMod(T)$ as well, and by hypothesis, $e_\sigma(\Phi) = 1$. Therefore, by completeness of RPL, $T_\mu \vdash_{RPL} \Phi$.

The other direction is similar. □

9 Some remarks on Bueno-Soler and Carnielli's approach

In the present paper, we have been concerned so far with a Łukasiewicz logic-based formalisation of probabilistic reasoning, and two approaches to paraconsistent inference from inconsistent theories in that context. An alternative approach considered by Bueno-Soler and Carnielli in [5] is to formalise a probability logic over a paraconsistent logic of events, that is, by replacing classical logic as internal logic of events by a suitable paraconsistent logic. In fact, the authors consider the logic of formal inconsistency **Ci** [7], which is an expansion of the positive fragment of classical propositional logic with two unary operators. Namely, a paraconsistency negation and a consistency operator, over which a suitable notion of probability is defined. The logic **Ci** can be basically introduced as follows:

(i) Let CPL^+ be the Positive fragment of classical propositional logic over the signature $\Sigma^+ = \{\wedge, \vee, \to\}$

(ii) Then the minimal paraconsistent logic **mbC** on the expanded signature $\Sigma = \Sigma^+ \cup \{\neg, \circ\}$ can be axiomatically defined as the extension CPL^+ plus the following two axioms:

(Ax10) $\varphi \vee \neg\varphi$

(bc1) $\circ\varphi \to (\varphi \to (\neg\varphi \to \psi))$

(iii) Finally, the logic **Ci** is defined as the extension of **mbC** with the following two additional axioms:

(ci) $\neg\circ\varphi \to (\varphi \wedge \neg\varphi)$

(cf) $\neg\neg\varphi \to \varphi$

However, as discussed in [6], **Ci** and other paraconsistent logics with non-deterministic semantics have the drawback that they do not satisfy the replacement property, and hence they are not self-extensional and not algebraizable in the sense of Blok and Pigozzi. To remedy this situation, in [6] the authors study the extension of those logics with suitable rules of replacement of equivalents.

(iv) The logic **RmbC** is defined as the extension of **mbC** with replacement rules for $\circ$ and $\neg$:

$$(R_\neg)\ \frac{\varphi \leftrightarrow \psi}{\neg\varphi \leftrightarrow \neg\psi} \qquad (R_\circ)\ \frac{\varphi \leftrightarrow \psi}{\circ\varphi \leftrightarrow \circ\psi}$$

(v) Then, the logic **RCi** is just the extension of **RmbC** with the axioms (ci) and (cf).

Then, in [7] the authors show that **RCi** is self-extensional, and it admits an algebraic semantics with respect to a subvariety of Boolean algebras with LFIs (BALFIs), i.e. Boolean algebras with two extra unary operators $\neg$ and $\circ$ satisfying the equations corresponding to (Ax10), (bc1), (ci) and (cf). In fact, a completeness result is proved with respect to the degree-preserving semantics for evaluations on the class of corresponding BALFIs, namely, $\Gamma \vdash_{RCi} \varphi$ iff either φ is a tautology or there exists a set $\{\psi_1, ..., \psi_n\} \subseteq \Gamma$ such that $\psi_1 \wedge ... \wedge \psi_n \to \varphi$ is a tautology in the corresponding subvariety of BALFI algebras.

Therefore, analogously to FP(Ł) or FP(RPL), one could consider the task of defining a fuzzy probability logic over events formalised as propositions in **RCi**. For this, one first needs to specify what a probability function over **RCi** is. In [5] (see also [40]) there is a general definition of a probability P over formulas for a given logic. Here we particularise that definition for the case of **RCi**. Let $\mathcal{L}$ be the set of **RCi** formulas, then a (paraconsistent) probability on $\mathcal{L}$ is a mapping $P : \mathcal{L} \to [0, 1]$ fulfilling the following properties:

- Non-negativity: $0 \leq P(\varphi) \leq 1$, for all $\varphi \in \mathcal{L}$
- Tautologicity: If $\vdash_{RCi} \varphi$, then $P(\varphi) = 1$
- Anti-tautologicity: If $\varphi \vdash_{RCi} \psi$ for all ψ then $P(\varphi) = 0$
- Comparison: If $\varphi \vdash_{RCi} \psi$, then $P(\varphi) \leq P(\psi)$
- Finite additivity: $P(\varphi \vee \psi) = P(\varphi) + P(\psi) - P(\varphi \vee \psi)$

Once we adopt this definition, one can proceed to axiomatically define the (fuzzy) probability logic over **RCi**, that we will denote FP(**RCi**,RPL), emphasizing the fact that the logic of events is **RCi** and the outer logic keeps being RPL. Everything is analogous to the case of FP(RPL) with the obvious changes. In particular, non-modal formulas will be those of **RCi** (φ, ψ, χ, etc.), and modal formulas are built from elementary modal formulas of the form $P\varphi$, where φ is a non-modal formula, using the connectives and truth-constants of RPL (Φ, Ψ, etc.). The axioms of the logic FP(**RCi**,RPL) will be the following:

(i) Axioms and rules of **RCi** for non-modal formulas

(ii) Axioms of RPL for modal formulas

(iii) Probabilistic modal axioms:

(FP0')	$P\varphi$, for φ being a theorem of **RCi**
(FP1')	$P\varphi \to P\psi$, for $\varphi \to \psi$ being a theorem of **RCi**
(FP2')	$\neg P(\varphi \wedge \neg\varphi \wedge \circ\varphi)$
(FP3')	$P(\varphi \vee \psi) \equiv P\varphi \oplus (P\psi \ominus P(\varphi \wedge \psi))$

The only deduction rule of FP(**RCi**,RPL) is *modus ponens*, both for non-modal formulas (wrt the $\to$ of **RCi**) and for modal formulas (wrt the $\to$ of RPL).

The semantics for modal formulas is as for FP(Ł) with the required modification: for each probability on **RCi**-formulas μ, the evaluation of basic modal formulas $P\varphi$ is defined as $e_\mu(P\varphi) = \mu(\varphi)$, and then it is extended to compound modal formulas using the RPL truth-functions. Then, it is clear that deductions from theories are sound wrt the intended semantics given by evaluations e_μ's. Indeed, an RPL-evaluation e is a model of axioms (FP0')-(FP3') iff the mapping $\mu : \mathcal{L} \to [0,1]$ defined as $\mu(\varphi) = e(P\varphi)$ is a probability on $\mathcal{L}$.

Since it is not currently known to the authors of this chapter whether **RCi** is locally finite or not, we can only make the following remarks regarding the issue of completeness for FP(**RCi**,RPL). If it is the case that **RCi** is locally finite, then the finite strong completeness proof for FP(Ł) easily generalises to FP(**RCi**,RPL).[6] Otherwise, if it is indeed the case that **RCi** is not locally finite, we can always extend FP(**RCi**,RPL) with the infinitary inference rule (IR) (see Section 3) and obtain a (infinitary) logical system strongly complete with respect to 'paraconsistent' probabilities on **RCi**.

10 Conclusions and future work

In this paper we have discussed some initial steps towards reasoning with inconsistent probabilistic theories over classical events, within the setting of the probabilistic logic FP(RPL) defined on top of the $[0,1]$-valued Łukasiewicz fuzzy logic enriched with rational truth-constants. We have explored two approaches. A first approach amounts to replace the logic RPL, that is explosive, by its paraconsistent companion RPL$^{\leq}$. A second one consists of suitably weakening the formulas in an inconsistent theory T, depending on the degree of inconsistency of T. We have also explored the possibility of using the fuzzy logic approach to reason about probability on top of the paraconsistent logic **RCi** in the line of [5].

As for future work, we plan, in particular, to generalise the above approaches to deal with inconsistent theories about conditional probabilities. In order to do this, one would need to replace the underlying Łukasiewicz logic by a more powerful system such as the $\text{Ł}\Pi\frac{1}{2}$ logic, which combines connectives from Łukasiewicz logic and Product fuzzy logic [29]. On the other hand, the question of whether these fuzzy logic-based approaches are able to deal with inconsistent theories in the frame of other uncertainty measures (like belief functions or necessity and possibility measures) deserves to be part of future work as well.

[6] Remember that Ł, and thus RPL as well, is not strong complete, it is only complete for deductions from finite theories; but they are locally finite and this is what is used in the completeness proof for FP(Ł) and FP(RPL).

Dedication

This contribution is a humble homage to our colleague and friend Walter Carnielli to celebrate his outstanding contributions in many aspects of logic along his successful academic career, and we are sure there are still many to come.

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