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On some Implication-free Fragments of Substructural and Fuzzy Logics

Àngel García-Cerdaña

Foreword by Francesc Esteva and Ventura Verdú

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A mi padre, Francisco García Ruiz,
mi primer maestro de matemáticas

A mi madre, Fernanda Cerdaña Nadal,
mi primera maestra de arte

A Àngela Martínez y Julia Rico

A mi hermana Pespe

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Foreword

Mathematical fuzzy logics and substructural logics share many common logical systems and ideas. Mathematical fuzzy logic is devoted mainly to the study of multiple-valued logical systems with two conjunctions which are complete with respect to the algebraic structures defined over [0,1] by a t-norm and its residuum. In this regard it follows the tradition of infinite multiple-valued systems defined in the 1950s, i.e., Łukasiewicz infinite valued and Gödel-Dummett logics, which are complete with respect to the algebras over [0,1] defined by the Łukasiewicz and the minimum t-norms respectively. Except for the Gödel logic, these systems are substructural since they do not satisfy the structural rule of contraction. In fact the paper [EGGC03], coauthored by the author of this book, was (to our knowledge) the first to relate the hierarchy of fuzzy logics with the hierarchy of logics without contraction given by H.Ono in his monograph [Ono]. But substructural logics contain a very large number of logical systems (in addition to the logical systems without contraction) since they also contain systems without the structural rules of exchange or weakening.

Recently, many papers on substructural and fuzzy logics have been published but little attention has been paid to the study of the fragments without implication of the substructural logical systems, which are the main goal of this monograph. In our opinion, this study has the interest of establishing the individual role of the connectives of fusion (or multiplicative conjunction) and negation in substructural and fuzzy logics. We highlight the study of the algebraization of the Gentzen systems obtained from the Full Lambek calculus without implication when adding some structural rules. The study of the algebras that are the corresponding algebraic counterpart includes a number of results concerning subreducts that are obtained using the method of ideal completion of algebras. These results are then used in order to characterize algebraically the fragments of the analyzed Gentzen systems and the fragments of their associated external deductive systems. Finally, we should mention a surprising result concerning fragments in the case of systems without contraction (but with exchange and weakening): their fragments without negation and without implication are exactly the same as those found in classical logic. In other words, the connectives of implication and negation are the ones that contain both the properly substructural and fuzzy aspects of the logics.

How he explains in the preface, Àngel was remote of the scientific work during twenty years in which he dedicated himself to the theater. His personal history shows that it is never too late to devote oneself to research. His excellent work is reflected in his publishing record and in this book. He has a solid background in logic and in algebra, an open mind, and an ability to work together with other researchers, as the list of the papers he has published demonstrates. Finally I would stress his skill in clarifying concepts, summarizing dispersed results, and unifying and clarifying the nomenclature. This ability to present the material in a very readable way will be of great benefit to readers. We trust that this monograph will become a reference in the topic and will stimulate new research.

Francesc Esteva and Ventura Verdú
Bellaterra, Barcelona, October 2008

Abstract

The main contents of this monograph consist of some contributions to the general study of basic substructural systems (Part II), and an algebraic analysis of some free-implication fragments of basic intuitionistic substructural logics (Part III). The tools used stand within the framework of Algebraic Logic and, partially, in the area of Abstract Algebraic Logic and the notion of algebraizable Gentzen system.

The general context of our research is placed in the basic intuitionistic substructural Gentzen systems $\mathcal{FL}_\sigma$ defined by the calculi $\mathbf{FL}_\sigma$, that is, the Full Lambek Calculus extended with the structural rules codified by the sequence σ, that is, *exchange* (e), *left weakening* (w_l), *right weakening* (w_r) or *contraction* (c). The main goal of our research is to analyze the fragments without implications of the external deductive systems $\mathfrak{e}\mathcal{FL}_\sigma$ associated with systems $\mathcal{FL}_\sigma$ in all the languages containing the connectives of additive disjunction $(\vee)$, multiplicative conjunction $(*)$, 0 and 1. Let Ψ be one of the languages $\langle \vee, *, 0, 1\rangle$, $\langle \vee, \wedge, *, 0, 1\rangle$, $\langle \vee, *, {}^{\backprime}, {}', 0, 1\rangle$, or $\langle \vee, \wedge, *, {}^{\backprime}, {}', 0, 1\rangle$ (connectives ${}^{\backprime}$ and ${}'$ are the right negation and the left negation, respectively). Fixed a sequence σ (possibly empty), let $\mathcal{FL}_\sigma[\Psi]$ be the Gentzen system defined by the calculus $\mathbf{FL}_\sigma[\Psi]$ obtained by dropping from $\mathbf{FL}_\sigma$ the rules for the connectives that are not in Ψ. We prove that these subsystems are fragments of $\mathcal{FL}_\sigma$ in the corresponding languages. This fact entails, as an immediate consequence, that the external systems $\mathfrak{e}\mathcal{FL}_\sigma[\Psi]$ associated with $\mathcal{FL}_\sigma[\Psi]$ are fragments of $\mathfrak{e}\mathcal{FL}_\sigma$. The method used to achieve our goal is the following:

- We introduce the varieties of algebras $\mathring{\mathbb{M}}^{s\ell}_\sigma$ (semilatticed pointed monoids), $\mathring{\mathbb{M}}^{\ell}_\sigma$ (latticed pointed monoids), $\mathbb{PM}^{s\ell}_\sigma$ (semilatticed pseudocomplemented monoids), $\mathbb{PM}^{\ell}_\sigma$ (latticed pseudocomplemented monoids). In these acronyms subindex σ is a subsequence of ew_lw_rc and symbols e, w_l, w_r and c codify what we refer to as (algebraic) exchange, right-weakening, left-weakening and contraction properties, respectively. Such properties, which are expressed by quasi-inequations, are equivalent, respectively, to the algebraic properties of commutativity, integrality, 0-boundednes, and increasing idempotency.

- Using the method of ideal-completion applied to the considered classes of algebras, we prove that, for every σ, these classes of algebras are the subreducts in the corresponding languages of the class $\mathbb{FL}_\sigma$, i.e.,the variety of pointed residuated lattices defined by the equations codified by σ.

- We prove that subsystems $\mathcal{FL}_\sigma[\vee, *, 0, 1]$, $\mathcal{FL}_\sigma[\vee, \wedge, *, 0, 1]$, $\mathcal{FL}_\sigma[\vee, *, \backprime, \prime, 0, 1]$ and $\mathcal{FL}_\sigma[\vee, \wedge, *, \backprime, \prime, 0, 1]$ are algebraizable, having as respective equivalent algebraic semantics the varieties $\mathring{\mathbb{M}}^{s\ell}_\sigma$, $\mathring{\mathbb{M}}^{\ell}_\sigma$, $\mathbb{PM}^{s\ell}_\sigma$, $\mathbb{PM}^{\ell}_\sigma$. We also prove that the system $\mathcal{FL}_\sigma$ is algebraizable with equivalent algebraic semantics the variety $\mathbb{FL}_\sigma$.

- Finally, using these results of algebraization and the ones concerning subreducts we obtain that the systems $\mathcal{FL}_\sigma[\vee, *, 0, 1]$, $\mathcal{FL}_\sigma[\vee, \wedge, *, 0, 1]$, $\mathcal{FL}_\sigma[\vee, *, \backprime, \prime, 0, 1]$ and $\mathcal{FL}_\sigma[\vee, \wedge, *, \backprime, \prime, 0, 1]$ are fragments of $\mathcal{FL}_\sigma$, and that the corresponding external deductive systems are fragments of $\mathfrak{e}\mathcal{FL}_\sigma$.

We also show that each system $\mathcal{FL}_\sigma$ is equivalent to its associated external deductive system. However, it is shown that the fragments considered are not equivalent to any deductive system. Moreover we show that $\mathfrak{e}\mathcal{FL}_\sigma[\vee, *, 0, 1]$, $\mathfrak{e}\mathcal{FL}_\sigma[\vee, \wedge, *, 0, 1]$, $\mathfrak{e}\mathcal{FL}_\sigma[\vee, *, \backprime, \prime, 0, 1]$ and $\mathfrak{e}\mathcal{FL}_\sigma[\vee, \wedge, *, \backprime, \prime, 0, 1]$ are not protoalgebraic but they have, respectively, the varieties $\mathring{\mathbb{M}}^{s\ell}_\sigma$, $\mathring{\mathbb{M}}^{\ell}_\sigma$, $\mathbb{PM}^{s\ell}_\sigma$, $\mathbb{PM}^{\ell}_\sigma$ as algebraic semantics with defining equation $1 \vee p \approx p$.

Finally, we analyze the fragments in the languages $\langle \vee, *, 0, 1 \rangle$, $\langle \vee, \wedge, *, 0, 1 \rangle$ and $\langle \vee, \wedge, 0, 1 \rangle$ in the context of $\mathcal{FL}_{ew}$. We obtain as main result that each one of these fragments coincides with the fragment in the same language of classical logic. Since MTL, the most general t-norm based fuzzy logic, is an axiomatic extension of $\mathfrak{e}\mathcal{FL}_{ew}$, we have as a corollary that the fragments in these three languages of all the t-norm based fuzzy logics are equal to the corresponding fragments of classical logic.

Volver

> *Volver con la frente marchita*
> *las nieves del tiempo platearon mi sien.*
> *Sentir que es un soplo la vida,*
> *que veinte años no es nada,*
> *que febril la mirada*
> *errante en la sombra te busca y te nombra.*
>
> Carlos Gardel y Alfredo Le Pera

Vaig acabar la llicenciatura de Matemàtiques el setembre de 1976. Estava en els darrers anys de carrera quan em vaig apuntar a les *Jornadas de Investigación en Matemáticas* que es van organitzar a la Universitat Politècnica de Catalunya els mesos de juliol dels anys 75 i 76 i fins i tot vaig arribar a parlar amb el doctor Sales, que era el professor que va impulsar els estudis de lògica a la Facultat de Matemàtiques de Barcelona, per tal que em dirigís una tesina.

En acabar la carrera, vaig començar a treballar com a professor de matemàtiques en una escola de batxillerat a Caldes de Montbui. En aquells temps, la feina i la vida no em deixaven temps per a més i, en conseqüència, l'assumpte de la tesina va quedar en llista d'espera. A més, jo he estat, i sóc, i seré sempre, un bohemi. Hi ha una part de mi, heretada de la meva mare i de la mare de la meva mare que era cupletista, que em fa estimar les bambalines i els llums dels teatres. Així que, a principis dels vuitanta, vaig fer el cor fort, vaig deixar l'escola de Caldes i em vaig presentar a les proves de l'Institut del Teatre de Barcelona. Volia ser actor. I em van acceptar. Vet aquí que l'assumpte de la tesina va perdre bastants llocs en la llista d'espera. La meva relació amb la lògica en els gairebé vint anys següents va quedar reduïda a algunes lectures i poca cosa més i, també, tot s'ha de dir, a l'obsessió per escriure i interpretar monòlegs de teatre en els quals jo feia de professor, amb el nom artístic d'*El Sueco*, fent servir una pissarra que omplia de teoremes amb les seves corresponents demostracions sobre les qüestions més variades. Però aquesta és una altra història...

En un moment de la meva vida, diguem-ne de crisi vital, allà cap a l'any 99, se'm va acudir que potser seria molt saludable recuperar les meves relacions amb la lògica, així que vaig informar-me i em vaig assabentar de l'existència d'un doctorat que impartia el Departament de Lògica, Història i Filosofia de la Ciència de la Universitat de Barcelona. Un bon dia, vaig tocar el timbre del departament i em va obrir la porta en Josep Maria

Font, un antic company de classe. Per allà hi havia en Ventura Verdú, que en l'època en què jo vaig fer el *selectivo*, ell feia tercer. Ens vam posar a xerrar i, llavors, va sortir del seu despatx en Toni Torrens, un altre company de classe. Poc després, atret per la xerrada que s'havia establert entre en Josep Maria, en Ventura, en Toni i jo, va sortir del seu despatx el meu admirat Josep Pla. Em van dir que m'havien vist alguna vegada a la tele i vam parlar dels temps en què jo era a la facultat i, és clar, vam parlar del mestre de tots, el doctor Sales. Els vaig explicar que, feia vint-i-quatre anys havia quedat amb ell per fer una tesina en lògica i que, al meu entendre, havia arribat el moment de tirar endavant el meu propòsit (potser van pensar que jo era una mica lent). De fet, el mestre Sales ja estava jubilat, però allà eren els seus alumnes pel que calgués, així que, després de la xerrada, en Josep Maria em va fer passar al seu despatx i em va informar dels cursos de doctorat i de tots els detalls però, per la raó que sigui, jo ja estava decidit i, si ell hagués estat un venedor de catifes n'hi hauria comprat, com a mínim, mitja dotzena.

Vaig fer els cursos de doctorat i, l'any següent, dos treballs d'investigació tutelats per Ventura Verdú i Francesc Esteva, a qui havia conegut vint-i-quatre anys enrere en les *Jornadas de Investigación Matemática* i que m'havia deixat fascinat perquè ell, en contuberni amb un altre matemàtic, Enric Trillas, flirtejava amb un tema que aleshores em sonava a ciència-ficció: la lògica borrosa! El primer dia que vaig veure en Francesc Esteva va ser a les *Jornadas* del 75, en una conferència sobre negacions en reticles i ara, vint-i-cinc anys més tard, seguia amb el mateix, *dale que dale*: de menú, la lògica borrosa, i tot regat amb el vi d'un afecte especial a les negacions. Per la seva banda, Ventura Verdú, que en el temps de les *Jornadas de Investigación* flirtejava amb les anomenades “lògiques abstractes”, estava ficat en el present en assumptes relacionats amb una cosa que ell anomenava “lògiques subestructurals”.

A hores d'ara, han passat vuit anys des d'aquella visita al departament. I la vida té aquestes coses... Avui em toca presentar una memòria, dirigida per Francesc Esteva i Ventura Verdú, que parla dels aspectes subestructurals de les lògiques borroses fent servir, en expressió del doctor Sales, “l'artilleria” de la Lògica Algebraica Abstracta i posant una especial atenció a les negacions... No!!! (Aquesta negació s'ha de llegir amb expressió de pànic.)

Per acabar aquest prefaci direm que, com és sabut d'aquells que ho saben, les lògiques subestructurals es diferencien de la lògica clàssica en la carència o restricció d'alguna de les seves anomenades regles estructurals, és a dir, les regles de contracció, debilitament i intercanvi. Doncs bé, a propòsit de la presentació d'aquesta memòria, intentarem fer-la *sense contracció* (relaxadament), *sense debilitament* i, sobretot, tenint molt present el meravellós *intercanvi* humà que ha significat per a mi, en aquests vuit anys, el contacte amb l'espècie juganera que són els lògics.

Àngel García Cerdaña *El Sueco*

Acknowledgments

Pero callad.
Quiero deciros algo.
Sólo quiero deciros que estamos todos juntos.
A veces, al hablar, alguno olvida
su brazo sobre el mío,
y yo aunque esté callado doy las gracias,
porque hay paz en los cuerpos y en nosotros.
Quiero deciros cómo trajimos
nuestras vidas aquí, para contarlas.
Largamente, los unos con los otros
en el rincón hablamos, tantos meses!
que nos sabemos bien, y en el recuerdo
el júbilo es igual a la tristeza.
Para nosotros el dolor es tierno.
Ay el tiempo! Ya todo se comprende.

Jaime Gil de Biedma,
de *Las Personas del Verbo*

Gràcies a tots els que m'han donat suport en aquesta aventura. Gràcies als meus pares, a la meva germana Esperança, al seu company Juan, i a dos àngels de la guarda que jo tinc: l'Àngela i la Julia.

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Al meu pare, Francisco García Ruiz, el meu primer professor i el més entusiasta.

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Als meus professors de la Facultat de Matemàtiques de la Universitat de Barcelona, on vaig estudiar entre els anys 1969 i 1976. D'entre tots ells recordo amb un afecte molt especial els doctors Nadal Batle, Pablo Bobillo, Josep Pla i Francesc d'Assís Sales, que em van contagiar l'interès pels fonaments de la matemàtica i l'amor a la lògica. També vull tenir un record per als meus companys d'estudis Pedro Acero, Pere Casanovas, José María Figuera (el Tato), Marta Garrido, Montse Fargas i la Neus.

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Molt especialment i de tot cor agraeixo la generositat amb què els meus companys Félix Bou i Carles Noguera m'han ajudat durant tots aquests anys, suplint les carències derivades del fet que jo hagués estat gairebé vint anys apartat de les matemàtiques. Ells són els meus *matemàtics de guàrdia*. En tot moment s'han prestat a qualsevol tipus d'ajuda sense escatimar temps ni esforços.

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Note of the author

The dedications of this monograph, the preface *Volver* and the previous Acknowledgments are an exact reproduction of those that turn up at the beginning of my doctoral dissertation.

Acknowledgments for this monograph

This monograph is a revised version of part of my doctoral dissertation, which was written in Catalan. I would like to thank the Artificial Intelligence Research Intitute (IIIA-CSIC) to give me the opportunity for presenting my work in English. I have to thank various persons who have made the publication of this monograph possible. In particular, I want to thank Núria Castellote, librarian of the IIIA and attendant of the publication of this collection, for having made things very easy for me; to Mario Jiménez who has translated three of the chapters from Catalan into English; and to Eva Armengol and Félix Bou who helped me improve the Introduction and Conclusions of this monograph in clarity and brevity. Finally, I want to thank very especially the supervisors of my thesis, Francesc Esteva and Ventura Verdú, who have encouraged me to carry out this version and have helped me with their very useful opinions and suggestions.

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Introduction

The contents of this book are placed in the framework of *many-valued logics* from two different but convergent perspectives: that of *substructural logics* and that of *mathematical fuzzy logic*. The present work is a contribution to the study of both the fragments of substructural logics and the fragments of fuzzy logics based on triangular norms, pseudonorms, uninorms, and also weakly implicative fuzzy logics. Mainly, we study some fragments without implication of the basic intuitionistic substructural logics. Since the formal systems belonging to mathematical fuzzy logic are axiomatic extensions of a basic substructural logic, one of the motivations of our study is to provide a base for the analysis of the mentioned fragments for substructural and fuzzy logics.

A *substructural logic* is a logic admitting a presentation in terms of sequents obtained by either dropping or restricting some of the structural rules of either intuitionistic or classical logic. The development of this field of research is intimately related to the book *Substructural Logics* [DSH93], edited by Kosta Došen and Peter Schroeder-Heister, and published in 1993. This book was the first monograph devoted to this family of logics. In the preface, the authors point out that the name *Substructural Logics* was used for first time at the *Seminar für natürlich-sprachliche Systeme* of the Tübingen University held in October 1990. The scope of substructural logics includes, among other logics, the following ones:

- *Intuitionistic Logic* is a substructural logic since it can be obtained by restricting the *weakening* rule of Classical Logic to sequents with the empty sequence or a sequence with only one formula in the consequent.

- *Relevance Logic*, that rejects the rule of *weakening*.

- The so-called *logics without contraction*, as the system H_{BCK} defined by Ono and Komori [OK85], commonly known as *Monoidal Logic* inside the mathematical fuzzy logic community.

- *Lineal Logic* (Girard, [Gir87]), that rejects both, the *weakening* and the *contraction* rules.

- *Lambek Calculus* [Lam58] for the analysis of syntactic structures, that rejects the rules of *exchange*, *weakening* and *contraction*.

From the early 90's the substructural logics have been a field of growing activity. Among the monographs devoted to a systematic study of these logics, the more prominent ones are those by Greg Restall [Res00], Francesco Paoli [Pao02] and, specially, the recent book *Residuated Lattices: An Algebraic Glimpse at Substructural Logics* by Galatos et altri [GJKO07].

Concerning research on fuzzy logics, the book *Metamathematics of Fuzzy Logic* [Háj98], by Petr Hájek, published in 1998, is the first book devoted to the study of continuous t-norm based fuzzy logics. This book introduces a systematic analysis of the deductive systems and the algebraic structures of fuzzy logics seen as many-valued logics. The author's goal is to show that fuzzy logic, as the logic of the vague propositions, admits a formal foundation. Thus the so called *fuzzy inference* may be seen as logical deduction. A logic is *fuzzy* and *t-norm based* if it is sound and complete with respect to the algebras over [0,1] given by a family of left continuous t-norms. The t-norm based fuzzy logics are extensions of the *Monoidal* T*-norm based fuzzy Logic* (MTL, for short) introduced by Francesc Esteva and Lluís Godo in [EG01]. MTL is the weaker of this family of logics and it can be obtained as an axiomatic extension of the Monoidal Logic (Höhle [Höh95]), which is a logic without contraction in the sense of [OK85] and therefore a substructural logic because –as Esteva, Godo, and García-Cerdaña point out in [EGGC03]– it coincides with the logic associated to the *Full Lambek calculus with exchange and weakening* ($\mathbf{FL}_{ew}$*-logic*) of Hiroakira Ono [Ono90, Ono93]. In other words, the Monoidal Logic is equal to the logic associated, as external deductive system, with the sequent calculus $\mathbf{FL}_{ew}$, and it is substructural because $\mathbf{FL}_{ew}$ can be obtained by dropping the structural rule of contraction of a version in terms of sequents of the Intuitionistic Logic. The article [EGGC03] is the first work in which there are pointed out the connections among both hierarchies of logics, substructural and fuzzy, by adding and combining different axioms used in both traditions.

On the other hand, inside the general framework of fuzzy logics, there are also logics such as *pseudo t-norms* based logics (see [Háj03b, Háj03a, JM03]), *uninorm* based logics (see [GM07, MM07]), or *weakly implicative fuzzy logics* (see [Cin04, Cin06]), that are also extensions of more general substructural logics. In the present work, we analyze the fragments without implication in the general framework of $\mathbf{FL}$, that is, the most general basic intuitionistic substructural logic (without *exchange*, *weakening* and *contraction*). The analysis of the fragments of a logic is important in order to characterize the contribution of each connective to the general properties of that logic. The study of the fragments *with* implication of the substructural logics has received attention in the literature (see for example [vAR04] and the references that are quoted there). Concerning the fragments *with* implication of the logics based on t-norms, an essential work is [CHH07]. Nevertheless, the study of the fragments *without* implication of both, substructural and fuzzy logics, has not received attention in the literature. One exception of this fact is the study of the fragments in the languages $\langle\vee\rangle$, $\langle\wedge\rangle$, $\langle\vee,\wedge\rangle$ and $\langle\vee,\wedge,\neg\rangle$ of the intuitionist logic (see [PW75, DP80, FGV91, FV91] for the fragments in the languages $\langle\vee\rangle$, $\langle\wedge\rangle$, $\langle\vee,\wedge\rangle$ and [RV93, RV94] for the $\langle\vee,\wedge,\neg\rangle$-fragment). A first step towards a general study of the fragments without implication of the substructural logics can be found in the papers [BGCV06, AGCV07]. In [BGCV06] the authors analyzed the fragments in the languages $\langle \vee, *, \neg, 0, 1\rangle$ and $\langle \vee, \wedge, *, \neg, 0, 1\rangle$ of the logic associated to the calculus $\mathbf{FL}_{ew}$ without contraction. In [AGCV07] the fragments corresponding to the languages $\langle \vee, *, 0, 1\rangle$, $\langle \vee, \wedge, *, 0, 1\rangle$ and $\langle \vee, \wedge, 0, 1\rangle$ are studied. The contents of the present monograph can be seen as a generalization of some the results of the papers mentioned above to a more general framework of substructural logics.

The present monograph corresponds to one of the parts of the doctoral dissertation of Àngel García-Cerdaña, *Lògiques basades en normes triangulars: una contribució a l'estudi dels seus aspectes subestructurals* [GC07], supervised by Francesc Esteva and Ventura Verdú, and defended on the 20th November of 2007 in the University of Barcelona.[1] The thesis can be divided in two research lines: one of them [GC07, Part II] analyzes the connections between *fuzzy logics based on t-norms* and the framework of *substructural logics.* The contributions of this part have been published in the papers [GGCB03, EGGC03, GCNE05]. The contributions of the second research line are in the field of basic intuitionistic substructural logics and they are the main contents of this monograph. In particular, we analyze algebraically some fragments without implications of that logics. Part of this research is already published by Bou, García-Cerdaña and Verdú in [BGCV06] and by Adillon, García-Cerdaña and Verdú in [AGCV07].

Outline of the monograph

The main contents of the present monograph is a general study of basic substructural systems (Part II), and an algebraic analysis of some free-implication fragments of basic intuitionistic substructural logics (Part III). The used tools stand within the framework of Algebraic Logic and, in some parts, in the area of Abstract Algebraic Logic and algebraizable Gentzen systems.

Part I consists of three chapters where concepts and basic results are introduced. Chapter 1 is devoted to the notions and preliminary results of Universal Algebra. In Chapters 2 and 3 we introduce the notions and basic results concerning deductive systems (i.e., sentential logics) and Gentzen systems, as well as the corresponding notions of Algebraic Logic.

Part II (Chapters 4 and 5) is a contribution to the systematization of concepts and results already known but disseminated in a disperse way in the literature of the field. In addition, this part also contains some contributions to the study of the basic substructural systems.

In Chapter 4 we recall the definitions of the basic substructural Gentzen systems $\mathcal{FL}_\sigma$ and its associated external deductive systems $\mathfrak{e}\mathcal{FL}_\sigma$. They are presented in a language with two implications and two negations. We define the notion of *mirror image* of a sequent and we prove the *mirror image principle* for the systems $\mathcal{FL}_\sigma[\Psi]$, where Ψ contains the two implications or the two negations. We characterize the sequential

[1] This thesis is available online at http://www.iiia.csic.es/~angel/PhDthesis-A-Garcia-C.pdf

Leibniz congruence of the theories of the systems $\mathcal{FL}[\Psi]$ and the Leibniz congruence of the theories of the external systems $\mathfrak{e}\mathcal{FL}[\Psi]$, where the language Ψ contains one of the implication connectives, and prove that these external systems are protoalgebraic. Finally we provide known Hilbert-style axiomatizations for the external deductive systems $\mathfrak{e}\mathcal{FL}_\sigma$.

In Chapter 5 we present and focus on the notions of *notational copy*, *definability of connectives*, *definitional expansion* and *definitional equivalence* in the context of Gentzen systems. We also obtain results related to these notions for certain classes of Gentzen systems including the systems $\mathcal{FL}_\sigma[\Psi]$. These results are used in order to formalize some notions and claims which receive an informal treatment in the substructural logics literature, such as *collapse* and *definability* of connectives in certain systems $\mathcal{FL}_\sigma[\Psi]$ or the comparison among the different versions of the same system.

Part III (Chapters 6 until 9) is dedicated to the study of some fragments *without* implication of the systems $\mathcal{FL}_\sigma$ and $\mathfrak{e}\mathcal{FL}_\sigma$.

In Chapter 6 we introduce the ordered, latticed and semilatticed algebraic structures that will constitute the semantic core of some of the fragments later studied in Chapter 9. After some basic notions and preliminary results, we introduce the notion of *pointed monoid*. A pointed monoid (ordered, semilatticed or latticed) is obtained by adding the constant symbol 0 to the type of similarity of a monoid (ordered, semilatticed or latticed): such symbol is interpreted as a fixed element but arbitrary of the universe of the structure. We define the varieties of algebras $\mathring{\mathbb{M}}^{s\ell}_\sigma$ and $\mathring{\mathbb{M}}^{\ell}_\sigma$, where subindex σ is a subsequence of the sequence ew_lw_rc and symbols e, w_l, w_r and c codify what we refer to as (algebraic) exchange, right-weakening, left-weakening and contraction properties, respectively. Such properties, which are expressed by quasi-inequations are equivalent, respectively, to the following properties: commutativity, integrality, 0-boundedness and increasing idempotency. In Chapter 9 we will state the connection among the varieties $\mathring{\mathbb{M}}^{s\ell}_\sigma$ and $\mathring{\mathbb{M}}^{\ell}_\sigma$ (subvarieties of $\mathring{\mathbb{M}}^{s\ell}$ and $\mathring{\mathbb{M}}^{\ell}$ defined by the equations codified by σ) and the fragments of the systems $\mathcal{FL}_\sigma$ and $\mathfrak{e}\mathcal{FL}_\sigma$ in the languages $\langle\vee,*,0,1\rangle$ and $\langle\vee,\wedge,*,0,1\rangle$. At the end of the chapter, we recall the notion of residuation and the definitions and properties of residuated lattices, $\mathbb{FL}$-algebras and $\mathbb{FL}_\sigma$-algebras.

In Chapter 7 we introduce the notion of *pseudocomplementation* in the framework of the *pointed po-monoids*. The notion of pseudocomplement with respect to the monoidal operation can be seen as a generalization of the same notion defined in the framework of the pseudocomplemented distributive lattices (see [BD74, Lak73]). We define the class $\mathbb{PM}^{\preccurlyeq}$ of the pseudocomplemented po-monoids, and the classes $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^{\ell}$ of the semilatticed and latticed pseudocomplemented monoids. We show that the classes $\mathbb{PM}^{\preccurlyeq}$ can be defined by means a set of inequations and thus the classes $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^{\ell}$ are varieties. We also analyze the case when the pseudocomplementation is with respect to the minimum element of the monoid. In the two last sections we study the class of weakly contractive pseudocomplemented monoids and the class of involutive peseudocomplemented monoids. The pseudocomplements constitute the algebraic counterpart of negations: in Chapter 9 we will state the connection between the varieties $\mathbb{PM}^{s\ell}_\sigma$ and

$\mathbb{PM}^{\ell}_{\sigma}$ (subvarieties of $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^{\ell}$ defined by the equations codified by σ) with the fragments of the Gentzen system $\mathcal{FL}_{\sigma}$ and the associated external deductive system $\mathfrak{e}\mathcal{FL}_{\sigma}$ in the languages $\langle \vee, *, \backslash, /, 0, 1\rangle$ and $\langle \vee, \wedge, *, \backslash, /, 0, 1\rangle$.

In Chapter 8 two kind of constructions of a complete $\mathbb{FL}$-algebra from any $\mathbb{FL}$-algebra are considered: the *Dedekind-MacNeille completion* (DM-completion, in short) and the *ideal-completion* (see [Ono93, Ono03a]). Both constructions allow to build a complete $\mathbb{FL}$-algebra from the monoidal reduct of a $\mathbb{FL}$-algebra in such a way that this algebra is embeddable into its completion. We show that the method of the ideal-completion also works if we start from an algebra in $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$, $\mathring{\mathbb{M}}^{\ell}_{\sigma}$, $\mathbb{PM}^{s\ell}_{\sigma}$ or $\mathbb{PM}^{\ell}_{\sigma}$ and we obtain that every algebra of these classes is embeddable into a complete $\mathbb{FL}_{\sigma}$-algebra. These embeddings have as a consequence that the classes $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$, $\mathring{\mathbb{M}}^{\ell}_{\sigma}$, $\mathbb{PM}^{s\ell}_{\sigma}$ and $\mathbb{PM}^{\ell}_{\sigma}$ are the classes of all the subreducts of the algebras in the class $\mathbb{FL}_{\sigma}$. However, we prove that the DM-completion, which works for $\mathbb{FL}_{\sigma}$-algebras, does not work when starting from the monoidal reduct of an algebra of the classes $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$, $\mathring{\mathbb{M}}^{\ell}_{\sigma}$, $\mathbb{PM}^{s\ell}_{\sigma}$ or $\mathbb{PM}^{\ell}_{\sigma}$ because in these cases the construction of Dedekind-MacNeille does in general not produce a $\mathbb{FL}_{\sigma}$-algebra. The reason is that to carry out this construction the monoidal operation must be residuated and this is not always the case.

In Chapter 9 we study the fragments in the languages $\langle \vee, *, 0, 1\rangle$, $\langle \vee, \wedge, *, 0, 1\rangle$ and $\langle \vee, *, \backslash, /, 0, 1\rangle$ of the systems $\mathcal{FL}_{\sigma}$ and their associated external deductive systems $\mathfrak{e}\mathcal{FL}_{\sigma}$. We prove that the subsystems $\mathcal{FL}_{\sigma}[\vee, *, 0, 1]$, $\mathcal{FL}_{\sigma}[\vee, \wedge, *, 0, 1]$, $\mathcal{FL}_{\sigma}[\vee, *, \backslash, /, 0, 1]$ and $\mathcal{FL}_{\sigma}[\vee, \wedge, *, \backslash, /, 0, 1]$ are algebraizables, having as respective equivalent algebraic semantics the varieties $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$, $\mathring{\mathbb{M}}^{\ell}_{\sigma}$, $\mathbb{PM}^{s\ell}_{\sigma}$, $\mathbb{PM}^{\ell}_{\sigma}$. We also prove that the system $\mathcal{FL}_{\sigma}$ is algebraizable with equivalent algebraic semantics the variety $\mathbb{FL}_{\sigma}$. Using these results and the ones concerning subreducts obtained in the previous chapter we obtain that the mentioned subsystems are fragments of $\mathcal{FL}_{\sigma}$, and that the corresponding external deductive systems are fragments of $\mathfrak{e}\mathcal{FL}_{\sigma}$. We also show that each system $\mathcal{FL}_{\sigma}$ is equivalent to its associated external deductive system. However, it is shown that the considered fragments are not equivalent to any deductive system. Moreover we show that $\mathfrak{e}\mathcal{FL}_{\sigma}[\vee, *, 0, 1]$, $\mathfrak{e}\mathcal{FL}_{\sigma}[\vee, \wedge, *, 0, 1]$, $\mathfrak{e}\mathcal{FL}_{\sigma}[\vee, *, \backslash, /, 0, 1]$ and $\mathfrak{e}\mathcal{FL}_{\sigma}[\vee, \wedge, *, \backslash, /, 0, 1]$ are not protoalgebraic but they have, respectively, the varieties $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$, $\mathring{\mathbb{M}}^{\ell}_{\sigma}$, $\mathbb{PM}^{s\ell}_{\sigma}$, $\mathbb{PM}^{\ell}_{\sigma}$ as algebraic semantics with defining equation $1 \vee p \approx p$. We also give decidability results for some of the fragments considered. In the last section we define the basic substructural systems with weak contraction $\mathcal{FL}_{\sigma\hat{c}}$ and characterize the fragments in the languages $\langle \vee, *, \backslash, /, 0, 1\rangle$ and $\langle \vee, \wedge, *, \backslash, /, 0, 1\rangle$ of these systems and their associated external deductive systems.

In Chapter 10 we analyze the fragments in the languages $\langle \vee, *, 0, 1\rangle$, $\langle \vee, \wedge, *, 0, 1\rangle$, and $\langle \vee, \wedge, 0, 1\rangle$ in the context of $\mathcal{FL}_{ew}$. We obtain as main result that each one of these fragments coincides with the fragment in the same language of classical logic. Since MTL, the most general t-norm based fuzzy logic, is an axiomatic extension of $\mathfrak{e}\mathcal{FL}_{ew}$, we have as a corollary that the fragments in these three languages of all the t-norm based fuzzy logics are equal to the corresponding fragments of classical logic.

Chapter 11 is devoted to conclusions and future work.

Part I

Preliminary Concepts

Chapter 1

Universal Algebra

In this chapter we recall the basic concepts, the notation and some well-known results of Universal Algebra necessary for the reading of this monograph. We assume a certain familiarity of the reader with the subject. There are two excellent reference textbooks by Burris and Sankkappanavar [BS81] (see also [BS00], for a disposable version on line) and by Grätzer [Grä79]. As a reference for the notions of first order logic that we will use we address the reader to [End00].

Algebraic languages. Algebras. An *algebraic language* is a pair $\mathcal{L} = \langle F, \tau \rangle$, where F is a set of *functional symbols* and τ is a mapping $\tau : F \to \omega$ (where ω denote the set of natural numbers) which is called *algebraic similarity type*. For every $f \in F$, $\tau(f)$ is called the *arity* of the functional symbol f. The functional symbols of arity 0 are also called *constant symbols*. If $\mathcal{L} = \langle \mathcal{F}, \tau \rangle$ is an algebraic language with a finite number of functionals we say that $\mathcal{L}$ is *finite*. In this case, if $F = \{f_1, \ldots, f_n\}$, we identify $\mathcal{L}$ with the sequence $\langle f_1, \ldots, f_n \rangle$ and say that $\mathcal{L}$ is the language $\langle f_1, \ldots, f_n \rangle$ of type $\langle \tau(f_1), \ldots, \tau(f_n) \rangle$.

If $\mathcal{L} = \langle F, \tau \rangle$ and $\mathcal{L}' = \langle F', \tau' \rangle$ are two algebraic languages such that $F \subseteq F'$ and $\tau = \tau' \restriction F$ (i.e, the restriction of τ' to F) then we say that $\mathcal{L}$ is a *sublanguage* of $\mathcal{L}'$ and we write $\mathcal{L} \leq \mathcal{L}'$. We also say that $\mathcal{L}'$ is an *expansion* of $\mathcal{L}$. If $\mathcal{L}'$ is an expansion of $\mathcal{L}$ with a finite set of new functionals $f_1, \ldots, f_n$, sometimes we will denote $\langle \mathcal{L}, f_1, \ldots, f_n \rangle$ the language $\mathcal{L}'$.

Given an algebraic language $\mathcal{L} = \langle F, \tau \rangle$, an algebra $\mathbf{A}$ of type $\mathcal{L}$ ($\mathcal{L}$-algebra, for short) is a pair $\langle A, \{f^{\mathbf{A}} : f \in F\} \rangle$, where A is a non-empty set called the *universe* of $\mathbf{A}$ and where, for every $f \in F$, if $\tau(f) = n$, then $f^{\mathbf{A}}$ is a n-ary operation on A (a 0-ary operation on A is an element of A). If $\mathcal{L}$ is a finite language $\langle f_1, \ldots, f_n \rangle$ of type $\langle \tau(f_1), \ldots, \tau(f_n) \rangle$, we write $\mathbf{A} = \langle A, f_1^{\mathbf{A}}, \ldots, f_n^{\mathbf{A}} \rangle$, and we say that $\mathbf{A}$ is an algebra of type $\langle \tau(f_1), \ldots, \tau(f_n) \rangle$. The superscripts in the operations will be omitted when they are clear from the context.

An algebra $\mathbf{A}$ is *finite* if A is finite, and is *trivial* if A has only one element. To denote classes of algebras we will use capital letters in blackboard boldface, e.g., $\mathbb{K}, \mathbb{M} \ldots$ The members of a class $\mathbb{K}$ of algebras will sometimes be called $\mathbb{K}$-algebras.

Algebra of formulae. An important example of algebra is the *algebra of formulae*. Let $\mathcal{L}$ be a countable (i.e., finite or enumerable) algebraic language and let X be an enumerable set. The set $Fm_{\mathcal{L}}(X)$ of $\mathcal{L}$-formulae (or $\mathcal{L}$-terms) over X is inductively defined in the following way:

1. For every $x \in X$, $x \in Fm_{\mathcal{L}}(X)$.
2. For every $c \in F$ with arity 0, $c \in Fm_{\mathcal{L}}(X)$.
3. For every $f \in F$ with arity $n > 0$, if $\varphi_1, \ldots, \varphi_n \in Fm_{\mathcal{L}}(X)$, then
$$f(\varphi_1, \ldots, \varphi_n) \in Fm_{\mathcal{L}}(X).$$

X is called the set of *variables*. The algebra of formulae:

$$\mathbf{Fm}_{\mathcal{L}}(X) = \langle Fm_{\mathcal{L}}(X), \{f^{\mathbf{Fm}_{\mathcal{L}}(X)} : f \in F\}\rangle,$$

is defined by:

- For every $c \in F$ with arity 0, $c^{\mathbf{Fm}_{\mathcal{L}}(X)} := c$.
- For every $f \in F$ with arity $n > 0$ and every $\varphi_1, \ldots, \varphi_n \in Fm_{\mathcal{L}}(X)$,
$$f^{\mathbf{Fm}_{\mathcal{L}}(X)}(\varphi_1, \ldots, \varphi_n) := f(\varphi_1, \ldots, \varphi_n).$$

Given another enumerable set Y of variables, the corresponding algebra of formulae $\mathbf{Fm}_{\mathcal{L}}(Y)$ is isomorphic to $\mathbf{Fm}_{\mathcal{L}}(X)$. Thus, to simplify the notation, the set of formulae will be denoted by $Fm_{\mathcal{L}}$. Note that $Fm_{\mathcal{L}}$ is also enumerable.

The algebra of formulae is defined in the same way when the set of variables is not countable, but since the number of variables occurring in a formula is always finite, in general it is enough to consider formula algebras built over an enumerable set of variables.

We will denote by Var a generic enumerable set of variables. To indicate that the variables occurring in the formula φ are in $\{x_1, \ldots, x_n\}$ we will write $\varphi(x_1, \ldots, x_n)$.

An example of algebra: the class of lattices. An algebra $\mathbf{A} = \langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}\rangle$ of type $\langle 2, 2\rangle$ is a *lattice* if, and only if, the operations $\wedge^{\mathbf{A}}$ and $\vee^{\mathbf{A}}$ are associative, commutative and idempotent and, for each $a, b \in A$, it holds:

- $a \wedge^{\mathbf{A}} (a \vee^{\mathbf{A}} b) = a$
- $a \vee^{\mathbf{A}} (a \wedge^{\mathbf{A}} b) = a$

A *bounded lattice* is an algebra $\mathbf{A} = \langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, 0^{\mathbf{A}}, 1^{\mathbf{A}} \rangle$ of type $\langle 2, 2, 0, 0 \rangle$ such that $\langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}} \rangle$ is a lattice and such that

- $a \wedge^{\mathbf{A}} 0^{\mathbf{A}} = 0^{\mathbf{A}}$
- $a \vee^{\mathbf{A}} 1^{\mathbf{A}} = 1^{\mathbf{A}}$

A lattice $\mathbf{A}$ is *distributive* if, and only if, for every $a, b, c \in A$, it holds:

- $a \wedge^{\mathbf{A}} (b \vee^{\mathbf{A}} c) = (a \wedge^{\mathbf{A}} b) \vee^{\mathbf{A}} (a \wedge^{\mathbf{A}} c)$

In every lattice $\mathbf{A}$ it is possible to define a partial order in the following way: for every $a, b \in A$, $a \leq b$ if, and only if, $a \wedge^{\mathbf{A}} b = a$ (or, equivalently, $a \vee^{\mathbf{A}} b = b$). In this partial order every set of two elements $\{a, b\}$ has an infimum (the element $a \wedge^{\mathbf{A}} b$) and a supremum (the element $a \vee^{\mathbf{A}} b$). Consequently, by induction we have that every finite subset of A has an infimum and a supremum. A lattice is called *complete* if, and only if, every subset (infinite or not) of the universe has an infimum and a supremum with respect to their associated partial order. For more information about lattices and distributive lattices see [Grä78] and [BD74].

Subalgebras. Let $\mathbf{A} = \langle A, \{f^{\mathbf{A}} : f \in F\} \rangle$ and $\mathbf{B} = \langle B, \{f^{\mathbf{B}} : f \in F\} \rangle$ be two algebras of the same type. We say that $\mathbf{A}$ is a *subalgebra* of $\mathbf{B}$, and we write $\mathbf{A} \subseteq \mathbf{B}$ if, and only if,

- $A \subseteq B$,
- for every $c \in F$ with arity 0, $c^{\mathbf{A}} = c^{\mathbf{B}}$, and
- for every $f \in F$ with arity $n > 0$, $f^{\mathbf{A}} = f^{\mathbf{B}} \upharpoonright A^n$.[1]

The universe of a subalgebra of $\mathbf{A}$ is called a *subuniverse* of $\mathbf{A}$. Since the set of subuniverses of $\mathbf{A}$ is closed under arbitrary intersections, for every nonempty set $B \subseteq A$, one can define the *subalgebra generated* by B as the algebra $\langle B \rangle_{\mathbf{A}}$ whose universe is $\bigcap \{C \subseteq A : C \text{ is a subuniverse of } \mathbf{A} \text{ and } B \subseteq C\}$.

Homomorphisms. A mapping $h : A \to B$ is a *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ if, and only if,

- for every $c \in F$ with arity 0, $h(c^{\mathbf{A}}) = c^{\mathbf{B}}$, and
- for every $f \in F$ with arity $n > 0$ and for every $a_1, \ldots, a_n \in A$,

$$h(f^{\mathbf{A}}(a_1, \ldots, a_n)) = f^{\mathbf{B}}(h(a_1), \ldots, h(a_n)).$$

[1] If X and Y are sets and $Z \subseteq X$, given a function $f : X \longrightarrow Y$, we denote by $f \upharpoonright Z$ the function $Z \longrightarrow Y$ defined by $f \upharpoonright Z\,(z) = f(z)$, for every $z \in Z$.

$\mathbf{B}$ is *homomorphic image* of $\mathbf{A}$ if, and only if, there is a surjective homomorphism from $\mathbf{A}$ to $\mathbf{B}$. An injective homomorphism is called an *embedding*. If there exists an embedding from $\mathbf{A}$ into $\mathbf{B}$ sometimes we will denote this fact by writing $\mathbf{A} \hookrightarrow \mathbf{B}$. A surjective and injective homomorphism is called an *isomorphism*. We say that $\mathbf{A}$ and $\mathbf{B}$ are *isomorphic* and we write $\mathbf{A} \cong \mathbf{B}$ if, and only if, there is an isomorphism from $\mathbf{A}$ to $\mathbf{B}$.

Congruences. Given an algebra $\mathbf{A} = \langle A, \{f^{\mathbf{A}} : f \in F\}\rangle$, a set $\theta \subseteq A^2$ is a *congruence* of $\mathbf{A}$ if, and only if, is an equivalence relation on A and, for every $a_1, \ldots, a_n$, $b_1, \ldots, b_n \in A$ and every $f \in F$ with arity $n > 0$, if $\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle \in \theta$, then

$$\langle f^{\mathbf{A}}(a_1, \ldots, a_n), f^{\mathbf{A}}(b_1, \ldots, b_n)\rangle \in \theta.$$

The set of all the congruences on $\mathbf{A}$ is denoted by $Con(\mathbf{A})$ and it is closed under arbitrary intersections. We define on $Con(\mathbf{A})$ an operation $\vee$ in the following way:

$$\text{for each } \theta_1, \theta_2 \in Con(\mathbf{A}),\ \theta_1 \vee \theta_2 = \cap\{\theta \in Con(\mathbf{A}) : \theta_1 \cup \theta_2 \subseteq \theta\}.$$

Then, the structure

$$\mathbf{Con}(\mathbf{A}) = \langle Con(\mathbf{A}), \cap, \vee, \Delta_{\mathbf{A}}, \nabla_{\mathbf{A}}\rangle,$$

where $\cap$ is the operation of intersection, $\Delta_{\mathbf{A}} = \{\langle a, a\rangle : a \in A\}$ and $\nabla_{\mathbf{A}} = A^2$, is a bounded lattice. This lattice has as order associated the inclusion relation between sets. Moreover, $\mathbf{Con}(\mathbf{A})$ is a complete lattice, where the infimum and the supremum of any set $\{\theta_i\}_{i \in I} \subseteq Con\mathbf{A}$ are given, respectively, by

$$\bigwedge_{\mathbf{Con}(\mathbf{A})} \{\theta_i\}_{i\in I} = \bigcap_{i\in I} \theta_i \qquad \text{and} \qquad \bigvee_{\mathbf{Con}(\mathbf{A})} \{\theta_i\}_{i\in I} = \bigcap\{\theta \in Con\mathbf{A} : \bigcup_{i\in I} \theta_i \subseteq \theta\}.$$

Since $\mathbf{Con}(\mathbf{A})$ is a complete lattice, we have that if $R \subseteq A^2$, then there exists the smallest congruence containing R, which is denotes by $\Theta(R)$; this congruence is called *generated congruence* per R. The congruences of the form $\Theta(\{\langle a, b\rangle\})$, are called *principal congruences*. If $\theta_1, \theta_2 \in Con(\mathbf{A})$, the *composition of* θ_1 *with* θ_2 is defined as the binary relation

$$\theta_1 \circ \theta_2 := \{\langle a, b\rangle : \langle a, c\rangle \in \theta_2 \text{ and } \langle c, b\rangle \in \theta_1 \text{ for some } c \in A\}.$$

We say that an algebra $\mathbf{A}$ is *simple* if, and only if, $Con(\mathbf{A}) = \{\Delta_{\mathbf{A}}, \nabla_{\mathbf{A}}\}$.

Quotient algebra. Relative congruences. Let $\mathbf{A} = \langle A, \{f^{\mathbf{A}} : f \in F\}\rangle$ be an algebra and let $\theta \in Con(\mathbf{A})$. Given $a \in A$, its equivalence class with respect to θ is denote as a/θ. The *quotient algebra* of $\mathcal{A}$ by θ is defined as $\mathbf{A}/\theta = \langle A/\theta, \{f^{\mathbf{A}/\theta} : f \in F\}\rangle$, where:

- $A/\theta = \{a/\theta : a \in \theta\}$,

- for every $c \in F$ with arity 0, $c^{\mathbf{A}/\theta} = c^{\mathbf{A}}/\theta$, i
- for every $f \in F$ with arity $n > 0$ and for each $a_1, \ldots, a_n \in A$,

$$f^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = f^{\mathbf{A}}(a_1, \ldots, a_n)/\theta.$$

Given an algebra $\mathbf{A}$, if there exists an element $e \in A$ such that, for every $\theta_1, \theta_2 \in Con(\mathbf{A})$, we have that $a/\theta_1 = a/\theta_2$ implies $\theta_1 = \theta_2$, then we say that $\mathbf{A}$ is *e-regular* and if $\mathbf{A}$ is e-regular for every $e \in A$, then we say that $\mathbf{A}$ is *regular*. If $\mathbb{K}$ is a class of algebras of the same similitude type and e is a constant symbol of this type, then we say that $\mathbb{K}$ is *e-regular* if every algebra in $\mathbb{K}$ is e-regular.

If $\mathbf{A}$ is a $\mathcal{L}$-algebra, $\theta \in Con(\mathbf{A})$ and $\mathbb{K}$ is a class of $\mathcal{L}$-algebras we will say that θ is a $\mathbb{K}$-*congruence of* $\mathbf{A}$ *relative to* $\mathbb{K}$ if, and only if, the quotient algebra $\mathbf{A}/\theta$ belongs to $\mathbb{K}$; the set of all the congruences of $\mathbf{A}$ relatives to $\mathbb{K}$ will be denoted by $Con_{\mathbb{K}}(\mathbf{A})$.

Product algebra. Given a family $\{\mathcal{A}_i : i \in I\}$ of algebras of the same type, we define the *product algebra* (or *direct product algebra*) of the members of the family as the algebra

$$\prod_{i\in I} \mathbf{A}_i = \langle \prod_{i\in I} A_i, \{f^{\prod_{i\in I}\mathbf{A}_i} : f \in F\}\rangle,$$

where:

- $\prod_{i\in I} A_i$ is the Cartesian product of the universes,
- for every $c \in F$ with arity, $c^{\prod_{i\in I}\mathbf{A}_i} = \langle c^{\mathbf{A}_i} : i \in I\rangle$, and
- for every $f \in F$ with arity $n > 0$ and, for every $\hat{a}_1, \ldots, \hat{a}_n \in \prod_{i\in I} A_i$,

$$f^{\prod_{i\in I}\mathbf{A}_i}(\hat{a}_1, \ldots, \hat{a}_n)(i) = f^{\mathbf{A}_i}(\hat{a}_1(i), \ldots, \hat{a}_n(i)),$$

  for every $i \in I$, where $\hat{b}(i)$ denotes i-th component of $\hat{b} \in \prod_{i\in I} A_i$.

If $j \in I$, the *j-th projection* is the homomorphism $\pi_j : \prod_{i\in I}\mathbf{A}_i \to \mathbf{A}_j$ defined as $\pi(\hat{a}) := \hat{a}(j)$. If $I = n \in \omega$, then the product algebra is also denoted by $\mathbf{A}_0 \times \ldots \times \mathbf{A}_{n-1}$.

Reduced product. Ultraproduct. A *filter* $\mathcal{F}$ *on a set* I is a family of subsets of I such that:

- $I \in \mathcal{F}$,
- if $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$, and
- if $X \in \mathcal{F}$ and $X \subseteq Y \subseteq I$, then $Y \in \mathcal{F}$.

$\mathcal{F}$ is *proper* if, and only if, $\emptyset \notin \mathcal{F}$ (i.e., if $\mathcal{P}(I)$ is the power of I, $\mathcal{F} \neq \mathcal{P}(\mathcal{I})$). Given a family $\{\mathcal{A}_i : i \in I\}$ of algebras of the same type and a proper filter $\mathcal{F}$ on I, the following binary relation is defined on $\prod_{i \in I} A_i$: for every $\hat{a}, \hat{b} \in \prod_{i \in I} A_i$, $\hat{a} \sim_{\mathcal{F}} \hat{b}$ if, and only if, $\{i \in I : \hat{a}(i) = \hat{b}(i)\} \in \mathcal{F}$. $\sim_{\mathcal{F}}$ is a congruence of $\prod_{i \in I} \mathbf{A}_i$. The *reduced product algebra* of $\{\mathcal{A}_i : i \in I\}$ w.r.t. $\mathcal{F}$ is the algebra $\prod_{i \in I} \mathbf{A}_i/\mathcal{F} = \langle \prod_{i \in I} A_i/\mathcal{F}, \{f^{\prod_{i \in I} \mathbf{A}_i/\mathcal{F}} : f \in F\}\rangle$ defined by:

- $\prod_{i \in I} A_i/\mathcal{F}$ is the quotient by $\sim_{\mathcal{F}}$ of the Cartesian product $\prod_{i \in I} A_i$,
- for every $c \in F$ with arity 0, $c^{\prod_{i \in I} \mathbf{A}_i/\mathcal{F}} = c^{\prod_{i \in I} \mathbf{A}_i}/\mathcal{F}$, and
- for every $f \in F$ with arity $n > 0$ and, for every $\hat{a}_1/\mathcal{F}, \ldots, \hat{a}_n/\mathcal{F} \in \prod_{i \in I} A_i/\mathcal{F}$,

$$f^{\prod_{i \in I} \mathbf{A}_i/\mathcal{F}}(\hat{a}_1/\mathcal{F}, \ldots, \hat{a}_n/\mathcal{F}) = f^{\prod_{i \in I} \mathbf{A}_i}(\hat{a}_1, \ldots, \hat{a}_n)/\mathcal{F}.$$

For the sake of simplicity, the reduced product will be also denoted as $\prod_{\mathcal{F}}^{I} \mathcal{A}_i$.

Let $\mathcal{F}$ be a proper filter on I. $\mathcal{F}$ is an *ultrafilter* if, and only if, satisfies any of the following equivalent conditions:

- For every $X \subseteq I$, $X \in \mathcal{F}$ if, and only if, $I \setminus X \notin \mathcal{F}$.
- For every $X, Y \subseteq I$, $X \cup Y \in \mathcal{F}$ if, and only if, $X \in \mathcal{F}$ or $Y \in \mathcal{F}$.
- $\mathcal{F}$ is maximal in the set of proper filters on I ordered by the inclusion.

The reduced product w.r.t. an ultrafilter is called *ultraproduct*.

Subdirect product. Subdirectly irreducible algebras. Given a family $\{\mathbf{A}_i : i \in I\} \cup \{\mathbf{A}\}$ of algebras of the same type, we say that $\mathbf{A}$ is a *subdirect product* of $\{\mathbf{A}_i : i \in I\}$ if, and only if:

1. $\mathbf{A} \subseteq \prod_{i \in I} \mathbf{A}_i$, and
2. for every $j \in I$, the restriction on $\mathcal{A}$ of the j-th projection of $\prod_{i \in I} \mathcal{A}_i$ is surjective.

$\mathbf{A}$ is *representable as a subdirect product* of $\{\mathbf{A}_i : i \in I\}$ if, and only if it is isomorphic to a subdirect product of $\{\mathbf{A}_i : i \in I\}$, i.e. there exists an embedding $\alpha : \mathbf{A} \hookrightarrow \prod_{i \in I} \mathbf{A}_i$ such that for every $j \in J$, $\pi_j \circ \alpha$ is surjective. In this case α is called a *representation* of $\mathcal{A}$. We say that the representation is finite when I is finite.

An algebra $\mathbf{A}$ is *(finitely) subdirectly irreducible* if, and only if, for every (finite) representation $\alpha : \mathcal{A} \hookrightarrow \prod_{i \in I} \mathcal{A}_i$ there exists $j \in J$ such that $\pi_j \circ \alpha$ is an isomorphism.

Proposition 1.1. *Let $\mathbf{A}$ be an algebra and suppose that $\theta \in Con(\mathbf{A}) \setminus \{\nabla_{\mathbf{A}}\}$. The following conditions are equivalent:*

(i) $\mathbf{A}/\theta$ *is subdirectly irreducible.*

(ii) θ *is* $\cap$*-completely irreducible.*

(iii) θ *is maximal relatively to a pair, i.e. there is a pair* $\langle a, b\rangle \in A^2$ *such that* θ *is maximal in the set of proper congruences not containing* $\langle a, b\rangle$*.*

Corollary 1.2. *A non trivial algebra* $\mathbf{A}$ *is subdirectly irreducible if, and only if,* $Con(\mathbf{A}) \setminus \{\Delta_{\mathbf{A}}\}$ *has a minimum element.*

Theorem 1.3 (Birkhoff [Bir44])**.** *Every algebra* $\mathbf{A}$ *is representable as a subdirect product of subdirectly irreducible algebras (which are homomorphic images of* $\mathbf{A}$*).*

Given a class of algebras $\mathbb{K}$, we denote the class of its subdirectly irreducible members by $\mathbb{K}_{SI}$ and the class of its finitely subdirectly irreducible members by $\mathbb{K}_{FSI}$.

Notice that simple algebras are subdirectly irreducible. An algebra is called *semisimple* if, and only if, it is representable as a subdirect product of simple algebras.

Algebraic operators. The operators over classes of algebras that give their isomorphic images, subalgebras, homomorphic images, products, reduced products and ultraproducts are denoted respectively as $\mathbf{I}$, $\mathbf{S}$, $\mathbf{H}$, $\mathbf{P}$, $\mathbf{P}_R$, $\mathbf{P}_U$. Given a class of algebras $\mathbb{K}$ of the same type and an operator $\mathbf{O} \in \{\mathbf{I}, \mathbf{S}, \mathbf{H}, \mathbf{P}, \mathbf{P}_R, \mathbf{P}_U\}$, the following hold:

1. $\mathbf{O}(\mathbb{K}) \subseteq \mathbf{IO}(\mathbb{K})$,
2. $\mathbf{IO}(\mathbb{K}) = \mathbf{OI}(\mathbb{K})$,
3. $\mathbf{IPS}(\mathbb{K}) \subseteq \mathbf{ISP}(\mathbb{K})$,
4. $\mathbf{IPH}(\mathbb{K}) \subseteq \mathbf{HP}(\mathbb{K})$,
5. $\mathbf{ISH}(\mathbb{K}) \subseteq \mathbf{IHS}(\mathbb{K})$, and
6. $\mathbf{ISP}_R(\mathbb{K}) = \mathbf{ISPP}_U(\mathbb{K})$.

If $\mathbb{K} = \{\mathbf{A}_1, \ldots, \mathbf{A}_n\}$, we write $\mathbf{O}(\mathbf{A}_1, \ldots, \mathbf{A}_n)$ instead of $\mathbf{O}(\{\mathbf{A}_1, \ldots, \mathbf{A}_n\})$.

Equations. Quasiequations. An algebraic similarity type $\mathcal{L}$ can be seen as a first order similarity type with equality and without relational symbols. The only atomic formulas that can be constructed in this class of languages (we denote the identity by the symbol $\approx$) are in the form $\varphi \approx \psi$, where φ and ψ are $\mathcal{L}$-terms. The rest of formulas are built in the usual way with the help of the quantifiers $\forall$ and $\exists$ and the boolean connectives that we will denote with the symbols $\sqcup, \sqcap, \supset, \equiv, \sim$ for the disjunction, the conjunction, the conditional, the biconditional and the negation, respectively. The atomic formulas $\varphi \approx \psi$ are called *$\mathcal{L}$-equations* and those in the form

$$\varphi_0 \approx \psi_0 \ \sqcap \ldots \sqcap \ \varphi_{n-1} \approx \psi_{n-1} \supset \varphi_n \approx \psi_n,$$

where $\varphi_i, \psi_i \in Fm_{\mathcal{L}}$ for each $i \leq n$, are called *$\mathcal{L}$-quasiequations*. The set of all $\mathcal{L}$-equations is denoted by $Eq_{\mathcal{L}}$ and that of all the $\mathcal{L}$-quasiequacions by $QEq_{\mathcal{L}}$. Note that $Eq_{\mathcal{L}} \subseteq QEq_{\mathcal{L}}$, since an equation is an quasiequation with $n = 0$.

We will say that a $\mathcal{L}$-quasiequation

$$\varphi_0 \approx \psi_0 \ \sqcap \ldots \sqcap \ \varphi_{n-1} \approx \psi_{n-1} \supset \varphi_n \approx \psi_n,$$

such that its variables are in $\{x_1, \ldots, x_m\}$ *holds*, or *is valid* in a $\mathcal{L}$-algebra $\mathbf{A}$ if, and only if, its universal closure

$$\forall x_1 \ldots \forall x_m \ \varphi_0 \approx \psi_0 \ \sqcap \ldots \sqcap \ \varphi_{n-1} \approx \psi_{n-1} \supset \varphi_n \approx \psi_n$$

it is a true sentence in $\mathbf{A}$, i.e., if, and only if, for each $a_1, \ldots, a_m \in A$, $\varphi_n^{\mathbf{A}}(a_1, \ldots, a_m) = \psi_n^{\mathbf{A}}(a_1, \ldots, a_m)$, whenever $\varphi_i^{\mathbf{A}}(a_1, \ldots, a_m) = \psi_i^{\mathbf{A}}(a_1, \ldots, a_m)$ for each $i < n$. [2] In this case, we will also say that $\mathbf{A}$ is a *model* of the quasiequation, or that $\mathbf{A}$ *satisfies* the quasiequation, and we will write

$$\mathbf{A} \models \varphi_0 \approx \psi_0 \ \sqcap \ldots \sqcap \ \varphi_{n-1} \approx \psi_{n-1} \supset \varphi_n \approx \psi_n.$$

A class of $\mathcal{L}$-algebras $\mathbb{K}$ satisfies a quasiequation ε, and we write $\mathbb{K} \models \varepsilon$, if, and only if, $\mathbf{A} \models \varepsilon$ for each $\mathbf{A} \in \mathbb{K}$. The class $\mathbb{K}$ satisfies a set of quasiequations $\Lambda \subseteq QEq_{\mathcal{L}}$ if, and only if, $\mathbb{K} \models \varepsilon$ for each $\varepsilon \in \Lambda$ and this is denoted for $\mathbb{K} \models \Lambda$. The equations and the quasiequations can also be understood, respectively, as formulas and rules in the framework of the 2-deductive systems (see Chapter 2, page 23).

Varieties. A class of algebras $\mathbb{K}$ of the same type is said to be a *variety* if, and only if, it is closed under $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$. We denote as $\mathbf{V}(\mathbb{K})$ the *variety generated* by $\mathbb{K}$, i.e. the smallest variety containing $\mathbb{K}$. By the conditions 2, 3, 4 and 5 (see page 15) it is clear that $\mathbf{V}(\mathbb{K}) = \mathbf{HSP}(\mathbb{K})$. A class $\mathbb{K}$ of $\mathcal{L}$-algebras is an *equational class* if there exists a set of equations $\Lambda \subseteq Eq_{\mathcal{L}}$ such that $\mathbb{K} = \{\mathbf{A} : \mathbf{A} \models \Lambda\}$.

Theorem 1.4 (Birkhoff's Theorem [Bir35])**.** *A class $\mathbb{K}$ of algebras of the same type is a variety if, and only if, it is an equational class.*

[2] Given a first order language $\mathcal{L}$ and a $\mathcal{L}$-term t such that all its variables are in $\{x_1, \ldots, x_m\}$, if $a_1, \ldots, a_m \in A$, we denote by $t^{\mathbf{A}}(a_1, \ldots, a_m)$ the interpretation of the $\mathcal{L}$-term t in the $\mathcal{L}$-structure $\mathbf{A}$ for every assignment $\bar{a}$ of the variables in $\mathbf{A}$ such that $\bar{a}(x_1) = a_1, \ldots, \bar{a}(x_m) = a_m$.

Congruent permutable, congruent distributive and arithmetic varieties. We say that an algebra $\mathbf{A}$ is *congruent permutable* if, and only if, for every $\theta_1, \theta_2 \in Con(\mathbf{A})$, $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$. A variety $\mathbb{K}$ is *congruent permutable* if, and only if, for every $\mathbf{A} \in \mathbb{K}$, $\mathcal{A}$ is congruent permutable. A variety $\mathbb{K}$ is *congruent distributive* if, and only if, for every $\mathbf{A} \in \mathbb{K}$, $\mathbf{Con}(\mathbf{A})$ is a distributive lattice. A variety is *arithmetic* if, and only if, it is congruent distributive and congruent permutable.

The finitely subdirectly irreducible members of a congruent distributive variety have a useful description, as the following result states.

Theorem 1.5 (Jónsson's Lemma). *Let $\mathbb{K}$ be a class of algebras of the same type such that $\mathbf{V}(\mathbb{K})$ is congruent distributive. If an algebra $\mathcal{A} \in \mathbf{V}(\mathbb{K})$ is finitely subdirectly irreducible, then $\mathcal{A} \in \mathbf{HSP}_U(\mathbb{K})$.*

It is well known that the varieties of lattices are congruent distributive (see [BS00, Section §12, Chapter II]). Thus Jónsson's Lemma is applicable to them.

Quasivarieties. A class of algebras $\mathbb{K}$ is said to be a *quasivariety* if, and only if, it is closed under $\mathbf{I}$, $\mathbf{S}$ and $\mathbf{P}_R$. We denote as $\mathbf{Q}(\mathbb{K})$ the quasivariety generated by $\mathbb{K}$, i.e. the smallest quasivariety containing $\mathbb{K}$. It is clear that $\mathbf{Q}(\mathbb{K}) = \mathbf{ISP}_R(\mathbb{K})$, i.e., $\mathbf{Q}(\mathbb{K}) = \mathbf{ISPP}_U(\mathbb{K})$. A class $\mathbb{K}$ of $\mathcal{L}$-algebras is a *quasiequational class* if there exists a set of quasiequations $\Lambda \subseteq QEq_{\mathcal{L}}$ such that $\mathbb{K} = \{\mathbf{A} : \mathbf{A} \models \Lambda\}$.

Theorem 1.6 (Mal'cev [Mal66]). *A class $\mathbb{K}$ of algebras of the same type is a quasivariety if, and only if, it is a quasiequational class.*

Since $Eq_{\mathcal{L}} \subseteq QEq_{\mathcal{L}}$, the last theorem implies that every variety is a quasivariety.

Reducts. Subreducts. Let $\mathcal{L}$ be an algebraic language with a set of functional symbols F and $\mathbf{A} = \langle A, \{f^{\mathbf{A}} : f \in F\}\rangle$ a $\mathcal{L}$-algebra. Let $\mathcal{L}'$ be a sublanguage of $\mathcal{L}$ with set of functional symbols F'. The $\mathcal{L}'$-algebra $\langle A, \{f^{\mathbf{A}} : f \in F'\}\rangle$ is called the *$\mathcal{L}'$-reduct* of $\mathbf{A}$. If $\mathbb{K}'$ is the class of all the $\mathcal{L}'$-reducts of the members of a class $\mathbb{K}$ of type $\mathcal{L}$, the members of the class $\mathbf{IS}(\mathbb{K}')$ (i.e., the class of all the $\mathcal{L}'$-algebras isomorphic to a subalgebra in $\mathbb{K}'$) are called the *$\mathcal{L}'$-subreducts* of $\mathbb{K}$.

Theorem 1.7 (Mal'cev [Mal71]). *Let $\mathbb{K}$ be a class of $\mathcal{L}$-algebras, $\mathcal{L}'$ be a sublanguage of $\mathcal{L}$, and $\mathbb{K}'$ be the class of the $\mathcal{L}'$-reducts of the members of $\mathbb{K}$. If $\mathbb{K}$ is a quasivariety, then the quasivariety generated by $\mathbb{K}'$ is the class of all the subreducts of the members of $\mathbb{K}'$, i.e., $\mathbf{Q}(\mathbb{K}') = \mathbf{IS}(\mathbb{K}')$.*

Partial subalgebras. Partial embeddability. Let $\mathcal{L}$ be an algebraic language, let $\mathbf{A}$ be an algebra of type $\mathcal{L}$ and let $B \subseteq A$ be a nonempty set. The *partial subalgebra* $\mathbf{B}$ of $\mathbf{A}$ with universe B is the structure[3] $\langle B, \{f^{\mathbf{B}} : f \in F\}\rangle$, for every k-ary functional

[3]Note that in general it is not an algebra, since the operations may not be defined around all the universe. These structures have sometimes been called *partial algebras*.

$f \in F$, and $b_1, \ldots, b_n \in B$, $f^{\mathbf{B}}(b_1, \ldots, b_n) = \begin{cases} f^{\mathbf{A}}(b_1, \ldots, b_n) & \text{if } f^{\mathbf{A}}(b_1, \ldots, b_n) \in B, \\ \text{undefined} & \text{otherwise.} \end{cases}$
If $\mathbf{B}$ is the partial subalgebra of $\mathbf{A}$ with universe B this fact is denoted by $\mathbf{B} \subseteq_p \mathbf{A}$.

Given two algebras $\mathbf{A}$ and $\mathbf{B}$ of the same language we say that $\mathbf{A}$ is *partially embeddable* into $\mathbf{B}$ when every finite partial subalgebra of $\mathcal{A}$ is embeddable into $\mathcal{B}$. Generalizing this notion to classes of algebras, we say that a class $\mathbb{K}$ of algebras is *partially embeddable into a class* $\mathbb{M}$ if every finite partial subalgebra of a member of $\mathbb{K}$ is embeddable into a member of $\mathbb{M}$. If the language is finite, this turns out to be equivalent to saying that $\mathbb{K}$ belongs to the universal class generated by $\mathbb{M}$ (see for instance [Got01]). That is, by recalling Łos' theorem (Cf. [BS00, Theorem V.2.20]) characterizing the universal classes generated by $\mathbb{M}$ as $\mathbf{ISP}_U(\mathbb{M})$, we have the following equivalence.

Proposition 1.8. (Cf. [Got01, Theorem 1.2.2]) *Let $\mathbb{K}$ and $\mathbb{M}$ be classes of algebras of the same finite language. Then the following conditions are equivalent:*

- $\mathbb{K}$ *is partially embeddable into* $\mathbb{M}$
- $\mathbb{K} \subseteq \mathbf{ISP}_U(\mathbb{M})$

Locally finite classes. Classes with FEP, SFMP, FMP. A class $\mathbb{K}$ of algebras is *locally finite* (LF, for short) if, and only if, for every $\mathcal{A} \in \mathbb{K}$ and for every finite set $B \subseteq A$, the subalgebra generated by B is also finite. Notice that this property is inherited by the subclasses of $\mathbb{K}$. Given a class $\mathbb{K}$ of algebras, we denote by $\mathbb{K}_{fin}$ the class of its finite members. A class $\mathbb{K}$ of algebras has the *finite embeddability property* (FEP, for short) if, and only if, it is partially embeddable into $\mathbb{K}_{fin}$. A class $\mathbb{K}$ of algebras of the same type has the *strong finite model property* (SFMP, for short) if, and only if, every quasiequation that fails to hold in $\mathbb{K}$ can be refuted in some member of $\mathbb{K}_{fin}$. A class $\mathbb{K}$ of algebras of the same type has the *finite model property* (FMP, for short) if, and only if, every equation that fails to hold in $\mathbb{K}$ can be refuted in some member of $\mathbb{K}_{fin}$. It is clear that a variety has the FMP if, and only if, it is generated by its finite members and a quasivariety has the SFMP if, and only if, it is generated (as a quasivariety) by its finite members. An universal class (i.e., closed under $\mathbf{ISP}_U$) has de FEP if, and only if, it is generated by its finite members.

Theorem 1.9. (Blok-Van Alten [BvA02, Theorem 3.1]) *Let $\mathcal{L}$ be a finite algebraic language and let $\mathbb{K}$ be a class of algebras of type $\mathcal{L}$ closed under finite products. Then, $\mathbb{K}$ has the FEP if, and only if, $\mathbb{K}$ has the SFMP.*

Moreover it is clear that, for every class of algebras $\mathbb{K}$, we have:

- If $\mathbb{K}$ is locally finite, then it has the FEP.
- If $\mathbb{K}$ has the FEP, then it has the SFMP.
- If $\mathbb{K}$ has the SFMP, then it has the FMP.

Chapter 2

Deductive Systems

The logical systems that we will consider in this study are k-dimensional deductive systems and Gentzen systems, being these last a generalization of the first. In particular we will deal with 1-dimensional deductive systems, that is, deductive systems (or sentential logics, or propositional logics) in the sense of Blok and Pigozzi [BP89], and a type of 2-dimensional deductive system: the equational logic associated with a class of algebras (for more information about k-dimensional deductive systems see [BP92]). In this chapter we introduce the notions and basic results concerning deductive systems and the corresponding notions of Abstract Algebraic Logic (AAL) which we will use in this work. Basic references on AAL are the classic *Algebraizable Logics* of Blok and Pigozzi [BP89], the handbook of Czelakowski [Cze01] and the excellent overview of Font, Jansana and Pigozzi [FJP03]. The notions and results concerning Gentzen Systems will be applied in the following chapter.

Closure operators. Consequence relations. Let A a set and let $\mathcal{P}(A)$ be its their power set. A *closure operator* on A is a mapping

$$C : \mathcal{P}(A) \to \mathcal{P}(A)$$

such that, for each $X, Y \subseteq A$, the following conditions are satisfied:

1. $X \subseteq C(X)$,
2. if $X \subseteq Y$, then $C(X) \subseteq C(Y)$,
3. $C(C(X)) \subseteq C(X)$.

A closure operator C on A is *finitary* if, for every $X \subseteq A$,

$$C(X) = \bigcup \{C(F) : F \subseteq X,\ F \text{ finite}\}.$$

We say that a subset $X \subseteq A$ is a C-*closed* if $C(X) = X$.

A *consequence relation* on A is a relation $\vdash$ between subsets of A and elements of A (we write $X \vdash a$ instead of $\langle X, a\rangle \in \vdash$) such that the mapping $C : \mathcal{P}(A) \to \mathcal{P}(A)$ defined by $C(X) = \{a \in A : X \vdash a\}$ is a closure operator. It is easy to see that this is equivalent to saying that the relation $\vdash$ satisfies the following:

1) $a \in X$ implies $X \vdash a$,

2) $X \vdash a$ and $X \subseteq Y$ implies $Y \vdash a$,

3) $X \vdash a$ and $Y \vdash b$ for every $b \in X$ implies $Y \vdash a$.

Notice that property (2) is a consequence of properties (1) and (3). Observe that if C is the closure operator associated to $\vdash$, then X is a C-closed if, and only if,

$$a \in X \quad \text{iff} \quad X \vdash a.$$

An expression of the form $X \vdash a$ is called an *inference* or *consecution*. To say that $X \vdash a$ is not satisfied, we write $X \nvdash a$. As usual, we write $X,\ a \vdash b$ instead of $X \cup \{a\} \vdash b$ and we write $a_0, \ldots, a_{n-1} \vdash b$ instead of $\{a_0, \ldots, a_{n-1}\} \vdash b$. If $X \vdash b$ for every $b \in Y$, we write $X \vdash Y$.

A consequence relation $\vdash$ on A is *finitary* if the associated closure operator is finitary or, equivalently,

$$\text{if } X \vdash a, \text{ then } X' \vdash a \text{ for some } X' \subseteq X, \;\; X' \text{ finite.}$$

It is easy to see that this condition it is equivalent to the fact that the associated closure operator is finitary.

Let $\mathbf{A}$ be an algebra of universe A. A closure operator on A is *structural* if, for every $h \in Hom(\mathbf{A}, \mathbf{A})$ and every $X \subseteq A$, $h[C(X)] \subseteq C(h[X])$.[1] A consequence relation on A is *structural* if the associated closure operator is structural or, equivalently, if, for every $h \in Hom(\mathbf{A}, \mathbf{A})$, $X \vdash a$ implies $h[X] \vdash h(a)$.

Deductive systems. A *propositional language* $\mathcal{L}$ is an algebraic language. The symbols of $\mathcal{L}$ are called *propositional connectives*. A *deductive system* (or *sentential logic*, or *propositional logic*) $\mathcal{S}$ on $\mathcal{L}$ is a pair $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$, where $\mathcal{L}$ is a propositional language and $\vdash_{\mathcal{S}}$ is a structural consequence relation on the set of $\mathcal{L}$-formulas, $Fm_{\mathcal{L}}$. A deductive system $\mathcal{S}$ is called *finitary* if $\vdash_{\mathcal{S}}$ is finitary. Two formulas φ and ψ are $\mathcal{S}$*-equivalent* (notation: $\varsigma \dashv\vdash_{\mathcal{S}} \psi$) if $\varphi \vdash_{\mathcal{S}} \psi$ and $\psi \vdash_{\mathcal{S}} \varphi$ are simultaneously satisfied. A formula φ is called $\mathcal{S}$*-derivable* (or derivable in $\mathcal{S}$) if $\emptyset \vdash_{\mathcal{S}} \varphi$. In this case we also say that φ is a *theorem* of $\mathcal{S}$.

The homomorphisms $\sigma \in Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{Fm}_{\mathcal{L}})$ are called $\mathcal{L}$*-substitutions* (or simply *substitutions* when for the context it is clear which the language of reference is). The set

[1] Given a function f, we will use the notation $f[X]$ to denote the image of a subset X of the domain.

$Hom(\mathbf{Fm}_{\mathcal{L}}, \mathbf{Fm}_{\mathcal{L}})$ of *$\mathcal{L}$-substitutions* will be denoted by $Sub_{\mathcal{L}}$. A consequence relation on $Fm_{\mathcal{L}}$ which is structural is also referred to as *invariant under substitutions.* Given a formula $\varphi(p_1, \ldots, p_n)$,[2] we denote by $\varphi(p_1|\alpha_1, \ldots, p_n|\alpha_n)$ and also by $\varphi(\alpha_1, \ldots, \alpha_n)$ the formula $\sigma(\varphi)$, where σ is any substitution such that $\sigma(p_1) = \alpha_1, \ldots, \sigma(p_n) = \alpha_n$.

Decidability. A deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ is called *decidable* if, and only if, their inferences of the form $\Gamma \vdash_{\mathcal{S}} \varphi$, with Γ finite, are decidable, that is, if there is a procedure which it allows, in a finite number of steps, to gauge whether $\Gamma \vdash_{\mathcal{S}} \varphi$ or $\Gamma \nvdash_{\mathcal{S}} \varphi$. In the cases in which this condition is satisfied only in the case $\Gamma = \emptyset$ we say that *the set of theorems of $\mathcal{S}$ is decidable* or that *$\mathcal{S}$ is decidable with respect to the theorems.*

Hilbert-style axiomatic calculus. A *$\mathcal{L}$-rule of inference* is a set $r \subseteq \mathcal{P}_{fin}(Fm_{\mathcal{L}}) \times Fm_{\mathcal{L}}$[3] which is obtained as the closure under substitutions of a pair $\langle \Gamma, \varphi \rangle$ such that Γ is a finite set of $\mathcal{L}$-formulas (i.e., $\Gamma \in \mathcal{P}_{fin}(Fm_{\mathcal{L}})$) and φ is una $\mathcal{L}$-formula, i.e.,

$$r = \{\langle \sigma[\Gamma], \sigma(\varphi) \rangle : \sigma \in Sub_{\mathcal{L}}\}.$$

The pair $\langle \Gamma, \varphi \rangle$ is called the *generator* of r and the elements of this set are called *instances* of r. In the case $\Gamma \neq \emptyset$ the rules are called *proper rules*. Sometimes, to denote a rule we will use the notation

$$\frac{\Gamma}{\varphi}.$$

In the case $\Gamma = \emptyset$, the rules are called *axioms* and they are, therefore, sets of the form

$$\{\langle \emptyset, \sigma(\varphi) \rangle : \sigma \in Sub_{\mathcal{L}}\},$$

where φ is a $\mathcal{L}$-formula. We will identify the former set with the set of substitution instances of the formula φ, i.e., the set

$$\{\sigma(\varphi) : \sigma \in Sub_{\mathcal{L}}\}.$$

Let $\langle \Gamma, \varphi \rangle$ be a generator instance of a $\mathcal{L}$-rule r and let $\mathcal{S}$ be a deductive system on $\mathcal{L}$. We will say that r is *derivable* in $\mathcal{S}$ if, and only if, $\Gamma \vdash_{\mathcal{S}} \varphi$. Taking into account this definition, by structurality we have that r is derivable in $\mathcal{S}$ if, and only if, $\sigma[\Gamma] \vdash_{\mathcal{S}} \sigma(\varphi)$ for every $\sigma \in Sub_{\mathcal{L}}$. If r is derivable in $\mathcal{S}$ we also say that $\mathcal{S}$ *satisfies* the rule r.

A *$\mathcal{L}$-Hilbert-style (axiomatic) calculus* is a set $\mathbf{H}$ of $\mathcal{L}$-rules (axioms and proper rules[4]). It is usual to present the axioms and rules of a Hilbert-style calculus $\mathbf{H}$ by means of *schemas*: if φ is a $\mathcal{L}$-formula such that its variables are in $\{p_1, \ldots, p_n\}$, the *schema of formulas* generated by φ is an expression of the form $\varphi(p_1|\alpha_1, \ldots, p_n|\alpha_n)$,

[2]In logical contexts it is usual to use the letters p and q (possibly with subscript) instead of the letters x and y to denote variables.

[3]Given a set C, we denote by $\mathcal{P}_{fin}(C)$ the set of all its finite subsets C.

[4]It is usual to reserve the denomination *rule* for the proper rules.

where $\alpha_1, \ldots, \alpha_n$ are metavariables representing arbitrary formulas. Thus, for example, if we say that $\varphi \to (\psi \to \varphi)$ is an axiom of the calculus $\mathbf{H}$, this means that the set $\{\sigma(p) \to (\sigma(q) \to \sigma(p)) : \sigma \in Sub_{\mathcal{L}}\}$ is an axiom of $\mathbf{H}$ and if we say that $\langle \{\varphi \to \psi, \varphi\}, \psi \rangle$ is a rule of $\mathbf{H}$ it means that the set $\{\langle \{\sigma(p) \to \sigma(q), \sigma(p)\}, \sigma(q)\rangle : \sigma \in Sub_{\mathcal{L}}\}$ is a rule of $\mathbf{H}$. Thus, when we present the rules $\langle \Gamma, \varphi \rangle$ of a Hilbert-style calculus, Γ is a finite set of schemas and φ is a schema.

Every Hilbert-style $\mathcal{L}$- calculus $\mathbf{H}$ determines a finitary deductive system $\mathcal{S}_{\mathbf{H}} = \langle \mathcal{L}, \vdash_{\mathbf{H}} \rangle$ in the following way: For every $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, $\Gamma \vdash_{\mathcal{S}_{\mathbf{H}}} \varphi$ if, and only if, there is a finite sequence $\varphi_0, \ldots, \varphi_{n-1}$ of $\mathcal{L}$-formulas –called *proof of* φ *from* Γ *in* $\mathbf{H}$– such that $\varphi_{n-1} = \varphi$ and, for each $i < n$, one of the following conditions holds:

- φ_i is an instance of an axiom of $\mathbf{H}$,
- $\varphi_i \in \Gamma$,
- φ_i is obtained from $\{\varphi_j : j < i\}$ by using a rule r of $\mathbf{H}$, that is, there exists an instance $\langle \{\psi_0, \ldots, \psi_{m-1}\}, \psi \rangle$ of a rule r of $\mathbf{H}$ such that $\psi = \varphi_i$ and $\{\psi_0, \ldots, \psi_m\} \subseteq \{\varphi_0, \ldots, \varphi_{i-1}\}$.

In this case we will say that $\mathcal{S}_{\mathbf{H}}$ is the deductive system determined by $\mathbf{H}$. Obviously, this deductive system is finitary. Reciprocally, from every finitary deductive system $\mathcal{S}$ we can define a Hilbert-style calculus, which we will denote by $\mathbf{H}_{\mathcal{S}}$, in the following way: the axioms of $\mathbf{H}_{\mathcal{S}}$ are all the sets

$$\{\sigma(\varphi) : \sigma \in Sub_{\mathcal{L}}\},$$

where φ is a $\mathcal{L}$-formula such that $\emptyset \vdash_{\mathcal{S}} \varphi$, and its rules are all the sets

$$\{\langle \{\sigma(\varphi_0), \ldots, \sigma(\varphi_{n-1})\}, \varphi \rangle : \sigma \in Sub_{\mathcal{L}}\},$$

where $\varphi_0, \ldots, \varphi_{n-1}$ are $\mathcal{L}$-formulas such that $\varphi_0, \ldots, \varphi_{n-1} \vdash_{\mathcal{S}} \varphi$.

It is easy to see that $\mathcal{S}_{\mathbf{H}_{\mathcal{S}}} = \mathcal{S}$ (Łos, Suszko [ŁS58]).

Theorem 2.1 (Cf. [ŁS58]). *Let $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ be a deductive system. Then, S is finitary if, and only if, there exists a Hilbert-style calculus $\mathbf{H}$ such that $\vdash_{\mathbf{H}} = \vdash_{\mathcal{S}}$.*

A finitary deductive system is *finitely axiomatizable* if, and only if, it is defined by a Hilbert-style calculus with a finite number of axioms and rules.

Expansions, fragments, extensions. Given two propositional languages $\mathcal{L}$ and $\mathcal{L}'$ such that $\mathcal{L} \leq \mathcal{L}'$ and two deductive systems $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ and $\mathcal{S}' = \langle \mathcal{L}' \vdash_{\mathcal{S}'} \rangle$, we will say that $\mathcal{S}'$ is an *expansion* of $\mathcal{S}$ if, and only if, $\vdash_{\mathcal{S}} \subseteq \vdash_{\mathcal{S}'}$. In the case $\mathcal{L} = \mathcal{L}'$ we will say that $\mathcal{S}'$ is a *extension* of $\mathcal{S}$. The expansion is *conservative* if, and only if, for each $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, $\Gamma \vdash_{\mathcal{S}} \varphi$ iff $\Gamma \vdash_{\mathcal{S}'} \varphi$; in this case we say that $\mathcal{S}$ is the *$\mathcal{L}$-fragment* of $\mathcal{S}'$. We will use the notation $\mathcal{L}$-$\mathcal{S}'$ for the $\mathcal{L}$-fragment of a system $\mathcal{S}'$. We will say that $\mathcal{S}'$

is an *axiomatic expansion* of $\mathcal{S}$ if, and only if, $\mathcal{S}$ and $\mathcal{S}'$ are respectively defined by two Hilbert-style calculi $\mathbf{H}$ and $\mathbf{H}'$ which have the same proper rules and such that every axiom of $\mathbf{H}$ is also an axiom of $\mathbf{H}'$ (i.e., $\mathbf{H}'$ is obtained from adding axioms to $\mathbf{H}$). If, moreover, $\mathcal{L} = \mathcal{L}'$, then we will say that $\mathcal{S}'$ is an *axiomatic extension* of $\mathcal{S}'$.

Separability of a calculus. Separation theorems. Let $\mathbf{H}$ be a Hilbert-style axiomatic calculus and let Ψ be a sublanguage of the language of $\mathbf{H}$. We say that $\mathbf{H}$ is *separable for* Ψ if the theorems of the Ψ-fragment of the deductive system defined by $\mathbf{H}$ are derivable from the axioms and rules of $\mathbf{H}$ such that their generators contain only the connectives in Ψ. If, moreover, all the inferences of the Ψ-fragment (not only the theorems) of the deductive system defined by $\mathbf{H}$ are derivable in $\mathbf{H}$, then we will say that $\mathbf{H}$ is *strongly separable for* Ψ.

We will say that a deductive system satisfies the *Separation Theorem* (ST, for short) for Ψ if it admits a Hilbert-style axiomatization separable for Ψ. We will say that the system satisfies the *Strong Separation Theorem* (SST, for short) for Ψ if it admits a Hilbert-style axiomatization strongly separable for Ψ.[5]

k-dimensional Deductive Systems. Let $k \in \omega$. A *k-dimensional deductive system* on $\mathcal{L}$ (see [BP0x]) is a pair $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, where $\vdash_{\mathcal{S}}$ is structural consequence relation on $Fm^k_{\mathcal{L}}$ ($Fm^k_{\mathcal{L}} = \{\langle \varphi_0, \ldots, \varphi_{k-1} \rangle : \varphi_i \in Fm_{\mathcal{L}}, i < k\}$). We will say that $\mathcal{S}$ is *finitary* if $\vdash_{\mathcal{S}}$ is finitary. The set $Fm^k_{\mathcal{L}}$ is called *the set of the $\langle \mathcal{L}, k\rangle$-formulas* or, simply, of the *k-formulas* when the language is clear by the context. Sometimes $\langle \varphi_0, \ldots, \varphi_{n-1} \rangle$ is denoted by $\vec{\varphi}$. The 1-dimensional deductive systems are the deductive systems.

The k-dimensional deductive systems can be presented by means of an axiomatic calculus in a way analogous to that used with the deductive systems. A $\langle \mathcal{L}, k\rangle$-rule, or k-rule, is the closure under substitution of a pair $\langle \vec{\Gamma}, \vec{\varphi} \rangle$, where $\vec{\Gamma} \cup \{\vec{\varphi}\} \subseteq Fm^k_{\mathcal{L}}$. When $\vec{\Gamma} = \emptyset$ the rule is called *$\langle \mathcal{L}, k\rangle$-axiom*, or *$k$-axiom*. An axiomatic $\langle \mathcal{L}, k\rangle$-calculus is a set of $\langle \mathcal{L}, k\rangle$-rules. The extension of the notion of *proof* to this kind of calculi is done in the obvious way as in the case of deductive systems. Every (axiomatic) $\langle \mathcal{L}, k\rangle$-calculus $\mathbf{H}$ determines a finitary k-dimensional deductive system $\mathcal{S}_{\mathbf{H}}$ in the following way: for every $\vec{\Gamma} \cup \{\vec{\varphi}\} \subseteq Fm^k_{\mathcal{L}}$, $\vec{\Gamma} \vdash_{\mathcal{S}_{\mathbf{H}}} \vec{\varphi}$ if, and only if, there is a proof of $\vec{\varphi}$ from $\vec{\Gamma}$. It is easy to see that the following generalization of Theorem 2.1 holds.

Theorem 2.2. *Let $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ be a k-dimensional deductive system. Then, S is finitary if, and only if, there exists a $\langle \mathcal{L}, k\rangle$-calculus $\mathbf{H}$ such that $\vdash_{\mathbf{H}} = \vdash_{\mathcal{S}}$.*

A finitary k-dimensional deductive system is *finitely axiomatizable* if, and only if, it is defined by means a k-calculus which has a finite number of axioms and rules. The notions of *expansion*, *fragment* and *extension* are defined in a way analogous to the case of deductive systems.

[5] A common practice in the literature is to say that a deductive system *satisfies the (strong) separation theorem* when its language contains one connective of implication $\to$ and the system is defined by a Hilbert-style calculus (strongly) separable for every sublanguage Ψ such that $\to \in \Psi$.

For any class $\mathbb{K}$ of algebras, an example of 2-dimensional deductive system is the pair $\langle \mathcal{L}, \models_{\mathbb{K}} \rangle$, where $\models_{\mathbb{K}}$ is the equational consequence relation associated to $\mathbb{K}$ which is defined in the following paragraph.

Equational consequence. Let $\mathcal{L}$ be an proposicional algebraic language and let $\mathbb{K}$ be a class of $\mathcal{L}$-algebras. Then, $\models_{\mathbb{K}}$ will denote the *equational consequence relation* determined by $\mathbb{K}$, which is defined as follows:

For every $\Lambda \cup \{\varphi \approx \psi\} \subseteq Eq_{\mathcal{L}}$, $\Lambda \models_{\mathbb{K}} \varphi \approx \psi$ if, and only if, for every $\mathbf{A} \in \mathbb{K}$ and every homomorphism $h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$, the following holds:

$$\text{If } h(\alpha) = h(\beta) \text{ for each } \alpha \approx \beta \in \Lambda, \text{ then } h(\varphi) = h(\psi).$$

The following facts can be proved:

- $\models_{\mathbb{K}}$ is a structural consequence relation on $Eq_{\mathcal{L}}$ (i.e., on $Fm^2_{\mathcal{L}}$).
- $\models_{\mathbb{K}} = \models_{\mathbf{ISP}(\mathbb{K})}$.
- If $\mathbf{P}_U(\mathbb{K}) \subseteq \mathbb{K}$, then $\models_{\mathbb{K}}$ is finitary.
- For every class of algebras $\mathbb{K}$, $\models_{\mathbb{K}}$ is finitary if, and only if, it coincides with the equational consequence relation determined by the quasivariety generated by$\mathbb{K}$, i.e., $\models_{\mathbb{K}} = \models_{\mathbf{Q}(\mathbb{K})}$.

Let $\models_{\mathbb{K}}$ be finitary and let $\Lambda \subseteq QEq_{\mathcal{L}}$ be such that $\mathbf{Q}(\mathbb{K}) = \{\mathbf{A} : \mathbf{A} \models \Lambda\}$. Then, by identifying the pair $\langle \varphi, \psi \rangle$ with the equation $\varphi \approx \psi$, we have that $\langle \mathcal{L}, \models_{\mathbb{K}} \rangle$ is the 2-dimensional deductive system axiomatized by the following axioms and rules (Cf. [BP0x]):

1. $\varphi \approx \varphi$
2. $\langle \{\varphi \approx \psi\}, \psi \approx \varphi \rangle$
3. $\langle \{\varphi \approx \psi, \psi \approx \gamma\}, \varphi \approx \gamma \rangle$
4. $\langle \{\varphi_0 \approx \psi_0, \ldots, \varphi_{n-1} \approx \psi_{n-1}\}, \iota(\varphi_0, \ldots, \varphi_{n-1}) \approx \iota(\psi_0, \ldots, \psi_{n-1}) \rangle$,
   for each n-ary connective ι,
5. $\varphi \approx \psi$, for each equation $\varphi \approx \psi \in \Lambda$, and
6. $\langle \{\varphi_0 \approx \psi_0, \ldots, \varphi_{n-1} \approx \psi_{n-1}\}, \varphi \approx \psi \rangle$,
   for every $n \geq 1$ and every $\varphi_0 \approx \psi_0 \& \ldots \& \varphi_{n-1} \approx \psi_{n-1} \Rightarrow \varphi \approx \psi \in \Lambda$.

Algebraic semantics. Equivalent algebraic semantics. Algebraization. Let $\mathcal{S}$ be a finitary deductive system and let $\mathbb{K}$ be a class of $\mathcal{L}$-algebras. $\mathbb{K}$ is an *algebraic semantics* for $\mathcal{S}$ if, and only if, there exists a finite set of $\mathcal{L}$-equations in one variable

$$\delta(p) \approx \varepsilon(p) := \{\delta_i(p) \approx \varepsilon_i(p) : i < n\},$$

called *system* or *set of defining equations* such that, for every $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$,

$$\Gamma \vdash_{\mathrm{S}} \varphi \quad \textit{iff} \quad \{\delta(\psi) \approx \varepsilon(\psi) : \psi \in \Gamma\} \models_{\mathbb{K}} \delta(\varphi) \approx \varepsilon(\varphi). \tag{2.1}$$

Proposition 2.3. (Blok-Pigozzi [BP89, Corollary 2.3]) *If $\mathbb{K}$ is an algebraic semantics for a finitary deductive system $\mathcal{S}$, then $\mathbf{Q}(\mathbb{K})$ is also an algebraic semantics for $\mathcal{S}$ with the same system of defining equations.*

An algebraic semantics for $\mathcal{S}$ which is a quasivariety is called *quasivariety semantics* for $\mathcal{S}$. If a deductive system has an algebraic semantics, then, by the last proposition it also has a quasivariety semantics.

Given a deductive system $\mathcal{S}$, we will say that a class $\mathbb{K}$ of $\mathcal{L}$-algebras is an *equivalent algebraic semantics* for $\mathcal{S}$ if, and only if is an algebraic semantics for $\mathcal{S}$ and there exists a finite set of formulas in two variables

$$\Delta(p,q) := \{\Delta_j(p,q) : j < m\},$$

called *set of equivalence formulas*, such that, for every $\varphi \approx \psi \in Eq_{\mathcal{L}}$,

$$\varphi \approx \psi \;=\!\!|\!\!\models_K \delta(\Delta(\varphi,\psi)) \approx \varepsilon(\Delta(\varphi,\psi)), \tag{2.2}$$

where $\delta \approx \varepsilon$ is the system of defining equations of $\mathbb{K}$ for $\mathcal{S}$ and $\delta(\Delta(\varphi,\psi)) \approx \varepsilon(\Delta(\varphi,\psi))$ is an abbreviation for $\{\delta_i(\Delta_j(\varphi,\psi)) \approx \epsilon_i(\Delta_j(\varphi,\psi)) : i < n, j < m\}$.

The joint satisfaction of conditions (2.1) and (2.2) is equivalent to the satisfaction of the following two conditions ([BP89, Corollary 2.9]).

- For every $\Lambda \cup \{\varphi \approx \psi\} \subseteq Eq_{\mathcal{L}}$,

$$\Lambda \models_K \varphi \approx \psi \quad iff \quad \{\Delta(\xi,\eta) : \xi \approx \eta \in \Lambda\} \vdash_{\mathcal{S}} \Delta(\varphi,\psi). \tag{2.3}$$

- For every $\varphi \in Fm_{\mathcal{L}}$,

$$\varphi \dashv\vdash_{\mathcal{S}} \Delta(\delta(\varphi), \varepsilon(\varphi)). \tag{2.4}$$

A finitary deductive system is called *algebraizable* if, and only if, it has an equivalent algebraic semantics.[6]

[6]The notion of algebraizable logic that we present here is as in[BP89], where the deductive systems considered are finitary and where the sets $\delta(p)$ and $\Delta(p,q)$ are finite. At present (see for example, [FJP03]) these notions are defined in a more general frame, since the deductive systems are not necessarily finitary and the mentioned sets can be infinite. An algebraizable deductive system such that the sets $\delta(p)$ and $\Delta(p,q)$ are finite it is called *finitely algebraizable*. Thus, in the present work *algebriaizable system* means *finitely algebraizable system* in the sense of [FJP03].

Proposition 2.4. (Blok-Pigozzi [BP89, Corollary 2.11]) *Let $\mathbb{K}$ be an algebraic semantics for a finitary deductive system $\mathcal{S}$. Then, $\mathbb{K}$ is equivalent for $\mathcal{S}$ if, and only if, $\mathbf{Q}(\mathbb{K})$ is.*

Theorem 2.5. (Blok-Pigozzi [BP89, Theorem 2.15]) *If $\mathbb{K}$ and $\mathbb{K}'$ are two equivalent algebraic semantics for $\mathcal{S}$, then $\mathbf{Q}(\mathbb{K}) = \mathbf{Q}(\mathbb{K}')$.*

This quasivariety is called *the equivalent quasivariety semantics* of $\mathcal{S}$. The following theorem describes a method to axiomatize the equivalent quasivariety semantics of an algebrizable deductive system $\mathcal{S}$ from any Hilbert-style axiomatization of $\mathcal{S}$.

Theorem 2.6. (Blok-Pigozzi [BP89, Corollary 2.17]) *Let $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ be an algebraizable deductive system and let $\mathbb{K}$ be its equivalent quasivariety semantics. Suppose that $\delta \approx \varepsilon$ is the system of defining equations, $\Delta(p, q)$ is the set of equivalence formulas, and $\mathcal{S}$ is defined by a Hilbert-style calculus $\mathbf{H}$ with set of axioms $AX_{\mathbf{H}}$ and set of proper rules $R_{\mathbf{H}}$. Then the equivalent quasivariety semantics of $\mathcal{S}$ is defined by the following quasiequations:*

1. $\delta(\varphi) \approx \varepsilon(\varphi)$, *for each* $\varphi \in AX_{\mathbf{H}}$,
2. $\delta(\Delta(\varphi, \varphi)) \approx \varepsilon(\Delta(\varphi, \varphi))$,
3. $\delta(\varphi_0) \approx \varepsilon(\varphi_0) \& \ldots \& \delta(\varphi_{n-1}) \approx \varepsilon(\varphi_{n-1}) \Rightarrow \delta(\varphi) \approx \varepsilon(\varphi)$,
   for each $\langle \{\varphi_0, \ldots, \varphi_{n-1}\}, \varphi \rangle \in R_{\mathbf{H}}$, *and*
4. $\delta(\Delta(\varphi, \psi)) \approx \varepsilon(\Delta(\varphi, \psi)) \Rightarrow \varphi \approx \psi$.

Algebraization of extensions, fragments and expansions. The following result states that every finitary extension of an algebraizable deductive system is algebraizable and that there is a dual order isomorphism between the lattice of subquasivarieties of its equivalent algebraic semantics and the lattice of its finitary extensions.

Theorem 2.7 (Cf. [BP89]). *Let $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ be an algebraizable deductive system and let $\mathbb{K}$ be its equivalent quasivariety semantics. Let $\delta \approx \varepsilon$ be the system of defining equations and $\Delta(p, q)$ the set of equivalence formulas. Then, every finitary extension of* S *is algebraizable (with the same defining equations and equivalence formulas) and there is a dual order isomorphism between the lattice of subquasivarieties of $\mathbb{K}$ and the lattice of finitary extensions of* S *defined in the following way:*

To each finitary extension $\mathcal{S}'$ of $\mathcal{S}$ with the finite sets AX of axioms and R of proper rules we assign its equivalent quasivariety semantics, i.e., the subquasivariety of $\mathbb{K}$ defined by the quasiequations

- $\delta(\varphi) \approx \varepsilon(\varphi)$, *for each* $\varphi \in AX$,
- $\delta(\varphi_0) \approx \varepsilon(\varphi_0) \& \ldots \& \delta(\varphi_{n-1}) \approx \varepsilon(\varphi_{n-1}) \Rightarrow \delta(\varphi) \approx \varepsilon(\varphi)$,
  for every $\langle \{\varphi_0, \ldots, \varphi_{n-1}\}, \varphi \rangle \in R$.

The inverse of this mapping assigns to each subquasivariety $\mathbb{K}' \subseteq \mathbb{K}$ defined by a set of quasiequations Λ the finitary extension $\mathcal{S}'$ of $\mathcal{S}$ obtained by adding to the rules of $\mathcal{S}$ the set of rules

$$\{\langle \Delta(\varphi_0, \psi_0), \ldots, \Delta(\varphi_n, \psi_n)\rangle : \varphi_0 \approx \psi_0 \& \ldots \& \varphi_{n-1} \approx \psi_{n-1} \Rightarrow \varphi_n \approx \psi_n \in \Lambda\}.$$

When the equivalent quasivariety semantics is a variety, by restricting the dual order isomorphism of the last theorem, we also obtain a bijective correspondence between axiomatic extensions and subvarieties.

The following result states that every fragment of an algebraizable deductive system containing in its language all the connectives which appear in the defining equations and in the equivalence formulas is also algebraizable (cf. Theorem 1.6).

Theorem 2.8. (Blok-Pigozzi [BP89, Corollary 2.12]) *Let $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$ be an algebraizable deductive system and let $\mathbb{K}$ be its equivalent quasivariety semantics. Let $\mathcal{L}' \leq \mathcal{L}$ be such that $\mathcal{L}'$ contains all the connectives appearing in the defining equations and the equivalence formulas. Then, the $\mathcal{L}'$-fragment of $\mathcal{S}$ is algebraizable with the same defining equations and equivalence formulas. Moreover, if $\mathbb{K}'$ is the class of all the $\mathcal{L}'$-reducts of members of $\mathbb{K}$, then $\mathbf{IS}(\mathbb{K}')$ is the equivalent quasivariety semantics of the $\mathcal{L}'$-fragment of $\mathcal{S}$.*

Some expansions of an algebraizable deductive system are also algebraizable as the following result establishes.

Theorem 2.9 (Cf. [BP89])**.** *Let $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$ be an algebraizable logic and let $\mathbb{K}$ be its equivalent quasivariety semantics. Let $\delta \approx \varepsilon$ be the system of defining equations and $\Delta(p, q)$ the set of equivalence formulas. Let $\mathcal{L}'$ be an expansion of $\mathcal{L}$ and $\mathcal{S}' = \langle \mathcal{L}', \vdash_{\mathcal{S}'}\rangle$ an expansion of $\mathcal{S}$ obtained by adding a set AX of axioms and a set R of proper rules to a set of axioms and rules for $\mathcal{S}$. Assume that, for every new n-ary connective $\iota \in \mathcal{L}' \backslash \mathcal{L}$, with $n \geq 1$, we have*

$$\Delta(p_1, q_1) \cup \cdots \cup \Delta(p_n, q_n) \vdash_{\mathcal{S}'} \Delta(\iota(p_1, \ldots, p_n), \iota(q_1, \ldots, q_n)).$$

Then,

a) $\mathcal{S}'$ is algebraizable with the same system of defining equations and the same set of equivalence formulas and its equivalent quasivariety semantics $\mathbb{K}'$ is axiomatized by a set of quasiequacions defining $\mathbb{K}$ plus the quasiequations:

i) $\delta(\varphi) \approx \epsilon(\varphi)$, for each $\varphi \in AX$,

ii) $\delta(\varphi_0) \approx \epsilon(\varphi_0) \,\&\ldots\&\, \delta(\varphi_n) \approx \epsilon(\varphi_n) \Longrightarrow \delta(\varphi) \approx \epsilon(\varphi)$,
for every $\langle\{\varphi_0, \ldots, \varphi_n\}, \varphi\rangle \in R$.

b) $\mathcal{S}'$ is a conservative expansion of $\mathcal{S}$ if, and only if, every algebra in $\mathbb{K}$ is a subreduct of $\mathbb{K}'$.

An intrinsic characterization for algebraizable systems. The following result provides an intrinsic characterization of the algebraizables logics, that is, a characterization that does not make reference to its semantics.

Theorem 2.10. (Blok-Pigozzi [BP89, Theorem 4.7])
A deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ is algebritzable if, and only if, there is a set of formulas in two variable $\Delta(p,q) \subseteq Fm_{\mathcal{L}}$ and a system of equations in one variable $\delta \approx \varepsilon \subseteq Eq_{\mathcal{L}}$ such that the following conditions hold:

For every $\varphi, \psi, \gamma \in Fm_{\mathcal{L}}$,

i) $\vdash_{\mathcal{S}} \Delta(\varphi, \varphi)$,

ii) $\Delta(\varphi, \psi) \vdash_{\mathcal{S}} \Delta(\psi, \varphi)$,

iii) $\Delta(\varphi, \psi) \cup \Delta(\psi, \gamma) \vdash_{\mathcal{S}} \Delta(\varphi, \gamma)$.

For every n-ary connective ι of $\mathcal{L}$,

iv) $\Delta(p_1, q_1) \cup \ldots \cup \Delta(p_n, q_n) \vdash_{\mathrm{S}} \Delta(\iota(p_1, \ldots, p_n), \iota(q_1, \ldots, q_n))$.

For each $\varphi \in Fm_{\mathcal{L}}$,

v) $\varphi \dashv\vdash_{\mathcal{S}} \Delta(\delta(\varphi), \varepsilon(\varphi))$.

In this case, Δ and $\delta \approx \varepsilon$ are a set of equivalence formulas and a system of defining equations for $\mathcal{S}$, respectively.

Theories, matrix, models and filters in deductive systems. Let $\mathcal{S} = \langle \mathcal{L} \vdash_{\mathcal{S}} \rangle$ a deductive system. An subset T of $Fm_{\mathcal{L}}$ is a $\mathcal{S}$*-theory* if $T \vdash_{\mathcal{S}} \varphi$ implies $\varphi \in T$. The set of all the $\mathcal{S}$-theories is denoted by $Th\,\mathcal{S}$. A $\mathcal{L}$*-matrix*, o simply a *matrix*, is a pair $\langle \mathbf{A}, F \rangle$ where $\mathbf{A}$ is a $\mathcal{L}$-algebra and $F \subseteq A$. Each $\mathcal{L}$-matrix defines a deductive system in the language $\mathcal{L}$: given a matrix $\langle \mathbf{A}, F \rangle$, $\models_{\langle \mathbf{A}, F \rangle}$ is the consequence relation defined by $\Gamma \models_{\langle \mathbf{A}, F \rangle} \varphi$ if, and only if, for each homomorphism h from $\mathbf{Fm}_{\mathcal{L}}$ into $\mathbf{A}$, if $h[\Gamma] \subseteq F$ then $h(\varphi) \in F$. This consequence relation is invariant under substitutions and thus $\langle \mathcal{L}, \models_{\langle \mathbf{A}, F \rangle} \rangle$ is a deductive system. In the case that $\vdash_{\mathcal{S}} \leq \models_{\langle \mathbf{A}, F \rangle}$ (i.e., for all $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, if $\gamma \vdash_{\mathcal{S}} \varphi$ then $\Gamma \models_{\langle \mathbf{A}, F \rangle} \varphi$) then we will say that $\langle \mathbf{A}, F \rangle$ is a $\mathcal{S}$*-model* and that F is a $\mathcal{S}$*-filter* of $\mathbf{A}$. If $\mathcal{S}$ is defined by means of a Hilbert-style calculus $\mathbf{H}$, it can be proved that F is a $\mathcal{S}$-filter if, and only if, F contains all the interpretations of the axioms of $\mathbf{H}$ and it is closed under all the rules of $\mathbf{H}$. It is also easy to see that T is a $\mathcal{S}$-theory if, and only if, T is a $\mathcal{S}$-filter of $\mathbf{Fm}_{\mathcal{L}}$.

The Leibniz operator. Let $\mathbf{A}$ be a $\mathcal{L}$-algebra. Given a $\mathcal{L}$-matrix $\langle \mathbf{A}, F\rangle$, the *Leibniz congruence* $\Omega_{\mathbf{A}}F$ of the matrix $\langle \mathbf{A}, F\rangle$ is the equivalence relation on A defined in the following way:

$$\langle a, b\rangle \in \Omega_{\mathbf{A}}F \textit{ iff } \begin{cases} \text{per a tot } k \in \omega,\ \varphi(p, q_0, \ldots, q_{k-1}) \in Fm_{\mathcal{L}} \text{ i } c_0, \ldots, c_{k-1} \in A, \\ \varphi^{\mathbf{A}}(a, c_0, \ldots, c_{k-1}) \in F \quad \Leftrightarrow \quad \varphi^{\mathbf{A}}(b, c_0, \ldots, c_{k-1}) \in F. \end{cases}$$

We emphasize that this definition does not depend on any deductive system. It is easy to show that $\Omega_{\mathbf{A}}F$ is characterized by the fact that it is the largest congruence of $\mathbf{A}$ that is compatible with F (i.e., if $a \in F$ and $\langle a, b\rangle \in \theta$, then $b \in F$). The mapping $\Omega_{\mathbf{A}} : \mathcal{P}(A) \longrightarrow Con(\mathbf{A})$ assigning to each $F \subseteq A$ the congruence $\Omega_{\mathbf{A}}(F)$ is called the *Leibniz operator* on $\mathbf{A}$.

If $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{G}}\rangle$ is a deductive system and T is a $\mathcal{S}$-theory, then $\langle \mathbf{Fm}_{\mathcal{L}}, T\rangle$ is a $\mathcal{L}$-matrix which is a $\mathcal{S}$-model. In the case $\mathbf{A} = \mathbf{Fm}_{\mathcal{L}}$ the Leibniz operator will be denoted by Ω instead of $\Omega_{\mathbf{Fm}_{\mathcal{L}}}$. Notice that, for this particular type of matrix, the Leibniz congruence is characterized in the following way:

$$\langle \alpha, \beta\rangle \in \Omega T \quad \text{sii} \quad \begin{cases} \text{for every } k \in \omega \text{ and every } \varphi(p, q_0, \ldots, q_{k-1}) \in Fm_{\mathcal{L}}, \\ \varphi(p\,|\alpha, q_0, \ldots, q_{k-1}) \in T \quad \Leftrightarrow \quad \varphi(p\,|\beta, q_0, \ldots, q_{k-1}) \in T. \end{cases}$$

Protoalgebricity. Given a deductive system there are several notions that measure the closeness to an equational logic. One example of this, introduced above, is the presence or absence of an algebraic semantics. Another example is the hierarchy developed in the Abstract Algebraic Logic framework [BP89, Cze01, FJP03]. This hierarchy classifies at different levels the deductive systems that enjoy a certain good correspondence with respect to equational logics. While algebraizability corresponds to the strongest relationship between the logical side and the algebraic side, protoalgebraicity corresponds to the weakest relationship (inside this hierarchy). A deductive system $\mathcal{S}$ is *protoalgebraic* when for every algebra $\mathbf{A}$ the Leibniz operator $\Omega_{\mathbf{A}}$ is monotone on the set of all $\mathcal{S}$-filters of $\mathbf{A}$, i.e., if F and G are $\mathcal{S}$-filters and $F \subseteq G$ then $\Omega_{\mathbf{A}}(F) \subseteq \Omega_{\mathbf{A}}(G)$ or, equivalently, if T_1 and T_2 are $\mathcal{S}$-theories and $T_1 \subseteq T_2$, then $\Omega T_1 \subseteq \Omega T_2$. It is known that a deductive system is protoalgebraic iff there is a set of formulas $\Delta(p, q)$ in at most two variables such that

1) for every formula $\delta(p, q) \in \Delta$, $\emptyset \vdash_S \delta(p, p)$,

2) $p, \Delta(p, q) \vdash_{\mathcal{S}} q$.

From this it follows that protoalgebraicity is preserved under extensions and conservative expansions (*monotonicity*). Another interesting property is that all algebraizable deductive systems are protoalgebraic. Finally, we notice that if a deductive system $\mathcal{S}$ is not protoalgebraic, then there is no binary connective $\rightarrow$ such that

1) $\emptyset \vdash_{\mathcal{S}} p \rightarrow p$ (Identity),

2) $p, p \to q \vdash_{\mathcal{S}} q$ (Modus Ponens).

Therefore, the protoalgebraicity of a deductive system, roughly speaking, means that there is no way of obtaining a natural implication inside it. The Hilbert-style axiomatizations for this kind of systems cannot be the usual ones based on the rule of *modus ponens*.

Selfextensional, extensional and intensional deductive systems. Given a deductive system $\mathcal{S}$ and a set Σ of formulas, the *Frege relation of* Σ *relative to* $\mathcal{S}$, in symbols $\mathbf{\Lambda}_{\mathcal{S}}\Sigma$, is the equivalence relation on $\mathbf{Fm}_{\mathfrak{L}}$ defined as follows:

$$\mathbf{\Lambda}_{\mathcal{S}}\Sigma := \{\langle \varphi, \psi \rangle : \Sigma, \varphi \vdash_{\mathcal{S}} \psi \text{ and } \Sigma, \psi \vdash_{\mathcal{S}} \varphi\}.$$

Thus, $\langle \varphi, \psi \rangle \in \mathbf{\Lambda}_{\mathcal{S}}\Sigma$ if and only if φ and ψ belong to the same $\mathcal{S}$-theories that extend Σ. $\mathcal{S}$ is a *selfextensional* deductive system if $\mathbf{\Lambda}_{\mathcal{S}}\emptyset$ is a congruence of the formula algebra. If additionally it holds that $\mathbf{\Lambda}_{\mathcal{S}}\Sigma$ is a congruence of the formula algebra for every set Σ of formulas, then $\mathcal{S}$ is an *extensional* (or *Fregean*) deductive system. The deductive systems that are not extensional are called *intensional* or *non Fregean.* The interest in selfextensional deductive systems comes from the work of Wójcicki [Wój88, Wój03], where they are characterized as referential (i.e., the deductive systems admitting a certain kind of Kripke semantics). For additional information on the notions of this paragraph see [FJP03] and the references therein.

Chapter 3

Gentzen Systems

Most of the literature on Gentzen systems, and on deductive systems, focuses only on their derivable sequents, i.e., on the sequents derivable without any hypothesis. In our approach we analyze the full consequence relation admitting hypotheses in the proofs. The reader should bear in mind this difference between our approach and the one commonly considered in the literature. In this chapter we recall the notions and results which, from this more general perspective, will be needed later. The results in this chapter are presented without any proofs; the reader interested in the proofs (or a more detailed presentation) can check [RV93, GTV97] or the monograph of Rebagliato and Verdú on Gentzen Systems [RV95].

Gentzen Systems. Let $\mathcal{L}$ be a propositional language. From now on, we will use the Greek uppercase letters $\Gamma, \Delta, \Sigma, \Pi$ and Λ for finite (maybe empty) sequences of $\mathcal{L}$-formulas. Given $m, n \in \omega$, an *$\mathcal{L}$-sequent of type $\langle m, n\rangle$* is a pair $\varsigma = \langle \Gamma, \Delta\rangle$ of finite sequences of $\mathcal{L}$-formulas such that the length of Γ is m and the length of Δ is n. While ς will refer to a $\mathcal{L}$-sequent, we will use the metavariable Φ for sets of $\mathcal{L}$-sequents. We will write $\emptyset$ for the empty sequence[1], φ for $\langle\varphi\rangle$, $\Gamma \Rightarrow \Delta$ for the sequent $\langle \Gamma, \Delta\rangle$, and $\varphi_0, \ldots, \varphi_{m-1} \Rightarrow \psi_0, \ldots, \psi_{n-1}$ instead of $\langle \varphi_0, \ldots, \varphi_{m-1}\rangle \Rightarrow \langle \psi_0, \ldots, \psi_{n-1}\rangle$. Given a set $\mathcal{T} \subseteq \omega \times \omega$ we will denote by $Seq^{\mathcal{T}}_{\mathcal{L}}$ the set of all $\mathcal{L}$-sequents with type belonging to $\mathcal{T}$. Let $\mathcal{L}$ and $\mathcal{L}'$ be two propositional languages and let $\mathcal{T} \subseteq \omega \times \omega$. Given a mapping $f : Fm_{\mathcal{L}} \to Fm_{\mathcal{L}}$, we will also denote by f the mapping from $Seq^{\mathcal{T}}_{\mathcal{L}}$ into $Seq^{\mathcal{T}}_{\mathcal{L}'}$ defined in the following way:

$$f(\varphi_0, \ldots, \varphi_{n-1} \Rightarrow \psi_0, \ldots, \psi_{m-1}) :=$$

$$:= \begin{cases} f(\varphi_0), \ldots, f(\varphi_{n-1}) \Rightarrow f(\psi_0), \ldots, f(\psi_{m-1}), & \text{if } n \neq 0, m \neq 0, \\ \emptyset \Rightarrow f(\psi_0), \ldots, f(\psi_{m-1}), & \text{if } n = 0, m \neq 0, \\ f(\varphi_0), \ldots, f(\varphi_{n-1}) \Rightarrow \emptyset, & \text{if } n \neq 0, m = 0, \\ \emptyset \Rightarrow \emptyset, & \text{if } \langle n, m\rangle = \langle 0, 0\rangle. \end{cases}$$

[1] The context will tell us if this symbol denotes the empty set or the empty sequence.

A *Gentzen system* is a triple $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language, $\mathcal{T}$ is a non-empty set of pairs of natural numbers, and $\vdash$ is a relation between subsets of $Seq_{\mathcal{L}}^{\mathcal{T}}$ and elements of $Seq_{\mathcal{L}}^{\mathcal{T}}$ satisfying the following conditions.

1) If $\varsigma \in \Phi$, then $\Phi \vdash \varsigma$.

2) If $\Phi \vdash \varsigma$ and for every $\varsigma' \in \Phi$, $\Phi' \vdash \varsigma'$, then $\Phi' \vdash \varsigma$.

3) If $\Phi \vdash \varsigma$ and $\Phi \subseteq \Phi'$, then $\Phi' \vdash \varsigma$.

4) If $\Phi \vdash \varsigma$, then $\sigma[\Phi] \vdash \sigma(\varsigma)$ for any substitution $\sigma \in Sub_{\mathcal{L}}$.

The first three conditions say that $\vdash$ is a *consequence relation* on the set $Seq_{\mathcal{L}}^{\mathcal{T}}$, and the last one is called *invariance under substitutions*. The Gentzen system is *finitary* if, moreover, it satisfies the following condition:

5) If $\Phi \vdash \varsigma$, then there is a finite subset Φ' of Φ such that $\Phi' \vdash \varsigma$.

For the sake of simplicity, we will only consider finitary Gentzen systems. Thus, we will refer to finitary Gentzen systems simply as Gentzen systems. As usual, we will write $\Phi, \varsigma \vdash \varsigma'$ instead of $\Phi \cup \{\varsigma\} \vdash \varsigma'$. The set $\mathcal{T}$ is called the *type* of $\mathcal{G}$. The components of a Gentzen system $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash \rangle$ sometimes will be written respectively as $\mathcal{L}(\mathcal{G})$, $\mathcal{T}(\mathcal{G})$ and $\vdash_{\mathcal{G}}$ since this avoids any ambiguity. Two sequents ς and ς' are *$\mathcal{G}$-equivalent* (notation: $\varsigma \dashv\vdash_{\mathcal{G}} \varsigma'$ or simply $\varsigma \dashv\vdash \varsigma'$) if it holds at the same time that $\varsigma \vdash_{\mathcal{G}} \varsigma'$ i $\varsigma' \vdash_{\mathcal{G}} \varsigma$. A sequent ς is said *$\mathcal{G}$-derivable* if $\emptyset \vdash_{\mathcal{G}} \varsigma$.

The definition of Gentzen system generalizes the notion of deductive system defined by Blok and Pigozzi in [BP89]. A *deductive system* $\mathcal{S}$ is no less than a Gentzen system of type $\{0\} \times \{1\}$, where the formula φ is identified with the sequent $\emptyset \Rightarrow \varphi$.

Decidability. A Gentzen system $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash_{\mathcal{S}} \rangle$ is said to be *decidable* if and only if its inferences of the form $\Phi \vdash_{\mathcal{G}} \varsigma$, with Φ finite, are decidable, that is, if and only if there exists a procedure which allows us, in a finite number of steps, to determine if $\Phi \vdash_{\mathcal{G}} \varsigma$ or if $\Phi \nvdash_{\mathcal{G}} \varsigma$. In the case that this condition is only satisfied only when $\Phi = \emptyset$ we say that *the set of derivable sequents of $\mathcal{G}$ is decidable* or that *$\mathcal{G}$ is decidable with respect to the derivable sequents.*

Sequent calculi. An *$\langle \mathcal{L}, \mathcal{T} \rangle$-rule* is a set $r \subseteq \mathcal{P}_{fin}(Seq_{\mathcal{L}}^{\mathcal{T}}) \times Seq_{\mathcal{L}}^{\mathcal{T}}$ that is obtained as the closure under substitutions of a pair $\langle \Phi, \varsigma \rangle$ such that Φ is a finite subset of $\mathcal{L}$-sequents (i.e., $\Phi \in \mathcal{P}_{fin}(Seq_{\mathcal{L}}^{\mathcal{T}})$) and ς is an $\mathfrak{L}$-sequent. We will use the pair $\langle \Phi, \varsigma \rangle$ as a name for the rule that it generates. The rules $\langle \emptyset, \varsigma \rangle$ are called *axioms*, and then ς is called an *instance* of the axiom. A rule $r = \langle \Phi, \varsigma \rangle$ is *derivable* in a Gentzen system $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash_{\mathcal{G}} \rangle$ if $\Phi \vdash_{\mathcal{G}} \varsigma$; in this case it is also said that $\mathcal{G}$ *satisfies* the rule r. A weaker notion than the derivability of a rule is its admissibility. A rule r is *admissible* in the Gentzen system $\mathcal{G}$ if for every $\langle \Upsilon, \eta \rangle \in r$ and every substitution e, the $\mathcal{G}$-derivability of all sequents in $\{e(\varsigma) : \varsigma \in \Upsilon\}$ implies the $\mathcal{G}$-derivability of the sequent $e(\eta)$. Generally we will write a rule $\langle \Phi, \varsigma \rangle$ as $\dfrac{\Phi}{\varsigma}$.

An $\langle \mathcal{L}, \mathcal{T} \rangle$-*sequent calculus* is a set of $\langle \mathcal{L}, \mathcal{T} \rangle$-rules. Every $\langle \mathcal{L}, \mathcal{T} \rangle$-sequent calculus $\mathcal{C}$ determines a Gentzen system $\mathcal{G}_{\mathcal{C}} = \langle \mathcal{L}, \mathcal{T}, \vdash_{\mathcal{C}} \rangle$ in the following way: given $\Phi \cup \{\varsigma\} \subseteq Seq_{\mathcal{L}}^{\mathcal{T}}$, $\Phi \vdash_{\mathcal{C}} \varsigma$ if and only if there is a finite sequence $\varsigma_0, \ldots, \varsigma_{n-1}$ of $Seq_{\mathcal{L}}^{\mathcal{T}}$ (which is called a *proof* in $\mathcal{C}$ of ς from Φ) such that $\varsigma_{n-1} = \varsigma$ and for each $i < n$ one of the following conditions hold:

- ς_i is an instance of an axiom of $\mathcal{C}$,
- $\varsigma_i \in \Phi$,
- ς_i is obtained from $\{\varsigma_j : j < i\}$ by using a rule r of $\mathcal{C}$.

In this case we will say that $\mathcal{G}_{\mathcal{C}}$ is the *Gentzen system determined by the sequent calculus* $\mathcal{C}$. Again we emphasize that we have used the rules of the calculus to obtain sequents from sets of sequents, not only from the empty set.

In this work we consider two kind of rules: *structural rules* and *rules of introduction for the connectives.*

In the *structural rules* the connectives of the language do not appear explicitly; their common characteristic is that they admit instances formed only by variables.

The calculi that we will consider have their set of types between $\omega \times \{1\}$, $\omega \times \{0,1\}$ or $\omega \times \omega$. Moreover, all the considered calculi will have, among their structural rules,[2], the axiom

$$\varphi \Rightarrow \varphi \quad (Axiom)$$

and the version of the rule (Cut) adequate to the corresponding set of types. Thus,

- the $\langle \mathcal{L}, \omega \times \omega \rangle$-rule (Cut) has the form

$$\frac{\Gamma \Rightarrow \Lambda, \varphi, \Theta \qquad \Sigma, \varphi, \Pi \Rightarrow \Delta}{\Sigma, \Gamma, \Pi \Rightarrow \Lambda, \Delta, \Theta},$$

  where all the Greek capital letters represent finite sequences of formulas of any length;

- the $\langle \mathcal{L}, \omega \times \{0,1\} \rangle$-rule (Cut) has the form

$$\frac{\Gamma \Rightarrow \varphi \qquad \Sigma, \varphi, \Pi \Rightarrow \Delta}{\Sigma, \Gamma, \Pi \Rightarrow \Delta},$$

[2]Strictly speaking, the rules that we present in what follows are families of rules and not only just individual rules.

where Γ, Σ and Π are finite sequences of formulas of any length and Δ is a sequence of, at most, one formula;

- the $\langle \mathcal{L}, \omega \times \{1\}\rangle$-rule (Cut) has the form

$$\frac{\Gamma \Rightarrow \varphi \qquad \Sigma, \varphi, \Pi \Rightarrow \psi}{\Sigma, \Gamma, \Pi \Rightarrow \psi},$$

where Γ, Σ and Π are finite sequences of formulas of any length.

Other structural rules that we will consider are the following ones:

	Left	Right
Exchange:	$\dfrac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta}\ (e \Rightarrow)$	$\dfrac{\Gamma \Rightarrow \Lambda, \varphi, \psi, \Theta}{\Gamma \Rightarrow \Lambda, \psi, \varphi, \Theta}\ (\Rightarrow e)$
Weakening:	$\dfrac{\Gamma, \Pi \Rightarrow \Delta}{\Gamma, \varphi, \Pi \Rightarrow \Delta}\ (w \Rightarrow)$	$\dfrac{\Gamma \Rightarrow \Lambda, \Theta}{\Gamma \Rightarrow \Lambda, \psi, \Theta}\ (\Rightarrow w)$
Contraction:	$\dfrac{\Gamma, \varphi, \varphi, \Pi \Rightarrow \Delta}{\Gamma, \varphi, \Pi \Rightarrow \Delta}\ (e \Rightarrow)$	$\dfrac{\Gamma \Rightarrow \Lambda, \varphi, \varphi, \Theta}{\Gamma \Rightarrow \Lambda, \varphi, \Theta}\ (\Rightarrow e)$

Note that if the set of types is $\omega \times \{0,1\}$ or $\omega \times \{1\}$ the rules $(\Rightarrow e)$ and $(\Rightarrow c)$ are not expressible. Observe also that if the set of types is $\omega \times \{1\}$ the rule $(\Rightarrow w)$ is not expressible. If the set of types is $\omega \times \{0,1\}$ the rule $(\Rightarrow w)$ takes the form

$$\frac{\Gamma \Rightarrow \emptyset}{\Gamma \Rightarrow \psi}\ (\Rightarrow w)$$

A *rule of introduction* is a rule $\langle \Phi, \varsigma\rangle$ such that every formula in Φ is a subformula of some formula in ς and, moreover, there is a formula in Φ that is a proper subformula of some formula in ς.

Definition 3.1 (Regular sequent calculus). *We say that an $\langle \mathcal{L}, \mathcal{T}\rangle$-sequent calculus $\mathcal{C}$ is* regular *if and only if $\langle 1, 1\rangle \in \mathcal{T}$ and $\mathcal{C}$ contains only structural rules and two families of rules of introductions for each connective such that, if k is the arity of a connective ι of $\mathcal{L}$, the generators of the rules of these two families are of the form*

$$\frac{S}{\Gamma, \iota(\varphi_1, \ldots, \varphi_k), \Pi \Rightarrow \Delta} \qquad \frac{T}{\Gamma \Rightarrow \Lambda, \iota(\varphi_1, \ldots, \varphi_k), \Theta}$$

in such a way that the sequences in the sequents of the sets S and T and the sequences Γ, Π, Δ, Λ and Θ contain only variables. We use for these two families of rules the labels $(\iota \Rightarrow)$ and $(\Rightarrow \iota)$ and we call them respectively rule of introduction to the left *nd* rule of introduction to the right *for the connective ι. We will say that a Gentzen system is* regular *if it is defined by a sequent calculus which is regular.*

Definition 3.2 (Restrictions of a regular system)**.** *Given a regular Gentzen system $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash_{\mathcal{G}} \rangle$, i.e., defined by a regular sequent calculus $\mathcal{C}$, if Ψ is a sublanguage of $\mathcal{L}$, we will denote by $\mathcal{C}[\Psi]$ the calculus in the language Ψ obtained by dropping of the calculus $\mathcal{C}$ the introduction rules for the connectives that do not belong to Ψ. We denote by $\mathcal{G}[\Psi]$ the Gentzen system defined by the calculus $\mathcal{C}[\Psi]$. If $\Psi = \langle \iota_1, \dots, \iota_n \rangle$, we will use the explicit notations $\mathcal{C}[\iota_1, \dots, \iota_n]$ i $\mathcal{G}[\iota_1, \dots, \iota_n]$ for $\mathcal{C}[\Psi]$ and $\mathcal{G}[\Psi]$, respectively. If $\Psi' = \langle \Psi, \iota_1, \dots, \iota_n \rangle$ is a sublanguage of $\mathcal{L}$ which is obtained by adding to Ψ the connectives $\iota_1, \dots, \iota_n$, we will use the notations $\mathcal{C}[\Psi, \iota_1, \dots, \iota_n]$ and $\mathcal{G}[\Psi, \iota_1, \dots, \iota_n]$ for $\mathcal{C}[\Psi']$ and $\mathcal{G}[\Psi']$, respectively.*

Expansions, extensions, fragments. Let $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash_{\mathcal{G}} \rangle$ and $\mathcal{G}' = \langle \mathcal{L}', \mathcal{T}', \vdash_{\mathcal{G}'} \rangle$ be such that $\mathcal{L} \leq \mathcal{L}'$ and $\mathcal{T} \subseteq \mathcal{T}'$. We will say that $\mathcal{G}'$ is an *expansion* of $\mathcal{G}$ if and only if $\vdash_{\mathcal{G}} \subseteq \vdash_{\mathcal{G}'}$. In the case that $\mathcal{L} = \mathcal{L}'$ and $\mathcal{T} = \mathcal{T}'$ we will say that $\mathcal{G}'$ is an *extension* of $\mathcal{G}$. An expansion $\mathcal{G}' = \langle \mathcal{L}', \mathcal{T}', \vdash_{\mathcal{G}'} \rangle$ of $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash_{\mathcal{G}} \rangle$ is an *axiomatic expansion* if and only if $\mathcal{T} = \mathcal{T}'$ and the systems $\mathcal{G}$ and $\mathcal{G}'$ they are respectively defined by two sequent calculi $\mathcal{C}$ and $\mathcal{C}'$ such that a) have the same proper rules, and b) every axiom of $\mathcal{C}$ is an axiom of $\mathcal{C}'$ (i.e., $\mathcal{C}'$ is obtained from $\mathcal{C}$ by adding only axioms). If, moreover, $\mathcal{L} = \mathcal{L}'$, then we say that $\mathcal{G}'$ is an *axiomatic extension* of $\mathcal{G}$.

An expansion $\mathcal{G}' = \langle \mathcal{L}', \mathcal{T}', \vdash_{\mathcal{G}'} \rangle$ of $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash_{\mathcal{G}} \rangle$ is *conservative* if and only if

$$\text{for every } \Phi \cup \{\varsigma\} \subseteq Seq_{\mathcal{L}}^{\mathcal{T}},\ \Phi \vdash_{\mathcal{G}} \varsigma \ \text{ sii } \ \Phi \vdash_{\mathcal{G}'} \varsigma.$$

If $\mathcal{G}'$ is a conservative expansion of $\mathcal{G}$, we say also that $\mathcal{G}$ is the $\langle \mathcal{L}, \mathcal{T} \rangle$*-fragment* of $\mathcal{G}'$. If $\mathcal{T} = \mathcal{T}'$, we say that $\mathcal{G}$ is the $\mathcal{L}$*-fragment* of $\mathcal{G}'$ and, if $\mathcal{L} = \mathcal{L}'$, we say that $\mathcal{G}$ is the $\mathcal{T}$*-fragment* of $\mathcal{G}'$. We will use the notations $\langle \mathcal{L}, \mathcal{T} \rangle$-$\mathcal{G}'$, $\mathcal{L}$-$\mathcal{G}'$ and $\mathcal{T}$-$\mathcal{G}'$ for the $\langle \mathcal{L}, \mathcal{T} \rangle$-fragment, the $\mathcal{L}$-fragment, and the $\mathcal{T}$-fragment of $\mathcal{G}'$, respectively.

Separability of a sequent calculus. The notions of separability for Hilbert-style calculi (see page 23) can be generalized in a natural way to Gentzen systems defined by structural rules and rules of introduction for the connectives. Thus, we will say that a sequent calculus $\mathcal{C}$ is *separable for* Ψ if the derivable sequents of the Ψ-fragment of the Gentzen system defined by $\mathcal{C}$ are derivable from the structural rules of $\mathcal{C}$ and from the axioms and rules of $\mathcal{C}$ such that the generators of these axioms and rules contain only the connectives in Ψ.

If, moreover, all the inferences (not only the derivable sequents) of the Ψ-fragment of the Gentzen system defined by $\mathcal{C}$ are derivable in $\mathcal{C}$, we will say that $\mathcal{C}$ is *strongly separable for* Ψ. We will say that a Gentzen sytem satisfies the ST (*Separation Theorem*) –respectively, the SST (*Strong Separation Theorem*)– for Ψ if it admits as axiomatization a separable sequent calculus –respectively, strongly separable.

Equivalence of Gentzen systems. Let $\mathcal{G}$ and $\mathcal{G}'$ be two Gentzen systems $\langle \mathcal{L}, \mathcal{T}, \vdash \rangle$ i $\langle \mathcal{L}', \mathcal{T}', \vdash' \rangle$ such that $\mathcal{L} = \mathcal{L}'$. A $\langle \mathcal{L}, \mathcal{T}, \mathcal{T}' \rangle$-*translation* is a map

$$\tau : Seq_{\mathcal{L}}^{\mathcal{T}} \longrightarrow \mathcal{P}_{fin}(Seq_{\mathcal{L}}^{\mathcal{T}'})$$

such that

- for every $\langle m, n \rangle \in \mathcal{T}$, if $\langle m, n \rangle \neq \langle 0, 0 \rangle$ then the sequents in the set $\tau(p_0, \ldots, p_{m-1} \Rightarrow q_0, \ldots, q_{n-1})$ only uses variables in $\{p_0, \ldots, p_{m-1}, q_0, \ldots, q_{n-1}\}$,
- for every $\langle m, n \rangle \in \mathcal{T}$, $\varphi_0, \ldots, \varphi_{m-1} \in Fm_{\mathcal{L}}$ and $\psi_0, \ldots, \psi_{n-1} \in Fm_{\mathcal{L}}$, if $\langle m, n \rangle \neq \langle 0, 0 \rangle$ then

$$\tau(\varphi_0, \ldots, \varphi_{m-1} \Rightarrow \psi_0, \ldots, \psi_{n-1}) = e[\tau(p_0, \ldots, p_{m-1} \Rightarrow q_0, \ldots, q_{n-1})],$$

  where e is the substitution such that $e(p_i) = \varphi_i$ i $e(q_i) = \psi_i$.

From the above conditions it is obvious that the map τ is determined[3] by the restriction of τ to the set

$$\{\tau(p_0, \ldots, p_{m-1} \Rightarrow q_0, \ldots, q_{n-1}) : \langle m, n \rangle \in \mathcal{T}\}.$$

It is said that the Gentzen systems $\mathcal{G}$ and $\mathcal{G}'$ are *equivalent* if there is a $\langle \mathcal{L}, \mathcal{T}, \mathcal{T}' \rangle$-translation $\tau : Seq_{\mathcal{L}}^{\mathcal{T}} \longrightarrow \mathcal{P}_{fin}(Seq_{\mathcal{L}}^{\mathcal{T}'})$ and a $\langle \mathcal{L}, \mathcal{T}', \mathcal{T} \rangle$-translation $\rho : Seq_{\mathcal{L}}^{\mathcal{T}'} \longrightarrow \mathcal{P}_{fin}(Seq_{\mathcal{L}}^{\mathcal{T}})$ such that

1) for all $\Phi \cup \{\varsigma\} \subseteq Seq_{\mathcal{L}}^{\mathcal{T}}$, it holds that $\Phi \vdash \varsigma$ iff $\tau[\Phi] \vdash' \tau(\varsigma)$,
2) for all $\Phi \cup \{\varsigma\} \subseteq Seq_{\mathcal{L}}^{\mathcal{T}'}$, it holds that $\Phi \vdash' \varsigma$ iff $\rho[\Phi] \vdash \rho(\varsigma)$,
3) for all $\varsigma \in Seq_{\mathcal{L}}^{\mathcal{T}'}$, it holds that $\varsigma \dashv\vdash' \tau\rho(\varsigma)$,
4) for all $\varsigma \in Seq_{\mathcal{L}}^{\mathcal{T}}$, it holds that $\varsigma \dashv\vdash \rho\tau(\varsigma)$.

It is known that the previous definition is redundant because the conjunction of 1) and 3) is equivalent to the conjunction of 2) and 4) [RV95, Proposition 2.1].

Theories, matrix, models and filters in a Gentzen system. Assume that a Gentzen system $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash \rangle$ is fixed. A subset Φ of $Seq_{\mathcal{L}}^{\mathcal{T}}$ is a $\mathcal{G}$-*theory* if $\Phi \vdash \varsigma$ implies $\varsigma \in \Phi$. The set of all $\mathcal{G}$-theories is denoted by $Th\,\mathcal{G}$. A $\langle \mathcal{L}, \mathcal{T} \rangle$-*matrix*, or simply a *matrix*, is a pair $\langle \mathbf{A}, R \rangle$ where $\mathbf{A}$ is an $\mathcal{L}$-algebra and $R \subseteq \bigcup\{A^m \times A^n : \langle m, n \rangle \in \mathcal{T}\}$. Every $\langle \mathcal{L}, \mathcal{T} \rangle$-matrix allows us to introduce a Gentzen system with language $\mathcal{L}$ and type $\mathcal{T}$: given a matrix $\langle \mathbf{A}, R \rangle$, just consider the consequence relation $\models_{\langle \mathbf{A}, R \rangle}$ defined by $\Phi \models_{\langle \mathbf{A}, R \rangle} \varsigma$ if and only if

[3] Strictly speaking, the map τ is *quasi* determined because the value $\tau(\Gamma \Rightarrow \Delta)$ is determined with the exception of the case in which Γ and Δ are both the empty sequence.

for every homomorphism h from $\mathbf{Fm}_{\mathcal{L}}$ into $\mathbf{A}$, if $h[\Phi] \subseteq R$ then $h(\varsigma) \in R$.

It is easily verified that it is invariant under substitutions. Hence, $\langle \mathcal{L}, \mathcal{T}, \models_{\langle \mathbf{A},R\rangle}\rangle$ is a Gentzen system (possibly not finitary). In the case that $\vdash \leq \models_{\langle \mathbf{A},R\rangle}$ (i.e., for every set $\Phi \cup \{\varsigma\}$ of sequents, if $\Phi \vdash \varsigma$ then $\Phi \models_{\langle \mathbf{A},R\rangle} \varsigma$) then it is said that $\langle \mathbf{A}, R\rangle$ is a *$\mathcal{G}$-model* and that R is a *$\mathcal{G}$-filter* of $\mathbf{A}$. It is well known that whenever $\mathcal{G}$ is defined by means of a sequent calculus, then R is a $\mathcal{G}$-filter if and only if R contains all the interpretations of the axioms and is closed under each of the rules. Another easy remark is that Φ is a $\mathcal{G}$-theory if and only if Φ is a $\mathcal{G}$-filter of $\mathbf{Fm}_{\mathcal{L}}$.

Algebraization of Gentzen systems. If $\mathbb{K}$ is a class of $\mathcal{L}$-algebras, then the equational logic $\models_{\mathbb{K}}$ can be seen as a Gentzen system with language $\mathcal{L}$ and set of types $\mathcal{T} = \{1\} \times \{1\}$ where we identify an equation $\varphi \approx \psi \in Eq_{\mathcal{L}}$ with the sequent $\varphi \Rightarrow \psi$. The class $\mathbb{K}$ is an *algebraic semantics* for a Gentzen system $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash\rangle$ in the case that there is a translation $\tau : Seq_{\mathcal{L}}^{\mathcal{T}} \longrightarrow \mathcal{P}(Eq_{\mathcal{L}})$ such that for all $\Phi \cup \{\varsigma\} \subseteq Seq_{\mathcal{L}}^{\mathcal{T}}$,

$$\Phi \vdash \varsigma \quad \text{iff} \quad \tau[\Phi] \models_{\mathbb{K}} \tau(\varsigma).$$

A Gentzen system $\mathcal{G}$ is said to be *algebraizable with equivalent algebraic semantics* $\mathbb{K}$ if $\mathcal{G}$ and $\models_{\mathbb{K}}$ are equivalent Gentzen systems.

It holds that if $\mathbb{K}$ is an equivalent algebraic semantics for $\mathcal{G}$, then so is the quasivariety $\mathbf{Q}(\mathbb{K})$ generated by $\mathbb{K}$ [GTV97, Corollary 4.2]. It is also known that if $\mathbb{K}$ and $\mathbb{K}'$ are equivalent algebraic semantics for $\mathcal{G}$, then $\mathbb{K}$ and $\mathbb{K}'$ generate the same quasivariety [GTV97, Corollary 4.4]. This quasivariety is called *the equivalent quasivariety semantic* for $\mathcal{G}$.

We notice that if $\mathcal{S}$ is a deductive system then the fact that it is algebraizable in the sense of [BP89] with the set of equivalence formulas $\Delta(p, q)$ and the set of defining equations $\Theta(p)$ coincides precisely with the fact of being algebraizable in the above sense under the translations $\tau(p) := \Theta(p)$ and $\rho(p \approx q) := \Delta(p, q)$. Hence, the algebraization of Gentzen systems generalizes the algebraization of deductive systems introduced in [BP89].

Now we state a result that we will need in Chapter 9. It gives a sufficient condition to prove the algebraization of a Gentzen system [RV95, Lemma 2.5] (see also [GTV97, Lemma 4.5]). In fact, it is also known that this condition is necessary (see [RV95, Lemma 2.24]).

Lemma 3.3. *Let $\mathcal{G}$ be a Gentzen system $\langle \mathcal{L}, \mathcal{T}, \vdash\rangle$ and $\mathbb{K}$ a quasivariety. Suppose that there are two translations $\tau : Seq_{\mathcal{L}}^{\mathcal{T}} \longrightarrow \mathcal{P}_{fin}(Eq_{\mathcal{L}})$ and $\rho : Eq_{\mathcal{L}} \longrightarrow \mathcal{P}_{fin}(Seq_{\mathcal{L}}^{\mathcal{T}})$ such that*

1) for all $\varsigma \in Seq_{\mathcal{L}}^{\mathcal{T}}$, $\varsigma \dashv\vdash_{\mathcal{G}} \rho\tau(\varsigma)$,

2) for all $\varphi \approx \psi \in Eq_{\mathcal{L}}$, $\varphi \approx \psi \;\; \dashv\models_{\mathbb{K}} \;\; \tau\rho(\varphi \approx \psi)$, *3) for all* $\mathbf{A} \in \mathbb{K}$*, the set*

$$R := \{\langle \bar{a}, \bar{b} \rangle \in A^m \times A^n : \langle m, n \rangle \in \mathcal{T},\ \mathbf{A} \models \tau(p_0, \ldots, p_{m-1} \Rightarrow q_0, \ldots, q_{n-1})[\![\bar{a}, \bar{b}]\!]\}$$

is a $\mathcal{G}$*-filter,*[4]

4) for all $\Phi \in Th\,\mathcal{G}$*, the relation*

$$\theta_\Phi := \{\langle \varphi, \psi \rangle \in Fm^2_{\mathcal{L}} : \rho(\varphi \approx \psi) \subseteq \Phi\},$$

is a congruence relative to the quasivariety $\mathbb{K}$*, i.e.,* $\mathbf{Fm}_{\mathcal{L}}/\theta_\Phi \in \mathbb{K}$*.*

Then, $\mathcal{G}$ *is algebraizable with equivalent algebraic semantics* $\mathbb{K}$*.*

The Leibniz operator. One interesting property of algebraizable Gentzen systems with respect to quasivarieties is the existence of a characterization of congruences relative to the quasivariety. To describe this characterization we need the notion of *Leibniz operator.* Let $\mathbf{A}$ be an $\mathcal{L}$-algebra, and let $\mathcal{T}$ be a set of types. If $m, n \in \omega$, $\langle \bar{x}, \bar{y} \rangle \in A^m \times A^n$ i $a, b \in A$, then $\langle \bar{x}, \bar{y} \rangle (a|b)$ will denote the result of replacing one occurrence (if it exists) of a in $\langle \bar{x}, \bar{y} \rangle$ with b. Given a $\langle \mathcal{L}, \mathcal{T} \rangle$-matrix $\langle \mathbf{A}, R \rangle$, la *congruència de Leibniz* $\Omega_{\mathbf{A}} R$ of the matrix $\langle \mathbf{A}, R \rangle$ is the equivalence relation on A defined in the following way: $\langle a, b \rangle \in \Omega_{\mathbf{A}} R$ if and only if, for every $\langle m, n \rangle \in \mathcal{T}$, $\langle \bar{x}, \bar{y} \rangle \in A^m \times A^n$, $k \in \omega$, $\varphi(p, q_0, \ldots, q_{k-1}) \in Fm_{\mathcal{L}}$ and $c, c_0, \ldots, c_{k-1} \in A$,

$$\langle \bar{x}, \bar{y} \rangle (c|\varphi^{\mathbf{A}}(a, c_0, \ldots, c_{k-1})) \in R \quad \Leftrightarrow \quad \langle \bar{x}, \bar{y} \rangle (c|\varphi^{\mathbf{A}}(b, c_0, \ldots, c_{k-1})) \in R.$$

We emphasize that the previous definition does not depend on any Gentzen system. It holds that $\Omega_{\mathbf{A}} : \bigcup\{A^m \times A^n : \langle m, n \rangle \in \mathcal{T}\} \longrightarrow Con(\mathbf{A})$. This map is known as the *Leibniz operator* on $\mathbf{A}$. It is easy to show that $\Omega_{\mathbf{A}} R$ is characterized by the fact that it is the largest congruence of $\mathbf{A}$ that is compatible with R (i.e., if $\langle \bar{x}, \bar{y} \rangle \in R$ and $\langle a, b \rangle \in \theta$, then $\langle \bar{x}, \bar{y} \rangle (a|b) \in R$). Let $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash_{\mathcal{G}} \rangle$ be a Gentzen system and let Φ be a $\mathcal{G}$-theory. Then, $\langle \mathbf{Fm}_{\mathcal{L}}, \Phi \rangle$ is a $\langle \mathcal{L}, \mathcal{T} \rangle$-matrix that is a $\mathcal{G}$-model. In this case we denote the Leibniz operator by Ω instead of $\Omega_{\mathbf{Fm}_{\mathcal{L}}}$.

Next we give the result mentioned previously which concerns the congruences. [RV93, Theorem 2.23] (see also [GTV97, Theorem 4.7]).

Theorem 3.4. *Let* $\mathcal{G}$ *be a Gentzen system and* $\mathbb{K}$ *a quasivariety. The following statements are equivalent.*

1) $\mathcal{G}$ *is algebraizable with equivalent algebraic semantics* $\mathbb{K}$*.*

2) For every $\mathcal{L}$*-algebra* $\mathbf{A}$*, the Leibniz operator* $\Omega_{\mathbf{A}}$ *is an isomorphism between the lattice of* $\mathcal{G}$*-filters of* $\mathbf{A}$ *and the lattice of* $\mathbb{K}$*-congruences of* $\mathbf{A}$*.*

3) The Leibniz operator Ω *is a lattice isomorphism between* $Th\,\mathcal{G}$ *and* $Con_{\mathbb{K}}\mathbf{Fm}_{\mathcal{L}}$*.*

[4] If $\bar{a} = \langle a_0, \ldots, a_{m-1} \rangle$ and $\bar{b} = \langle b_0, \ldots, b_{n-1} \rangle$, we denote by $\tau(p_0, \ldots, p_{m-1} \Rightarrow q_0, \ldots, q_{n-1})[\![\bar{a}, \bar{b}]\!]$ the result obtained by applying the assignation such that $p_0^{\mathbf{A}} = a_0, \ldots, p_{m-1}^{\mathbf{A}} = a_{m-1}, q_0^{\mathbf{A}} = b_0, \ldots, q_{n-1}^{\mathbf{A}} = b_{n-1}$ to the equations of the set $\tau(p_0, \ldots, p_{m-1} \Rightarrow q_0, \ldots, q_{n-1})$.

Deductive systems associated to a Gentzen system. Let $\mathcal{G}$ be a Gentzen system $\langle \mathcal{L}, \mathcal{T}, \vdash \rangle$. G There are at least two methods in the literature used to associate a deductive system with $\mathcal{G}$. The common method is based on considering the derivable sequents. Specifically, $\Sigma \vdash_{\mathcal{I}(\mathcal{G})} \varphi$ holds when

there is a finite subset $\{\varphi_0, \ldots, \varphi_{n-1}\}$ of Σ such that $\emptyset \vdash \varphi_0, \ldots, \varphi_{n-1} \Rightarrow \varphi$.

This approach yields a deductive system, called *internal* if and only if the Gentzen system satisfies the following structural rules: $(Axiom)$, (Cut), exchange, left weakening and contraction. Another method, which works for all the Gentzen systems such that $\langle 0, 1 \rangle \in \mathcal{T}$, yields to the so-called external deductive system.[5] The *external deductive system* associated to $\mathcal{G}$ is defined as the deductive system $\mathcal{E}(\mathcal{G})$ such that

$$\Sigma \vdash_{\mathcal{E}(\mathcal{G})} \varphi \quad \text{iff} \quad \{\, \emptyset \Rightarrow \psi : \psi \in \Sigma\} \vdash \ \emptyset \Rightarrow \varphi.$$

Since we have restricted ourselves to finitary Gentzen systems it is clear that $\mathcal{E}(\mathcal{G})$ is finitary.

[5]We use the names *internal* and *external* following Avron (see [Avr88]).

Part II

Substructural Systems

Chapter 4

Basic Intuitionistic Substructural Systems

In this chapter we recall the definitions of the basic substructural Gentzen systems $\mathcal{FL}_\sigma$ and their associated external deductive systems $\mathfrak{e}\mathcal{FL}_\sigma$. The calculi $\mathbf{FL}_\sigma$ defining the systems $\mathcal{FL}_\sigma$ are presented with sequents of type $\omega \times \{0,1\}$ and in the language $\langle \vee, \wedge, *, \backslash, /, {}^\backprime, {}', 0, 1\rangle$ which includes two negations ${}^\backprime$ and ${}'$ and a constant symbol 0. The systems $\mathcal{FL}_\sigma$ also allow a presentation which is definitionally equivalent in sequents of type $\omega \times \{1\}$ and in the language without negations $\langle \vee, \wedge, *, \backslash, /, 0, 1\rangle$ (see Section 5.7). We have decided to adopt the foregoing since in Chapter 9 we will study certain fragments without implications that contain such negations. In Section 4.2 we define the notion of *mirror image of a sequent* and set out the *Law of Mirror Images* for the systems $\mathcal{FL}_\sigma[\Psi]$ such that Ψ contains the two implications or the two negations. In Section 4.3 we characterize the sequential Leibniz congruence of the theories of the systems $\mathcal{FL}[\Psi]$ and the Leibniz congruence of the theories of their associated external systems $\mathfrak{e}\mathcal{FL}[\Psi]$ for the case in which language Ψ contains one of the implication connectives and prove that these external systems are protoalgebraic. The results related to characterizations are obtained as a consequence of other more general results established in larger classes of Gentzen systems to which systems $\mathcal{FL}_\sigma[\Psi]$ belong to. In Section 4.4 we present Hilbert-style axiomatizations known in the literature for the external deductive systems $\mathfrak{e}\mathcal{FL}_\sigma$.

4.1 The Calculi $\mathbf{FL}_\sigma$ with Negations

We will now recall the definition of the intuitionistic basic substructural calculus **FL** in its version with sequents of type $\omega \times \{0,1\}$ and with the two negations, which will be denoted by symbols ${}^\backprime$ and ${}'$ hereafter, as primitive connectives. It is well known that the calculus **FL** has the variety $\mathbb{FL}$ of the pointed residuated lattices as algebraic counterpart: in Chapter 9 we will establish this relation in the following strong sense:

the system defined by **FL** and the equational system associated with the variety $\mathbb{FL}$ are equivalent as Gentzen systems. The definition of **FL**, introduced by Hiroakira Ono, appears for the first time in [Ono90] (cf. also [Ono90, Ono93, Ono98, Ono03b, GO06]).

Definition 4.1 (Full Lambek Calculus). *Let $\mathfrak{L}$ be the propositional language $\langle \vee, \wedge, *, \backslash, /, ', ', 0, 1\rangle$ of type $\langle 2, 2, 2, 2, 2, 1, 1, 0, 0\rangle$. Let φ, ψ $\mathfrak{L}$-formulas; Γ, Π, Σ finite sequences (possibly empty) of $\mathfrak{L}$-formulas and Δ a sequence with at the most one formula. The* Full Lambek Calculus **FL** *is the $\langle \mathfrak{L}, \omega \times \{0,1\}\rangle$-calculus defined by the following axioms and rules:*

$$\varphi \Rightarrow \varphi \quad (Axiom) \qquad \frac{\Gamma \Rightarrow \varphi \qquad \Sigma, \varphi, \Pi \Rightarrow \Delta}{\Sigma, \Gamma, \Pi \Rightarrow \Delta} \ (Cut)$$

$$\frac{\Sigma, \varphi, \Gamma \Rightarrow \Delta \qquad \Sigma, \psi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi \vee \psi, \Gamma \Rightarrow \Delta} \ (\vee \Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi} \ (\Rightarrow \vee_1) \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi} \ (\Rightarrow \vee_2)$$

$$\frac{\Sigma, \varphi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi \wedge \psi, \Gamma \Rightarrow \Delta} \ (\wedge_1 \Rightarrow) \qquad \frac{\Sigma, \psi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi \wedge \psi, \Gamma \Rightarrow \Delta} \ (\wedge_2 \Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi} \ (\Rightarrow \wedge)$$

$$\frac{\Sigma, \varphi, \psi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi * \psi, \Gamma \Rightarrow \Delta} \ (* \Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi \qquad \Pi \Rightarrow \psi}{\Gamma, \Pi \Rightarrow \varphi * \psi} \ (\Rightarrow *)$$

$$\frac{\Gamma \Rightarrow \varphi \qquad \Sigma, \psi, \Pi \Rightarrow \Delta}{\Sigma, \Gamma, \varphi \backslash \psi, \Pi \Rightarrow \Delta} \ (\backslash \Rightarrow) \qquad \frac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \backslash \psi} \ (\Rightarrow \backslash)$$

$$\frac{\Gamma \Rightarrow \varphi \quad \Sigma, \psi, \Pi \Rightarrow \Delta}{\Sigma, \psi / \varphi, \Gamma, \Pi \Rightarrow \Delta} \ (/ \Rightarrow) \qquad \frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \psi / \varphi} \ (\Rightarrow /)$$

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma, \varphi' \Rightarrow \emptyset} \ (' \Rightarrow) \qquad \frac{\varphi, \Gamma \Rightarrow \emptyset}{\Gamma \Rightarrow \varphi'} \ (\Rightarrow ')$$

$$\frac{\Gamma \Rightarrow \varphi}{{}'\varphi, \Gamma \Rightarrow \emptyset} \ (' \Rightarrow) \qquad \frac{\Gamma, \varphi \Rightarrow \emptyset}{\Gamma \Rightarrow {}'\varphi} \ (\Rightarrow ')$$

$$\frac{\Sigma, \Gamma \Rightarrow \Delta}{\Sigma, 1, \Gamma \Rightarrow \Delta} \ (1 \Rightarrow) \qquad \emptyset \Rightarrow 1 \quad (\Rightarrow 1)$$

$$0 \Rightarrow \emptyset \quad (0 \Rightarrow) \qquad \frac{\Gamma \Rightarrow \emptyset}{\Gamma \Rightarrow 0} \ (\Rightarrow 0)$$

Nomenclature 4.2. The connectives of the language $\mathfrak{L}$ of **FL** are denoted by the name stated in Table 4.1.

We now define the extensions of the calculus **FL** with different combinations for the (right and left) exchange, weakening and contraction structural rules.

Definition 4.3. (The Calculi $\mathbf{FL}_\sigma$).

$\mathbf{FL}_e$ *is the calculus obtained by adding to the rules of* **FL** *the $\langle \mathfrak{L}, \omega \times \{0,1\}\rangle$-rule of exchange:*

$$\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta} \ (e \Rightarrow).$$

Symbol	Name
$\vee$	Disjunction
$\wedge$	Conjunction (or additive conjunction)
$*$	Fusion (or multiplicative conjunction)
$\backslash$	Right implication
$/$	Left implication
$`$	Right negation
$'$	Left negation
0	Falsity (or Zero)
1	Truth (or One)

Table 4.1: Connectives of **FL**

$\mathbf{FL}_{w_l}$ *is the calculus obtained by adding to the rules of* **FL** *the* $\langle \mathfrak{L}, \omega \times \{0,1\}\rangle$*-rule of left-weakening:*

$$\frac{\Sigma, \Gamma \Rightarrow \Delta}{\Sigma, \varphi, \Gamma \Rightarrow \Delta} \quad (w \Rightarrow).$$

$\mathbf{FL}_{w_r}$ *is the calculus obtained by replacing in* **FL** *the rule* $(\Rightarrow 0)$ *with the* $\langle \mathfrak{L}, \omega \times \{0,1\}\rangle$*-rule of right weakening:*

$$\frac{\Gamma \Rightarrow \emptyset}{\Gamma \Rightarrow \varphi} \quad (\Rightarrow w).$$

$\mathbf{FL}_{w}$ *is the calculus obtained by adding to* **FL** *the rules* $(w \Rightarrow)$ *and* $(\Rightarrow w)$*.*

$\mathbf{FL}_{c}$ *is the calculus obtained by adding to* **FL** *the* $\langle \mathfrak{L}, \omega \times \{0,1\}\rangle$*-rule of contraction:*

$$\frac{\Sigma, \varphi, \varphi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi, \Gamma \Rightarrow \Delta} \quad (c \Rightarrow).$$

Let σ *be a subsequence of the sequence* ew_lw_rc*. If in* σ *there appears the sequence* w_lw_r*, we will denote it in short by* w*. We will refer by* $\mathbf{FL}_\sigma$ *to the calculus obtained by adding to the rules of* **FL** *the* $\langle \mathfrak{L}, \omega \times \{0,1\}\rangle$*-rules codified by the letters appearing in* σ *and if* σ *is the empty sequence, then* $\mathbf{FL}_\sigma$ *is the calculus* **FL***. We will call the* $\mathbf{FL}_\sigma$ *calculi as* basic substructural calculi*.*

So, for example, $\mathbf{FL}_{w_rc}$ is the calculus obtained by adding to the rules of **FL** the structural rules $(\Rightarrow w)$ and $(c \Rightarrow)$.

Observation 4.4. Note that in the calculi with the rule of left-weakening, $(1 \Rightarrow)$ is an instance of $(w \Rightarrow)$ and that in calculi with the rule of right-weakening, $(\Rightarrow 0)$ is an instance of $(\Rightarrow w)$.

Definition 4.5. (Systems $\mathcal{FL}_\sigma$ and $\mathfrak{e}\mathcal{FL}_\sigma$). *Let* $\sigma \leq ew_lw_rc$.[1] *The Gentzen system defined by a calculus* $\mathbf{FL}_\sigma$ *will be denoted by* $\mathcal{FL}_\sigma$*. The systems* $\mathcal{FL}_\sigma$ *will be called* basic

[1] In order to denote that a sequence of symbols σ_1 is a subsequence of another sequence σ_2, we will write $\sigma_1 \leq \sigma_2$.

substructural Gentzen systems. *The external deductive system associated with $\mathcal{FL}_\sigma$ will be denoted by $\mathfrak{e}\mathcal{FL}_\sigma$.*

Cut Elimination Theorem. Subformula Property. It is well known that the calculi $\mathbf{FL}_\sigma$, except for $\mathbf{FL}_c$, satisfy the cut elimination theorem and the subformula property (see, for example, [Ono98, Theorem 6, Lemma 7] or, also, [Ono03b]). Let us recall that a calculus with the cut rule (*Cut*) satisfies the *Cut Elimination Theorem* if, for any sequent ς derivable in the calculus, there exists a proof of ς in which the cut rule is not used. A calculus satisfies the *subformula property* if, for any sequent ς derivable in the calculus, there exists a derivation of ς in which every formula included in it is a subformula of any formula in ς.

Theorem 4.6 ([Ono98])**.** *The calculi $\mathbf{FL}_\sigma$, with $\sigma \neq c$, satisfy the Cut Elimination Theorem. The calculus $\mathbf{FL}_c$ does not satisfy the Cut Elimination Theorem. The calculi $\mathbf{FL}_\sigma$, with $\sigma \neq c$, satisfy the Subformula Property. The calculus $\mathbf{FL}_c$ does not satisfy the Subformula Property.*

As a direct consequence we have that the calculi $\mathbf{FL}_\sigma[\Psi]$,[2] Ψ being a sublanguage of $\mathfrak{L}$, also satisfy the Cut Elimination Theorem and the Subformula Property, except for $\sigma = c$.

Corollary 4.7. *Let Ψ be any sublanguage of $\mathfrak{L}$ whatsoever. The calculi $\mathbf{FL}_\sigma[\Psi]$, with $\sigma \neq c$, satisfy the Cut Elimination Theorem and and the Subformula Property.*

Observation 4.8 (About Subsystems and Fragments)**.** At this point we stress that we do not know a priori if $\mathcal{FL}_\sigma[\Psi]$ coincides or not with Ψ-$\mathcal{FL}_\sigma$ (i.e., the Ψ-fragment of $\mathcal{FL}_\sigma$). The Cut Elimination Theorem is not applicable in order to prove that the referred subsystems are fragments in the appropriate sublanguages, because this theorem and, therefore, the subformula property, are solely applicable to derivable sequents in such calculi. This fact can be applied to the external systems $\mathfrak{e}\mathcal{FL}_\sigma[\Psi]$ and the Ψ-fragments of $\mathfrak{e}\mathcal{FL}_\sigma$. It is, however, clear that if $\mathcal{FL}_\sigma[\Psi]$ and Ψ-$\mathcal{FL}_\sigma$ coincide, then $\mathfrak{e}\mathcal{FL}_\sigma[\Psi]$ and Ψ-$\mathfrak{e}\mathcal{FL}_\sigma$ will coincide as well.

Calculi without exchange in which this rule is derivable. In the results included below we show that the exchange rule is derivable in $\mathbf{FL}_{w_l c}[\Psi]$ if Ψ contains one of the conjunction connectives.

Lemma 4.9. (Cf. [Ono03b]) *Let Ψ be such that $\langle * \rangle \leq \Psi \leq \mathfrak{L}$. The rule ($e \Rightarrow$) is derivable in $\mathbf{FL}_{w_l c}[\Psi]$ and, consequently, in $\mathbf{FL}_{wc}[\Psi]$.*

Proof: Firstly, we will see that sequent $\psi, \varphi \Rightarrow \varphi * \psi$ is derivable in $\mathbf{FL}_{w_l c}[\Psi]$.

[2] See notations described in Definition 3.2.

$$
\begin{prooftree}
\AxiomC{$\psi \Rightarrow \psi$}
\AxiomC{$\varphi \Rightarrow \varphi$}
\RightLabel{$(\Rightarrow *)$}
\BinaryInfC{$\psi, \varphi \Rightarrow \psi * \varphi$}
\AxiomC{$\varphi \Rightarrow \varphi$}
\RightLabel{$(w \Rightarrow)$}
\UnaryInfC{$\psi, \varphi \Rightarrow \varphi$}
\RightLabel{$(* \Rightarrow)$}
\UnaryInfC{$\psi * \varphi \Rightarrow \varphi$}
\AxiomC{$\psi \Rightarrow \psi$}
\RightLabel{$(w \Rightarrow)$}
\UnaryInfC{$\psi, \varphi \Rightarrow \psi$}
\RightLabel{$(* \Rightarrow)$}
\UnaryInfC{$\psi * \varphi \Rightarrow \psi$}
\RightLabel{$(\Rightarrow *)$}
\BinaryInfC{$\psi * \varphi, \psi * \varphi \Rightarrow \varphi * \psi$}
\RightLabel{$(c \Rightarrow)$}
\UnaryInfC{$\psi * \varphi \Rightarrow \varphi * \psi$}
\RightLabel{(Cut)}
\BinaryInfC{$\psi, \varphi \Rightarrow \varphi * \psi$}
\end{prooftree}
$$

By using this fact, we now observe that the exchange rule is derivable in $\mathbf{FL}_{w_l c}[\Psi]$.

$$
\begin{prooftree}
\AxiomC{$\psi, \varphi \Rightarrow \varphi * \psi$}
\AxiomC{$\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta$}
\RightLabel{$(* \Rightarrow)$}
\UnaryInfC{$\Gamma, \varphi * \psi, \Pi \Rightarrow \Delta$}
\RightLabel{(Cut)}
\BinaryInfC{$\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta$}
\end{prooftree}
$$

Therefore, $\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta \vdash_{\mathbf{FL}_{w_l c}[\Psi]} \Gamma, \psi, \varphi, \Pi \Rightarrow \Delta$. $\square$

Lemma 4.10. *Let* Ψ *be such that* $\langle \wedge \rangle \leq \Psi \leq \mathfrak{L}$. *The rule* $(e \Rightarrow)$ *is derivable in* $\mathbf{FL}_{w_l c}[\Psi]$ *and, consequently, in* $\mathbf{FL}_{wc}[\Psi]$.

Proof: The sequent $\psi, \varphi \Rightarrow \varphi \wedge \psi$ is derivable in $\mathbf{FL}_{w_l c}[\Psi]$. In fact:

$$
\begin{prooftree}
\AxiomC{$\varphi \Rightarrow \varphi$}
\LeftLabel{$(w \Rightarrow)$}
\UnaryInfC{$\psi, \varphi \Rightarrow \varphi$}
\AxiomC{$\psi \Rightarrow \psi$}
\RightLabel{$(w \Rightarrow)$}
\UnaryInfC{$\psi, \varphi \Rightarrow \psi$}
\RightLabel{$(\Rightarrow \wedge)$}
\BinaryInfC{$\psi, \varphi \Rightarrow \varphi \wedge \psi$}
\end{prooftree}
$$

Now we will employ this sequent to prove that the exchange rule is derivable in $\mathbf{FL}_{w_l c}[\Psi]$:

$$
\begin{prooftree}
\AxiomC{$\psi, \varphi \Rightarrow \varphi \wedge \psi$}
\AxiomC{$\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta$}
\RightLabel{$(\wedge_1 \Rightarrow)$}
\UnaryInfC{$\Gamma, \varphi \wedge \psi, \psi, \Pi \Rightarrow \Delta$}
\RightLabel{$(\wedge_2 \Rightarrow)$}
\UnaryInfC{$\Gamma, \varphi \wedge \psi, \varphi \wedge \psi, \Pi \Rightarrow \Delta$}
\RightLabel{$(c \Rightarrow)$}
\UnaryInfC{$\Gamma, \varphi \wedge \psi, \Pi \Rightarrow \Delta$}
\RightLabel{(Cut)}
\BinaryInfC{$\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta$}
\end{prooftree}
$$

Thereby, $\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta \vdash_{\mathbf{FL}_{w_l c}[\Psi]} \Gamma, \psi, \varphi, \Pi \Rightarrow \Delta$. $\square$

Corollary 4.11. *Let* $\Psi \leq \mathfrak{L}$ *be such that* $\langle * \rangle \leq \Psi$ *or* $\langle \wedge \rangle \leq \Psi$. *Then,*

a) $\mathcal{FL}_{w_l c}[\Psi] = \mathcal{FL}_{e w_l c}[\Psi]$,

b) $\mathcal{FL}_{wc}[\Psi] = \mathcal{FL}_{ewc}[\Psi]$.

Proof: $a)$: As $(e \Rightarrow)$ is derivable in $\mathbf{FL}_{w_l c}[\Psi]$, this calculus and $\mathbf{FL}_{e w_l c}[\Psi]$ define the same Gentzen system, i.e., $\mathcal{FL}_{w_l c}[\Psi] = \mathcal{FL}_{e w_l c}[\Psi]$.

$b)$: Direct consequence of $a)$. $\square$

4.2 Mirror Images

The notion of *mirror image* comes from the framework of the residuated structures (see, for example, [JT02]), where we have a partial order $\leq$, an operation $*$ compatible with such order and two operations $\backslash$ and $/$ satisfying the Law of Residuation: for every a, b, c from the universe, $a * b \leq c$ iff $b \leq a\backslash c$ iff $a \leq c/b$. The *mirror image of a formula* is the formula obtained by replacing the terms in the form $x * y$ with $y * x$ and the terms in the form $x\backslash y$ and x/y with y/x i $y\backslash x$, respectively. An essential fact which is satisfied in these residuated structures is that, according to the *Law of Mirror Images*, if $\varphi \leq \psi$ is satisfied in a residuated structure, then $\varphi' \leq \psi'$ will also be satisfied, where φ' and ψ' are the mirror images of φ and ψ, respectively.

In this section we introduce the notion of mirror image in the context of the $\mathfrak{L}$-sequents and establish a similar result to the mentioned mirror image principle applied to derivations in the calculi $\mathbf{FL}_\sigma$.

Definition 4.12. *Let $\varphi \in Fm_{\mathfrak{L}}$. We define a mapping μ from $Fm_{\mathfrak{L}}$ into $Fm_{\mathfrak{L}}$ according to the following:*

$$\mu(\varphi) := \begin{cases} \varphi, & \text{if } \varphi \in Var \text{ or } \varphi \in \{0, 1\} \\ \mu(\alpha) \vee \mu(\beta), & \text{if } \varphi = \alpha \vee \beta, \\ \mu(\alpha) \wedge \mu(\beta), & \text{if } \varphi = \alpha \wedge \beta, \\ \mu(\beta) * \mu(\alpha), & \text{if } \varphi = \alpha * \beta, \\ \mu(\alpha)\backslash\mu(\beta), & \text{if } \varphi = \beta/\alpha, \\ \mu(\beta)/\mu(\alpha), & \text{if } \varphi = \alpha\backslash\beta, \\ \mu(\alpha)^{`}, & \text{if } \varphi = {}'\alpha, \\ {}'\mu(\alpha), & \text{if } \varphi = \alpha^{`}. \end{cases}$$

Suppose that $\mu(\varphi)$ is the mirror image *of φ. Let Γ be a finite sequence of $\mathfrak{L}$-formulas. We define the* mirror image $\mu(\Gamma)$ *of Γ as follows:*

$$\mu(\Gamma) := \begin{cases} \mu(\varphi_{m-1}), \ldots, \mu(\varphi_0), & \text{if } \Gamma = \varphi_0, \ldots, \varphi_{m-1}; \\ \emptyset, & \text{if } \Gamma = \emptyset. \end{cases}$$

We define the mirror image of a sequent $\Gamma \Rightarrow \Delta$ *as the sequent*

$$\mu(\Gamma \Rightarrow \Delta) := \mu(\Gamma) \Rightarrow \mu(\Delta).$$

Note that $\mu \circ \mu$ is the identity.

In the following lemma we prove that the mirror image of a rule related to a calculus $\mathbf{FL}_\sigma$ is a rule derived from this calculus. This result will allow us to establish the mirror image principle which states that the mirror image of a derivation in $\mathbf{FL}_\sigma$ is a derivation as well and, in particular, that the mirror image of a derivable sequent in $\mathbf{FL}_\sigma$ is also derivable.

Lemma 4.13. *If $\langle \Phi, s \rangle$ is an instance of a rule of $\mathbf{FL}_\sigma$, then its mirror image $\langle \mu[\Phi], \mu(s) \rangle$ is an instance of a rule of $\mathbf{FL}_\sigma$. More specifically,*

- *If $\langle \Phi, s\rangle$ is an instance of a structural rule, then $\langle \mu[\Phi], \mu(s)\rangle$ is an instance of such rule.*
- *If $\langle \Phi, s\rangle$ is an instance of an introduction rule for any of the connectives in $\{\vee, \wedge, *, 0, 1\}$, then $\langle \mu[\Phi], \mu(s)\rangle$ is an instance of such rule.*
- *If $\langle \Phi, s\rangle$ is an instance of an introduction rule for the connective $\backslash$ (from the connective $/$), then $\langle \mu[\Phi], \mu(s)\rangle$ is an instance of the introduction rule for the connective $/$ (from the connective $\backslash$).*
- *If $\langle \Phi, s\rangle$ is an instance of the introduction rule for the connective ${}^{\backprime}$ (for the connective ${}'$), then $\langle \mu[\Phi], \mu(s)\rangle$ is an instance of the introduction rule for the connective ${}'$ (for the connective${}^{\backprime}$).*

Proof: This is a simple and routine proof. As an example we will consider the case concerning the rule $(\backslash \Rightarrow)$:

$$\frac{\Gamma \Rightarrow \varphi \qquad \Sigma, \psi, \Pi \Rightarrow \Delta}{\Sigma, \Gamma, \varphi\backslash\psi, \Pi \Rightarrow \Delta}$$

We have $\mu(\Gamma \Rightarrow \varphi) = \mu(\Gamma) \Rightarrow \mu(\varphi)$ and $\mu(\Sigma, \psi, \Pi \Rightarrow \Delta) = \mu(\Pi), \mu(\psi), \mu(\Sigma) \Rightarrow \mu(\Delta)$. By applying the rules $(/ \Rightarrow)$ to the sequents $\mu(\Gamma) \Rightarrow \mu(\varphi)$ and $\mu(\Pi), \mu(\psi), \mu(\Sigma) \Rightarrow \mu(\Delta)$ we obtain the sequent

$$\mu(\Pi), \mu(\psi)/\mu(\varphi), \mu(\Gamma), \mu(\Sigma) \Rightarrow \mu(\Delta)$$

and this sequent is exactly $\mu(\Sigma, \Gamma, \varphi\backslash\psi, \Pi \Rightarrow \Delta)$. □

Theorem 4.14 (Law of Mirror Images). *Let $\Upsilon \cup \{\varsigma\} \subseteq Seq_{\mathfrak{L}}^{\omega\times\{0,1\}}$. It holds that:*

$$\textit{if } \Upsilon \vdash_{\mathbf{FL}_\sigma} \varsigma, \textit{ then } \mu[\Upsilon] \vdash_{\mathbf{FL}_\sigma} \mu(\varsigma).$$

Specifically, if $\varsigma_1, \ldots, \varsigma_n$ is a proof of ς from Υ in $\mathbf{FL}_\sigma$, then $\mu(\varsigma_1), \ldots, \mu(\varsigma_n)$ is a proof of $\mu(\varsigma)$ from $\mu[\Upsilon]$ in $\mathbf{FL}_\sigma$.

In particular, if ς is derivable in $\mathbf{FL}_\sigma$, then $\mu(\varsigma)$ is derivable in $\mathbf{FL}_\sigma$ as well.

Proof: By induction on n. If ς is an axiom, then $\mu(\varsigma)$ is also an axiom of $\mathbf{FL}_\sigma$, since $\mu(\varphi) \Rightarrow \mu(\varphi)$ is an instance of (*Axiom*), $\mu(0 \Rightarrow \emptyset)$ is $0 \Rightarrow \emptyset$ and $\mu(\emptyset \Rightarrow 1)$ is $\emptyset \Rightarrow 1$. If $\varsigma \in \Upsilon$ is obvious.

Suppose that ς is obtained by applying an instance $\langle\{\varsigma_i, \varsigma_j\}, \varsigma\rangle$, with $i \le j < n$, of a rule of $\mathbf{FL}_\sigma$. Then, for Lemma 4.13, we have that $\langle\{\mu(\varsigma_i), \mu(\varsigma_j)\}, \mu(\varsigma)\rangle$ is an instance of a rule of $\mathbf{FL}_\sigma$. Moreover, according to the induction hypothesis $\mu(\varsigma_1), \ldots, \mu(\varsigma_i)$ is a prof of $\mu(\varsigma_i)$ and $\mu(\varsigma_1), \ldots, \mu(\varsigma_j)$ is a proof of $\mu(\varsigma_j)$. Therefore, $\mu(\varsigma_1), \ldots, \mu(\varsigma_n)$ is a proof of $\mu(\varsigma)$. □

Observation 4.15. Note that the same result obtained in Theorem 4.14 is also valid for the calculi $\mathbf{FL}_\sigma[\Psi]$ obtained as a restriction in the rules of $\mathbf{FL}_\sigma$ to a sublanguage Ψ of $\mathfrak{L}$ such that $\langle \backslash, / \rangle \le \Psi$ or so that $\langle {}^{\backprime}, {}' \rangle \le \Psi$ and, of course, for all sublanguages having neither implications nor negations. We have then the following more general result.

Theorem 4.16. *Let Ψ be a sublanguage of $\mathfrak{L}$ such that $\langle \backslash, / \rangle \leq \Psi$ or such that $\langle ', ' \rangle \leq \Psi$ or such that it contains neither implications nor negations. Then, for any $\Upsilon \cup \{\varsigma\} \subseteq Seq_{\Psi}^{\omega \times \{0,1\}}$, it holds:*

$$\textit{if } \Upsilon \vdash_{\mathbf{FL}_\sigma[\Psi]} \varsigma, \textit{ then } \mu[\Upsilon] \vdash_{\mathbf{FL}_\sigma[\Psi]} \mu(\varsigma).$$

In particular, if ς is derivable in $\mathbf{FL}_\sigma[\Psi]$, then $\mu(\varsigma)$ is derivable in $\mathbf{FL}_\sigma[\Psi]$ as well.

4.3 Characterization of the Leibniz Congruence

In this section we give a characterization of the sequential Leibniz congruence $\Omega\Phi$ associated with every $\mathcal{FL}[\Psi]$-theory Φ for all languages $\Psi \leq \mathfrak{L}$. We also give a characterization of the Leibniz congruence ΩT associated with every $\mathfrak{e}\mathcal{FL}[\Psi]$-theory T for all languages $\Psi \leq \mathfrak{L}$ which contain the connective $\backslash$ or the connective $/$. Firstly, we will establish some results that embrace a more general scope of Gentzen systems. The characterizations presented will be a consequence of these results.

In Proposition 2.21 of [RV95] it a characterization of the Leibniz congruence of a class of matrix $\langle \mathbf{A}, R\rangle$ is given, where $\mathbf{A}$ is an algebra of any type $\mathcal{L}$ and following conditions are complied with:

a) R contains the interpretation of (*Axiom*) and is closed under the $\langle \mathcal{L}, \omega \times \omega\rangle$-rules of left and right exchange and the next restricted version of the cut rule:

$$\frac{\Gamma \Rightarrow \Lambda, \varphi \qquad \varphi, \Pi \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Lambda, \Delta}$$

b) The set $\Theta_R = \{\langle a, b\rangle \in A^2 : \langle a, b\rangle \in R \text{ and } \langle b, a\rangle \in R\}$ is a congruence.

Under these conditions $\Omega_{\mathbf{A}} R = \Theta_R$ holds. A direct consequence of this characterization is that if $\mathcal{G}$ is a Gentzen system satisfying (*Axiom*) and also satisfying the adaptations to its type of sequents of the right and left exchange rules and the previous restricted version of the cut rule, then if Φ is a $\mathcal{G}$-theory and the set

$$\Theta_\Phi = \{\langle \varphi, \psi\rangle \in Fm_{\mathcal{L}}^2 : \varphi \Rightarrow \psi \in \Phi \text{ i } \psi \Rightarrow \varphi \in \Phi\}$$

is a congruence, we have that $\Omega\Phi = \Theta_\Phi$ (see [RV95, Corollary 2.22]). Note that this result is not applicable in order to characterize the Leibniz congruence of a $\mathcal{FL}$-theory, since the Gentzen system $\mathcal{FL}$ does not fall within the scope of the systems that are considered in the result reported.

Even so, in the following result we prove that if we start from the condition in which R contains the interpretation of (*Axiom*) and is closed under the most general version of the $\langle \mathcal{L}, \omega \times \omega\rangle$-rule of Cut, i.e., the rule

$$\frac{\Gamma \Rightarrow \Lambda, \varphi, \Theta \qquad \Sigma, \varphi, \Pi \Rightarrow \Delta}{\Sigma, \Gamma, \Pi \Rightarrow \Lambda, \Delta, \Theta},$$

then it is not necessary to impose that R must be closed under the exchange rules. As a corollary, we will have that if a system $\mathcal{G}$ satisfies $(Axiom)$ and (Cut), Φ is a $\mathcal{G}$-theory and the set Θ_Φ is a congruence, then $\Omega\Phi = \Theta_\Phi$ and, therefore, the result will be applicable for characterizing the Leibniz congruence of a $\mathcal{FL}$-theory.

Theorem 4.17. *Let $\mathcal{L}$ be any language, $\mathcal{T} \subseteq \omega \times \omega$ and $\langle \mathbf{A}, R\rangle$ a $\langle \mathcal{L}, \mathcal{T}\rangle$-matrix. We consider the relation*

$$\Theta_R = \{\langle a, b\rangle \in A^2 : \langle a, b\rangle \in R \text{ and } \langle b, a\rangle \in R\}.$$

It holds that:

i) If Θ_R is reflexive, then $\Omega_{\mathbf{A}}R \subseteq \Theta_R$.

ii) If $\Theta_R \in Con(\mathbf{A})$ and R is closed under the cut rule, then $\Theta_R = \Omega_{\mathbf{A}}R$.

Proof: $i)$: Let $\langle a, b\rangle \in \Omega_{\mathbf{A}}R$. As $\Omega_{\mathbf{A}}$ is compatible with R and $\langle a, a\rangle \in R$, we have $\langle a, a\rangle(a|b) \in R$ and, therefore, $\langle a, b\rangle \in R$ and $\langle b, a\rangle \in R$, i.e., $\langle a, b\rangle \in \Theta_R$.

$ii)$: Suppose $\langle \bar{x}, \bar{y}\rangle \in R$ and $\langle a, b\rangle \in \Theta_R$. We assume that $\bar{x} = \bar{x}_1, a, \bar{x}_2$. So, $\langle\langle \bar{x}_1, a, \bar{x}_2\rangle, \bar{y}\rangle \in R$ but, as $\langle b, a\rangle \in R$ and R is closed under the cut rule, we obtain $\langle\langle \bar{x}_1, b, \bar{x}_2\rangle, \bar{y}\rangle \in R$. Similarly, if $\bar{y} = \bar{y}_1, a, \bar{y}_2$, it is proved that $\langle \bar{x}, \langle \bar{y}_1, b, \bar{y}_2\rangle\rangle \in R$. □

Corollary 4.18. *Let $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash_{\mathcal{G}}\rangle$ be a Gentzen system such that $\langle 1, 1\rangle \in \mathcal{T}$. Suppose that $\mathcal{G}$ satisfies the structural rules $(Axiom)$ and (Cut). Given a $\mathcal{G}$-theory Φ, the set*

$$\Theta_\Phi = \{\langle \varphi, \psi\rangle \in Fm^2_{\mathcal{L}} : \varphi \Rightarrow \psi \in \Phi \text{ and } \psi \Rightarrow \varphi \in \Phi\}$$

is an equivalence relation and, if $\Theta_\Phi \in Con(\mathbf{Fm}_{\mathcal{L}})$, then $\Omega\Phi = \Theta_\Phi$.

Proof: Θ_Φ is an equivalence relation:

- *Reflexivity:* For each formula φ, we have that $\langle \varphi, \varphi\rangle \in \Theta_\Phi$, since $\varphi \Rightarrow \varphi$ is an instance of $(Axiom)$.
- *Symmetry:* According to the definition of Θ_Φ.
- *Transitivity:* If $\varphi \Rightarrow \psi \in \Phi$ and $\psi \Rightarrow \gamma \in \Phi$, then by applying (Cut) we obtain $\varphi \Rightarrow \gamma \in \Phi$.

If $\Theta_\Phi \in Con(\mathbf{Fm}_{\mathcal{L}})$, by Theorem 4.17, we have that $\Omega\Phi = \Theta_\Phi$. □

Corollary 4.19. *Let $\mathcal{G}$ be as in Corollary 4.18 and suppose that $\Theta_\Phi \in Con(\mathbf{Fm}_{\mathcal{L}})$, for every $\mathcal{G}$-theory Φ. Thus, it is satisfied that:*

For each $\Phi_1, \Phi_2 \in Th\,\mathcal{G}$, if $\Phi_1 \subseteq \Phi_2$, then $\Omega\Phi_1 \subseteq \Omega\Phi_2$.

Proof: If $\Phi_1, \Phi_2 \in Th\,\mathcal{G}$ are like $\Phi_1 \subseteq \Phi_2$ i $\langle\varphi,\psi\rangle \in \Theta_{\Phi_1}$, then $\varphi \Rightarrow \psi \in \Phi_2$ i $\psi \Rightarrow \varphi \in \Phi_2$ and, therefore, $\langle\varphi,\psi\rangle \in \Theta_{\Phi_2}$. So, $\Theta_{\Phi_1} \subseteq \Theta_{\Phi_2}$, that is, for the Corollary 4.18, $\Omega\Phi_1 \subseteq \Omega\Phi_2$. □

Theorem 4.20. *Let $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash_\mathcal{G}\rangle$ be a Gentzen system with $\langle 0,1\rangle, \langle 1,1\rangle \in \mathcal{T}$. Suppose that $\mathcal{G}$ satisfies the structural rules $(Axiom)$ and (Cut) and that $\mathcal{L}$ has a connective $\backslash$ whereby the following rules are satisfied:*

$$\frac{\Gamma \Rightarrow \varphi \qquad \Sigma, \psi, \Pi \Rightarrow \Delta}{\Sigma, \Gamma, \varphi\backslash\psi, \Pi \Rightarrow \Delta}\ (\backslash \Rightarrow) \qquad\qquad \frac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi\backslash\psi}\ (\Rightarrow \backslash)$$

If T is a theory of the external deductive system $\mathcal{E}(\mathcal{G})$, let Φ_T be the $\mathcal{G}$-theory generated by $\{\emptyset \Rightarrow \alpha : \alpha \in T\}$ and let Θ_{Φ_T} be the equivalence relation of Corollary 4.18 defined by Φ_T. It holds:

i) $\Theta_{\Phi_T} = \{\langle\varphi,\psi\rangle \in Fm^2_\mathcal{L} : \varphi\backslash\psi \in T \text{ and } \psi\backslash\varphi \in T\}$.

ii) *If $\Theta_{\Phi_T} \in Con(\mathbf{Fm}_\mathcal{L})$, then $\Omega T = \Omega\Phi_T$.*

Proof: i): $\langle\varphi,\psi\rangle \in \Theta_{\Phi_T}$ is equivalent to $\{\varphi \Rightarrow \psi, \varphi \Rightarrow \psi\} \subseteq \Phi_T$. By using $(Axiom)$, (Cut), $(\backslash \Rightarrow)$ i $(\Rightarrow \backslash)$, it is easy to see that

$$\{\varphi \Rightarrow \psi, \psi \Rightarrow \varphi\} \dashv\vdash_\mathcal{G} \{\emptyset \Rightarrow \varphi\backslash\psi, \emptyset \Rightarrow \psi\backslash\varphi\}.$$

Consequently, $\langle\varphi,\psi\rangle \in \Theta_{\Phi_T}$ is equivalent to $\{\emptyset \Rightarrow \varphi\backslash\psi, \emptyset \Rightarrow \psi\backslash\varphi\} \subseteq \Phi_T$. However, due to the definition of Φ_T, this is equivalent to

$$\{\emptyset \Rightarrow \alpha : \alpha \in T\} \vdash_\mathcal{G} \{\emptyset \Rightarrow \varphi\backslash\psi, \emptyset \Rightarrow \psi\backslash\varphi\},$$

which, due to the definition of $\mathcal{E}(\mathcal{G})$, is equivalent to $T \vdash_{\mathcal{E}(\mathcal{G})} \{\varphi\backslash\psi, \psi\backslash\varphi\}$ that, as T is a theory of $\mathcal{E}(\mathcal{G})$, is equivalent to $\{\varphi\backslash\psi, \psi\backslash\varphi\} \subseteq T$.

ii): If Θ_{Φ_T} is a congruence, due to Corollary 4.18, we have that $\Omega\Phi_T = \Theta_{\Phi_T}$. If $\langle\varphi,\psi\rangle \in \Omega\Phi_T$ and $\alpha \in T$, then $\emptyset \Rightarrow \alpha \in \Phi_T$ and, for the compatibility of $\Omega\Phi_T$ with Φ_T, we have that $(\emptyset \Rightarrow \alpha)(\varphi|\psi) \in \Phi_T$. So, $\emptyset \Rightarrow \alpha(\varphi|\psi) \in \Phi_T$, that is, $\alpha(\varphi|\psi) \in T$. Therefore, Θ_{Φ_T} is compatible with T. Now let ϑ be a congruence compatible with T and suppose $\langle\varphi,\psi\rangle \in \vartheta$. We have that $\varphi\backslash\varphi \in T$, since $\emptyset \Rightarrow \varphi\backslash\varphi$ is obtained from $\varphi \Rightarrow \varphi$ by applying $(\Rightarrow \backslash)$. Therefore, for the compatibility of ϑ with T, we have $\varphi\backslash\psi \in T$ and $\psi\backslash\varphi \in T$, i.e., $\langle\varphi,\psi\rangle \in \Theta_{\Phi_T}$. In short, Θ_{Φ_T} is the largest congruence compatible with T. □

Corollary 4.21. *Let $\mathcal{G}$ be as in Theorem 4.20 and suppose that $\Theta_{\Phi_T} \in Con(\mathbf{Fm}_\mathcal{L})$, for every $\mathcal{E}(\mathcal{G})$-theory T. Thus, for each $T_1, T_2 \in Th\,\mathcal{E}(\mathcal{G})$, if $T_1 \subseteq T_2$, then $\Omega T_1 \subseteq \Omega T_2$, that is, $\mathcal{E}(\mathcal{G})$ is protoalgebraic.*

Proof: If $T_1 \subseteq T_2$, then, obviously, $\Phi_{T_1} \subseteq \Phi_{T_2}$. Therefore, for Corollary 4.19, we have that $\Omega\Phi_{T_1} \subseteq \Omega\Phi_{T_2}$ but this, for *ii)* of Theorem 4.20, means that $\Omega T_1 \subseteq \Omega T_2$. □

Similarly, we obtain the following results.

Theorem 4.22. *Let $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash_{\mathcal{G}} \rangle$ be a Gentzen system with $\langle 0,1\rangle, \langle 1,1\rangle \in \mathcal{T}$. Suppose that $\mathcal{G}$ satisfies the structural rules (Axiom) and (Cut) and that $\mathcal{L}$ contains a connective / whereby the following rules are satisfied:*

$$\frac{\Gamma \Rightarrow \varphi \quad \Sigma, \psi, \Pi \Rightarrow \Delta}{\Sigma, \psi/\varphi, \Gamma, \Pi \Rightarrow \Delta}\ (/\Rightarrow) \qquad \frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \psi/\varphi}\ (\Rightarrow /)$$

If T is a theory of the external deductive system $\mathcal{E}(\mathcal{G})$, let Φ_T be the $\mathcal{G}$-theory generated by $\{\emptyset \Rightarrow \alpha : \alpha \in T\}$ and let Θ_{Φ_T} be the equivalence relation in Corollary 4.18 defined by Φ_T. The following holds:

i) $\Theta_{\Phi_T} = \{\langle \varphi, \psi\rangle \in Fm_{\mathcal{L}}^2 : \psi/\varphi \in T$ *and* $\varphi/\psi \in T\}$.

ii) If $\Theta_{\Phi_T} \in Con(\mathbf{Fm}_{\mathcal{L}})$*, then* $\Omega T = \Omega \Phi_T$.

Corollary 4.23. *Let $\mathcal{G}$ be as in Theorem 4.22 and suppose that $\Theta_{\Phi_T} \in Con(\mathbf{Fm}_{\mathcal{L}})$, for every $\mathcal{E}(\mathcal{G})$-theory T. Thus, for each $T_1, T_2 \in Th\,\mathcal{E}(\mathcal{G})$, if $T_1 \subseteq T_2$, then $\Omega T_1 \subseteq \Omega T_2$, that is, $\mathcal{E}(\mathcal{G})$ is protoalgebraic.*

With the above results, we have all we need to establish the characterizations of the Leibniz congruence of $\mathcal{FL}[\Psi]$-theories for any $\Psi \leq \mathfrak{L}$, and of the theories of the external deductive systems $\mathfrak{e}\mathcal{FL}[\Psi]$ for all the sublanguages Ψ of $\mathfrak{L}$ that contain one of the implication connectives.

Lemma 4.24. *Let Ψ be any sublanguage of language $\mathfrak{L}$ of* **FL**. *For every $\mathcal{FL}[\Psi]$-theory Φ, the set*

$$\Theta_\Phi = \{\langle \varphi, \psi\rangle \in Fm_\Psi : \varphi \Rightarrow \psi \in \Phi \ i\ \psi \Rightarrow \varphi \in \Phi\}$$

is a congruence of $\mathbf{Fm}_\Psi$.

Proof: Suppose $\langle \varphi_1, \psi_1\rangle, \langle \varphi_2, \psi_2\rangle \in \theta_\Phi$, that is,

$$\{\varphi_1 \Rightarrow \psi_1, \psi_1 \Rightarrow \varphi_1, \varphi_2 \Rightarrow \psi_2, \psi_2 \Rightarrow \varphi_2\} \subseteq \Phi.$$

By using the introduction rules of each binary connective $\odot \in \Psi$ it is easy to see that

$$\{\varphi_1 \odot \varphi_2 \Rightarrow \psi_1 \odot \psi_2, \psi_1 \odot \psi_2 \Rightarrow \varphi_1 \odot \varphi_2\} \subseteq \Phi.$$

On the other hand, if $\langle {}^{\backprime} \rangle \leq \Psi$ or $\langle {}' \rangle \leq \Psi$, by using the introduction rules for the negations it is easy to prove that if $\{\varphi \Rightarrow \psi, \psi \Rightarrow \varphi\} \subseteq \Phi$, then

$$\{\varphi^{\backprime} \Rightarrow \psi^{\backprime}, \psi^{\backprime} \Rightarrow \varphi^{\backprime}\} \subseteq \Phi \quad \text{or} \quad \{{}'\varphi \Rightarrow {}'\psi, {}'\psi \Rightarrow {}'\varphi\} \subseteq \Phi.$$

$\square$

Corollary 4.25. *Let Ψ be any sublanguage of the language $\mathfrak{L}$ of* **FL**. *For every $\mathcal{FL}[\Psi]$-theory Φ,*

$$\Omega\Phi = \{\langle\varphi,\psi\rangle \in Fm_\Psi : \varphi \Rightarrow \psi \in \Phi \textit{ and } \psi \Rightarrow \varphi \in \Phi\}.$$

Proof: The set Θ_Φ of Lemma 4.24 is a congruence. Therefore, since the Gentzen system $\mathcal{FL}[\Psi]$ determined by $\mathbf{FL}[\Psi]$ complies with the conditions of the Gentzen systems considered in Corollary 4.18, we have that $\Theta_\Phi = \Omega\Phi$. □

Corollary 4.26. *Let Ψ be a sublanguage of $\mathfrak{L}$ such that $\langle\backslash\rangle \leq \Psi$. Then, for every $\mathfrak{e}\mathcal{FL}[\Psi]$-theory T, $\Omega T = \{\langle\varphi,\psi\rangle \in Fm^2_\Psi : \varphi\backslash\psi \in T$ i $\psi\backslash\varphi \in T\}$.*

Proof: Let Φ_T be the $\mathfrak{e}\mathcal{FL}[\Psi]$-theory generated by $\{\emptyset \Rightarrow \varphi : \varphi \in T\}$. System $\mathcal{FL}[\Psi]$ complies with the conditions of the Gentzen systems considered in Theorem 4.20. Then, for *i*) of Theorem 4.20, we have that

$$\Theta_{\Phi_T} = \{\langle\varphi,\psi\rangle \in Fm^2_\Psi : \varphi\backslash\psi \in T \text{ and } \psi\backslash\varphi \in T\}.$$

Nevertheless, for Lemma 4.24, Θ_{Φ_T} is a congruence and, for *ii*) of Theorem 4.20, $\Omega T = \Omega\Phi_T$. □

Similarly, by using Theorem 4.22, we obtain the following result:

Corollary 4.27. *Let Ψ be a sublanguage of $\mathfrak{L}$ such that $\langle/\rangle \leq \Psi$. Then, for every $\mathfrak{e}\mathcal{FL}[\Psi]$-theory T, $\Omega T = \{\langle\varphi,\psi\rangle \in Fm^2_\Psi : \psi/\varphi \in T$ i $\varphi/\psi \in T\}$.*

Finally, we obtain that the external systems $\mathfrak{e}\mathcal{FL}[\Psi]$, for a sublanguage of $\mathfrak{L}$ that contains at least one of the two implications, are protoalgebraic.

Corollary 4.28. *If Ψ is a sublanguage of $\mathfrak{L}$ such that $\langle\backslash\rangle \leq \Psi$ or such that $\langle/\rangle \leq \Psi$, then $\mathfrak{e}\mathcal{FL}[\Psi]$ is protoalgebraic.*

Proof: By Corollaries 4.21 i 4.23. □

4.4 Hilbert-style Axiomatization for Systems $\mathfrak{e}\mathcal{FL}_\sigma$

In this section we present some axiomatizations for the external systems $\mathfrak{e}\mathcal{FL}_\sigma$ known in the literature.

Definition 4.29. ([GJKO07, Section 2.5.1]) **HFL** *is the deductive system with language $\langle\vee,\wedge,*,\backslash,/,0,1\rangle$ of type $\langle 2,2,2,2,2,0,0\rangle$ defined by the following axioms and rules:*

($\backslash$-*id*)	$\varphi\backslash\varphi$	*(identity)*
($\backslash$-*pf*)	$(\varphi\backslash\psi)\backslash[(\gamma\backslash\varphi)\backslash(\gamma\backslash\varphi)]$	*(prefixing)*
($\backslash$-*as*)	$\varphi\backslash[(\psi/\varphi)\backslash\psi]$	*(assertion)*
(a)	$[(\psi\backslash\gamma)/\varphi]\backslash[\psi\backslash(\gamma/\varphi)]$	*(associativity)*
($*\backslash/$)	$[(\psi * (\psi\backslash\varphi))/\psi]\backslash(\varphi/\psi)$	*(fusion implications)*
($*\wedge$)	$[(\varphi\wedge 1) * (\psi\wedge 1)]\backslash(\varphi\wedge\psi)$	*(fusion conjunction)*

$(\wedge_1\backslash)$	$(\varphi \wedge \psi)\backslash\varphi$	*(conjunction implication 1)*
$(\wedge_2\backslash)$	$(\varphi \wedge \psi)\backslash\psi$	*(conjunction implication 2)*
$(\backslash\wedge)$	$[(\gamma\backslash\varphi) \wedge (\gamma\backslash\psi)] \backslash [\gamma\backslash(\varphi \wedge \psi)]$	*(implication conjunction)*
$(\backslash\vee_1)$	$\varphi\backslash(\varphi \vee \psi)$	*(implication disjunction 1)*
$(\backslash\vee_2)$	$\psi\backslash(\varphi \vee \psi)$	*(implication disjunction 2)*
$(\vee\backslash)$	$[(\varphi\backslash\gamma) \wedge (\psi\backslash\gamma)] \backslash [(\varphi \vee \psi)\backslash\gamma]$	*(disjunction implication)*
$(\backslash *)$	$\psi\backslash(\varphi\backslash(\varphi * \psi))$	*(implication fusion)*
$(*\backslash)$	$[\psi\backslash(\varphi\backslash\gamma)] \backslash [(\varphi * \psi)\backslash\gamma]$	*(fusion implication)*
(1)	1	*(unit)*
$(1\backslash)$	$1\backslash(\varphi\backslash\varphi)$	*(unit implication)*
$(\backslash 1)$	$\varphi\backslash(1\backslash\varphi)$	*(implication unit)*
$(\backslash\text{-}mp)$	$\langle\{\varphi, \varphi\backslash\psi\}, \psi\rangle$	*($\backslash$-modus ponens)*
(adj_u)	$\langle\{\varphi\}, \varphi \wedge 1\rangle$	*(adjunction unit)*
$(\backslash\text{-}pn)$	$\langle\{\varphi\}, \psi\backslash(\varphi * \psi)\rangle$	*($\backslash$-product normality)*
$(/\text{-}pn)$	$\langle\{\varphi\}, (\psi * \varphi)/\psi)\rangle$	*($/$-product normality)*

It is easy to see that, if we add to the axiomatization set out in Definition 4.29 the schemata

$(\backslash\text{-}\,def_1)$ $\quad \varphi^{\backslash}\backslash(\varphi\backslash 0)$
$(\backslash\text{-}\,def_2)$ $\quad (\varphi\backslash 0)\backslash\varphi^{\backslash}$

$(/\text{-}\,def_1)$ $\quad {}'\varphi/(0/\varphi)$
$(/\text{-}\,def_2)$ $\quad (0/\varphi)/{}'\varphi$

the deductive system which defines the new axiomatization is a definitional expansion, in the language $\langle\vee, \wedge, *, \backslash, /, {}^{\backslash}, {}', 0, 1\rangle$, of the system **HFL**.

Definition 4.30. ([GJKO07, Section 2.5.1]) $\mathbf{HFL}_e$ *is the deductive system in the language* $\langle\vee, \wedge, *, \to, 0, 1\rangle$ *of type* $\langle 2, 2, 2, 2, 0, 0\rangle$ *defined by the following axioms and rules:*[3]

(id)	$\varphi \to \varphi$	*(identity)*
(pf)	$(\varphi \to \psi) \to ((\gamma \to \varphi) \to (\gamma \to \varphi))$	*(prefixing)*
(per)	$(\varphi \to (\psi \to \gamma)) \to (\psi \to (\varphi \to \gamma))$	*(permutation)*
$(*\wedge)$	$((\varphi \wedge 1) * (\psi \wedge 1)) \to \varphi \wedge \psi$	*(fusion conjunction)*
$(\wedge_1 \to)$	$\varphi \wedge \psi \to \varphi$	*(conjunction implication 1)*
$(\wedge_2 \to)$	$\varphi \wedge \psi \to \psi$	*(conjunction implication 2)*
$(\to \wedge)$	$(\gamma \to \varphi) \wedge (\gamma \to \psi) \;\to\; (\gamma \to \varphi \wedge \psi)$	*(implication conjunction)*

[3] It is understood that the connective $\to$ establishes a less strong binding than the other connectives.

$(\to \vee_1)$	$\varphi \to \varphi \vee \psi$	*(implication disjunction 1)*
$(\to \vee_2)$	$\psi \to \varphi \vee \psi$	*(implication disjunction 2)*
$(\vee \to)$	$(\varphi \to \gamma) \wedge (\psi \to \gamma) \to (\varphi \vee \psi \to \gamma)$	*(disjunction implication)*
$(\to *)$	$\psi \to (\varphi \to \varphi * \psi)$	*(implication fusion)*
$(* \to)$	$(\psi \to (\varphi \to \gamma)) \to (\varphi * \psi \to \gamma)$	*(fusion implication)*
(1)	1	*(unit)*
$(1 \to)$	$1 \to (\varphi \to \varphi)$	*(unit implication)*
(mp)	$\langle \{\varphi, \varphi \to \psi\}, \psi \rangle$	*(modus ponens)*
(adj_u)	$\langle \{\varphi\}, \varphi \wedge 1 \rangle$	*(adjunction unit)*

If we add to an axiomatization of $\mathbf{HFL}_e$ the schemata

$(\neg\text{-}def_1)$ $\neg\varphi \to (\varphi \to 0)$
$(\neg\text{-}def_2)$ $(\varphi \to 0) \to \neg\varphi$

the deductive system which defines the new axiomatization is a definitional expansion, in language $\langle \vee, \wedge, *, \to, \neg, 0, 1 \rangle$, of the system $\mathbf{HFL}_e$.

Definition 4.31 (Systems $\mathbf{HFL}_\sigma$). *We consider the schemata:*

(w_l) $\varphi \backslash (\psi \backslash \varphi)$
(w_r) $0 \backslash \varphi$
(c) $(\varphi \backslash (\varphi \backslash \psi)) \backslash (\varphi \backslash \psi)$

If $\sigma \leq w_l w_r c$, $\mathbf{HFL}_\sigma$ denotes the extension of $\mathbf{HFL}$ with the schemata codified by σ. If $e \leq \sigma \leq e w_l w_r c$, $\mathbf{HFL}_\sigma$ denotes the extension of $\mathbf{HFL}$ with the schemata codified by σ. In this case, we substitute connective $\backslash$ for connective $\to$ in the referred schemata.

Theorem 4.32. (Cf. [GJKO07, Section 2.5]) *Let σ be a subsequence, may be empty, of $e w_l w_r c$. The following is satisfied:*

$$\mathfrak{e}\mathcal{FL}_\sigma = \mathbf{HFL}_\sigma.$$

4.4.1 Strongly Separable Axiomatizations

As the authors of [GO06] point out, there is a strongly separable axiomatization for **HFL** in the manuscript [GO], which we will not present here because for the moment we do not have access to this manuscript. In [vAR04] Van Alten and Raftery present a strongly separable axiomatization for the fragment without exponents, without 0 and without additive constants of the *intuitionist linear logic* (see [Tro92, p.67]), that is, for the logic of commutative residuated lattices. This presentation, when we add to language symbol 0 without adding any schema where this symbol expressly appears, is a strongly separable axiomatization of the deductive system $\mathbf{HFL}_e$ (see first paragraph of [GJKO07, Section 2.5]). We transcribe it below:

(id)	$\varphi \to \varphi$
(pf)	$(\varphi \to \psi) \to [(\gamma \to \varphi) \to (\gamma \to \varphi)]$
(per)	$[\varphi \to (\psi \to \gamma)] \to [\psi \to (\varphi \to \gamma)]$
(mp)	$\langle\{\varphi, \varphi \to \psi\}, \psi\rangle$
$(\to \vee_1)$	$\varphi \to \varphi \vee \psi$
$(\to \vee_2)$	$\psi \to \varphi \vee \psi$
(dis)	$\langle\{\varphi \to \gamma, \psi \to \gamma\}, \varphi \vee \psi \to \gamma\rangle$
$(\wedge_1 \to)$	$\varphi \wedge \psi \to \varphi$
$(\wedge_2 \to)$	$\varphi \wedge \psi \to \psi$
$(\to \wedge)$	$(\gamma \to \varphi) \wedge (\gamma \to \psi) \to (\gamma \to \varphi \wedge \psi)$
(adj)	$\langle\{\varphi, \psi\}, \varphi \wedge \psi\rangle$
$(\to *)$	$(\psi \to (\varphi \to \gamma)) \to (\varphi * \psi \to \gamma)$
$(* \to)$	$\psi \to (\varphi \to (\varphi * \psi))$
(1)	1
$(1 \to)$	$1 \to (\varphi \to \varphi)$

Note 4.33. The system $\mathbf{HFL}_{ew}$ is definitionally equivalent to the Monoidal Logic (see [Höh95, Got01, GGCB03]), also called Intuitionistic Logic without contraction in [Adi01, AV00, BGCV06]. If we add to the calculus of Van Alten and Raftery the schema $\varphi \to (\psi \to \varphi)$, then we obtain an axiomatization for $\mathbf{HFL}_{ew_l}$ where one can prove that the rule (dis) may be replaced by the schema $(\vee \to)$, the rule (adj), by the schema $\varphi \to (\psi \to \varphi \wedge \psi)$, and the schemata (1) and $(1 \to)$, by the schema $\varphi \to 1$. If in this axiomatization for $\mathbf{HFL}_{ew_l}$ we add schema $0 \to \varphi$, we obtain an axiomatization for the Monoidal Logic in language $\langle \vee, \wedge, *, \to, 0, 1\rangle$ which is strongly separable.

Chapter 5

Definability of Connectives and Definitional Expansions

In Section 5.1 we present and focus on the notions of *notational copy*, *definability of connectives*, *definitional expansion* and *definitional equivalence* in the context of the Gentzen systems. In Section 5.2 we obtained results related to these notions for certain general sorts of Gentzen systems which belong to systems $\mathcal{FL}_\sigma[\Psi]$. These results are used in the following sections in order to establish exactly the notions and affirmations which receive an informal treatment in the substructural logics literature, such as collapse and definability of connectives in certain systems $\mathcal{FL}_\sigma[\Psi]$ or the comparison of the different versions of the same system.

5.1 Motivation and definitions

In the substructural logics literature, it is usual to address the issue of the definability of connectives in an informal manner. One example of this sort of treatment may be found when we say that, in presence of the exchange rule, the two implications are "basically the same connective", or are "equivalents", or "they behave in the same way", since the sequents $\varphi\backslash\psi \Rightarrow \psi/\varphi$ and $\psi/\varphi \Rightarrow \varphi\backslash\psi$ are derivable in $\mathbf{FL}_e$ and, therefore, due to the presence of the cut rule "every implication satisfies the introduction rules of the other implication". Expressions such as the ones in inverted commas are generally accepted, even though there are notions such as "essential equality of connectives", or "connective equivalence", or "connective behaviour", or "satisfaction of a rule by a connective" are used, which have not been defined previously, but rather they refer to intuitions, reasonable -of course- but, essentially, only to intuitions.

Another instance of the same sort of informal treatment may be found in the comparison between the calculus $\mathbf{FL}_{ewc}$ and the Gentzen calculus of LJ for the intuitionistic logic. $\mathbf{FL}_{ewc}$ is obtained from LJ by adding to the connectives of this calculus a connective $*$ that satisfies the rules

$$\frac{\Sigma, \varphi, \psi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi * \psi, \Gamma \Rightarrow \Delta}\ (* \Rightarrow) \qquad\qquad \frac{\Gamma \Rightarrow \varphi \qquad \Pi \Rightarrow \psi}{\Gamma, \Pi \Rightarrow \varphi * \psi}\ (\Rightarrow *).$$

By using the rules $(e \Rightarrow)$, $(w \Rightarrow)$ and $(c \Rightarrow)$ it can be proved that sequents $\varphi * \psi \Rightarrow \psi \wedge \varphi$ and $\psi \wedge \varphi \Rightarrow \varphi * \psi$ are derivable. Then, from this fact we may conclude that in this context the two conjunctions have the same behaviour and that, therefore, $\mathbf{FL}_{ewc}$ and LJ are equivalent or that $\mathbf{FL}_{ewc}$ is a redundant version of LJ. This is generally accepted but, under what general conditions can we agree that two systems expressed in different languages are "equivalent", or that a system is a "redundant version" of another one.

It is also common, when comparing logic systems presented through calculi of sequents defined by two different languages, to employ expressions such as "the system A is *basically* the same as system B", where this notion of *essential equality* is a clearly informal notion. An instance of this fact is assumed when comparing the two frequent versions of the *Full Lambek Calculus*: on the one hand, the version in a language with negations (and, from time to time, with zero) and sequents that permit the empty sequence or a formula in the consequent and, on the other hand, the version in a language without negations and with zero and sequents which exactly accept a formula in the consequent. Negations are dealt with in two ways in the *Full Lambek Calculus*: a) they are included in the language; b) we have a constant 0 in the language and consider negations as abbreviations of the formulas $\varphi\backslash 0$ and $0/\varphi$. It is generally agreed that these two forms are equivalent which that the right negation is definable based on the right implication and zero and that the left negation is so based on the left implication and zero. However, in any case, these considerations do not rely on a strict definition of *definable connective* and when we say that the two versions are equivalent, it is not a reference to the notion of *equivalence* between systems expressed in different languages which have been defined previously, but rather that it is an implicit recognition of the form "all roads lead to Rome".

From our point of view, the appropriate concepts for the informal notions we have just considered are the following:

- the concept of *notational copy* for the notion of connective equivalence,
- a concept of *definable connective* based on the substitutability through the sequential Leibniz congruence,
- the concepts of *expansion* and *definitional equivalence* for the notions of equivalence among systems expressed in different languages.

These concepts, which we address below, are generalizations of the analogue notions introduced by Wójcicki in the framework of sentential logics (see [Wój88, Chapter 1]).[1]

[1] These notions are considered by Adillon, in the context of sentential logics as well, in [Adi01], where they are adapted for the language of the Abstract Algebraic Logic.

Definition 5.1 (Notational Copy). *Let $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash_{\mathcal{G}} \rangle$ and $\mathcal{G}' = \langle \mathcal{L}', \mathcal{T}', \vdash_{\mathcal{G}'} \rangle$ be two Gentzen systems such that $\mathcal{L}$ and $\mathcal{L}'$ have the same type of similarity and such that $\mathcal{T} = \mathcal{T}'$. We will say that $\mathcal{G}'$ is a* notational copy *of $\mathcal{G}$ if there is an isomorphism $f : \mathbf{Fm}_{\mathcal{L}} \longrightarrow \mathbf{Fm}_{\mathcal{L}'}$ such that $f \upharpoonright Var$ is the identity and, for every $\Upsilon \cup \{\varsigma\} \subseteq Seq_{\mathcal{L}}^{\mathcal{T}}$,*

$$\Upsilon \vdash_{\mathcal{G}} \varsigma \qquad \text{iff} \qquad f[\Upsilon] \vdash_{\mathcal{G}'} f(\varsigma).$$

In this case, if $\iota \in \mathcal{L}$ and $\iota' \in \mathcal{L}'$ of arity k are such that $\iota \neq \iota'$ but, for every $\alpha_1, \ldots, \alpha_k \in Fm_{\mathcal{L}}$, we have that $f(\iota(\alpha_1, \ldots, \alpha_k) = \iota'(f(\alpha_1), \ldots, f(\alpha_k))$, then we will say that the connectives ι and ι' are the same up to notation.

Of course, the relation of being a notational copy is an equivalence relation. Two systems are notational copy if, and only if, each connective of $\mathcal{L}$ is, either equal to a connective of $\mathcal{L}'$, or equal to a connective of $\mathcal{L}'$ up to notation.

Observation 5.2. Note that if $\mathcal{G}$ and $\mathcal{G}'$ are notational copy and $\langle 0, 1 \rangle \in \mathcal{T}$, then the external deductive systems $\mathcal{E}(\mathcal{G})$ and $\mathcal{E}(\mathcal{G}')$ are notational copy as well.

Definition 5.3 (Definable connective). *Let $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash \rangle$ be a Gentzen system and let Ψ be a sublanguage of $\mathcal{L}$. We assume that a connective $\iota \in \mathcal{L}$ of arity k is* definable in *$\mathcal{G}$ in terms of the connectives of Ψ if there is a formula $\eta \in Fm_{\Psi}$ such that, for every $\Phi \in Th\,\mathcal{G}$ and every $\alpha_1, \ldots, \alpha_k \in Fm_{\mathcal{L}}$,*

$$\langle \iota(\alpha_1, \ldots, \alpha_k)\, , \eta(p_1|\alpha_1, \ldots, p_k|\alpha_k) \rangle \in \Omega\Phi. \tag{5.1}$$

In this case, we will say that the connective ι is definable by means the formula *η.*

Note 5.4. A congruence θ of an algebra $\mathbf{A}$ is *fully invariant* if for every $a, b \in A$ and every endomorphism $\sigma \in Hom(\mathbf{A}, \mathbf{A})$, if $\langle a, b \rangle \in \theta$, then $\langle \sigma(a), \sigma(b) \rangle \in \theta$. It is easy to see that, for each theory Φ, $\Omega\Phi$ is a fully invariant congruence. By applying this fact, it is deduced, in order to prove (5.1), that it is enough to show that $\langle \iota(p_1, \ldots, p_k), \eta \rangle \in \Omega\Phi$.

Definition 5.5 (Definitional Expansion). *Let $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash_{\mathcal{G}} \rangle$ and $\mathcal{G}' = \langle \mathcal{L}', \mathcal{T}', \vdash_{\mathcal{G}'} \rangle$ such that $\mathcal{L} \leq \mathcal{L}'$ and $\mathcal{T} \subseteq \mathcal{T}'$. We say that $\mathcal{G}'$ is a* definitional expansion *of $\mathcal{G}$ if it is conservative (i.e., $\mathcal{G}$ is the $\langle \mathcal{L}, \mathcal{T} \rangle$-fragment of $\mathcal{G}'$) and every connective of $\mathcal{L}'$ is definable in $\mathcal{G}'$ in terms of the connectives of $\mathcal{L}$.*

The concept of definitional equivalence relies on the notions of *notational copy* and *definitional expansion*.

Definition 5.6 (Definitional Equivalence). *The two Gentzen systems $\mathcal{G}_1$ and $\mathcal{G}_2$ are* definitionally equivalent *if there is a Gentzen system that is a definitional expansion of $\mathcal{G}_1$ or a notational copy of $\mathcal{G}_1$ and of $\mathcal{G}_2$ or a notational copy of $\mathcal{G}_2$.*

Observation 5.7. Obviously, if $\mathcal{G}$ is a definitional expansion of $\mathcal{G}'$, then $\mathcal{G}$ and $\mathcal{G}'$ are definitionally equivalent.

According to these notions, the informal affirmations that we referred at the beginning of this section may be formulated as follows.

- Saying that in $\mathbf{FL}_e$ the two implications are "the same connective" and also the two negations may be precisely expressed by establishing that the systems $\mathcal{FL}_e[\vee, \wedge, *, \backslash, ^{\backprime}, 0, 1]$ and $\mathcal{FL}_e[\vee, \wedge, *, /, ', 0, 1]$ are a notational copy.
- The fact that the calculi $\mathbf{FL}_e$ and $\mathbf{FL}_e[\vee, \wedge, *, \backslash, ^{\backprime}, 0, 1]$ are equivalent may be stated with precision by proving that the system $\mathcal{FL}_e$ is a definitional expansion of $\mathcal{FL}_e[\vee, \wedge, *, \backslash, ^{\backprime}, 0, 1]$ and that the left implication is definable in terms of the right implication by means of the formula $p\backslash q$ (see Paragraph I in Section 5.3).
- The fact that $\mathbf{FL}_{ewc}$ is a "redundant version" of LJ can be formulated as follows: the Gentzen system defined by $\mathbf{FL}_{ewc}$ is a definitional expansion of the Gentzen system defined by LJ (we can also say that both are definitionally equivalent) and that the connective $*$ is definable in $\mathbf{FL}_{ewc}$ terms of the connective $\wedge$ by means the formula $p \wedge q$ (cf. Paragraph II in Section 5.3).
- Let us denote by $\mathbf{FL}$ the *Full Lambek Calculus* in its version in sequents of type $\omega \times \{0, 1\}$ and language $\langle \vee, \wedge, *, \backslash, /, ^{\backprime}, ', 0, 1\rangle$, and by $\mathbf{FL}'$ the *Full Lambek Calculus* in its version in sequents of type $\omega \times \{1\}$ and language $\langle \vee, \wedge, *, \backslash, /, 0, 1\rangle$. If we denote $\mathcal{FL}$ and $\mathcal{FL}'$ by the Gentzen systems defined by $\mathbf{FL}$ and $\mathbf{FL}'$ respectively, then, expressions such as "$\mathbf{FL}$ and $\mathbf{FL}'$ are equivalent", or "basically the same" can be formulated exactly establishing that $\mathcal{FL}$ *is a definitional expansion of* $\mathcal{FL}'$ (we may also say that both systems are definitionally equivalent) and that the right (left) negation connective is definable in terms of the right (left) implication connective and the zero by the formula $p\backslash 0$ (the formula $0/p$) (see Section 5.7).

Note 5.8. The notions of definitional expansion and definitional equivalence between deductive systems or Gentzen systems can be seen as generalizations of the analogue notions that are commonly used in the Universal Algebra for establishing equivalences between classes of algebra of different kinds of similarity (or among their associated equational consequences). Some terms used by this kind of equivalences are the *definitional equivalence* (see [McN76, Definition 1.11] and references offered therein), *polynomial equivalence* (see [BS00, Definition 13.3]) and also *term-equivalence* (see, e.g., [Pał04]).

5.2 Some general results about Definability and Definitional Expansions

By applying the characterizations made in Section 4.3 and the notions considered in the previous section, in this section we present some results related to the definability of connectives and definitional expansions. The two results specified below will be used repeatedly in following sections and are established for the general class of Gentzen systems considered in Section 4.3 which comply with the requirement of being regular (see Definition 3.1). Note that in these results we employed the notations relative to restriction to sublanguages introduced in Definition 3.2. Both theorems are applicable to the systems $\mathcal{FL}_\sigma[\Psi]$, because they belong to the aforesaid general class.

Theorem 5.9. *Let $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash_\mathcal{G}\rangle$ be a regular Gentzen system defined by a calculus $\mathcal{C}$ that has (Axiom) and (Cut) within its structural rules and assume that, for every $\mathcal{G}$-theory Φ, the set Θ_Φ is a congruence. Let Ψ and Ψ' be two sublanguages of $\mathcal{L}$ such that $\Psi \leq \Psi'$ and let $\iota_1, \ldots, \iota_n$ be the connectives of Ψ' which are not in Ψ. Suppose that the following conditions are satisfied:*

i) For every j, $1 \leq j \leq n$, the connective ι_j is definable in $\mathcal{G}[\Psi']$ in terms of the connectives of Ψ by means of a formula η_j.

ii) For every j, $1 \leq j \leq n$, the two introduction rules for the connective ι_j are derivable by using only the rules of the calculus $\mathcal{C}[\Psi]$ and the sequents

$$\iota_j(\varphi_1, \ldots, \varphi_{k_j}) \Rightarrow \eta_j(p_1|\varphi_1, \ldots, p_{k_j}|\varphi_{k_j}), \text{ and}$$
$$\eta_j(p_1|\varphi_1, \ldots, p_{k_j}|\varphi_{k_j}) \Rightarrow \iota_j(\varphi_1, \ldots, \varphi_{k_j}),$$

where k_j is the arity of the connective ι_j.

Then, the following holds:

a) The calculus obtained by replacing the introduction rules of the connectives $\iota_1, \ldots, \iota_n$ with the sequents of the condition ii) is an alternative axiomatization for $\mathcal{G}[\Psi']$.

b) If f is the mapping from $Fm_{\Psi'}$ into Fm_Ψ defined by

$$f(\varphi) := \begin{cases} \varphi, & \text{if } \varphi \in Var, \\ \iota(f(\alpha_1), \ldots, f(\alpha_k)), & \text{if } \varphi = \iota(\alpha_1, \ldots, \alpha_k) \text{ and } \iota \in \Psi, \\ \eta_j(p_1|f(\alpha_1), \ldots, p_{k_j}|f(\alpha_{k_j})), & \text{if } \varphi = \iota_j(\alpha_1, \ldots, \alpha_{k_j}), \end{cases}$$

for every $\Upsilon \cup \{\varsigma\} \subseteq Seq^{\mathcal{T}}_{\Psi'}$, if $\Upsilon \vdash_{\mathcal{G}[\Psi']} \varsigma$ then $f[\Upsilon] \vdash_{\mathcal{G}[\Psi]} f(\varsigma)$.

c) $\mathcal{G}[\Psi]$ is the Ψ-fragment of $\mathcal{G}[\Psi']$.

d) $\mathcal{G}[\Psi']$ is a definitional expansion of $\mathcal{G}[\Psi]$.

Proof: a): As $\mathcal{G}$ is regular, it should be enough to see that the sequents of the condition ii) are derivable in $\mathcal{G}[\Psi']$. Due to condition i) we have that, for every $\mathcal{G}[\Psi']$-theory Φ,

$$\langle \iota_j(\varphi_1, \ldots, \varphi_{k_j}), \eta_i(p_1|\varphi_1, \ldots, p_{k_j}|\varphi_{k_j})\rangle \in \Omega\Phi.$$

As Θ_Φ is a congruence, by Corollary 4.18, we have that $\Omega\Phi = \Theta_\Phi$ and, therefore, the sequents of the condition ii) belong to every $\mathcal{G}[\Psi']$-theory and, in particular, to the smallest theory, i.e., they are derivable in $\mathcal{G}[\Psi']$.

b): Observe that if $\langle \Phi, \varrho \rangle$ is an instance of a rule of the alternative axiomatization of $\mathcal{G}[\Psi']$ in *a*), then $\langle f[\Phi], f(\varrho) \rangle$ is an instance of a rule of the calculus $\mathcal{C}[\Psi']$: it is evident in the case of the common rules for both calculi and, if ϱ is an instance of one of the sequents of the condition *ii*), then $f(\varrho)$ is an instance of (*Axiom*). Now let $\Upsilon \cup \{\varsigma\} \subseteq Seq^{\mathcal{T}}_{\Psi'}$ and suppose that $\Upsilon \vdash_{\mathcal{G}[\Psi']} \varsigma$. We will use induction on the length of the proof of ς from Υ. If ς is an instance of an axiom, according to the one we have just seen, $f(\varsigma)$ is an instance of an axiom of $\mathcal{G}[\Psi]$. If $\varsigma \in \Upsilon$, obviously $f(\varsigma) \in f[\Upsilon]$ and, therefore, $f[\Upsilon] \vdash_{\mathcal{G}[\Psi]} f(\varsigma)$. If $n > 1$ and ς are obtained by applying an instance $\langle \Phi, \varsigma \rangle$ of a rule, as we have seen, we have that $\langle f[\Phi], f(\varsigma) \rangle$ is an instance of a rule of the calculus $\mathcal{C}[\Psi]$ and, by the induction hypothesis, we have $f[\Upsilon] \vdash_{\mathcal{G}[\Psi]} f[\Phi]$. Consequently, $f[\Upsilon] \vdash_{\mathcal{G}[\Psi]} f(\varsigma)$.

c): We should notice that, for every $\Upsilon \cup \{\varsigma\} \subseteq Seq^{\mathcal{T}}_{\Psi}$, $\Upsilon \vdash_{\mathcal{G}[\Psi']} \varsigma$ iff $\Upsilon \vdash_{\mathcal{G}[\Psi]} \varsigma$. Suppose $\Upsilon \vdash_{\mathcal{G}[\Psi']} \varsigma$. By applying *b*) we obtain $f[\Upsilon] \vdash_{\mathcal{G}[\Psi]} f(\varsigma)$. But, as $\Upsilon \cup \{\varsigma\} \subseteq Seq^{\mathcal{T}}_{\Psi'}$, we have $f[\Upsilon] = \Upsilon$ and $f(\varsigma) = \varsigma$. Thus, $\Upsilon \vdash_{\mathcal{G}[\Psi]} \varsigma$. Reciprocally, if $\Upsilon \vdash_{\mathcal{G}[\Psi]} \varsigma$, then it is evident that $\Upsilon \vdash_{\mathcal{G}[\Psi']} \varsigma$, since $\mathcal{G}[\Psi]$ is a subsystem of $\mathcal{G}[\Psi']$.

d): By condition *i*) and item *c*) we have that $\mathcal{G}[\Psi']$ is a definitional expansion of $\mathcal{G}[\Psi]$.

□

Theorem 5.10. *Let $\mathcal{G} = \langle \mathcal{L}, \mathcal{T}, \vdash_{\mathcal{G}} \rangle$ be as in Theorem 5.9. Let ι_1 and ι_2 be two connectives of $\mathcal{L}$ of the same arity k and let Ψ be a sublanguage of $\mathcal{L}$ such that ι_1 and ι_2 are not in Ψ. Suppose that the following two conditions are satisfied:*

i) *ι_2 is definable in terms of ι_1 by the formula $\iota_1(p_1, \ldots, p_k)$.*

ii) *The two introduction rules for connective ι_2 are derivable by using only rules of the calculus $\mathcal{C}[\Psi, \iota_1]$ and the sequents*

$$\iota_1(\varphi_1, \ldots, \varphi_k) \Rightarrow \iota_2(\varphi_1, \ldots, \varphi_k) \;\; \textit{and} \;\; \iota_2(\varphi_1, \ldots, \varphi_k) \Rightarrow \iota_1(\varphi_1, \ldots, \varphi_k).$$

Then $\mathcal{G}[\Psi, \iota_1, \iota_2]$ is a definitional expansion of $\mathcal{G}[\Psi, \iota_1]$ and $\mathcal{G}[\Psi, \iota_2]$. Furthermore, $\mathcal{G}[\Psi, \iota_1]$ and $\mathcal{G}[\Psi, \iota_2]$ are notational copy, i.e., the connectives ι_1 and ι_2 are the same up to notation.

Proof: On the one hand, by Theorem 5.9 we have that $\mathcal{G}[\Psi, \iota_1, \iota_2]$ is a definitional expansion of $\mathcal{G}[\Psi, \iota_1]$ and $\mathcal{G}[\Psi, \iota_2]$. And, on the other hand, we should observe that there is an isomorphism h between $Fm_{\langle \Psi, \iota_1 \rangle}$ and $Fm_{\langle \Psi, \iota_2 \rangle}$ such that, for every $\Upsilon \cup \{\varsigma\} \subseteq Seq^{\mathcal{T}}_{\langle \Psi, \iota_1 \rangle}$,

$$\Upsilon \vdash_{\mathcal{G}[\Psi, \iota_1]} \varsigma \quad \text{iff} \quad h[\Upsilon] \vdash_{\mathcal{G}[\Psi, \iota_2]} h(\varsigma).$$

Suppose $\Upsilon \vdash_{\mathcal{G}[\Psi, \iota_1]} \varsigma$. By Theorem 5.9, given the mapping f from $Fm_{\langle \Psi, \iota_1, \iota_2 \rangle}$ into $Fm_{\langle \Psi, \iota_2 \rangle}$ defined by

$$f(\varphi) := \begin{cases} \varphi, & \text{if } \varphi \in Var, \\ \iota(f(\alpha_1), \ldots, f(\alpha_k)), & \text{if } \varphi = \iota(\alpha_1, \ldots, \alpha_k) \text{ and } \iota \in \langle \Psi, \iota_1 \rangle, \\ \iota_2(p_1 | f(\alpha_1), \ldots, p_k | f(\alpha_k)), & \text{if } \varphi = \iota_1(\alpha_1, \ldots, \alpha_k), \end{cases}$$

we have that $f[\Upsilon] \vdash_{\mathcal{G}[\Psi,\iota_2]} f(\varsigma)$. If we define $h := f \restriction Fm_{\langle\Psi,\iota_2\rangle}$, it is clear that h is an isomorphism and, of course, $h[\Upsilon] \vdash_{\mathcal{G}[\Psi,\iota_2]} h(\varsigma)$. Reciprocally, suppose $h[\Upsilon] \vdash_{\mathcal{G}[\Psi,\iota_2]} h(\varsigma)$. Once again by Theorem 5.9, given the mapping g from $Fm_{\langle\Psi,\iota_1,\iota_2\rangle}$ into $Fm_{\langle\Psi,\iota_1\rangle}$ defined by

$$g(\varphi) := \begin{cases} \varphi, & \text{if } \varphi \in Var, \\ \iota(f(\alpha_1),\ldots,f(\alpha_k)), & \text{if } \varphi = \iota(\alpha_1,\ldots,\alpha_k) \text{ and } \iota \in \langle\Psi,\iota_2\rangle, \\ \iota_1(p_1|f(\alpha_1),\ldots,p_k|f(\alpha_k)), & \text{if } \varphi = \iota_2(\alpha_1,\ldots,\alpha_k), \end{cases}$$

if we assume $\bar{h} := g \restriction Fm_{\langle\Psi,\iota_1\rangle}$, we will have $(\bar{h} \circ h)[\Upsilon] \vdash_{\mathcal{G}[\Psi,\iota_1]} (\bar{h} \circ h)(\varsigma)$. However, it is obvious that $\bar{h}$ is the inverse application of h, i.e., $\bar{h} \circ h$ is the identity and, therefore, $\Upsilon \vdash_{\mathcal{G}[\Psi,\iota_1]} \varsigma$. □

In the following result the necessary conditions are met in order to ensure that if a system $\mathcal{G}'$ is a definitional expansion of a system $\mathcal{G}$, there will be the same relation between the external systems $\mathcal{E}(\mathcal{G}')$ and $\mathcal{E}(\mathcal{G})$. Note that in this result we do not impose that the Gentzen systems are regular.

Theorem 5.11. *Let $\mathcal{G} = \langle\mathcal{L}, \mathcal{T}, \vdash_{\mathcal{G}}\rangle$ be a Gentzen system with $\langle 0,1\rangle, \langle 1,1\rangle \in \mathcal{T}$. Suppose that $\mathcal{G}$ satisfies the structural rules (Axiom) and (Cut) and that $\mathcal{L}$ contains a connective $\backslash$ (a connective $/$) for which the rules ($\backslash \Rightarrow$) and ($\Rightarrow \backslash$) (rules ($/ \Rightarrow$) and ($\Rightarrow /$)) are satisfied. Let $\mathcal{G}' = \langle\mathcal{L}', \mathcal{T}', \vdash_{\mathcal{G}'}\rangle$ be a conservative expansion of $\mathcal{G}$. If, for every $\mathcal{G}'$-theory Φ, the set $\Theta_\Phi = \{\langle\varphi,\psi\rangle \in Fm^2_{\mathcal{L}} : \{\varphi \Rightarrow \psi, \psi \Rightarrow \varphi\} \subseteq \Phi\}$ is a congruence, then the following is satisfied:*

- *a) If ι is a connective of $\mathcal{L}'$ that is not in $\mathcal{L}$ and that is definable in $\mathcal{G}'$ in terms of the connectives of $\mathcal{L}$ by a formula η, then ι is also definable in the external system $\mathcal{E}(\mathcal{G}')$ in terms of the connectives of $\mathcal{L}$ by the same formula η.*
- *b) Si $\mathcal{G}'$ is a definitional expansion of $\mathcal{G}$, then $\mathcal{E}(\mathcal{G}')$ is a definitional expansion of $\mathcal{E}(\mathcal{G})$.*

Proof: $a)$: Let T be a theory of $\mathcal{E}(\mathcal{G})$ and let Φ_T be a $\mathcal{G}$-theory generated by the set $\{\emptyset \Rightarrow \alpha : \emptyset \Rightarrow \alpha \in T\}$. As ι is definable in $\mathcal{G}'$ in terms of the connectives of $\mathcal{L}$ by η we have that if k is the arity of ι, then, for each $\alpha_1, \ldots, \alpha_k \in Fm_{\mathcal{L}'}$,

$$\langle\iota(\alpha_1,\ldots,\alpha_k), \eta(p_1|\alpha_1,\ldots,p_k|\alpha_k)\rangle \in \Omega\Phi_T.$$

Nevertheless, as Θ_Φ is a congruence, by Theorem 4.20 (by Theorem 4.22 for the case of the connective $/$) we have that $\Omega\Phi_T = \Omega T$ and, therefore,

$$\langle\iota(\alpha_1,\ldots,\alpha_k), \eta(p_1|\alpha_1,\ldots,p_k|\alpha_k)\rangle \in \Omega T.$$

In short, ι is definable in $\mathcal{E}(\mathcal{G})$ in terms of the connectives of Ψ by means of η.

$b)$: If $\mathcal{G}'$ is a definitional expansion of $\mathcal{G}$ then, by definition, we have that

- $i)$ the connectives of $\mathcal{L}'$ which are not in $\mathcal{L}$ are definable in $\mathcal{G}'$ in terms of the connectives of $\mathcal{L}$.

ii) $\mathcal{G}$ is the $\mathcal{L}$-fragment of $\mathcal{G}'$.

By *a*) we have that the connectives of $\mathcal{L}'$ which are not in $\mathcal{L}$ are definable in $\mathcal{G}'$ in terms of the connectives of $\mathcal{L}$. On the other hand, it is obvious that if $\mathcal{G}$ is the $\mathcal{L}$-fragment of $\mathcal{G}'$, then $\mathcal{E}(\mathcal{G})$ is the $\mathcal{L}$-fragment of $\mathcal{E}(\mathcal{G}')$. Thus, $\mathcal{E}(\mathcal{G}')$ is a definitional expansion of $\mathcal{E}(\mathcal{G})$. □

Corollary 5.12. *Let $\sigma \leq ew_l w_r c$ and let Ψ and Ψ' be two sublanguages of $\mathfrak{L}$ such that $\langle\backslash\rangle \leq \Psi \leq \Psi'$ or such that $\langle/\rangle \leq \Psi \leq \Psi'$. Then, the following is satisfied:*

a) *If a connective $\iota \in \Psi'$ is definable in $\mathcal{FL}_\sigma[\Psi']$ in terms of the connectives of Ψ by a formula η, then ι is definable in $\mathfrak{e}\mathcal{FL}_\sigma[\Psi']$ in terms of the connectives of Ψ by η.*

b) *If $\mathcal{FL}_\sigma[\Psi']$ is a definitional expansion of $\mathcal{FL}_\sigma[\Psi]$, then $\mathfrak{e}\mathcal{FL}_\sigma[\Psi']$ is a definitional expansion of $\mathfrak{e}\mathcal{FL}_\sigma[\Psi]$.*

Proof: Let Φ be a $\mathcal{FL}_\sigma[\Psi]$-theory. Φ is also a $\mathcal{FL}[\Psi]$-theory and, thus, for Lemma 4.24, we have that Θ_Φ is a congruence. On the other hand, the Gentzen system $\mathcal{FL}_\sigma[\Psi]$ satisfies the conditions of Theorem 5.11. □

5.3 About basic substructural calculi with the exchange rule

In this section we consider the basic substructural systems which satisfy the exchange rule and study them by applying the notions of definability of the two previous sections.

I. Collapse of the two implications and the two negations in presence of the exchange rule. As we see in this paragraph, given a sublanguage $\Psi \leq \mathfrak{L}$ which contains the two implications or the two negations, in the systems $\mathcal{FL}_\sigma[\Psi]$ with the exchange rule the two implication connectives are the same except for the notation; the same happens with the two negation connectives.

Lemma 5.13. *The following is satisfied:*

a) *The right (left) implication connective is definable in $\mathcal{FL}_e[\,\backslash, /\,]$ in terms of the left (right) implication connective by the formula q/p ($p\backslash q$).*

b) *The right (left) negation connective is definable in $\mathcal{FL}_e[\,^{\backprime}, {}'\,]$ in terms of the left (right) negation connective by the formula ${}'p$ ($p^{\backprime}$).*

Proof:

a): We have to prove that, given a $\mathcal{FL}_e[\,\backslash, /\,]$-theory Φ, it holds $\langle\varphi\backslash\psi, \psi/\varphi\rangle \in \Omega\Phi$ for each pair of $\langle\,\backslash, /\,\rangle$-formulas φ and ψ. By the characterization of Corollary 4.25, this is equivalent to proving that $\{\varphi\backslash\psi \Rightarrow \psi/\varphi, \psi/\varphi \Rightarrow \varphi\backslash\psi\} \subseteq \Phi$. We will show that these sequents are derivable in $\mathbf{FL}_e[\backslash, /]$ (therefore, they will belong to Φ). Taking into account the mirror image principle, it must be enough to prove that the sequent $\varphi\backslash\psi \Rightarrow \psi/\varphi$ is derivable. Consider the following derivation:

$$\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi \qquad \psi \Rightarrow \psi}{\varphi, \varphi\backslash\psi \Rightarrow \psi}\,(\backslash \Rightarrow)}{\varphi\backslash\psi, \varphi \Rightarrow \psi}\,(e \Rightarrow)}{\varphi\backslash\psi \Rightarrow \psi/\varphi}\,(\Rightarrow /)$$

b): In this case, it will be enough to prove that, for every $\langle {}^{\backprime}, {}' \rangle$-formula φ, the sequent $\varphi^{\backprime} \Rightarrow {}'\varphi$ is derivable in $\mathbf{FL}_e[{}^{\backprime}, {}']$. In fact:

$$\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi, \varphi^{\backprime} \Rightarrow \emptyset}\,({}^{\backprime} \Rightarrow)}{\varphi^{\backprime}, \varphi \Rightarrow \emptyset}\,(e \Rightarrow)}{\varphi^{\backprime} \Rightarrow {}'\varphi}\,(\Rightarrow {}')$$

□

Lemma 5.14. *The following is satisfied:*

a) The left (right) implication rules are derivable by using the right (left) implication rules, the exchange rule, the cut rule and the sequents

$$\varphi\backslash\psi \Rightarrow \psi/\varphi,\ \psi/\varphi \Rightarrow \varphi\backslash\psi.$$

b) The left (right) negation rules are derivable by using the right (left) negation rules, the exchange rule, the cut rule and the sequents

$$\varphi^{\backprime} \Rightarrow {}'\varphi,\ {}'\varphi \Rightarrow \varphi^{\backprime}.$$

Proof: Consider the following derivations, where n is the length of the sequence Γ and $(R)^n$ denotes the application of n consecutive times of the rule (R):

$$\dfrac{\dfrac{\psi/\varphi \Rightarrow \varphi\backslash\psi \qquad \dfrac{\Gamma \Rightarrow \varphi \qquad \Sigma, \psi, \varphi, \Pi \Rightarrow \Delta}{\Sigma, \Gamma, \varphi\backslash\psi, \Pi \Rightarrow \Delta}\,(\backslash \Rightarrow)}{\Sigma, \Gamma, \psi/\varphi, \Pi \Rightarrow \Delta}\,(Cut)}{\Sigma, \psi/\varphi, \Gamma, \Pi \Rightarrow \Delta}\,(e \Rightarrow)^n$$

$$\dfrac{\dfrac{\dfrac{\Gamma, \varphi \Rightarrow \psi}{\varphi, \Gamma \Rightarrow \psi}\,(e \Rightarrow)^n}{\Gamma \Rightarrow \varphi\backslash\psi}\,(\Rightarrow \backslash) \qquad \varphi\backslash\psi \Rightarrow \psi/\varphi}{\Gamma \Rightarrow \psi/\varphi}\,(Cut)$$

$$\dfrac{{}'\varphi \Rightarrow \varphi^{\backprime} \qquad \dfrac{\Gamma \Rightarrow \varphi}{\Gamma, \varphi^{\backprime} \Rightarrow \emptyset}\,({}^{\backprime}\Rightarrow)}{\dfrac{\Gamma, {}'\varphi \Rightarrow \emptyset}{{}'\varphi, \Gamma \Rightarrow \emptyset}\,(e \Rightarrow)^n}\,(Cut) \qquad\qquad \dfrac{\dfrac{\dfrac{\Gamma, \varphi \Rightarrow \emptyset}{\varphi, \Gamma \Rightarrow \emptyset}\,(e \Rightarrow)^n}{\Gamma \Rightarrow \varphi^{\backprime}}\,(\Rightarrow {}^{\backprime}) \qquad \varphi^{\backprime} \Rightarrow {}'\varphi}{\Gamma \Rightarrow {}'\varphi}\,(Cut)$$

The derivations corresponding to the introduction rules or the connectives $\backslash$ and ${}^{\backprime}$ by using the rules for $/$ and $'$, the rule $(e \Rightarrow)$, the cut rule and the relevant sequents are analogous.

□

Theorem 5.15. *Let $\sigma \leq w_l w_r c$. The following is satisfied:*

a) For every $\Psi \leq \langle \vee, \wedge, *, {}^{\backprime}, ', 0, 1 \rangle$, the system $\mathcal{FL}_{e\sigma}[\Psi, \backslash, /]$ is a definitional expansion of $\mathcal{FL}_{e\sigma}[\Psi, \backslash]$ and $\mathcal{FL}_{e\sigma}[\Psi, /]$ and these two systems are notational copy.

b) For every $\Psi \leq \langle \vee, \wedge, *, \backslash, /, 0, 1 \rangle$, the system $\mathcal{FL}_{e\sigma}[\Psi, {}^{\backprime}, ']$ is a definitional expansion of $\mathcal{FL}_{e\sigma}[\Psi, {}^{\backprime}]$ and $\mathcal{FL}_{e\sigma}[\Psi, ']$ and these two systems are notational copy.

Proof: The considered systems belong to the class of Gentzen systems of Theorem 5.10 and Lemmas 5.13 and 5.14 ensure that they satisfy the conditions *i*) and *ii*) of this theorem.

□

Note 5.16. In the literature it is usual to present the calculus $\mathbf{FL}_e$ or the calculi $\mathbf{FL}_\sigma$, with $e \leq \sigma$, in a language of type $\langle 2, 2, 2, 2, 1, 0, 0 \rangle$ with only one implication and one negation. The introduction rules of the implication and negation are those from $\backslash$ and ${}^{\backprime}$ although denoting these connectives, respectively, by $\rightarrow$ and $\neg$:

$$\dfrac{\Gamma \Rightarrow \varphi \qquad \Sigma, \psi, \Pi \Rightarrow \Delta}{\Sigma, \Gamma, \varphi \rightarrow \psi, \Pi \Rightarrow \Delta}\,(\rightarrow\Rightarrow) \qquad\qquad \dfrac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \rightarrow \psi}\,(\Rightarrow\rightarrow)$$

$$\dfrac{\Gamma \Rightarrow \varphi}{\Gamma, \neg\varphi \Rightarrow \emptyset}\,(\neg \Rightarrow) \qquad\qquad \dfrac{\varphi, \Gamma \Rightarrow \emptyset}{\Gamma \Rightarrow \neg\varphi}\,(\Rightarrow \neg)$$

We will use $\mathfrak{L}_e$ to denote the language $\langle \vee, \wedge, *, \rightarrow, \neg, 0, 1 \rangle$. □

II. Collapse of the two conjunctions in calculi with left-weakening, contraction and exchange rules. We will see that, due to the simultaneous presence of rules $(e \Rightarrow)$, $(w \Rightarrow)$ and $(c \Rightarrow)$, given a language $\Psi \leq \mathfrak{L}$ which contains the two conjunctions (multiplicative and additive), in systems $\mathcal{FL}_{ew_lc}[\Psi]$ and $\mathcal{FL}_{ew_lc}[\Psi]$ the two implication connectives are the same up to notation.

Lemma 5.17. *The connective of multiplicative (additive) conjunction is definable in the system $\mathcal{FL}_{ew_lc}[\wedge, *]$ in terms of the connective of additive (multiplicative) conjunction by the formula $p \wedge q$ $(p * q)$.*

Proof: We have to prove that, for each $\varphi, \psi \in Fm_{\langle \wedge, * \rangle}$ and every $\mathcal{FL}_{ew_lc}[\wedge, *]$-theory Φ, we have $\langle \varphi \wedge \psi, \varphi * \psi \rangle \in \Omega\Phi$, that is, $\varphi \wedge \psi \Rightarrow \varphi * \psi \in \Phi$ and $\varphi * \psi \Rightarrow \varphi \wedge \psi \in \Phi$. Consider the following derivations:

$$\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi, \psi \Rightarrow \varphi}(w \Rightarrow) \quad \dfrac{\psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \psi}(w \Rightarrow)}{\varphi, \psi \Rightarrow \varphi \wedge \psi}(\Rightarrow \wedge)}{\varphi * \psi \Rightarrow \varphi \wedge \psi}(* \Rightarrow) \qquad \dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi \wedge \psi \Rightarrow \varphi}(\wedge_1 \Rightarrow) \quad \dfrac{\psi \Rightarrow \psi}{\varphi \wedge \psi \Rightarrow \psi}(\wedge_2 \Rightarrow)}{\varphi \wedge \psi, \varphi \wedge \psi \Rightarrow \varphi * \psi}(\Rightarrow *)}{\varphi \wedge \psi \Rightarrow \varphi * \psi}(c \Rightarrow)$$

So, these sequents are derivable in $\mathbf{FL}_{ew_lc}[\wedge, *]$ and, therefore, they belong to Φ. $\square$

Lemma 5.18. *The introduction rules for the multiplicative (additive) conjunction are derivable by using only the rules for the additive (multiplicative) conjunction, the rules $(e \Rightarrow)$, $(w \Rightarrow)$, $(c \Rightarrow)$, (Cut) and the sequents $\varphi * \psi \Rightarrow \psi \wedge \varphi$ and $\varphi \wedge \psi \Rightarrow \varphi * \psi$.*

Proof: Consider the following derivations, where n and m are the length of the sequences Γ and Π, respectively.

• Rules $(* \Rightarrow)$ and $(\Rightarrow *)$:

$$\dfrac{\varphi * \psi \Rightarrow \varphi \wedge \psi \qquad \dfrac{\dfrac{\dfrac{\Sigma, \varphi, \psi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi \wedge \psi, \psi, \Gamma \Rightarrow \Delta}(\wedge_1 \Rightarrow)}{\Sigma, \varphi \wedge \psi, \varphi \wedge \psi, \Gamma \Rightarrow \Delta}(\wedge_2 \Rightarrow)}{\Sigma, \varphi \wedge \psi, \Gamma \Rightarrow \Delta}(c \Rightarrow)}{\Sigma, \varphi * \psi, \Gamma \Rightarrow \Delta}(Cut)$$

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \varphi}{\Gamma, \Pi \Rightarrow \varphi}(w \Rightarrow)^n \quad \dfrac{\Pi \Rightarrow \psi}{\Gamma, \Pi \Rightarrow \psi}(w \Rightarrow)^m}{\Gamma, \Pi \Rightarrow \varphi \wedge \psi}(\Rightarrow \wedge) \qquad \varphi \wedge \psi \Rightarrow \varphi * \psi}{\Gamma, \Pi \Rightarrow \varphi * \psi}(Cut)$$

• Rules $(\wedge_1 \Rightarrow)$, $(\wedge_2 \Rightarrow)$ and $(\Rightarrow \wedge)$:

$$\dfrac{\varphi \wedge \psi \Rightarrow \varphi * \psi \qquad \dfrac{\dfrac{\Sigma, \varphi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi, \psi, \Gamma \Rightarrow \Delta}(w \Rightarrow)}{\Sigma, \varphi * \psi, \Gamma \Rightarrow \Delta}(* \Rightarrow)}{\Sigma, \varphi \wedge \psi, \Gamma \Rightarrow \Delta}(Cut) \qquad \dfrac{\varphi \wedge \psi \Rightarrow \varphi * \psi \qquad \dfrac{\dfrac{\Sigma, \psi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi, \psi, \Gamma \Rightarrow \Delta}(w \Rightarrow)}{\Sigma, \varphi * \psi, \Gamma \Rightarrow \Delta}(* \Rightarrow)}{\Sigma, \varphi \wedge \psi, \Gamma \Rightarrow \Delta}(Cut)$$

In the derivation of $(\Rightarrow \wedge)$, we will use the following derived rule, which is proved by using $(e \Rightarrow)$ and $(c \Rightarrow)$:

$$\Gamma, \Gamma \Rightarrow \Delta \vdash \Gamma \Rightarrow \Delta \tag{5.2}$$

Proof of (5.2) (suppose $\Gamma = \gamma_1, \dots, \gamma_n$):

$$\dfrac{\dfrac{\gamma_1, \dots, \gamma_n, \gamma_1, \dots, \gamma_n \Rightarrow \Delta}{\gamma_1, \gamma_1, \dots, \gamma_i, \gamma_i, \dots, \gamma_n, \gamma_n}(e \Rightarrow)^{\frac{(n-1).n}{2}}}{\gamma_1, \dots, \gamma_n \Rightarrow \Delta}(c \Rightarrow)^n$$

Derivation of $(\Rightarrow \wedge)$:

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \varphi \qquad \Gamma \Rightarrow \psi}{\Gamma, \Gamma \Rightarrow \varphi * \psi}(\Rightarrow *)}{\Gamma \Rightarrow \varphi * \psi}(5.2) \qquad \varphi * \psi \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \varphi \wedge \psi}(Cut)$$

□

Theorem 5.19. *Let $\sigma \in \{w_l, w\}$. If $\Psi \leq \langle \vee, \rightarrow, \neg, 0, 1\rangle$, then the following is satisfied:*

a) $\mathcal{FL}_{e\sigma c}[\Psi, \wedge, *]$ *is a definitional expansion of* $\mathcal{FL}_{e\sigma c}[\Psi, *]$ *and* $\mathcal{FL}_{e\sigma c}[\Psi, \wedge]$.

b) $\mathcal{FL}_{e\sigma c}[\Psi, *]$ *and* $\mathcal{FL}_{e\sigma c}[\Psi, \wedge]$ *are notational copies.*

Proof: The referred systems belong to the class of Gentzen systems in Theorem 5.10 and Lemmas 5.17 and 5.18 ensure that conditions $i)$ and $ii)$ of the above theorem are satisfied. □

Note 5.20. Observe that the calculus $\mathbf{FL}_{ewc}[\vee, \wedge, \rightarrow, \neg, 0, 1]$ is a version of the calculus LJ from Gentzen [Gen35] for the intuitionistic logic. As we have already seen, systems $\mathcal{FL}_{ewc}[\vee, \wedge, \rightarrow, \neg, 0, 1]$ and $\mathcal{FL}_{ewc}[\vee, *, \rightarrow, \neg, 0, 1]$ are equal up the notation. This corresponds to the well-known fact thet the Gentzen system determined by the calculus LJ admits as an alternative axiomatization the one resulting from replacing the rules $(\wedge_1 \Rightarrow)$, $(\wedge_2 \Rightarrow)$ and $(\Rightarrow \wedge)$ with the rules

$$\frac{\Sigma, \varphi, \psi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow)_m \qquad\qquad \frac{\Gamma \Rightarrow \varphi \qquad \Pi \Rightarrow \psi}{\Gamma, \Pi \Rightarrow \varphi \wedge \psi}\ (\Rightarrow \wedge)_m$$

5.4 Definability of Zero in $\mathbf{FL}_\sigma$

In this section we show that if Ψ is a sublanguage of $\langle \vee, \wedge, *, \backslash, /, {}^{\backprime}, {}', 1\rangle$ which contains at least one of the negation connectives, then for every $\sigma \leq ew_lw_rc$, the system $\mathcal{FL}_\sigma[\Psi, 0]$ is a definitional extension of $\mathcal{FL}_\sigma[\Psi]$.

Lemma 5.21. *The connective* 0 *is definable in* $\mathcal{FL}[{}^{\backprime}, 0, 1]$ *(in* $\mathcal{FL}[{}', 0, 1]$*) in terms of the connectives* ${}^{\backprime}$ *and* 1 *(*${}'$ *and* 1*) by the formula* $1^{\backprime}$ *(*${}'1$*).*

Proof: We have to prove that $\{\langle 0, 1^{\backprime}\rangle, \langle 0, {}'1\rangle\} \subseteq \Omega\Phi$ for every $\mathcal{FL}[{}^{\backprime}, 0, 1]$-theory Φ. We will prove that the sequents $0 \Rightarrow 1^{\backprime}$ and $1^{\backprime} \Rightarrow 0$ are derivable. Then, pursuant to the mirror images law, sequents $0 \Rightarrow {}'1$ and ${}'1 \Rightarrow 0$ will be derivable as well. Consider the following derivations:

$$\dfrac{\dfrac{\dfrac{}{0 \Rightarrow \emptyset}\,(0 \Rightarrow)}{1, 0 \Rightarrow 0}\,(1 \Rightarrow)}{0 \Rightarrow 1^{\backprime}}\,(\Rightarrow {}^{\backprime}) \qquad \dfrac{\dfrac{\dfrac{}{\emptyset \Rightarrow 1}\,(\Rightarrow 1)}{1^{\backprime} \Rightarrow \emptyset}\,({}^{\backprime} \Rightarrow)}{1^{\backprime} \Rightarrow 0}\,(\Rightarrow 0)$$

□

Lemma 5.22. *Rules* $(0 \Rightarrow)$ *and* $(\Rightarrow 0)$ *are derivable only by using the right (left) negation rules, rules of* 1 *the cut rule and the sequents* $0 \Rightarrow {}^{\backprime}1$ *and* ${}^{\backprime}1 \Rightarrow 0$ *(*$0 \Rightarrow 1'$ *and* $1' \Rightarrow 0$*).*

Proof: Consider the following derivations:

$$
\dfrac{0 \Rightarrow 1^{\backprime} \qquad \dfrac{\emptyset \Rightarrow 1}{1^{\backprime} \Rightarrow \emptyset}\,({}^{\backprime}\Rightarrow)}{0 \Rightarrow \emptyset}\,(Cut)
\qquad
\dfrac{\dfrac{\Gamma \Rightarrow \emptyset}{1, \Gamma \Rightarrow \emptyset}\,(1 \Rightarrow) \qquad {}^{\backprime}1 \Rightarrow 0}{\Gamma \Rightarrow 0}\,(Cut)
$$

The derivations corresponding to the introduction rules for 0 using left negation rules, rules of l’ 1, cut rule and sequents $0 \Rightarrow 1'$ and $1' \Rightarrow 0$ are analogous. □

Theorem 5.23. *Let $\Psi \leq \langle \vee, \wedge, *, \backslash, /, {}^{\backprime}, {}', 1 \rangle$ such that $\langle {}^{\backprime} \rangle \leq \Psi$ or $\langle {}' \rangle \leq \Psi$. Then $\mathcal{FL}_\sigma[\Psi, 0]$ is a definitional expansion of $\mathcal{FL}_\sigma[\Psi]$.*

Proof: The aforesaid systems belong to the class of Gentzen system considered in Theorem 5.9. Lemmas 5.21 and 5.22 ensure that conditions *i*) and *ii*) of this theorem are satisfied. □

5.5 Definability of the negations in $\mathbf{FL}_\sigma$

Now we will see that the right (left) negation connective is definable in terms of the right (left) implication connective and zero. Besides, we show that, if Ψ is a sublanguage of $\mathfrak{L}$ such that $\langle \backslash, {}^{\backprime}, 0 \rangle \leq \Psi$ or $\langle /, {}', 0 \rangle \leq \Psi$ and $\widehat{\Psi}$ is the language obtained by eliminating in Ψ the negation connectives it may contain, then $\mathcal{FL}_\sigma[\Psi]$ is a definitional expansion of $\mathcal{FL}_\sigma[\widehat{\Psi}]$.

Lemma 5.24.

a) *The right negation connective is derivable in $\mathcal{FL}[\backslash, {}^{\backprime}, 0]$ in terms of connectives $\backslash$ and 0 by the formula $p\backslash 0$.*

b) *The left negation connective is definable in $\mathcal{FL}[/, {}', 0]$ in terms of connectives $/$ and 0 by formula $0/p$.*

Proof: *a*): We should see that $\langle \varphi^{\backprime}, \varphi\backslash 0 \rangle \in \Omega\Phi$, for every $\mathcal{FL}[\backslash, {}^{\backprime}, 0]$-theory Φ. It will be enough to prove that sequents $\varphi^{\backprime} \Rightarrow \varphi\backslash 0$ and $\varphi\backslash 0 \Rightarrow \varphi^{\backprime}$ are derivable. Consider the following derivations:

$$
\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi, \varphi^{\backprime} \Rightarrow \emptyset}\,({}^{\backprime}\Rightarrow)}{\varphi, \varphi^{\backprime} \Rightarrow 0}\,(\Rightarrow 0)}{\varphi^{\backprime} \Rightarrow \varphi\backslash 0}\,(\Rightarrow \backslash)
\qquad
\dfrac{\dfrac{\varphi \Rightarrow \varphi \qquad 0 \Rightarrow \emptyset}{\varphi, \varphi\backslash 0 \Rightarrow \emptyset}\,(\backslash \Rightarrow)}{\varphi\backslash 0 \Rightarrow \varphi^{\backprime}}\,(\Rightarrow {}^{\backprime})
$$

b): As they are the mirror images of the previous ones, the sequents ${}'\varphi \Rightarrow 0/\varphi$ and $0/\varphi \Rightarrow {}'\varphi$ are derivable in $\mathbf{FL}[/, {}', 0]$. Therefore, $\langle {}'\varphi, 0/\varphi \rangle \in \Omega\Phi$, for every $\mathcal{FL}[/, {}', 0]$-theory Φ. □

Lemma 5.25.

- *The introduction rules for the right negation are derivable only by using the right implication rules, the rules of* 0*, the cut rule and the sequents*

$$\varphi^{\backprime} \Rightarrow \varphi\backslash 0 \quad \text{and} \quad \varphi\backslash 0 \Rightarrow \varphi^{\backprime}. \tag{5.3}$$

- *The introduction rules for the left negation are derivable only by using the left implication rules, the rules of* 0*, the cut rule and the sequents*

$$'\varphi \Rightarrow 0/\varphi \quad \text{and} \quad 0/\varphi \Rightarrow {}'\varphi. \tag{5.4}$$

Proof: We will make only the derivations corresponding to the rules $(^{\backprime} \Rightarrow)$ and $(\Rightarrow {}^{\backprime})$ (the derivations corresponding to the rules $(' \Rightarrow)$ and $(\Rightarrow ')$ are analogous).

$$\dfrac{\varphi^{\backprime} \Rightarrow \varphi\backslash 0 \qquad \dfrac{\Gamma \Rightarrow \varphi \qquad \dfrac{}{0 \Rightarrow \emptyset}\,(0 \Rightarrow)}{\Gamma, \varphi\backslash 0 \Rightarrow \emptyset}\,(\backslash \Rightarrow)}{\Gamma, \varphi^{\backprime} \Rightarrow \emptyset}\,(Cut) \qquad \dfrac{\dfrac{\dfrac{\varphi, \Gamma \Rightarrow \emptyset}{\varphi, \Gamma \Rightarrow 0}\,(\Rightarrow 0)}{\Gamma \Rightarrow \varphi\backslash 0}\,(\Rightarrow \backslash) \qquad \varphi\backslash 0 \Rightarrow \varphi^{\backprime}}{\Gamma \Rightarrow \varphi^{\backprime}}\,(Cut)$$

□

Notation 5.26. We will denote by $\widehat{\mathfrak{L}}$ the language $\langle \vee, \wedge, *, \backslash, /, 0, 1 \rangle$.

Theorem 5.27. *Let* $\Psi \leq \mathfrak{L}$ *such that* $\langle \backslash, {}^{\backprime}, 0 \rangle \leq \Psi$ *or* $\langle /, ', 0 \rangle \leq \Psi$ *and let* $\widehat{\Psi}$ *be the language obtained by eliminating from* Ψ *the negation connectives it may contain. The system* $\mathcal{FL}_\sigma[\Psi]$ *is a definitional expansion of* $\mathcal{FL}_\sigma[\widehat{\Psi}]$*. In particular,* $\mathcal{FL}_\sigma$ *is a definitional expansion of* $\mathcal{FL}_\sigma[\widehat{\mathfrak{L}}]$*.*

Proof: These systems belong to the class of Gentzen systems considered in Theorem 5.9 and Lemmas 5.24 and 5.25 ensure that conditions $i)$ and $ii)$ of this theorem are satisfied. □

Corollary 5.28. *If* $\Psi \leq \langle \vee, \wedge, *, 1 \rangle$*, then*

a) $\mathcal{FL}_\sigma[\Psi, \backslash, 0]$ *and* $\mathcal{FL}_\sigma[\Psi, \backslash, {}^{\backprime}]$ *are definitionally equivalent.*

b) $\mathcal{FL}_\sigma[\Psi, /, 0]$ *and* $\mathcal{FL}_\sigma[\Psi, /, ']$ *are definitionally equivalent.*

c) $\mathcal{FL}_\sigma[\Psi, \backslash, /, 0]$ *and* $\mathcal{FL}_\sigma[\Psi, \backslash, /, {}^{\backprime}, ']$ *are definitionally equivalent.*

Proof: We prove only this item $c)$. Consider the system $\mathcal{FL}_\sigma[\Psi, \backslash, /, {}^{\backprime}, ', 0]$. By Theorem 5.27 we have that such system is a definitional expansion of $\mathcal{FL}_\sigma[\Psi, \backslash, /, 0]$ and by Theorem 5.23 we have that it is a definitional expansion of $\mathcal{FL}_\sigma[\Psi, \backslash, /, {}^{\backprime}, ']$. Therefore, $\mathcal{FL}_\sigma[\Psi, \backslash, /, {}^{\backprime}, ', 0]$ is a common definitional expansion for $\mathcal{FL}_\sigma[\Psi, \backslash, /, 0]$ and $\mathcal{FL}_\sigma[\Psi, \backslash, /, {}^{\backprime}, ']$. □

5.6 Definability of 1 in systems with left-weakening

In the following we show that if Ψ is a sublanguage of $\langle \vee, \wedge, *, \backslash, /, `, ', 0\rangle$ which contains at least one of the implication connectives, then for every σ such that $w_l \leq \sigma$, the system $\mathcal{FL}_\sigma[\Psi, 1]$ is a definitional extension of $\mathcal{FL}_\sigma[\Psi]$.

Lemma 5.29. *The connective* 1 *is definable in* $\mathcal{FL}_{w_l}[\backslash, 1]$ *(en* $\mathcal{FL}_{w_l}[/, 1]$*) in terms of the right (left) implication by the formula* $p\backslash p$ *(*p/p*).*

Proof: Sequent $1 \Rightarrow \varphi\backslash\varphi$ is derivable in **FL**: it is obtained by applying $(1 \Rightarrow)$ to $\emptyset \Rightarrow \varphi\backslash\varphi$ and this is obtained by applying $(\Rightarrow \backslash)$ to $\varphi \Rightarrow \varphi$. Sequent $\varphi\backslash\varphi \Rightarrow 1$ is obtained by applying $(w \Rightarrow)$ to $\emptyset \Rightarrow 1$. According to the mirror image principle we have that $1 \Rightarrow \varphi\backslash\varphi$ and $\varphi\backslash\varphi \Rightarrow 1$ are derivable as well. □

Lemma 5.30. *Rules* $(1 \Rightarrow)$ *and* $(\Rightarrow 1)$ *are derivable only by using the right (left) implication rules, the left-weakening rule, the cut rule and the sequents* $1 \Rightarrow \varphi\backslash\varphi$ *and* $\varphi\backslash\varphi \Rightarrow 1$ *(*$1 \Rightarrow \varphi/\varphi$ *and* $\varphi/\varphi \Rightarrow 1$*).*

Proof: $(1 \Rightarrow)$ is an instance of $(w \Rightarrow)$. Consider the following derivation of $(\emptyset \Rightarrow 1)$:

$$\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\emptyset \Rightarrow \varphi\backslash\varphi}\,(\Rightarrow \backslash) \qquad \varphi\backslash\varphi \Rightarrow 1}{\emptyset \Rightarrow 1}\,(Cut)$$

The derivations corresponding to the rules of 1 using the left-implication rules, the left-weakening rule, the cut rule and sequents $1 \Rightarrow \varphi/\varphi$ i $\varphi/\varphi \Rightarrow 1$ are analogous. □

Theorem 5.31. *Let* σ *be such that* $w_l \leq \sigma$*. Let* $\Psi \leq \langle \vee, \wedge, *, \backslash, /, `, ', 0\rangle$ *be such that* $\langle \backslash \rangle \leq \Psi$ *or* $\langle / \rangle \leq \Psi$*. Then* $\mathcal{FL}_\sigma[\Psi, 1]$ *is a definitional expansion of* $\mathcal{FL}_\sigma[\Psi]$*.*

Proof: These systems belong to a class of Gentzen systems for Theorem 5.9 and Lemmas 5.29 and 5.30 ensure that conditions $i)$ and $ii)$ of the referred theorem are satisfied. □

5.7 Versions of FL and FL$_e$ without negations and with sequents of the form $\omega \times \{1\}$

In the literature it is usual to present the calculus of sequents **FL** (**FL**$_e$) in the language without negations $\widehat{\mathfrak{L}} := \langle \vee, \wedge, *, \backslash, /, 0, 1\rangle$ ($\widehat{\mathfrak{L}}_e := \langle \vee, \wedge, *, \rightarrow, 0, 1\rangle$) and with sequents of the form $\omega \times \{1\}$, that is, with sequents that exactly have a formula in the consequent. In the interim, these versions will be denoted by **FL**$'$ and **FL**$'_e$. **FL**$'$ comes from the following rules:

$$\varphi \Rightarrow \varphi \quad \text{(Axiom)} \qquad\qquad \dfrac{\Gamma \Rightarrow \varphi \qquad \Sigma, \varphi, \Pi \Rightarrow \xi}{\Sigma, \Gamma, \Pi \Rightarrow \xi}\ \text{(Cut)}$$

$$\frac{\Sigma, \varphi, \Gamma \Rightarrow \xi \qquad \Sigma, \psi, \Gamma \Rightarrow \xi}{\Sigma, \varphi \vee \psi, \Gamma \Rightarrow \xi}\ (\vee \Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi}\ (\Rightarrow \vee_1) \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi}\ (\Rightarrow \vee_2)$$

$$\frac{\Sigma, \varphi, \Gamma \Rightarrow \xi}{\Sigma, \varphi \wedge \psi, \Gamma \Rightarrow \xi}\ (\wedge_1 \Rightarrow) \qquad \frac{\Sigma, \psi, \Gamma \Rightarrow \xi}{\Sigma, \varphi \wedge \psi, \Gamma \Rightarrow \xi}\ (\wedge_2 \Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}\ (\Rightarrow \wedge)$$

$$\frac{\Sigma, \varphi, \psi, \Gamma \Rightarrow \xi}{\Sigma, \varphi * \psi, \Gamma \Rightarrow \xi}\ (* \Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi \qquad \Pi \Rightarrow \psi}{\Gamma, \Pi \Rightarrow \varphi * \psi}\ (\Rightarrow *)$$

$$\frac{\Gamma \Rightarrow \varphi \qquad \Sigma, \psi, \Pi \Rightarrow \xi}{\Sigma, \Gamma, \varphi \backslash \psi, \Pi \Rightarrow \xi}\ (\backslash \Rightarrow) \qquad \frac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \backslash \psi}\ (\Rightarrow \backslash)$$

$$\frac{\Gamma \Rightarrow \varphi \qquad \Sigma, \psi, \Pi \Rightarrow \xi}{\Sigma, \psi / \varphi, \Gamma, \Pi \Rightarrow \xi}\ (/ \Rightarrow) \qquad \frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \psi / \varphi}\ (\Rightarrow /)$$

$$\frac{\Sigma, \Gamma \Rightarrow \xi}{\Sigma, 1, \Gamma \Rightarrow \xi}\ (1 \Rightarrow) \qquad \emptyset \Rightarrow 1 \ \ (\Rightarrow 1)$$

The standard presentation of $\mathbf{FL}'_e$, in language $\mathfrak{L}_e$, is obtained from the previous one by adding the version in sequents of the form $\omega \times \{1\}$ of the exchange rule, i.e.,

$$\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \xi}{\Gamma, \psi, \varphi, \Pi \Rightarrow \xi}\ (e \Rightarrow)$$

and by replacing the rules of the two implications with the following rules:

$$\frac{\Gamma \Rightarrow \varphi \qquad \Sigma, \psi, \Pi \Rightarrow \xi}{\Sigma, \Gamma, \varphi \to \psi, \Pi \Rightarrow \xi}\ (\to\Rightarrow) \qquad \frac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\ (\Rightarrow\to)$$

Observation 5.32. Observe that in calculi $\mathbf{FL}'$ and $\mathbf{FL}'_e$ there are no rules referring to the constant symbol 0.

Let σ be a sublanguage of $w_l w_r c$. Similar to the procedure we have followed to define the calculi $\mathbf{FL}_\sigma$ and $\mathbf{FL}_{e\sigma}$, we define the calculi $\mathbf{FL}'_\sigma$ and $\mathbf{FL}'_{e\sigma}$ by using the versions in sequents of type $\omega \times \{1\}$ of the exchange, left and right-weakening and contraction rules:

$$\frac{\Sigma, \Gamma \Rightarrow \xi}{\Sigma, \varphi, \Gamma \Rightarrow \xi}\ (w \Rightarrow) \qquad 0 \Rightarrow \varphi \ \ (\Rightarrow w)_0 \qquad \frac{\Sigma, \varphi, \varphi, \Gamma \Rightarrow \xi}{\Sigma, \varphi, \Gamma \Rightarrow \xi}\ (c \Rightarrow)$$

A relevant feature is that the role of the right-weakening rule is carried out in this case through the axiom $0 \Rightarrow \varphi$.

5.7.1 Equivalence of the two versions for calculi $\mathbf{FL}_\sigma$

Let $\sigma \leq e w_l w_r c$. We will refer to the Gentzen system defined by the calculus $\mathbf{FL}'_\sigma$ as $\mathcal{FL}'_\sigma$. In the following we will show that $\mathcal{FL}_\sigma[\widehat{\mathfrak{L}}]$ and $\mathcal{FL}'_\sigma$ are equivalent as Gentzen systems. We implement the proofs based on languages $\mathfrak{L}$ and $\widehat{\mathfrak{L}}$ with two implications.[2]

[2] Of course, the results are also valid for calculi with the exchange rule presented in languages $\langle \vee, \wedge, *, \to, \neg, 0, 1 \rangle$ and $\langle \vee, \wedge, *, \to, 0, 1 \rangle$.

Definition 5.33. *We define the translations τ of $\omega\times\{0,1\}$-sequents in $\omega\times\{1\}$-sequents and ρ of $\omega\times\{1\}$-sequents in $\omega\times\{0,1\}$-sequents as follows:*

$$\tau(\Gamma \Rightarrow \Delta) := \Gamma \Rightarrow \delta, \textit{ where } \delta = \varphi \textit{ if } \Delta = \varphi \textit{ and } \delta = 0 \textit{ if } \Delta = \emptyset.$$

$$\rho(\Gamma \Rightarrow \varphi) := \{\Gamma \Rightarrow \varphi\}.$$

Lemma 5.34. *Let τ be as in Definition 5.33. The following conditions hold:*

a) If $\langle \Phi, \varsigma\rangle$ is an instance for the rule of $\mathbf{FL}_\sigma[\widehat{\mathfrak{L}}]$, then $\langle \tau[\Phi], \tau(\varsigma)\rangle$ is an instance of a rule of $\mathbf{FL}'_\sigma$.

b) For every $\Upsilon \cup \{\varsigma\} \subseteq Seq_{\widehat{\mathfrak{L}}}^{\omega\times\{0,1\}}$, if $\Upsilon \vdash_{\mathbf{FL}_\sigma[\widehat{\mathfrak{L}}]} \varsigma$, then $\tau[\Upsilon] \vdash_{\mathbf{FL}'_\sigma} \tau(\varsigma)$.

Proof: *a*) If $\langle \Phi, \varsigma\rangle$ is an instance of (*Axiom*), (*Cut*) or –in the event that the letters codifying them appear in the sequence σ– of one of the structural rules $(e \Rightarrow)$, $(w_l \Rightarrow)$ or $(c \Rightarrow)$, then it is clear that $\langle \tau[\Phi], \tau(\varsigma)\rangle$ is an instance of the rule with the same label in $\mathbf{FL}'_\sigma$.

If $\langle \Phi, \varsigma\rangle$ is an instance of $(0 \Rightarrow)$ we have $\tau(0 \Rightarrow \emptyset) = 0 \Rightarrow 0$, which is in turn an instance of (*Axiom*).

If $\langle \Phi, \varsigma\rangle$ is an instance of $(\Rightarrow 0)$, we have to prove that $\tau(\Gamma \Rightarrow \emptyset) \vdash_{\mathbf{FL}'_\sigma} \tau(\Gamma \Rightarrow 0)$ and this is immediate, since it is the same as proving $\Gamma \Rightarrow 0 \vdash_{\mathbf{FL}'_\sigma} \Gamma \Rightarrow 0$.

If $w_r \leq \sigma$ and $\langle \Phi, \varsigma\rangle$ is an instance of $(\Rightarrow w)$, we must see that $\tau(\Gamma \Rightarrow \emptyset) \vdash_{\mathbf{FL}'_\sigma} \tau(\Gamma \Rightarrow \varphi)$, that is, $\Gamma \Rightarrow 0 \vdash_{\mathbf{FL}'_\sigma} \Gamma \Rightarrow \varphi$. In fact, from $\Gamma \Rightarrow 0$ and $0 \Rightarrow \varphi$ (axiom $(\Rightarrow w)_0$) we obtain sequent $\Gamma \Rightarrow \varphi$ by applying (*Cut*).

b) By induction on length of the proof in $\mathbf{FL}_\sigma[\widehat{\mathfrak{L}}]$ of ς from Υ, using *a*), we obtain this result. □

Theorem 5.35. *The Gentzen systems $\mathcal{FL}'_\sigma$ and $\mathcal{FL}_\sigma[\widehat{\mathfrak{L}}]$ are equivalent with the translations τ and ρ from Definition 5.33.*

Proof: We will show that the following conditions are satisfied:

a) For every $\Upsilon \cup \{\varsigma\} \subseteq Seq_{\widehat{\mathfrak{L}}}^{\omega\times\{1\}}$, $\Upsilon \vdash_{\mathbf{FL}'_\sigma} \varsigma$ iff $\rho[\Upsilon] \vdash_{\mathbf{FL}_\sigma[\widehat{\mathfrak{L}}]} \rho(\varsigma)$,

b) For every $\varsigma \in Seq_{\widehat{\mathfrak{L}}}^{\omega\times\{0,1\}}$, $\varsigma \dashv\vdash_{\mathbf{FL}_\sigma[\widehat{\mathfrak{L}}]} \rho\tau(\varsigma)$.

a): Taking into account the definition of the translation ρ, we have to prove that

$$\Upsilon \vdash_{\mathbf{FL}'_\sigma} \varsigma \quad \text{iff} \quad \Upsilon \vdash_{\mathbf{FL}_\sigma[\widehat{\mathfrak{L}}]} \varsigma.$$

Suppose $\Upsilon \vdash_{\mathbf{FL}'_\sigma} \varsigma$. Then, by Lemma 5.34, we have $\tau[\Upsilon] \vdash_{\mathbf{FL}'_\sigma} \tau(\varsigma)$. But, as $\Upsilon \cup \{\varsigma\} \subseteq Seq_{\widehat{\mathfrak{L}}}^{\omega\times\{1\}}$, we have $\tau[\Upsilon] = \Upsilon$ i $\tau(\varsigma) = \varsigma$. Therefore, $\Upsilon \vdash_{\mathbf{FL}_\sigma[\widehat{\mathfrak{L}}]} \varsigma$. Reciprocally, if $\Upsilon \vdash_{\mathbf{FL}_\sigma[\widehat{\mathfrak{L}}]} \varsigma$, then it is clear that $\Upsilon \vdash_{\mathbf{FL}'_\sigma} \varsigma$, since every instance of a rule of $\mathbf{FL}_\sigma[\widehat{\mathfrak{L}}]$ is an instance of a $\mathbf{FL}'_\sigma$ rule as well.

b): It is immediate, since for every $\varsigma \in Seq_{\widehat{\mathfrak{L}}}^{\omega\times\{0,1\}}$, we have $\rho\tau(\varsigma) = \varsigma$. □

Corollary 5.36. $\mathfrak{e}\mathcal{FL}'_\sigma = \mathfrak{e}\mathcal{FL}_\sigma[\widehat{\mathfrak{L}}]$.

Proof: Let $\Gamma \cup \{\varphi\} \subseteq Fm_{\widehat{\mathfrak{L}}}$. By applying the definition of the external system and item a) of Theorem 5.35, we have:

$$\Gamma \vdash_{\mathfrak{e}\mathcal{FL}'_\sigma} \varphi \quad \text{iff} \quad \{\emptyset \Rightarrow \gamma : \gamma \in \Gamma\} \vdash_{\mathbf{FL}'_\sigma} \emptyset \Rightarrow \varphi \quad \text{iff}$$

$$\{\emptyset \Rightarrow \gamma : \gamma \in \Gamma\} \vdash_{\mathbf{FL}_\sigma[\widehat{\mathfrak{L}}]} \emptyset \Rightarrow \varphi \quad \text{iff} \quad \Gamma \vdash_{\mathfrak{e}\mathcal{FL}_\sigma[\widehat{\mathfrak{L}}]} \varphi.$$

□

Corollary 5.37. *$\mathcal{FL}'_\sigma$ is the $\omega \times \{1\}$-fragment of $\mathcal{FL}_\sigma[\widehat{\mathfrak{L}}]$.*

Proof: It is the one proved in item a) of Theorem 5.35. □

Corollary 5.38. *$\mathcal{FL}'_\sigma$ is the $\langle\widehat{\mathfrak{L}}, \omega \times \{1\}\rangle$-fragment of $\mathcal{FL}_\sigma$. Therefore, $\mathcal{FL}_\sigma$ is a definitional expansion of $\mathcal{FL}'_\sigma$.*

Proof: By Theorem 5.27 and Corollary 5.37. □

Note 5.39. In the literature, the denotation $\mathbf{FL}_\sigma$ is used for both versions of calculi, the one with the language without negations and sequents of type $\omega \times \{1\}$ and the one with the language with negations and sequents of type $\omega \times \{0, 1\}$. In this study we will maintain the difference in notations in order to avoid any confusion.

Observation 5.40. Note that the previous results are also valid if we start from two sublanguages Ψ and $\widehat{\Psi}$ such that $\langle \backslash, ^\backprime, 0\rangle \leq \Psi$ or $\langle /, ', 0\rangle \leq \Psi$ and $\widehat{\Psi}$ is obtained from Ψ by deleting the negation connectives it contains. Therefore, we have the following result that generalize the previous results.

Theorem 5.41. *Let Ψ be such that $\langle \backslash, ^\backprime, 0\rangle \leq \Psi$ or $\langle /, ', 0\rangle \leq \Psi$ and let $\widehat{\Psi}$ be the language obtained by deleting in Ψ the negation connectives that it may contain. Then, it is satisfied:*

a) The Gentzen systems $\mathcal{FL}'_\sigma[\widehat{\Psi}]$ and $\mathcal{FL}_\sigma[\widehat{\Psi}]$ are equivalent.

b) $\mathfrak{e}\mathcal{FL}'_\sigma[\widehat{\Psi}] = \mathfrak{e}\mathcal{FL}_\sigma[\widehat{\Psi}]$.

c) $\mathcal{FL}'_\sigma[\widehat{\Psi}]$ is the $\omega \times \{1\}$-fragment of $\mathcal{FL}_\sigma[\widehat{\Psi}]$.

d) $\mathcal{FL}'_\sigma[\widehat{\Psi}]$ is the $\langle\widehat{\Psi},\, \omega \times \{1\}\rangle$-fragment of $\mathcal{FL}_\sigma[\Psi]$.

e) $\mathcal{FL}_\sigma[\Psi]$ is a definitional expansion of $\mathcal{FL}'_\sigma[\widehat{\Psi}]$.

Part III

Fragments without Implication

Chapter 6

Basic Ordered Algebraic Structures

In this chapter we present the ordered, latticed and semilatticed algebraic structures that will constitute the semantic core of some of the fragments seen in Chapter 9. In the first two sections we show the notions and preliminary results. In Section 6.3 we introduce the notion of *pointed monoid*. A pointed monoid (ordered, semilatticed or latticed) is obtained by adding the constant symbol 0 to the type of similarity of a monoid (ordered, semilatticed or latticed): this symbol is interpreted as a fixed element but arbitrary of the universe of the structure. We define the varieties of algebras $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$ and $\mathring{\mathbb{M}}^{\ell}_{\sigma}$, where subindex σ is a subsequence of the sequence ew_lw_rc and symbols e, w_l, w_r and c codify what we refer to as (algebraic) exchange, right-weakening, left-weakening and contraction properties, respectively. Theese properties, which are expressed by quasi-inequations are equivalent, respectively, to the following properties: commutativity, integrality, 0-boundedness and increasing idempotency. As we see in Chapter 9, once a sequence σ is fixed, the classes $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$ and $\mathring{\mathbb{M}}^{\ell}_{\sigma}$ are equivalent, respectively, to the $\langle\vee, *, 0, 1\rangle$-fragment and to the $\langle\vee, \wedge, *, 0, 1\rangle$-fragment of the Gentzen system $\mathcal{FL}_{\sigma}$ and they are algebraic semantics, respectively, of the $\langle\vee, *, 0, 1\rangle$-fragment and of the $\langle\vee, \wedge, *, 0, 1\rangle$-fragment of the external system $\mathfrak{e}\mathcal{FL}_{\sigma}$. In Section 6.4 we recall the notion of residuation and the definitions and properties of residuated lattices and $\mathbb{FL}$-algebras and its subvarieties $\mathbb{FL}_{\sigma}$.

6.1 Order-Algebras

We will call *order-algebra* any pair $\mathcal{A} = \langle \mathbf{A}, \leq\rangle$, where $\mathbf{A}$ is an algebra and $\leq$ is a partial order defined in the universe of $\mathbf{A}$. If $\mathbf{A}$ is a $\mathcal{L}$-algebra, we will say that $\mathcal{A}$ is an $\mathcal{L}$-*order-algebra*. We say that $\mathbf{A}$ is the *algebraic reduct* of $\mathcal{A}$. If $\mathcal{L} = \langle \mathcal{F}, \tau\rangle$ is finite with $F = \{f_1, \ldots, f_n\}$, we say that $\mathcal{A}$ is a $\langle f_1, \ldots, f_n, \leq\rangle$-algebra of algebraic type $\langle \tau(f_1), \ldots, \tau(f_n)\rangle$. As in the case of algebras, we will use capital letters in blackboard boldface, e.g., $\mathbb{K}, \mathbb{M} \ldots$ to denote the generic classes of order-algebras.

If $\mathcal{L} = \langle \mathcal{F}, \tau \rangle$, we consider the first-order language with equality

$$\mathcal{L}^{\preccurlyeq} = \langle F \cup \{\approx, \preccurlyeq\}, \tau' \rangle,$$

where its type of similarity τ' is given by $\tau' \restriction F = \tau$ and $\tau'(\preccurlyeq) = 2$. This language, then, besides the equality symbol, contains a proper single binary relational symbol $\preccurlyeq$ of arity 2 and a set of functional symbols. A $\mathcal{L}$-order-algebra can be seen as a $\mathcal{L}^{\preccurlyeq}$-structure, where the functional interpretation is $f^{\mathcal{A}} = f^{\mathbf{A}}$, for every $f \in F$, and the relational interpretation is $\preccurlyeq^{\mathcal{A}} = \leq$. The atomic formulas of this language are in the form $\varphi \approx \psi$ and $\varphi \preccurlyeq \psi$, where φ and ψ are $\mathcal{L}$-terms and the rest of the $\mathcal{L}^{\preccurlyeq}$-formulas, as in the case of the $\mathcal{L}$-algebras, are generated as usual with the help of the quantifiers $\forall$ and $\exists$ and the boolean connectives $\sqcup, \sqcap, \supset, \sim$.

We will call $\mathcal{L}$-*inequation* any $\mathcal{L}^{\preccurlyeq}$-atomic formula in the form $\varphi \preccurlyeq \psi$ and will call $\mathcal{L}$-*quasi-inequation* any $\mathcal{L}^{\preccurlyeq}$-formula in the form

$$\varphi_0 \preccurlyeq \psi_0 \ \sqcap \ldots \sqcap \ \varphi_{n-1} \preccurlyeq \psi_{n-1} \supset \varphi_n \preccurlyeq \psi_n,$$

where $\varphi_i, \psi_i \in Fm_{\mathcal{L}}$ for every $i \leq n$. Note that every inequation is a quasi-inequation with $n = 0$.

A *Horn formula* is a first-order formula of one of the three types included below:

i) β,

ii) $\alpha_1 \sqcap \ldots \sqcap \alpha_n \supset \beta$,

iii) $\sim (\alpha_1 \sqcup \ldots \sqcup \alpha_n)$,

where $\alpha_1, \ldots, \alpha_n, \beta$ are atomic formulas. A *universal Horn sentence* is a universal closure of a Horn formula and we classify it as *strict* if it is the universal closure of a Horn formula in the forms *i)* or *ii)*. So, the $\mathcal{L}$-equations and $\mathcal{L}$-quasi-equations as well as the $\mathcal{L}$-inequations and $\mathcal{L}$-quasi-inequations are Horn formulas of the language $\mathcal{L}^{\preccurlyeq}$. We will say that a $\mathcal{L}$-quasi-inequation δ *is satisfied*, or *is valid*, in a $\langle \mathcal{L}, \leq \rangle$-algebra $\mathcal{A}$ if the corresponding strict universal Horn sentence is valid in $\mathcal{A}$ in the usual sense and we will annotate $\mathcal{A} \models \delta$ and $\mathbb{K} \models \delta$ if it is valid for all the members belonging to a class $\mathbb{K}$ of order-algebras.

Given a $\langle \mathcal{L}, \leq \rangle$-algebra $\mathcal{A} = \langle A, \leq \rangle$, the order is *definable* in terms of the functionals if there is a $\mathcal{L}$-formula δ of first-order with its variables in $\{x_1, x_2\}$ such that

$$\{\langle a, b \rangle \in A^2 : a \leq b\} = \{\langle a, b \rangle \in A^2 : \mathbf{A} \models \delta^{\mathbf{A}}(a, b)\}.$$

Monotonicity, antimonotonicity. Let ι be an operation of arity $k \geq 1$ defined on an ordered set $\langle A, \leq \rangle$ and let i such that $1 \leq i \leq k$. We will say that ι is *monotonous in the argument i-th* with respect to the order if, for every $a, b, c_1, \ldots, c_k \in A$,

$$\iota(c_1, \ldots, c_{i-1}, a, c_{i+1}, \ldots, c_k) \leq \iota(c_1, \ldots, c_{i-1}, b, c_{i+1}, \ldots, c_k) \text{ when } a \leq b.$$

If the operation ι is monotonous in all the arguments, we will say, simply, that it is *monotonous*. In this case it is said also that the operation is *compatible* with the order. With this expression we indicate that a structure containing such an operation is not only a structure with operations and an order, but there is a nexus established between one of the operations and the order.

We will say that ι is *antimonotonous in the argument i-th* with respect to the order if, for every $a, b, c_1, \ldots, c_k \in A$,

$$\iota(c_1, \ldots, c_{i-1}, b, c_{i+1}, \ldots, c_k) \leq \iota(c_1, \ldots, c_{i-1}, a, c_{i+1}, \ldots, c_k) \text{ when } a \leq b.$$

If the operation ι is antimonotonous in all the arguments, we will say, simply, that it is *antimonotonous*. In this case we say that this operation is *dually compatible* with the order.

6.2 Partially Ordered Monoids and Groupoids

We define below some classes of order-algebras that are characterized by the fact of containing a binary operation which is monotonous in the two arguments with respect to the order. The contents of this section are based on [DJLC53] and [Bir73].

- Let $\langle * \rangle$ be an algebraic language of type $\langle 2 \rangle$. An order-algebra $\mathcal{A} = \langle A, *, \leq \rangle$ is a *partially ordered groupoid* or, in short, *po-groupoid*, if the operation $*$ is monotonous with respect to the order, i.e., for every $a, b, c \in A$,

  r) if $a \leq b$, then $a * c \leq b * c$,

  l) if $a \leq b$, then $c * a \leq c * b$.

  or, equivalently, if $a \leq c$ and $b \leq d$, then $a * b \leq c * d$.

- Let $\langle *, 1 \rangle$ be an algebraic language with the type $\langle 2, 0 \rangle$. An order-algebra $\mathcal{A} = \langle A, *, 1, \leq \rangle$ is a *po-monoid* if $\langle A, *, \leq \rangle$ is a po-groupoid and the algebraic reduct is a monoid, i.e.,

  i) the operation $*$ is associative,

  ii) 1 is the identity element.[1]

- A *commutative po-groupoid (po-monoid)* is a po-groupoid (po-monoid) such that the operation $*$ is commutative. In this case, conditions *r*) and *l*) of monotonicity are equivalent.

[1] The *identity* element or the *unit* element of a groupoid $\mathbf{G} = \langle G, * \rangle$ is an element $1 \in G$ such that, for every $a \in G$, it is satisfied $a * 1 = 1 * a = 1$. If there is such an element, then it is unique.

In the literature it is usual to refer by *partially ordered algebraic structures* (or, just, *ordered*) to the order-algebras containing a po-groupoid as a reduct. In this context, the name "ordered" means the presence of a binary operation compatible with the order.

- A *semilatticed groupoid*, or *$s\ell$-groupoid*, is an algebra $\mathbf{A} = \langle A, \vee, * \rangle$ of type $\langle 2, 2 \rangle$ where the following equations are satisfied:
    1. A set of equations defining the $\vee$-semilattices (i.e., idempotency, associativity and commutativity of the operation $\vee$).
    2. The distributivity laws of the operation $*$ with respect to the operation $\vee$, i.e.,

       r) $\quad (x \vee y) * z \approx (x * z) \vee (y * z),$

       l) $\quad z * (x \vee y) \approx (z * x) \vee (z * x).$
- A *latticed groupoid*, or *ℓ-groupoid*, is an algebra $\mathbf{A} = \langle A, \vee, \wedge, * \rangle$ with the type $\langle 2, 2, 2 \rangle$ such that in $\mathbf{A}$ a set of equations is satisfied defining the $\langle \vee, \wedge \rangle$-lattices and the equations *r)* and *l)*.
- A *semilatticed monoid*, or *$s\ell$-monoid*, is an algebra $\mathbf{A} = \langle A, \vee, *, 1 \rangle$ with the type $\langle 2, 2, 0 \rangle$ such that $\langle A, \vee, * \rangle$ is a $s\ell$-groupoid and such that its $\langle *, 1 \rangle$-reduct is a monoid.
- A *semilatticed monoid*, or *ℓ-monoid* is an algebra $\mathbf{A} = \langle A, \vee, \wedge, *, 1 \rangle$ with the type $\langle 2, 2, 2, 0 \rangle$ such that $\langle A, \vee, \wedge, * \rangle$ is a ℓ-groupoid and such that its $\langle *, 1 \rangle$-reduct is a monoid.

It is easy to prove that equations *r)* and *l)* of the definition of semilatticed groupoid are equivalent to the equation:

$$(x \vee y) * (z \vee t) \approx (x * z) \vee (x * t) \vee (y * z) \vee (y * t). \tag{6.1}$$

If the operation $*$ is commutative, then the term *commutative* is added to the name of the corresponding class of algebras. Of course, in the commutative classes, the conditions *r)* and *l)* are equivalent. Every semilatticed groupoid $\mathbf{A} = \langle A, \vee, * \rangle$ defines an order-algebra $\mathcal{A} = \langle \mathbf{A}, \leq \rangle$, where $\leq$ is the order defined by the semilattice by means of the atomic formula $x \vee y \approx y$, in such a way that the structure $\langle A, *, \leq \rangle$ is a po-groupoid as is shown in the following proposition.

Proposition 6.1. *In every $s\ell$-groupoid $\mathbf{A}$ monotonicity conditions are satisfied.*

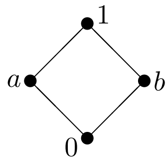
Proof: Let $a, b, c, d \in A$ and suppose that $a \leq c$ and $b \leq d$. Then, by applying (6.1) and the properties of the semilattices, we have:

$c*d = (a \vee c)*(b \vee d) = (a*b) \vee (a*d) \vee (c*b) \vee (c*d) = (a*b) \vee (c*b) \vee (a*d) \vee (c*d) = (a*b) \vee (c*b) \vee ((a \vee c)*d) = (a*b) \vee (c*b) \vee (c*d) = (a*b) \vee (c*(b \vee d)) = (a*b) \vee (c*d)$ and, therefore, $a * b \leq c * d$. $\square$

Thus, the order-algebra $\langle A, *, \leq\rangle$, where $\leq$ is the order of the semilattice (or of the lattice), is a partially ordered groupoid. So, every $s\ell$-groupoid defines naturally a po-groupoid. It is easy to see that $\mathbf{A} = \langle A, \vee, *\rangle$, $\mathcal{A} = \langle \mathbf{A}, \leq\rangle = \langle A, \vee, *, \leq\rangle$ and $\langle A, *, \leq\rangle$ are definitionally equivalent structures. However, it is not true that every po-groupoid $\mathcal{A} = \langle A, *, \leq\rangle$ being a semilattice under its relation of partial order defines a semilatticed groupoid, that is, it is not true that a $s\ell$-groupoid being a po-groupoid is a semilattice under its relation of partial order. In the following proposition it is shown that distributivity is a stronger condition than monotonicity.

Proposition 6.2. *There are po-monoids (and therefore, po-groupoids) $\langle A, *, 1, \leq\rangle$ that are lattices (and hence, semilattices) under their relation of partial order which do not satisfy the distributivity condition of the operation $*$ with respect to the operation $\vee$ associated with the semilattice.*

Proof: Let $\mathcal{A} = \langle A, *, \leq\rangle$, where $A = \{0, a, b, 1\}$ and where the order and the operation $*$ are defined in the following diagram and table:



$*$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
1	0	a	b	1

Note that the operation $*$ is associative, commutative and that 1 is the unit element. The order is latticed and the operation $*$ is obviously monotonous with respect to the order. On the other hand, the distributivity of $*$ with respect to $\vee$itis not satisfied:

$$(a \vee b) * b = 1 * b = b \neq 0 = 0 \vee 0 = (a * b) \vee (b * b).$$

So we have a po-monoid that is a lattice under its relation of partial order but does not satisfy the distributivity condition. $\square$

Observe that all the binary operations of the semilatticed and latticed groupoids and monoids are monotonous. So, if we consider, for example, the structure $\langle A, \vee, \wedge, *, \leq\rangle$ associated with a latticed groupoid $\langle A, \vee, \wedge, *\rangle$, we have that the reducts $\langle A, \vee, \leq\rangle$, $\langle A, \wedge, \leq\rangle$ and $\langle A, *, \leq\rangle$ are po-groupoids.

6.3 The varieties $\mathring{\mathbb{M}}^{s\ell}_\sigma$ and $\mathring{\mathbb{M}}^{\ell}_\sigma$

If we expand the algebraic type of the structures considered in the previous section with the constant symbol 0 which will be interpreted as a fixed element but arbitrary in the universe, we obtain the structure classes known as *pointed*.

- A *pointed po-groupoid* is an order-algebra $\mathcal{A} = \langle A, *, 0, \leq\rangle$ of algebraic type $\langle 2, 0\rangle$, where $\langle A, *, \leq\rangle$ is a po-groupoid and 0 is a fixed but element arbitrary of A (a *distinguished* element).

- A *pointed po-monoid* is an order-algebra $\mathcal{A} = \langle A, *, 0, 1, \leq \rangle$ of algebraic type $\langle 2, 0, 0 \rangle$ such that $\langle A, *, 1, \leq \rangle$ is a po-monoid and 0 is a fixed but arbitrary element of A.

- A *$s\ell$-pointed monoid* is an algebra $\mathbf{A} = \langle A, \vee, *, 0, 1 \rangle$ of type $\langle 2, 2, 0, 0 \rangle$ such that $\mathbf{A} = \langle A, \vee, *, 1 \rangle$ is a $s\ell$-monoid and 0 is a fixed but arbitrary element of A. We will denote the class of $s\ell$-pointed monoids by $\mathring{\mathbb{M}}^{s\ell}$.

- A *ℓ-pointed monoid* is an algebra $\mathbf{A} = \langle A, \vee, \wedge, *, 0, 1 \rangle$ of type $\langle 2, 2, 2, 0, 0 \rangle$ such that $\mathbf{A} = \langle A, \vee, \wedge, *, 1 \rangle$ is a ℓ-monoid and 0 is a fixed but arbitrary element of A. We will denote the class of ℓ-pointed monoids by $\mathring{\mathbb{M}}^{\ell}$.

Obviously, the classes $\mathring{\mathbb{M}}^{s\ell}$ and $\mathring{\mathbb{M}}^{\ell}$ are varieties.

Proposition 6.3. *$\mathring{\mathbb{M}}^{s\ell}$ is the equational class of the algebras in the language $\langle \vee, *, 0, 1 \rangle$ of type $\langle 2, 2, 0, 0 \rangle$ that satisfies: a) any set of equations defining the class of the $\vee$-semilattices, b) any set of $\langle *, 1 \rangle$-equations defining the class of the monoids, and c) the distributivity equations of the monoidal operation with respect to the operation $\vee$.*

Proposition 6.4. *$\mathring{\mathbb{M}}^{\ell}$ is the equational class of the algebras in the language $\langle \vee, \wedge, *, 0, 1 \rangle$ of type $\langle 2, 2, 2, 0, 0 \rangle$ that satisfies: a) any set of $\langle \vee, \wedge \rangle$-equations defining the class of the lattices, b) any set of $\langle *, 1 \rangle$-equations defining the class of the monoids, and c) the distributivity equations of the monoidal operation with respect to the operation $\vee$.*

The varieties $\mathring{\mathbb{M}}^{s\ell}$ and $\mathring{\mathbb{M}}^{\ell}$, as we will see in Chapter 9, constitute the algebraic counterpart of the $\langle \vee, *, 0, 1 \rangle$-fragment and the $\langle \vee, \wedge, *, 0, 1 \rangle$-fragment of the Gentzen system $\mathcal{FL}$, respectively. We define below the varieties $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$ and $\mathring{\mathbb{M}}^{\ell}_{\sigma}$, where σ is a subsequence of the sequence $ew_l w_r c$, and where the symbols e, w_l, w_r and c codify the (algebraic) properties that we will call *exchange*, *right-weakening*, *left-weakening* and *contraction*, respectively. As we show below, these properties have the form of quasi-inequations, and their satisfaction in a pointed po-monoid is equivalent, respectively, to the satisfaction of the following properties: commutativity, integrality, 0-boundedness and increasing idempotency. As we also see in Chapter 9, once a sequence σ is fixed, the classes $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$ and $\mathring{\mathbb{M}}^{\ell}_{\sigma}$ are the algebraic counterpart of the $\langle \vee, *, 0, 1 \rangle$-fragment and the $\langle \vee, \wedge, *, 0, 1 \rangle$-fragment of the Gentzen system $\mathcal{FL}_{\sigma}$, respectively.

Definition 6.5 (Exchange property)**.** *We say that a po-groupoid $\mathcal{A}$ satisfies the* exchange property *if the following quasi-inequation is satisfied:*

$$x * y \preccurlyeq z \supset y * x \preccurlyeq z \qquad (e \preccurlyeq)$$

Lemma 6.6. *Let $\mathcal{A}$ be a po-groupoid and let u, v, t be terms of its language. The following is equivalent:*

i) $\mathcal{A} \models u \preccurlyeq t \ \supset \ v \preccurlyeq t$.

ii) $\mathcal{A} \models v \preccurlyeq u$.

Proof: Suppose that the variables of u, v and t are in $\{x_1, \ldots, x_m\}$.

$i) \Rightarrow ii)$: Let $a_1, \ldots, a_n \in A$. Since $i)$, as $u^{\mathcal{A}}(a_1, \ldots, a_n) \leq u^{\mathcal{A}}(a_1, \ldots, a_n)$, we have $v^{\mathcal{A}}(a_1, \ldots, a_n) \leq u^{\mathcal{A}}(a_1, \ldots, a_n)$.

$ii) \Rightarrow i)$: Let $a_1, \ldots, a_n \in A$. Suppose that $u^{\mathcal{A}}(a_1, \ldots, a_n) \leq t^{\mathcal{A}}(a_1, \ldots, a_n)$. By $ii)$, $v^{\mathcal{A}}(a_1, \ldots, a_n) \leq u^{\mathcal{A}}(a_1, \ldots, a_n)$. Then we have that $v^{\mathcal{A}}(a_1, \ldots, a_n) \leq t^{\mathcal{A}}(a_1, \ldots, a_n)$.

□

As we show below, the property $(e \preccurlyeq)$ is equivalent to the commutativity of the groupoid operation and, therefore, the po-groupoids satisfying $(e \preccurlyeq)$ are, precisely, the commutative po-groupoids.

Proposition 6.7. *Let $\mathcal{A}$ be a po-groupoid. The following conditions are equivalent:*

i) $\mathcal{A}$ satisfies the quasi-inequation $(e \preccurlyeq)$.

ii) $\mathcal{A}$ satisfies the inequation $x * y \preccurlyeq y * x$.

iii) $\mathcal{A}$ satisfies the equation $x * y \approx y * x$.

Proof: $i)$ and $ii)$ are equivalent due to Lemma 6.6. The equivalence between $ii)$ and $iii)$ is evident. □

Definition 6.8 (Left-weakening property). *We say that a po-monoid $\mathcal{A}$ satisfies the* left-weakening property *if the following quasi-inequation is satisfied:*

$$x * y \preccurlyeq z \supset x * t * y \preccurlyeq z \qquad (w \preccurlyeq)$$

Definition 6.9 (Integral po-monoid). *We say that a po-monoid $\mathcal{A}$ is* integral *if the unit element is the maximum with respect to the order, that is, if $\mathcal{A} \models x \preccurlyeq 1$.*

In a po-monoid the left-weakening property is equivalent to the integrality and and it is also equivalent to the fact that the result of operating two elements of the monoid is always the same as or lesser than eithero element.

Proposition 6.10. *Let $\mathcal{A}$ be a po-monoid. The following conditions are equivalent:*

i) $\mathcal{A} \models x * y \preccurlyeq x$,

ii) $\mathcal{A} \models x \preccurlyeq 1$,

iii) $\mathcal{A} \models x * y \preccurlyeq y$,

iv) $\mathcal{A} \models x * z * y \preccurlyeq x * y$,

v) $\mathcal{A}$ satisfies the quasi-inequation $(w \preccurlyeq)$.

Proof: Let $a, b, c \in A$.

$i) \Rightarrow ii)$: By using that 1 is the unit element and the condition $i)$ we have $a = 1*a \leq 1$.

$ii) \Rightarrow iii)$: Since $ii)$ we have $a \leq 1$. By applying monotonicity we obtain $a*b \leq 1*b = b$.

$iii) \Rightarrow iv)$: Since $iii)$ we have $c * b \leq b$ and, by applying monotonicity, we obtain $a * c * b \leq a * b$.

$iv) \Rightarrow v)$: Due to Lemma 6.6.

$v) \Rightarrow i)$: As 1 is the unit element we have $a * 1 = a$ and, therefore, $a * 1 \leq a$. Hence, by using $v)$ we obtain $a * b * 1 \leq a$, that is, $a * b \leq a$. □

Therefore, the po-monoids satisfying the property $(w \preccurlyeq)$ are, precisely, the integral po-monoids.

Definition 6.11 (Right-weakening property). *We say that a pointed po-groupoid satisfies the* right-weakening *property if the following quasi-inequation satisfies:*

$$x \preccurlyeq 0 \supset x \preccurlyeq y \qquad (\preccurlyeq w)$$

In a pointed po-groupoid the right-weakening property is equivalent to the fact that the distinguished element 0 is the minimum with respect to the order.

Proposition 6.12. *Let $\mathcal{A}$ be a pointed po-groupoid. The following is equivalent:*

i) $\mathcal{A}$ satisfies the quasi-inequation $(\preccurlyeq w)$.

ii) $\mathcal{A} \models 0 \preccurlyeq x$.

Proof: Let $a, b \in A$.

$i) \Rightarrow ii)$: Given that $0 \leq 0$, by applying $i)$ we obtain $0 \leq a$.

$ii) \Rightarrow i)$: If $a \leq 0$, given that $0 \leq b$, due to the transitivity we have that $a \leq b$. □

Definition 6.13 (Contraction Property). *We say that a po-groupoid satisfies the* contraction *property if the following quasi-inequation is satisfied:*

$$x * x \preccurlyeq y \supset x \preccurlyeq y \qquad (c \preccurlyeq)$$

We say that a po-grupoid has the *increasing idempotency* property if every element is equal to or less than the result of operating this element with itself. Next we show that this property is equivalent to the contraction property.

Proposition 6.14. *Let $\mathcal{A}$ be a po-groupoid. The following conditions are equivalent:*

i) $\mathcal{A}$ satisfies the quasi-inequation $(c \preccurlyeq)$.

ii) $\mathcal{A} \models x \preccurlyeq x * x$.

Proof: Due to Lemma 6.6. □

Now we define the algebra classes $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$ and $\mathring{\mathbb{M}}^{\ell}_{\sigma}$. Let $\lambda \in \{s\ell, \ell\}$. We will use the following denotations:

- $\mathring{\mathbb{M}}^{\lambda}_{e}$ is the class of pointed λ-monoids satisfying $(e \preccurlyeq)$, that is, of the commutative pointed λ-monoids.
- $\mathring{\mathbb{M}}^{\lambda}_{w_l}$ is the class of pointed λ-monoids satisfying $(w \preccurlyeq)$, that is, of the integral pointed λ-monoids.
- $\mathring{\mathbb{M}}^{\lambda}_{w_r}$ is the class of pointed λ-monoids satisfying $(\preccurlyeq w)$, that is, of the pointed λ-monoids with lower bound 0.
- $\mathring{\mathbb{M}}^{\lambda}_{w}$ is the class of pointed λ-monoids satisfying $(w \preccurlyeq)$ and $(\preccurlyeq w)$.
- $\mathring{\mathbb{M}}^{\lambda}_{c}$ is the class of λ-pointed monoids satisfying $(c \preccurlyeq)$ or, in a similar manner, the increasing idempotency, which we will refer to as *contractives.*

Let σ be a subsequence (possibly) of the sequence ew_lw_rc. If in σ there is the sequence w_lw_r, in short we will replace it with w. We will denote by $\mathring{\mathbb{M}}^{\lambda}_{\sigma}$ the class of pointed λ-monoids satisfying the properties codified by the letters appearing in σ and if σ is the empty sequence, then $\mathring{\mathbb{M}}^{\lambda}_{\sigma}$ is the class $\mathring{\mathbb{M}}^{\lambda}$. So, for example, $\mathring{\mathbb{M}}^{\lambda}_{w_rc}$ is the class of the pointed λ-monoids satisfying $(\preccurlyeq w)$ and $(c \preccurlyeq)$.

The classes $\mathring{\mathbb{M}}^{\lambda}_{\sigma}$ are subvarieties of $\mathring{\mathbb{M}}^{\lambda}$, since as we have seen the quasi-inequations $(e \preccurlyeq)$, $(w \preccurlyeq)$, $(\preccurlyeq w)$ and $(c \preccurlyeq)$ are equivalent to inequations and, as in the classes considered the order is definable by the equation $x \vee y \approx y$, the inequations are equivalent to equations. So,

- $\mathring{\mathbb{M}}^{\lambda}_{e}$ is the subvariety of $\mathring{\mathbb{M}}^{\lambda}$ defined by the equation $x * y \approx y * x$.
- $\mathring{\mathbb{M}}^{\lambda}_{w_l}$ is the subvariety of $\mathring{\mathbb{M}}^{\lambda}$ defined by the equation $x \vee 1 \approx 1$ or, equivalently, by the equation $(x * y) \vee x \approx x$.
- $\mathring{\mathbb{M}}^{\lambda}_{w_r}$ is the subvariety of $\mathring{\mathbb{M}}^{\lambda}$ defined by the equation $0 \vee x \approx x$.
- $\mathring{\mathbb{M}}^{\lambda}_{c}$ is the subvariety of $\mathring{\mathbb{M}}^{\lambda}$ defined by the equation $x \vee (x * x) \approx x * x$.

By combining these equations we obtain all the subvarieties $\mathring{\mathbb{M}}^{\lambda}_{\sigma}$.

Corollary 6.15. *Classes $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$ and $\mathring{\mathbb{M}}^{\ell}_{\sigma}$ are varieties.*

Notation 6.16. From now on, in the context of the semilatticed and latticed algebras, given two terms t_1 and t_2, we will use the expression $t_1 \preccurlyeq t_2$ as an abbreviation for the equation $t_1 \vee t_2 \approx t_2$. Note that the expression $t_1 \preccurlyeq t_2$ may also be seen as an atomic formula of the language of the order-algebra associated with every semilatticed monoid.

Proposition 6.17. *In a 0-bounded integral po-monoid the minimum element is the zero of the monoid.*[2]

Proof: Let $\mathcal{A}$ be a 0-bounded integral po-monoid. If $a \in A$, due to integrality, we have $a \leq 1$ and, due to monotonicity, $a * 0 \leq 1 * 0 = 0$ and $0 * a \leq 0 * 1 = 0$. But, as 0 is the minimum, $a * 0 = 0 * a = 0$. □

Proposition 6.18. *Let* $\mathbf{A} \in \mathring{\mathbb{M}}^{\ell}$. *The following conditions are equivalent:*

i) $\mathbf{A} \models x \preccurlyeq 1$,

ii) $\mathbf{A} \models x * y \preccurlyeq x \wedge y$.

Proof: By Proposition 6.10 we have that the equation $x \preccurlyeq 1$ is equivalent to the equations $x * y \preccurlyeq x$ and $x * y \preccurlyeq y$ but in a latticed structure these two equations are equivalent to the equation $x * y \preccurlyeq x \wedge y$. □

Proposition 6.19. *Let* $\mathbf{A} \in \mathring{\mathbb{M}}^{s\ell}_{w_l}$. *Then, the following are equivalent:*

i) $\mathbf{A} \models x * x \approx x$ (**A** *is idempotent),*

ii) $\mathbf{A} \models x \preccurlyeq x * x$ (**A** *is contractive),*

iii) For every $a, b \in A$, *the infimum of* a *and* b *exists and is equal to* $a * b$,

iv) $\mathbf{A} \models x \preccurlyeq y$ *iff* $\mathbf{A} \models x * y \approx x$.

Proof: We only prove *ii)* $\Rightarrow$ *iii)*. The other implications are trivial. Let $a, b, c \in A$. Due to integrality we have that $a*b$ is a common lower bound to a and b. Suppose now that c is a common lower bound to a and b. Due to monotonicity we have: $c*c \leq a*b$. But by *ii)* we have $c \leq c * c$ and, therefore, $c \leq a * b$. □

Similarly, for the latticed varieties we have the following proposition.

Proposition 6.20. *Let* $\mathbf{A} \in \mathring{\mathbb{M}}^{\ell}_{w_l}$. *The following conditions are equivalent:*

i) $\mathbf{A} \models x * x \approx x$ (**A** *is idempotent),*

ii) $\mathbf{A} \models x \preccurlyeq x * x$ (**A** *is contractive),*

iii) $\mathbf{A} \models x * y \approx x \wedge y$,

iv) $\mathbf{A} \models x \preccurlyeq y$ *iff* $\mathbf{A} \models x * y \approx x$.

[2]Remember that the *zero* element or the *absorbent* element of a groupoid $\mathbf{G} = \langle G, * \rangle$ is an element $0 \in G$ such that, for every $a \in G$, $a * 0 = 0 * a = 0$ is satisfied. If there is such an element, then it is unique.

Proposition 6.21. *Let* $\mathbf{A}$ *be in* $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$ *or in* $\mathring{\mathbb{M}}^{\ell}_{\sigma}$, *with* $w_l c \leq \sigma$. *Then* $\mathbf{A} \models x * y \approx y * x$. *Therefore,* $\mathring{\mathbb{M}}^{s\ell}_{w_l c} = \mathring{\mathbb{M}}^{s\ell}_{ew_l c}$, $\mathring{\mathbb{M}}^{s\ell}_{wc} = \mathring{\mathbb{M}}^{s\ell}_{ewc}$, $\mathring{\mathbb{M}}^{\ell}_{w_l c} = \mathring{\mathbb{M}}^{\ell}_{ew_l c}$ *and* $\mathring{\mathbb{M}}^{\ell}_{wc} = \mathring{\mathbb{M}}^{\ell}_{ewc}$.

Proof: In these varieties it is clear that the operation $*$ is commutative since $a * b$ is the infimum of $\{a, b\}$. □

Observation 6.22. Note that $\mathring{\mathbb{M}}^{s\ell}_{ewc}$ is the variety of the bounded distributive lattices (cf.[BD74, Chapter II], where this class is denoted by $\mathbf{D_{01}}$) and $\mathring{\mathbb{M}}^{s\ell}_{ew_l c}$ is the variety of the upper bounded distributive lattices.

Observation 6.23. The varieties $\mathring{\mathbb{M}}^{s\ell}_{ew_l c}$ and $\mathring{\mathbb{M}}^{\ell}_{ew_l c}$ are definitionally equivalent and the operations $*$ and $\wedge$ are the same. This can be also applied to the varieties $\mathring{\mathbb{M}}^{s\ell}_{ewc}$ and $\mathring{\mathbb{M}}^{\ell}_{ewc}$.

6.4 Residuated Structures

In this section we present the residuated structures related to the systems introduced in Chapter 4. We have based our argument on [DJLC53], [Bir73], [JT02] and [GJKO07]. As a novelty we should emphasize that we use the notion of *relative pseudocomplement*. This notion is a generalization of the same notion used traditionally within the framework of lattices. Remember that, given a lattice $\mathbf{L}$ and two elements $a, b \in L$, if the largest element in L such that $a \wedge x \leq b$ exists, this element is denoted by $a \to b$ and is referred to as *relative pseudocomplement of a with respect to b*. We should remember that a Heyting algebra is a lattice $\mathbf{L}$ with minimum element such that the relative pseudocomplement of a with respect to b exists for every $a, b \in L$ (Cf. [BD74, Chapter IX]). For the introduction of the relative pseudocomplement notion we start from an ordered set $\langle A, \leq \rangle$ in which we have defined a binary operation $*$.

Definition 6.24 (Relative Pseudocomplement)**.** *Let* $*$ *be a binary operation defined in an ordered set* $\langle A, \leq \rangle$. *Given* $a, b \in A$, *if the largest element* $x \in A$ *such that* $a * x \leq b$ $(x * a \leq b)$ *exists, we say that this element is the* $*$-right (left) relative pseudocomplement of a with respect to b. *If the operation is commutative, then the notions of left and right relative pseudocomplement coincide and we talk, simply, about the relative pseudocomplement of a with respect to b.*

Note that if the order of the ordered set is a $\wedge$-semilattice or a lattice, then the notions of $\wedge$-relative pseudocomplement and the traditional one of relative pseudocomplement coincide.

Notation 6.25. Given a binary operation $*$ defined in an ordered set, when the context avoids any confusion we will use the name of (right or left) relative pseudocomplement instead of relative $*$-pseudocomplement.

We now recall the notion of *residuation* and establish the link between this notion and the one related to the relative pseudocomplement.

Definition 6.26 (Residuated operation). *A binary operation $*$ defined in a partially ordered set $\langle A, \leq\rangle$ is called* residuated *if there are two binary operations $\backslash$ and $/$ defined in A such that, for every $a, b, c \in A$,*

$$a * b \leq c \quad \textit{iff} \quad b \leq a\backslash c \quad \textit{iff} \quad a \leq c/b. \tag{LR}$$

This condition is called law of residuation, *and the operations $\backslash$ and $/$ are called* right residual *and* left residual *of the operation $*$, respectively. Given $a, b \in A$, the element $a\backslash b$ will be referred to as the* right residual of a relative to b *and the element b/a will be referred to as the* left residual of a relative to b. *In the event that the operation is commutative, the two residuals coincide and the corresponding operation is called the* residual *of the operation $*$. In this case we will use the symbol $\rightarrow$ and annotate $a \rightarrow b$ instead of $a\backslash b$ or b/a.*

A *basic residuated structure* is an order-algebra $\mathcal{A} = \langle A, *, \backslash, /, \leq\rangle$ with the algebraic type $\langle 2, 2, 2\rangle$ such that $*$ is a residuated operation with respect to the order and such that the operations $\backslash$ and $/$ are the right and left residuals of the operation $*$, respectively. We will call *residuated structure* any structure having a basic residuated structure as a reduct. Residuated structures are characterized by the fact that the operation $*$ is monotonous with respect to the order and by the fact that, for every a, b in the universe, there exist the right and left pseudocomplements of a relative to b. This last point implies the uniqueness of the residual.

Proposition 6.27. *Let $\langle A, \leq\rangle$ be a partially ordered set and let $*$ be a binary operation defined in A. For every $a, b \in A$, we define the sets*

$$\mathrm{R}^b(a) = \{x \in A : a * x \leq b\}, \qquad \mathrm{L}^b(a) = \{x \in A : x * a \leq b\}.$$

Then, the following conditions are equivalent:

i) The operation $*$ is residuated.

ii) The operation $*$ is monotonous and, for every $a, b \in A$, there exist the right and the left pseudocomplements of a with respect to b, i.e., the sets $\mathrm{R}^b(a)$ and $\mathrm{L}^b(a)$ have a maximum element.

Under these conditions, for every $a, b \in A$, $a\backslash b = max\, \mathrm{R}^b(a)$ and $b/a = max\, \mathrm{L}^b(a)$.

Proof:

i) $\Rightarrow$ *ii)*: Suppose that $*$ is residuated and let $a, b \in A$. We will see that $a\backslash b$ is the maximum element of $\mathrm{R}^b(a)$. By applying (LR), from $a\backslash b \leq a\backslash b$ we obtain $a * (a\backslash b) \leq b$ and, therefore, $a\backslash b \in \mathrm{R}^b(a)$. Suppose now that $c \in \mathrm{R}^b(a)$: then we have $a * c \leq b$ that, by (LR), is equivalent to $c \leq a\backslash b$. Therefore, $a\backslash b$ is the maximum of $\mathrm{R}^b(a)$. Similarly we obtain that b/a is the maximum of $\mathrm{L}^b(a)$. Now we will see that monotonicity is satisfied. Let $a, b, c, d \in A$ and suppose that $a \leq c$ and $b \leq d$. From $c * d \leq c * d$ we obtain $b \leq d \leq c\backslash(c * d)$ and therefore $c * b \leq c * d$. Based on this we obtain $a \leq c \leq (c * d)/b$ and, consequently, $a * b \leq c * d$.

ii) $\Rightarrow$ *i)*: For every $a, b \in A$ we define

$$a \backslash b := \max \mathrm{R}^b(a), \qquad a/b := \max \mathrm{L}^a(b).$$

Given that for every $a, b \in A$ the sets $\mathrm{R}^b(a)$ and $\mathrm{L}^a(b)$ have a maximum, the operations $\backslash$ and $/$ are well defined. Let $c \in A$ and suppose that $a * c \leq b$. Then $c \in \mathrm{R}^b(a)$ and, therefore, $c \leq a\backslash b$. Suppose $c \leq a\backslash b$. According to left monotonicity, we obtain $a * c \leq a * (a\backslash b)$. However, as $a\backslash b = \max \mathrm{R}^b(a)$, in particular $a\backslash b \in \mathrm{R}^b(a)$ and, hence, $a * (a\backslash b) \leq b$. So, $a * c \leq b$. Similarly, from $c * a \leq b$ we obtain $c \leq b/a$ and, from $c \leq b/a$, by applying the right monotonicity, we obtain $c * a \leq b$. □

Corollary 6.28 (Uniqueness of the residuals). *If a binary operation defined in an ordered set $\langle A, \leq\rangle$ is residuated, then there are precisely two binary operations $\backslash$ and $/$ that satisfy the law of residuation.*

We observe that, as a consequence of Proposition 6.27, we have that in the framework of the po-groupoids, if the operation is residuated, notions of residual and relative pseudocomplement coincide.

Corollary 6.29. *Let $\langle A, *, \leq\rangle$ be a po-groupoid. Then there are equivalent:*

i) The operation $*$ is residuated.

ii) For every $a, b \in A$, there are right and left relative pseudocomplements of a with respect to b.

Under these conditions, for every $a, b \in A$, the right residual of a relative to b is the right relative pseudocomplement of a with respect to b and the left residual of a relative to b is the left relative pseudocomplement of a with respect to b.

Definition 6.30 (Residuated po-groupoid). *A* residuated po-groupoid *is an order-algebra $\mathcal{A} = \langle A, *, \backslash, /, \leq\rangle$, where $\langle A, *, \leq\rangle$ is a po-groupoid, the operation $*$ is residuated and the operations $\backslash$ and $/$ are its residuals.*

Nomenclature 6.31. In accordance with the above results we have that the basic residuated structures are precisely the residuated po-groupoids. In the nomenclature we use here for practical reasons we will dispense with the name *partially ordered* or the prefix *po* because the notion of *residuation* implicitly entails the presence of a partial order in the structure in such a way that the operation $*$ is monotonous with respect to that order. Residuals constitute a generalization of the division operation in the groups. In concordance with this idea $a\backslash b$ is read as "a under b" and b/a is read as "b above a". In both cases we may say that b is the *numerator* and a the *denominator*.

As the operation $*$ of a po-groupoid is compatible with the order, the residuals of a residuated groupoid are connected as well with the order in the following sense: the right (left) residual is antimonotonous in the first (second) argument and monotonous in the second (first) argument.

Proposition 6.32. *In every residuated groupoid $\mathcal{A}$ the following conditions, for any $a, b, c \in A$, are satisfied:*

i) *if $a \leq b$, then $c\backslash a \leq c\backslash b$ and $b\backslash c \leq a\backslash c$,*

ii) *if $a \leq b$, then $a/c \leq b/c$ and $c/b \leq c/a$.*

Proof: *i)*: Suppose $a \leq b$. According to reflexivity, we have that $c\backslash a \leq c\backslash a$. Hence, by applying (LR) we obtain $c * (c\backslash a) \leq a$ and, therefore, $c * (c\backslash a) \leq b$ and, again for (LR), we obtain $c\backslash a \leq c\backslash b$. On the other hand, by applying monotonicity, from $a \leq b$ we obtain $a * (b\backslash c) \leq b * (b\backslash c)$ and from $b\backslash c \leq b\backslash c$, since (LR), we obtain $b * (b\backslash c) \leq c$ and, thus, $a * (b\backslash c) \leq c$ which is equivalent to $b\backslash c \leq a\backslash c$.

ii): Proved in the same manner. □

In a residuated groupoid the operation $*$ preserves the existing suprema in each argument and the residuals preserve all the existing infima in the numerator and turn the existing suprema into infima in the denominator, as is shown in the following propositions.

Notation 6.33. If $\{a_i : i \in I\}$ is a family of elements of a partially ordered set $\langle A, \leq\rangle$, then the supremum and the infimum (if they exist) in A of the family will be denoted by $\bigvee_{i\in I} a_i$ and $\bigwedge_{i\in I} a_i$, respectively.

Proposition 6.34 (Generalized distributivity)**.** *Let $\mathcal{A} = \langle A, *, \backslash, /, \leq\rangle$ be a residuated groupoid and $\{a_i : i \in I\}$ and $\{b_j : j \in J\}$ two families of elements in A. If $\bigvee_{i\in I} a_i$ and $\bigvee_{j\in J} b_j$ exist, then there exists $\bigvee_{\langle i,j\rangle\in I\times J} a_i * b_j$ and the following holds:*

$$\bigvee_{i\in I} a_i * \bigvee_{j\in J} b_j = \bigvee_{\langle i,j\rangle\in I\times J} a_i * b_j.$$

Proof: According to monotonicity it is clear that, for every $\langle i, j\rangle \in I \times J$, $\bigvee_{i\in I} a_i * \bigvee_{j\in J} b_j$ is an upper bound of $a_i * b_j$. Suppose that $a_i * b_j \leq c$. Then, by applying (LR) we have $b_j \leq a_i\backslash c$ and, therefore, $\bigvee_{j\in J} b_j \leq a_i\backslash c$. Once again by (LR) we obtain $a_i * \bigvee_{j\in J} b_j \leq c$ and, hence, $a_i \leq c/\bigvee_{j\in J} b_j$ and, thus, $\bigvee_{i\in I} a_i \leq c/\bigvee_{j\in J} b_j$ that, again by (LR), allow us to conclude $\bigvee_{i\in I} a_i * \bigvee_{j\in J} b_j \leq c$. □

Proposition 6.35. *Let $\mathcal{A} = \langle A, *, \backslash, /, \leq\rangle$ be a residuated groupoid and $\{a_i : i \in I\}$ and $\{b_j : j \in J\}$ two families of elements of A. If $\bigvee_{i\in I} a_i$ and $\bigwedge_{j\in J} b_j$ exist, then, for every $c \in A$, there exist $\bigwedge_{i\in I} a_i\backslash c$, $\bigwedge_{j\in J} c\backslash b_j$, $\bigwedge_{i\in I} c/a_i$ and $\bigwedge_{j\in J} b_j/c$ and it is satisfied:*

$$(\bigvee_{i\in I} a_i)\backslash c = \bigwedge_{i\in I} a_i\backslash c\,;\; c\backslash(\bigwedge_{j\in J} b_j) = \bigwedge_{j\in J} c\backslash b_j\,;\; c/(\bigvee_{i\in I} a_i) = \bigwedge_{i\in I} c/a_i\,;\; (\bigwedge_{j\in J} b_j)/c = \bigwedge_{j\in J} b_j/c.$$

Proof: From $a_i \leq \bigvee_{i \in I} a_i$, due to the antimonotonicity in the first argument of the right residual, we obtain $(\bigvee_{i \in I} a_i)\backslash c \leq a_i\backslash c$. Therefore, $(\bigvee_{i \in I} a_i)\backslash c$ is a lower bound of $a_i\backslash c$. Let $d \in A$ and suppose that, for every $i \in I$, $d \leq a_i\backslash c$. This is equivalent to $a_i * d \leq c$ that is equivalent in turn to $a_i \leq c/d$. Hence, we obtain $(\bigvee_{i \in I} a_i) \leq c/d$, that is equivalent to $(\bigvee_{i \in I} a_i) * d \leq c$ and, therefore, to $d \leq (\bigvee_{i \in I} a_i)\backslash c$. Consequently, $(\bigvee_{i \in I} a_i)\backslash c$ is the infimum of $a_i\backslash c$.

From $\bigwedge_{j \in J} b_j \leq b_j$, due to the monotonicity in the second argument of the right residual, we obtain $c\backslash(\bigwedge_{j \in J} b_j) \leq c\backslash b_j$. Suppose that, for every $j \in J$, $d \leq c\backslash b_j$. This is equivalent to $c * d \leq b_j$. Hence we obtain $c * d \leq (\bigwedge_{j \in J} b_j)$, that is equivalent to $d \leq c\backslash(\bigwedge_{j \in J} b_j)$. Therefore, $(\bigwedge_{j \in J} b_j)$ is the infimum of $c\backslash b_j$.

The other two equalities are proved similarly using (LR), the antimonotonicity in the second argument and the monotonicity in the first argument of the left residual. □

In Section 6.2 we have seen that every $s\ell$-groupoid define a *po*-groupoid (Proposition 6.1) and also we have seen that distributivity is a stronger condition than monotonicity (Proposition 6.2) and, therefore, it is not true in general that a *po*-groupoid being a semilattice under its partial order relation is a $s\ell$-groupoid. However, this will be true whenever the operation $*$ is residuated:

Proposition 6.36. *Let $\langle A, *, \leq \rangle$ be a residuated groupoid that is a semilattice under its partial order relation. We define $x \vee y =: \bigvee\{x, y\}$. Then $\langle A, \vee, * \rangle$ is a $s\ell$-groupoid.*

Proof: It is a direct consequence of Proposition 6.34. □

Definition 6.37 (Residuated monoid)**.** *A* residuated monoid *is an order-algebra $\mathcal{A} = \langle A, *, \backslash, /, 1, \leq \rangle$ such that $\langle A, *, \backslash, /, \leq \rangle$ is a residuated groupoid and such that $\langle A, *, 1 \rangle$ is a monoid.*

Definition 6.38 (Residuated lattice)**.** *A* residuated lattice *is an algebra $\mathbf{A} = \langle A, \vee, \wedge, *, \backslash, /, 1 \rangle$ such that $\langle A, \vee, \wedge \rangle$ is a lattice and $\langle A, *, \backslash, /, 1, \leq \rangle$, where $\leq$ is the order of the lattice, is a residuated monoid. We will denote the class of residuated lattices by $\mathbb{RL}$.*

Definition 6.39 (Pointed residuated lattice)**.** *A* pointed residuated lattice *is an algebra $\mathbf{A} = \langle \vee, \wedge, *, \backslash, /, 0, 1 \rangle$ such that its $\langle \vee, \wedge, *, \backslash, /, 1 \rangle$-reduct is a residuated lattice and such that* 0 *is a fixed element, but arbitrary, of A.*

Pointed residuated lattices are called *Full Lambek algebras* according to Ono (see for instance [Ono93]) on account of their connection with sequent calculus **FL**. We will denote by $\mathbb{FL}$ the class of pointed residuated lattices and will call its members $\mathbb{FL}$-*algebras.*

Observe that (pointed) residuated lattices are (pointed) ℓ-monoids such that their monoidal operation is residuated. Residuated lattices and $\mathbb{FL}$-algebras can be understood as order-algebras $\langle \mathbf{A}, \leq \rangle$, where $\leq$ is the order defined by the lattice.

Definition 6.40 (Mirror image.). *If t is a term of an algebraic language $\mathcal{L}$ such that $\langle *, \backslash, / \rangle \leq \mathcal{L} \leq \langle \vee, \wedge, *, \backslash, /, 0, 1 \rangle$, we define its* mirror image $\mu(t)$ *inductively on the complexity of t:*[3]

$$\mu(t) := \begin{cases} t, & \text{if } t \in Var \text{ or } t\, t \in \{0,1\}, \\ \mu(u) \vee \mu(v), & \text{if } t = u \vee v, \\ \mu(u) \wedge \mu(v), & \text{if } t = u \wedge v, \\ \mu(v) * \mu(u), & \text{if } t = u * v, \\ \mu(u) \backslash \mu(v), & \text{if } t = v/u, \\ \mu(v)/\mu(u), & \text{if } t = u \backslash v. \end{cases}$$

We define the mirror image of a formula *of the first order language with equality $\mathcal{L}^{\preccurlyeq} = \langle \mathcal{L}, \preccurlyeq \rangle$ as the formula obtained after replacing all the existing terms therein with their mirror images.*

Lemma 6.41. *Let $\mathbb{K}$ be the class of the residuated groupoids, of the residuated monoids, of the residuated lattices or of the $\mathbb{FL}$-algebras. We denote their algebraic language by $\mathcal{L}_{\mathbb{K}}$ and we denote by $\mathcal{L}_{\mathbb{K}}^{\preccurlyeq}$ the first order language with equality $\langle \mathcal{L}_{\mathbb{K}}, \preccurlyeq \rangle$. If $\mathcal{A} \in \mathbb{K}$, consider the $\mathcal{L}_{\mathbb{K}}^{\preccurlyeq}$-structure $\mathcal{A}'$, with a universe equal to the one of $\mathcal{A}$, with the same order as $\mathcal{A}$, and where the operations and constants of $\mathcal{A}'$ being in $\{\vee, \wedge, 0, 1\}$ are the same as in $\mathcal{A}$ and the remaining operations are defined as follows: for every $a, b \in A$,*

$$a *^{\mathcal{A}'} b := b *^{\mathcal{A}} a, \quad a \backslash^{\mathcal{A}'} b := b /^{\mathcal{A}} a, \quad b /^{\mathcal{A}'} a := a \backslash^{\mathcal{A}} b.$$

Then,

i) *$\mathcal{A}'$, which we will name as* opposite *of $\mathcal{A}$, belongs to $\mathbb{K}$*

ii) *for every term t of $\mathcal{L}_{\mathbb{K}}$, $\mu(t)^{\mathcal{A}'} = t^{\mathcal{A}}$ is satisfied.*

Proof: *i)*: It is easy to see that the operation $*^{\mathcal{A}'}$ is residuated and that $\backslash^{\mathcal{A}'}$ and $/^{\mathcal{A}'}$ are its right and left residual, respectively.

ii): By induction on the complexity of the term t. Suppose that the variables appearing in t are in $\{x_1, \ldots, x_n\}$. We must prove that for every assignment $\bar{a}$ of the variables in A, if this assignment is such that $\bar{a}(x_i) = a_i$, for every $1 \leq i \leq n$, then $\mu(t)^{\mathcal{A}'}(a_1, \ldots, a_n) = t^{\mathcal{A}}(a_1, \ldots, a_n)$. If t is a variable or $t \in \{u \vee v, u \wedge v, 0, 1\}$, then it is obvious. Suppose $t = u \backslash v$, where u and v are $\langle *, \backslash, / \rangle$-terms. Then we have:

$$\mu(u \backslash v)^{\mathcal{A}'}(a_1, \ldots, a_n) = \mu(v)/\mu(u))^{\mathcal{A}'}(a_1, \ldots, a_n) =$$

$$= \mu(v)^{\mathcal{A}'}(a_1, \ldots, a_n) /^{\mathcal{A}'} \mu(u)^{\mathcal{A}'}(a_1, \ldots, a_n),$$

[3] *Complexity of a term* means the number of functional occurrences of arity $k \geq 1$ in this term.

and by applying the induction hypothesis:

$$\mu(v)^{\mathcal{A}'}(a_1,\dots,a_n)/^{\mathcal{A}'}\mu(u)^{\mathcal{A}'}(a_1,\dots,a_n) = v^{\mathcal{A}}(a_1,\dots,a_n)/^{\mathcal{A}'}u^{\mathcal{A}}(a_1,\dots,a_n) =$$

$$= u^{\mathcal{A}}(a_1,\dots,a_n)\backslash^{\mathcal{A}}v^{\mathcal{A}}(a_1,\dots,a_n) = (u\backslash v)^{\mathcal{A}}(a_1,\dots,a_n).$$

Cases $t = v/u$ and $t = u * v$ are similar. □

Lemma 6.42. *Let $\mathcal{A}$ be a residuated groupoid (residuated monoid, residuated lattice, $\mathbb{FL}$-algebra) and let $\mathcal{A}'$ be its opposite. Then, the following is satisfied for every formula φ of the first order language of the residuated groupoids (residuated monoids, residuated lattices, $\mathbb{FL}$-algebras):*

$$\mathcal{A} \vDash \varphi \quad \textit{iff} \quad \mathcal{A}' \vDash \mu(\varphi).$$

Proof: Let $\mathbb{K}$ and $\mathcal{L}_{\mathbb{K}}^{\preccurlyeq}$ be as in Lemma 6.41 and let φ be a $\mathcal{L}_{\mathbb{K}}^{\preccurlyeq}$-formula. We will see that if $\mathcal{A} \in \mathbb{K}$, then $\mathcal{A} \nvDash \varphi$ if, and only if, $\mathcal{A}' \nvDash \mu(\varphi)$. The proof is made by induction on the complexity of φ.[4] If φ is an atomic formula, it will be an equation or an inequation. If it is in $t_1 \preccurlyeq t_2$ or $t_1 \approx t_2$ and variables in terms t_1 and t_2 are in $\{x_1,\dots,x_m\}$, then we have that $\mathcal{A} \nvDash \varphi$ is equivalent to the fact that there are elements $a_1,\dots,a_m$ such that

$$t_1^{\mathcal{A}}(a_1,\dots,a_m) > t_2^{\mathcal{A}}(a_1,\dots,a_m) \text{ or } t_1^{\mathcal{A}}(a_1,\dots,a_m) \neq t_2^{\mathcal{A}}(a_1,\dots,a_m),$$

which, according to Lemma 6.41, this is equivalent to

$$\mu(t_1)^{\mathcal{A}'}(a_1,\dots,a_m) > \mu(t_2)^{\mathcal{A}'}(a_1,\dots,a_m) \text{ or } \mu(t_1)^{\mathcal{A}'}(a_1,\dots,a_m) \neq \mu(t_2)^{\mathcal{A}'}(a_1,\dots,a_m),$$

that is, $\mathcal{A}' \nvDash \varphi$. The remaining proof is a simple and routine task. □

Theorem 6.43 (Law of Mirror Images)**.** *A formula is valid in the class of the residuated groupoids (residuated monoids, residuated lattices, $\mathbb{FL}$-algebras) if and only if it is its mirror image.*

Proof: Due to Lemma 6.42. □

Corollary 6.44. *A quasi-inequation (inequation, quasi-equation, equation) is valid in the class of the residuated groupoids (residuated monoids, residuated lattices, $\mathbb{FL}$-algebras) if and only if it is its mirror image.*

Observation 6.45. Note that every subclass of the considered classes is defined by a set of formulas and their mirror images satisfy the Law of Mirror Images.

Properties of residuated monoids. In the following proposition, we give some properties of residuated monoids which are easy to prove.

Proposition 6.46. *In all residuated monoids, the following inequations and equations (and their mirror images) are satisfied:*

[4] *Complexity of a first order formula* means the number of occurrences of the boolean operators and the quantifiers.

a) $x * (x\backslash y) \preccurlyeq y$,

b) $1 \preccurlyeq x\backslash x$,

c) $(x\backslash y) * z \preccurlyeq x\backslash(y * z)$,

d) $x\backslash y \preccurlyeq (z * x)\backslash(z * y)$,

e) $(x\backslash y) * (y\backslash z) \preccurlyeq x\backslash z$,

f) $(x * y)\backslash z \approx y\backslash(x\backslash z)$,

g) $x\backslash(y/z) \approx (x\backslash y)/z$,

h) $(x\backslash 1) * y \preccurlyeq x\backslash y$.

i) $x * (x\backslash x) \approx x$,

j) $(x\backslash x) * (x\backslash x) \approx x\backslash x$.

In the following proposition we give some properties of residuated groupoids (and, therefore, of residuated monoids) with a minimum element.

Proposition 6.47. *If a residuated groupoid $\mathcal{A}$ has a minimum element $\perp$, then the element $\perp\backslash\perp$ ($\perp/\perp$) is the maximum element of A. Furthermore, for every $a \in A$, we have*

$$i)\quad a * \perp = \perp = \perp * a\,, \qquad ii)\quad \perp\backslash a = \top = a\backslash\top, \qquad iii)\quad a/\perp = \top = \top/a\,,$$

where we denote by $\top$ the element $\perp\backslash\perp = \perp/\perp$.

Proof: Let $a \in A$. As $\perp$ is the minimum element, we have $\perp \leq a\backslash\perp$ and, by applying (LR), this is equivalent to $a * \perp \leq \perp$ which in turn is equal to $a \leq \perp/\perp$. On the other hand, $a \leq \perp\backslash\perp$ is obtained as mirror image. So, $\top := \perp\backslash\perp = \perp/\perp$ is the maximum element of A.

i) Given that $a \leq \perp\backslash\perp$ we have $a * \perp \leq \perp$ and, as $\perp$ is the minimum, $a * \perp = \perp$. According to the mirror image principle, $\perp * a = \perp$ is obtained.

ii) As $\perp * (\perp\backslash\perp) \leq \perp$, we have $\perp * \top \leq \perp$ and, therefore, for every $a \in A$, $\perp * \top \leq a$ that is equal to $\top \leq \perp\backslash a$. Thus, $\perp\backslash a = \top$. Based on the fact that $\top$ is the maximum we have $a * \top \leq \top$, which is equivalent to $\top \leq a\backslash\top$. Therefore, $\top = a\backslash\top$.

iii) The referred equalities are mirror images of the equalities in *iii*). □

The class of the integral residuated monoids, that is, of the residuated monoids where the unit element of the monoid is the maximum element with respect to the order, is definitionally equivalent to the class formed by all its algebraic reducts as is seen in the following proposition.

Proposition 6.48. *If $\mathcal{A}$ is an integral residuated monoid, then the following conditions are equivalent:*

i) $\mathcal{A} \models x \preccurlyeq y$,

ii) $\mathcal{A} \models x \backslash y \approx 1$,

iii) $\mathcal{A} \models y / x \approx 1$.

Proof: Pursuant to the mirror image principle, the proof of equivalence of the two first items will be enough. Let $a, b \in A$ be such that $a \leq b$. We have $a * 1 \leq b$ and this, according to the law of residuation, is equivalent to $1 \leq a \backslash b$ but due to integrality, this is equal to $1 = a \backslash b$. $\square$

So then, the class of integral residuated monoid can be defined as a class of algebras $\mathbf{A} = \langle A, *, \backslash, /, 1 \rangle$ of type $\langle 2, 2, 2, 0 \rangle$. Obviously, it is a quasivariety. The class of commutative integral residuated monoids is known in the literature by the acronym $\mathbb{POCRIM}$ (*partially-ordered commutative residuated integral monoids*). It is a quasivariety not being a variety (see [Hig84]), since it is not a class closed by homomorphic images. As a consequence of this fact we have that the quasivariety corresponding to the non commutative case will not be a variety either.

Properties of residuated lattices. In the following proposition we give some properties for the residuated lattices which are a consequence of (i) from Proposition 6.34.

Proposition 6.49. *In every residuated lattice* $\mathbf{A}$ *the following equations are satisfied:*

1. $(x \vee y) \backslash z \approx (x \backslash z) \wedge (y \backslash z)$,
2. $z / (x \vee y) \approx (z / x) \wedge (z / y)$,
3. $z \backslash (x \wedge y) \approx (z \backslash x) \wedge (z \backslash y)$,
4. $(x \wedge y) / z \approx (x / z) \wedge (y / z)$.

The class of residuated lattices is a variety. Below we give an equational base (see [JT02]).

Theorem 6.50 (Equational presentation of $\mathbb{RL}$). *$\mathbb{RL}$ is the equational class of algebras* $\mathbf{A} = \langle A, \vee, \wedge, *, \backslash, /, 1 \rangle$ *of type* $\langle 2, 2, 2, 2, 2, 0 \rangle$ *that satisfies:*[5]

1. *Any set of equations defining the class of lattices,*
2. *Any set of equations defining the class of the monoids with an identity element 1,*
3. *(r)* $x * ((x \backslash z) \wedge y) \preccurlyeq z$; *(l)* $((z / x) \wedge y) * x \preccurlyeq z$,
4. *(r)* $y \preccurlyeq x \backslash ((x * y) \vee z)$; *(l)* $y \preccurlyeq ((y * x) \vee z) / x$.

[5] Recall that we use the inequation $t_1 \preccurlyeq t_2$ as an abbreviation for the equation $t_1 \vee t_2 \approx t_2$.

The varieties $\mathbb{FL}_\sigma$. In this work we use the presentation of the class of the $\mathbb{FL}$-algebras in the language $\langle \vee, \wedge, *, \backslash, /, ^{\backprime}, {}', 0, 1\rangle$ including the right and left negation operations as primitives connectives. The reason for this is that in Chapter 9 we will study some fragments without implication and with negation of the logic systems in question.

For future reference we now give an equational presentation of the class $\mathbb{FL}$ in language $\langle \vee, \wedge, *, \backslash, /, ^{\backprime}, {}', 0, 1\rangle$.

Theorem 6.51 (Equational presentation of $\mathbb{FL}$). *$\mathbb{FL}$ is the equational class of the algebras $\mathbf{A} = \langle A, \vee, \wedge, *, \backslash, /, ^{\backprime}, {}', 0, 1\rangle$ of type $\langle 2, 2, 2, 2, 2, 1, 1, 0, 0\rangle$ that satisfies:*

1. *Any set of equations defining the lattices,*
2. *Any set of equations defining the monoids with an identity element 1,*
3. *(r) $x * ((x\backslash z) \wedge y) \preccurlyeq z$; (l) $((z/x) \wedge y) * x \preccurlyeq z$,*
4. *(r) $y \preccurlyeq x\backslash((x * y) \vee z)$; (l) $y \preccurlyeq ((y * x) \vee z)/x$,*
5. *(r) $x^{\backprime} \approx x\backslash 0$; (l) ${}'x \approx 0/x$.*

Much as we did with pointed monoids, we will define classes $\mathbb{FL}_\sigma$, that is, subclasses defined by the properties $(e \preccurlyeq)$, $(w \preccurlyeq)$, $(\preccurlyeq w)$ and $(c \preccurlyeq)$. We observe that the $\langle \vee, *, 0, 1\rangle$-reduct of a $\mathbb{FL}$-algebra is a $\mathring{\mathbb{M}}^{s\ell}$-algebra, since residuated lattices satisfy the distributivity of the operation $*$ with respect to operation $\vee$.

Definition 6.52 ($\mathbb{FL}_\sigma$-algebra). *A $\mathbb{FL}_\sigma$-algebra is a $\mathbb{FL}$-algebra such that its $\langle \vee, *, 0, 1\rangle$-reduct is a $\mathring{\mathbb{M}}^{s\ell}_\sigma$-algebra.*

Note that in the $\mathbb{FL}_e$-algebras, the equations $x\backslash y \approx y/x$ and $x^{\backprime} \approx {}'x$ are satisfied on account of the commutativity of the monoidal operation. For this reason the class $\mathbb{FL}_e$ is presented having only one residual and one negation that, in accordance with its logic interpretation, are denoted by $\rightarrow$ and $\neg$, respectively. For future reference we give a equational presentation of the class $\mathbb{FL}_e$ below.

Theorem 6.53 (Equational presentation of $\mathbb{FL}_e$). *$\mathbb{FL}_e$ is the equational class of algebras $\mathbf{A} = \langle A, \vee, \wedge, *, \rightarrow, \neg, 0, 1\rangle$ of type $\langle 2, 2, 2, 2, 1, 0, 0\rangle$ that satisfies:*

1. *Any set of equations defining the lattices,*
2. *Any set of equations defining the monoids with an identity element 1,*
3. *$x * ((x \rightarrow z) \wedge y) \preccurlyeq z$,*
4. *$y \preccurlyeq x \rightarrow ((x * y) \vee z)$,*
5. *$\neg x \approx x \rightarrow 0$.*

Obviously, all the classes $\mathbb{FL}_\sigma$ are varieties. For the equational presentation of a class $\mathbb{FL}_\sigma$ we will add to the equations defining $\mathbb{FL}$ or $\mathbb{FL}_e$ the following equations:

- $x \preccurlyeq 1$ or $x * y \preccurlyeq x$ if $w_l \leq \sigma$,
- $0 \preccurlyeq x$ if $w_r \leq \sigma$,
- $x \preccurlyeq x * x$ if $c \leq \sigma$.

Note 6.54. Class $\mathbb{FL}_{ewc}$ is definitionally equivalent to the class of Heyting algebras, the semantic counterpart of the intuitionistic logic. For more details about Heyting algebras, refer to [BD74].

Chapter 7

Adding Negation Operators: Pseudocomplemented Structures

In this chapter we present the notion of *pseudocomplementation* in the framework of the *pointed grupoids* and we define the class $\mathbb{PM}^{\preccurlyeq}$ of the pseudocomplemented po-monoids and the classes $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^{\ell}$ of the semilatticed and latticed pseudocomplemented monoids. The notion of pseudocomplement with respect to the monoidal operation can be seen as a generalization of the same notion defined in the framework of the pseudocomplemented distributive lattices (see [BD74, Lak73]). We show that the classes $\mathbb{PM}^{\preccurlyeq}$ can be defined by means a set of inequations and thus the classes $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^{\ell}$ are varieties (Sections 7.2 and 7.3). Section 7.4 analyzes the case when the pseudocomplementation is with respect to the minimum element of the monoid. In Section 7.5 the classes of weakly contractive pseudocomplemented monoids are presented and characterized. Finally, in Section 7.6 we analyze the involutive peseudocomplemented monoids.

The pseudocomplements constitute the algebraic counterpart of negations: in Chapter 9 we will state the connection between the varieties $\mathbb{PM}^{s\ell}_{\sigma}$ and $\mathbb{PM}^{\ell}_{\sigma}$ (subvarieties of $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^{\ell}$ defined by the equations codified by σ) with the fragments of the Gentzen system $\mathcal{FL}_{\sigma}$ and the associated external deductive system $\mathfrak{e}\mathcal{FL}_{\sigma}$ in the languages $\langle \vee, *, ^{\backprime}, ', 0, 1\rangle$ and $\langle \vee, \wedge, *, ^{\backprime}, ', 0, 1\rangle$.

7.1 The operations of pseudocomplementation

In the following we introduce the operations of left and right pseudocomplementation in the general context of the pointed po-grupoids.

Proposition 7.1. *Let $\mathcal{A} = \langle A, *, 0, \leq\rangle$ be a pointed po-grupoid. The following conditions are equivalent:*

i) For every $a \in A$, there exist the right and left relative pseudocomplements of a with respect to 0.

ii) There exist two unary operations ${}^{\backprime}$ *and* ${}'$ *defined on* A *such that, for every* $a, b \in A$,

$$a * b \leq 0 \quad \textit{iff} \quad b \leq a^{\backprime} \quad \textit{iff} \quad a \leq {}'b. \tag{LP}$$

Given these conditions, for each $a \in A$, $a^{\backprime}$ *is the right relative pseudocomplement of* a *with respect to* 0 *and* ${}'a$ *is the left relative pseudocomplement of* a *with respect to* 0. *Hence, there are exactly two operations* ${}^{\backprime}$ *and* ${}'$, *which we will call* right pseudocomplement *and* left pseudocomplement of the operation $*$, *satisfying condition* (LP). *This condition will be called* law of pseudocomplementation.

Proof: $i) \Rightarrow ii)$: For each $c \in A$, let us consider the sets

$$\mathrm{R}^0(c) = \{x \in A : c * x \leq 0\} \quad \text{and} \quad \mathrm{L}^0(c) = \{x \in A : x * c \leq 0\}.$$

By $i)$ we have that these sets both have minimum element. Then, for each $c \in A$, let us define $c^{\backprime} := \max \mathrm{R}^0(c)$ and ${}'c := \max \mathrm{L}^0(c)$. Let $a, b \in A$ and suppose $a * b \leq 0$. Then, obviously, $b \leq a^{\backprime}$ and $a \leq {}'b$. On the other hand, if $b \leq a^{\backprime}$, by left monotonicity we have $a * b \leq a * a^{\backprime}$ but, since $a^{\backprime} \in \mathrm{R}^0(a)$, we have $a * b \leq 0$; if $a \leq {}'b$, by right monotonicity, we have $a * b \leq {}'b * b$ and, since ${}'b \in \mathrm{L}^0(b)$, we obtain $a * b \leq 0$.

$ii) \Rightarrow i)$: Let $a \in A$. We want to see that $a^{\backprime}$ is the maximum element of $\mathrm{R}^0(a)$. Observe that, by $ii)$, $a^{\backprime} \leq a^{\backprime}$ is equivalent to $a * a^{\backprime} \leq 0$ and so $a \in \mathrm{R}^0(a)$. Suppose now that $c \in \mathrm{R}^0(a)$: then $a * c \leq 0$ which, by $ii)$, is equivalent to $c \leq a^{\backprime}$. Analogously we can prove that ${}'a$ is the maximum element of $\mathrm{L}^0(a)$. □

Note 7.2. If the operation $*$ is commutative then both pseudocomplements coincide and the corresponding operation receives the name of *pseudocomplement* of $*$. In this case we will use the symbol $\neg$ and we will write $\neg a$ instead of $a^{\backprime}$ or ${}'a$.

Definition 7.3 (Pseudocomplemented po-grupoids)**.** *A* pseudocomplemented po-grupoid *is an order-algebra* $\mathcal{A} = \langle A, *, {}^{\backprime}, {}', 0, \leq\rangle$ *with algebraic type* $\langle 2, 1, 1, 0\rangle$ *such that* $\langle A, *, 0, \leq\rangle$ *is a pointed po-grupoid and the operations* ${}^{\backprime}$ *and* ${}'$ *are the left and right pseudocomplements of the operation* $*$, *respectively. We refer by* pseudocomplemented *to every structure having as reduct a pseudocomplemented po-grupoid.*

Observation 7.4. Note that every $\mathbb{FL}$-algebra is a pseudocomplemented structure since its $\langle \leq, *, {}^{\backprime}, {}', 0, 1\rangle$-reduct, where $\leq$ is the order associated to the lattice, is a pseudocomplemented po-grupoid.

The following result is a reformulation of Corollary 6.29.

Proposition 7.5. *Let* $\mathcal{A}$ *be a po-grupoid. The following conditions are equivalent:*

i) For each $b \in A$, *there are two unary operations* ${}^{\backprime\mathcal{A}_b}$ *and* ${}'^{\mathcal{A}_b}$ *definid on* A *such that the structure* $\mathcal{A}_b = \langle A, *, {}^{\backprime\mathcal{A}_b}, {}'^{\mathcal{A}_b}, b, \leq\rangle$ *is a pseudocomplemented po-grupoid.*

ii) The operation $*$ *is residuated.*

Under these conditions, for each $a, b \in A$, the right residual of a relative to b is the right relative pseudocomplement of a with respect to b and the left residual of a relative to b is the left relative pseudocomplement of a with respect to b.

Proof: $i) \Rightarrow ii)$: Let us define on A two binary operations $\backslash$ and $/$ in the following way: for each $a, b \in A$, $a\backslash b := a^{\backprime_{\mathcal{A}_b}}$ and $b/a := {}^{\prime_{\mathcal{A}_b}}a$. Since, for each $b \in A$, the operations $*$, ${}^{\backprime_{\mathcal{A}_b}}$ i ${}^{\prime_{\mathcal{A}_b}}$ satisfy the condition (LP), we have that, for each $a, b \in A$ the operations $*$, $\backslash$ and $/$ satisfy the condition (LR). Hence, $*$ is residuated.

$ii) \Rightarrow i)$: Let $\backslash$ and $/$ the left and right residuum of the operation $*$ and let $b \in A$. We define on A two binary operations ${}^\backprime$ and ${}^\prime$ in the following way: for each $a \in A$, $a^\backprime := a\backslash b$ and ${}'a := b/a$. Then, by (LR) we have that the operations $*$, ${}^\backprime$ and ${}^\prime$ satisfy the condition (LP). Thus, $\mathcal{A}_b = \langle A, *, {}^\backprime, {}^\prime, b, \leq\rangle$ is a pseudocomplemented po-grupoid.

□

Definition 7.6 (Pseudocomplemented po-monoid)**.** *A pseudocomplemented po-monoid is an order-algebra $\mathcal{A} = \langle A, *, {}^\backprime, {}^\prime, 0, 1, \leq\rangle$ such that $\langle A, *, 1\rangle$ is a monoid and $\langle A, *, {}^\backprime, {}^\prime, 0, \leq\rangle$ is a pseudocomplemented po-groupoid.*

Definition 7.7 (Pseudocomplemented $s\ell$-monoid)**.** *A pseudocomplemented $s\ell$-monoid is an algebra $\mathbf{A} = \langle A, \vee, *, {}^\backprime, {}^\prime, 0, 1\rangle$ such that $\langle A, \vee, *, 1\rangle$ is a $s\ell$-monoid and $\langle A, *, {}^\backprime, {}^\prime, 0, \leq\rangle$, where $\leq$ is the semilattice order, is a pseudocomplemented po-groupoid.*

Definition 7.8 (Pseudocomplemented ℓ-monoid)**.** *A pseudocomplemented ℓ-monoid is an algebra $\mathbf{A} = \langle A, \vee, \wedge, *, {}^\backprime, {}^\prime, 0, 1\rangle$ such that $\langle A, \vee, \wedge, *, 1\rangle$ is a ℓ-monoid and $\langle A, *, {}^\backprime, {}^\prime, 0, \leq\rangle$, where $\leq$ is the order of the lattice, is a pseudocomplemented po-groupoid.*

Definition 7.9 (The classes $\mathbb{PM}^{\preccurlyeq}_\sigma$, $\mathbb{PM}^{s\ell}_\sigma$, $\mathbb{PM}^{\ell}_\sigma$)**.** *We will denote by $\mathbb{PM}^{\preccurlyeq}$, $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^{\ell}$, the classes of the pseudocomplemented po-monoids, of the pseudocomplemented $s\ell$-monoids and of the pseudocomplemented ℓ-monoids, respectively. Let $\lambda \in \{\preccurlyeq, s\ell, \ell\}$ and let σ be a subsequence, possibly empty, of ew_lw_rc. We define the classes $\mathbb{PM}^{\lambda}_\sigma$ as the substructures of $\mathbb{PM}^{\lambda}$ satisfying the properties in the set $\{(e \preccurlyeq), (w \preccurlyeq), (\preccurlyeq w), (c \preccurlyeq)\}$ codified by the sequence σ and, if σ is the empty sequence, then $\mathbb{PM}^{\lambda}_\sigma$ is $\mathbb{PM}^{\lambda}$. The members of $\mathbb{PM}^{s\ell}_\sigma$ ($\mathbb{PM}^{\ell}_\sigma$) are called $\mathbb{PM}^{s\ell}$-algebras ($\mathbb{PM}^{\ell}$-algebras).*

Definition 7.10 (Mirror image)**.** *Let t be a term of an algebraic language $\mathcal{L}$ such that $\langle *, {}^\backprime, {}^\prime\rangle \leq \mathcal{L} \leq \langle \vee, \wedge, *, {}^\backprime, {}^\prime, 0, 1\rangle$. We define its* mirror image *$\mu(t)$ inductively on the complexity of t:*

$$\mu(t) := \begin{cases} t, & \text{if } t \in Var \text{ or } t \in \{0, 1\}, \\ \mu(u) \vee \mu(v), & \text{if } t = u \vee v, \\ \mu(u) \wedge \mu(v), & \text{if } t = u \wedge v, \\ \mu(v) * \mu(u), & \text{if } t = u * v, \\ \mu(u)^\backprime, & \text{if } t = u^\backprime, \\ {}'\mu(u), & \text{if } t = {}'u. \end{cases}$$

We define the mirror image of a formula *of the first order language with equality $\mathcal{L}^{\preccurlyeq} = \langle \mathcal{L}, \preccurlyeq \rangle$ as the formula obtained by substituting all the terms occurring in the formula by their mirror images.*

Lemma 7.11. *We will denote by $\mathbb{K}$ the class of the pseudocomplemented po-grupoids (po-monoids, sℓ-monoids, ℓ-monoids). Let us denote by $\mathcal{L}_{\mathbb{K}}$ its algebraic language and by $\mathcal{L}_{\mathbb{K}}^{\preccurlyeq}$ the first order language with equality $\langle \mathcal{L}_{\mathbb{K}}, \preccurlyeq \rangle$. Let $\mathcal{A} \in \mathbb{K}$, and let $\mathcal{A}^{\mathbf{o}}$ be the $\mathcal{L}_{\mathbb{K}}^{\preccurlyeq}$-structure defined in the following way:*

- *i) the universe and the order of $\mathcal{A}^{\mathbf{o}}$ are as in $\mathcal{A}$,*
- *ii) the operations and constants of $\mathcal{A}^{\mathbf{o}}$ belonging to $\{\vee, \wedge, 0, 1\}$ are the same than in $\mathcal{A}$, and*
- *iii) the rest of operations are defined in the following way: for each $a, b \in A$,*

$$a *^{\mathcal{A}^{\mathbf{o}}} b := b *^{\mathcal{A}} a, \quad a^{\backslash \mathcal{A}^{\mathbf{o}}} := {}^{/\mathcal{A}}a, \quad {}^{/\mathcal{A}^{\mathbf{o}}} a := a^{\backslash \mathcal{A}}.$$

Then,

- *i) $\mathcal{A}^{\mathbf{o}}$, which will be called the* opposite *of $\mathcal{A}$, belongs to $\mathbb{K}$,*
- *ii) for every term t of $\mathcal{L}_{\mathbb{K}}$, holds that $\mu(t)^{\mathcal{A}^{\mathbf{o}}} = t^{\mathcal{A}}$.*

Proof: $i)$: It is easy to see that the operations $*^{\mathcal{A}^{\mathbf{o}}}$, ${}^{\backslash \mathcal{A}^{\mathbf{o}}}$ and ${}^{/\mathcal{A}^{\mathbf{o}}}$ satisfy (LP).

$ii)$: By induction on the complexity of t. □

Lemma 7.12. *Let $\mathcal{A}$ be a pseudocomplemented po-grupoid (po-monoid, sℓ-monoid, ℓ-monoid) and let $\mathcal{A}^{\mathbf{o}}$ be its opposite. Then, for every first order formula φ of the language of the pseudocomplemented po-grupoids (po-monoids, sℓ-monoids, ℓ-monoids), the following holds:*

$$\mathcal{A} \vDash \varphi \quad \textit{if, and only if,} \quad \mathcal{A}^{\mathbf{o}} \vDash \mu(\varphi).$$

Proof: Let $\mathbb{K}$ and $\mathcal{L}_{\mathbb{K}}^{\preccurlyeq}$ be as in Lemma 7.11. Given $\mathcal{A} \in \mathbb{K}$ and a $\mathcal{L}_{\mathbb{K}}^{\preccurlyeq}$-formula φ, it is easy to see by induction on the complexity of φ that $\mathcal{A} \nvDash \varphi$ if, and only if, $\mathcal{A}^{\mathbf{o}} \nvDash \mu(\varphi)$.

□

Theorem 7.13 (Law of Mirror Images)**.** *A formula is valid in the class of the pseudocomplemented po-grupoids (po-monoids, sℓ-monoids, ℓ-monoids) if, and only if, its mirror image is also valid in the class.*

Proof: By Lemma 7.12. □

Corollary 7.14. *A quasi-inequation is valid in the class of the pseudocomplemented po-grupoids (po-monoids, sℓ-monoids, ℓ-monoids) if, and only if, its mirror image is valid in the class.*

Observation 7.15. Observe that every subclass of the considerate classes defined by a set of formulas and their mirror images satisfies the law of mirror images.

Proposition 7.16. *In every pseudocomplemented po-grupoid $\mathcal{A}$ the following inequations and quasi-inequations are satisfied:*

i)	$x * x^{\backprime} \preccurlyeq 0$,	*i′)*	${}'x * x \preccurlyeq 0$,
ii)	$x \preccurlyeq y \supset x * y^{\backprime} \preccurlyeq 0$,	*ii′)*	$x \preccurlyeq y \supset {}'y * x \preccurlyeq 0$,
iii)	$x \preccurlyeq y \supset y^{\backprime} \preccurlyeq x^{\backprime}$,	*iii′)*	$x \preccurlyeq y \supset {}'y \preccurlyeq {}'x$,
iv)	$x \preccurlyeq {}'(x^{\backprime})$,	*iv′)*	$x \preccurlyeq ({}'x)^{\backprime}$,
v)	$x^{\backprime} \approx ({}'(x^{\backprime}))^{\backprime}$,	*v′)*	${}'x \approx {}'(({}'x)^{\backprime})$.

Proof: By the law of mirror images, it will be sufficient to prove one of the two inequations or quasi-inequations in each file. Let $a, b \in A$.

i) By reflexivity we have $a^{\backprime} \leq a^{\backprime}$ and, by (LP), this is equivalent to $a * a^{\backprime} \leq 0$.

ii) Suppose $a \leq b$. By monotonicity and by *i)* we have $a * b^{\backprime} \leq b * b^{\backprime} \leq 0$.

iii) By *ii*), if $a \leq b$, then $a * b^{\backprime} \leq 0$ and by applying (LP), we obtain $b^{\backprime} \leq a^{\backprime}$.

iv) From $a * a^{\backprime} \leq 0$ we obtain $a \leq {}'(a^{\backprime})$ by applying (LP).

v) By *iv*) we have $a \leq {}'(a^{\backprime})$; from this by *iii*) we obtain $({}'(a^{\backprime}))^{\backprime} \leq a^{\backprime}$, which by *iv′*) gives $a^{\backprime} \leq ({}'(a^{\backprime}))^{\backprime}$. Hence, $a^{\backprime} = ({}'(a^{\backprime}))^{\backprime}$. □

Thus, by *iii*) and *iii′*) of the last proposition we have that the left and right pseudocomplements are antimonotonous operations.

Proposition 7.17. *Let $\mathcal{A} = \langle A, *, {}^{\backprime}, {}', 0, \leq \rangle$ be a pseudocomplemented po-grupoid and let $\{a_i : i \in I\}$ be a family of elements in A. If $\bigvee_{i\in I} a_i$ exists, then there exist $\bigwedge_{i\in I} a_i{}^{\backprime}$ and $\bigwedge_{i\in I} {}'a_i$ and the following holds:*

$$(\bigvee_{i\in I} a_i)^{\backprime} = \bigwedge_{i\in I} a_i{}^{\backprime} \ ; \ {}'(\bigvee_{i\in I} a_i) = \bigwedge_{i\in I} {}'a_i.$$

Proof: Since $a_i \leq \bigvee_{i\in I} a_i$, by the antimonotonicity of the right pseudocomplement, we obtain $(\bigvee_{i\in I} a_i)^{\backprime} \leq a_i{}^{\backprime}$. Therefore, $(\bigvee_{i\in I} a_i)^{\backprime}$ is a lower bound of the set $\{a_i{}^{\backprime} : i \in I\}$. Let $b \in A$ and suppose that b is also a lower bound of that set. By (LP), $b \leq a_i{}^{\backprime}$ is equivalent to $a_i * b \leq 0$, which is equivalent to $a_i \leq {}'b$. Therefore, $(\bigvee_{i\in I} a_i) \leq {}'b$, which is equivalent to $b \leq (\bigvee_{i\in I} a_i)^{\backprime}$. Consequently, $(\bigvee_{i\in I} a_i)^{\backprime}$ is the infimum of the set $\{a_i{}^{\backprime} : i \in I\}$. The other identity is analogously proved by using the antimonotonicity of the left pseudocomplement and (LP). □

7.2 Characterization of the class $\mathbb{PM}^{\preccurlyeq}$

In this section we present a set of inequalities which, together with the condition of antimonotonicity of the operations $^\backprime$ and $'$, characterize the law of pseudocomplementation (LP) in the class $\mathbb{PM}^{\preccurlyeq}$.

Theorem 7.18. *Let $\mathcal{A} = \langle A, *, ^\backprime, ', 0, 1, \leq\rangle$ be an order algebra with algebraic type $\langle 2, 1, 1, 0, 0\rangle$ and such that $\langle A, *, 1, \leq\rangle$ is a po-monoid. The following conditions are equivalent:*

i) $\mathcal{A}$ is a pseudocomplemented po-monoid.

ii) The operations $^\backprime$ and $'$ are antimonotonous and $\mathcal{A}$ is a model of the inequations

$$\begin{array}{llllll} r_1) & 1^\backprime \preccurlyeq 0 & r_2) & 1 \preccurlyeq 0^\backprime & r_3) & x * (y * x)^\backprime \preccurlyeq y^\backprime \\ l_1) & {}'1 \preccurlyeq 0 & l_2) & 1 \preccurlyeq {}'0 & l_3) & {}'(x * y) * x \preccurlyeq {}'y \end{array}$$

Proof: *i)* $\Rightarrow$ *ii)*: By Proposition 7.16, the operations $^\backprime$ and $'$ are antimonotonous. Now we will show that (r_1), (r_2) i (r_3) are satisfied. This will be sufficient since (l_1), (l_2) i (l_3) are their respective mirror images. We will use the fact that in every pseudocomplemented monoid $x * x^\backprime \preccurlyeq 0$ holds. We have that $1^\backprime \leq 1 * 1^\backprime \leq 0$. Thus, since $0 * 1 = 0 \leq 0$, by (LP) we obtain $1 \leq 0^\backprime$. Let $a, b \in A$. Then $b * (a * (b * a)^\backprime) \leq (b * a) * (b * a)^\backprime \leq 0$. Thus, by (LP) we can conclude $a * (b * a)^\backprime \leq b^\backprime$.

ii) $\Rightarrow$ *i)*: Let $a, b \in A$. If $a * b \leq 0$, then, by the antimonotonicity of $^\backprime$, we have $0^\backprime \leq (a * b)^\backprime$ and, therefore, $1 \leq (a * b)^\backprime$; now w have $b = b * 1 \leq b * (a * b)^\backprime \leq a^\backprime$. Suppose now $b \leq a^\backprime$. Then, $a * b \leq a * a^\backprime = a * (1 * a)^\backprime \leq 1^\backprime \leq 0$. To prove the equivalence between $a * b \leq 0$ and $a \leq {}'b$ we can proceed analogously. Therefore, $\mathcal{A}$ satisfies (LP). □

In the commutative case this result takes the following form.

Theorem 7.19. *Let $\mathcal{A} = \langle A, *, \neg, 0, 1, \leq\rangle$ be an order-algebra with algebraic type $\langle 2, 1, 1, 0, 0\rangle$ and such that $\langle A, *, 1, \leq\rangle$ is a* po-*monoid. The following conditions are equivalent:*

i) $\mathcal{A}$ is a pseudocomplemented po-monoid.

ii) The operation $\neg$ is antimonotonous and the following inequations are valid in $\mathcal{A}$:

$$p_1)\quad \neg 1 \preccurlyeq 0 \qquad p_2)\quad 1 \preccurlyeq \neg 0 \qquad p_3)\quad x * \neg(y * x) \preccurlyeq \neg y$$

As corollaries of the previous results we obtain the following characterizations for the classes $\mathbb{PM}^{\preccurlyeq}$ i $\mathbb{PM}_e^{\preccurlyeq}$.

Corollary 7.20. *An order-algebra $\mathcal{A} = \langle A, *, ^\backprime, ', 0, 1, \leq\rangle$ with algebraic type $\langle 2, 1, 1, 0, 0\rangle$ is a pseudocomplemented monoid if, and only if,*

1. *$\langle A, *, 1, \leq\rangle$ is a po-monoid.*

2. *The operations* $^{\backprime}$ *and* $'$ *are antimonotonous with respect to the partial order.*

3. $\mathcal{A}$ *satisfies the inequations:*

$$\begin{array}{llllll} r_1) & 1^{\backprime} \preccurlyeq 0 & r_2) & 1 \preccurlyeq 0^{\backprime} & r_3) & x * (y * x)^{\backprime} \preccurlyeq y^{\backprime} \\ l_1) & {}'1 \preccurlyeq 0 & l_2) & 1 \preccurlyeq {}'0 & l_3) & {}'(x * y) * x \preccurlyeq {}'y \end{array}$$

Corollary 7.21. *An order-algebra* $\mathcal{A} = \langle A, *, \neg, 0, 1, \leq\rangle$ *with algebraic type* $\langle 2, 1, 0, 0\rangle$ *is a commutative pseudocomplemented monoid if, and only if,*

1. $\langle A, *, 1, \leq\rangle$ *is a commutative po-monoide commutatiu.*

2. *The operation* $\neg$ *is antimonotonous with respect to the partial order.*

3. $\mathcal{A}$ *satisfies the inequations:*

$$p_1)\quad \neg 1 \preccurlyeq 0 \qquad p_2)\quad 1 \preccurlyeq \neg 0 \qquad p_3)\quad x * \neg(y * x) \preccurlyeq \neg y$$

In the following results we give some equations and inequations that are satisfied in the classes $\mathbb{PM}^{\preccurlyeq}$ and $\mathbb{PM}_e^{\preccurlyeq}$.

Proposition 7.22. *In every pseudocomplemented po-monooid* $\mathcal{A}$ *the following equations and inequations are satisfied:*

$$\begin{array}{llll} i) & 0 \approx 1^{\backprime} & i') & 0 \approx {}'1 \\ ii) & x \preccurlyeq {}'(y * (x * y)^{\backprime}) & ii') & x \preccurlyeq ({}'(y * x) * y)^{\backprime} \end{array}$$

Proof: It is sufficient to prove *i)*, *ii)*, and *iii)*, since *i')*, *ii')*, and *iii')*, are their respective mirror images.

i) We have $1 * 0 = 0 \leq 0$. Thus, by applying (LP), $0 \leq 1^{\backprime}$; this and (r_1) imply $0 \approx 1^{\backprime}$.

ii) If $a, b \in A$, since $(a * b) * (a * b)^{\backprime} \leq 0$, by associativity, $a * (b * (a * b)^{\backprime}) \leq 0$ and from this, by applying (LP), $a \leq {}'(b * (a * b)^{\backprime})$. □

In the commutative case this result takes the following form.

Proposition 7.23. *In every commutative pseudocomplemented po-monoid the following conditions are satisfied:*

$$i)\quad 0 \approx \neg 1 \qquad ii)\quad x \preccurlyeq \neg(y * \neg(x * y))$$

7.3 The classes $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^{\ell}$ are varieties

In this section, we characterize the classes $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^{\ell}$ as equational classes.

Lemma 7.24. *Let* $\mathbf{A} = \langle A, \vee\rangle$ *be a semilattice and let* ι *be an unary operation defined on* A. *The following conditions are equivalent:*

i) The operation ι *is antimonotonous with respect to the order of the semilattice.*

ii) $\mathbf{A} \models \iota(x \vee y) \vee \iota(x) \approx \iota(x)$.

Proof: Suppose $a, b \in A$ and let $\leq$ be the semilattice order.

i) $\Rightarrow$ *ii)*: Since the fact that $a \leq a \vee b$, by the antimonotonicity of the operation ι we obtain $\iota(a \vee b) \leq \iota(a)$, i.e., $\iota(a \vee b) \vee \iota(a) = \iota(a)$.

ii) $\Rightarrow$ *i)*: Suppose $a \leq b$. Then $a \vee b = b$ and so, by using *ii)*, we have $\iota(b) = \iota(a \vee b) \leq \iota(a)$. $\square$

Theorem 7.25. $\mathbb{PM}^{s\ell}$ *is the equational class of the algebras* $\mathbf{A} = \langle A, \vee, *, ^{\backprime}, ', 0, 1\rangle$ *of type* $\langle 2, 2, 1, 1, 0, 0\rangle$ *satisfying:*

1. *Any set of equations defining the class of* $s\ell$*-monoids.*
2. *The equations (which characterize the pseudocomplementation law):*

$$r_1)\ \ 1^{\backprime} \approx 0 \qquad r_2)\ \ 1 \vee 0^{\backprime} \approx 0^{\backprime} \qquad r_3)\ \ (x * (y * x)^{\backprime}) \vee y^{\backprime} \approx y^{\backprime}$$

$$l_1)\ \ {}'1 \approx 0 \qquad l_2)\ \ 1 \vee {}'0 \approx {}'0 \qquad l_3)\ \ ({}'(x * y) * x) \vee {}'y \approx {}'y$$

3. *The equations (which characterize the antimonotonicity of the pseudocomplements):*

$$r_a)\ \ (x \vee y)^{\backprime} \vee x^{\backprime} \approx x^{\backprime}$$

$$l_a)\ \ {}'(x \vee y) \vee {}'x \approx {}'x$$

Proof: By the definition, the fact that $\mathbf{A}$ is a pseudocomplemented $s\ell$-monoid is equivalent to the facts that $\langle A, \vee, *, 1\rangle$ is a $s\ell$-monoid and that, if $\leq$ is the order of the semilattice, then $\mathcal{A} = \langle A, *, ^{\backprime}, ', 0, 1, \leq\rangle$ is a pseudocomplemented po-monoid and, by the characterization of Theorem 7.18, this is equivalent to saying that the operations $^{\backprime}$ and $'$ are antimonotonous and that in $\mathcal{A}$ the inequations $(r_1), \ldots, (l_3)$ of the mentioned theorem are satisfied. Observe that by substituting, given two terms t_1 and t_2, the inequalities $t_1 \preccurlyeq t_2$ by the equations $t_1 \vee t_2 \approx t_2$ (bearing in mind that (r_1) and (l_1) they can be substituted by the equations $1^{\backprime} \approx 0$ and ${}'1 \approx 0$) we obtain the equations $(r_1), \ldots, (l_3)$ of the present theorem. Moreover, by Lemma 7.24, the antimonotonicity of the operations $^{\backprime}$ and $'$ is equivalent to the fact that the equations (r_a) and (l_a) are satisfied in $\mathbf{A}$. $\square$

Analogously, we have the following characterization of $\mathbb{PM}^{\ell}$ as equational class.

Theorem 7.26. $\mathbb{PM}^{\ell}$ *is the equational class of the algebras* $\mathbf{A} = \langle A, \vee, \wedge, *, ^{\backprime}, ', 0, 1\rangle$ *of type* $\langle 2, 2, 2, 1, 1, 0, 0\rangle$ *satisfying any set of equations defining the class of the* ℓ*-monoids and the equations* $(r_1), \ldots, (l_3)$*,* (r_a) *and* (l_a) *of Theorem 7.25.*

Corollary 7.27. *The classes* $\mathbb{PM}^{s\ell}_{\sigma}$ *and* $\mathbb{PM}^{\ell}_{\sigma}$*, with* $\sigma \leq ew_lw_rc$*, are varieties.*

Proof: Any class $\mathbb{PM}^{s\ell}_\sigma$ ($\mathbb{PM}^{\ell}_\sigma$) is obtained by adding some of the equations

$$x * y \approx y * x,\ x \vee 1 \approx 1,\ 0 \vee x \approx x,\ x \vee (x * x) \approx x * x,$$

to the set of equations characterizing the class $\mathbb{PM}^{s\ell}$ ($\mathbb{PM}^{\ell}$). □

Next, by adapting the notation, we give an equational characterization to the commutative classes $\mathbb{PM}^{s\ell}_e$ and $\mathbb{PM}^{\ell}_e$.

Theorem 7.28. *$\mathbb{PM}^{s\ell}_e$ is the equational class of the algebras $\mathbf{A} = \langle A, \vee, *, \neg, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ satisfying*

1. *Any set of equations defining the class of the commutative $s\ell$-monoids,*
2. *p_1) $\neg 1 \approx 0$, p_2) $1 \vee \neg 0 \approx \neg 0$, p_3) $(x * \neg(y * x)) \vee \neg y \approx \neg y$, a) $\neg(x \vee y) \vee \neg x \approx \neg x$.*

Theorem 7.29. *$\mathbb{PM}^{\ell}_e$ is the equational class of the algebras $\mathbf{A} = \langle A, \vee, \wedge, *, \neg, 0, 1\rangle$ of type $\langle 2, 2, 2, 1, 0, 0\rangle$ satisfying any set of equations defining the commutative ℓ-monoids and the equations (p_1), (p_2), (p_3) and (a) of Theorem 7.28.*

7.4 Pseudocomplementation with respect to the minimum

In this section we show that in the framework of pseudocomplemented po-monoids, when the distinguished element 0 is the minimum with respect to the partial order, the law of pseudocomplementation is equivalent to the inequalities (r_1), (r_2), (r_3), (l_1), (l_2) and (l_3) without the need for adding the condition of antimonotonicity of the pseudocomplements to these inequalities. We also give an alternative set of inequalities which, together with the antimonotonicity of the pseudocomplements, characterize (LP) in this class of pseudocomplemented po-monoids. We also analyze the case in which the structures of this kind are integral.

Theorem 7.30. *Let $\mathcal{A} = \langle A, *, ^{\backprime}, {}', 0, 1, \leq\rangle$ be an order algebra with algebraic type $\langle 2, 1, 1, 0, 0\rangle$ such that $\langle A, *, 1, \leq\rangle$ is a* po*-monoid. Let us suppose that* 0 *is the minimum element with respect to the partial order. Then, the following conditions are equivalent:*

i) $\mathcal{A}$ is a pseudocomplemented po-monoid.

ii) $\mathcal{A}$ satisfies the inequations: [1]

$$\begin{array}{llllll} r_1) & 1^{\backprime} \preccurlyeq 0 & r_2) & 1 \preccurlyeq 0^{\backprime} & r_3) & x * (y * x)^{\backprime} \preccurlyeq y^{\backprime} \\ l_1) & {}'1 \preccurlyeq 0 & l_2) & 1 \preccurlyeq {}'0 & l_3) & {}'(x * y) * x \preccurlyeq {}'y \end{array}$$

iii) The operations $^{\backprime}$ and $'$ are antimonotonous and $\mathcal{A}$ satisfies the inequations:

$$\begin{array}{llll} r_4) & x * x^{\backprime} \preccurlyeq 0 & r_5) & x \preccurlyeq {}'(y * (x * y)^{\backprime}) \\ l_4) & {}'x * x \preccurlyeq 0 & l_5) & x \preccurlyeq ({}'(y * x) * y)^{\backprime} \end{array}$$

[1] The inequations r_1) and r_2) can be substituted by the equations $1^{\backprime} \approx 0$ and $'1 \approx 0$.

Proof: $i) \Leftrightarrow ii)$: As we have seen in Theorem 7.18, in every pseudocomplemented po-monoid the inequalities $(r_1), \ldots, (l_3)$ are satisfied and, reciprocally, their validity and the antimonotonicity of the operations ${}^\backprime$ and ${}'$ allow us to prove (LP). Therefore, to state this equivalence it is sufficient to prove that when $(r_1), \ldots, (l_3)$ hold in $\mathcal{A}$, the operations ${}^\backprime$ and ${}'$ are antimonotonous.

First observe that the validity of (r_1) and (r_3) in $\mathcal{A}$ allows us to prove that $x * x^\backprime \preccurlyeq 0$ is also valid in $\mathcal{A}$: if $a \in A$, then $a * a^\backprime = a * (1 * a)^\backprime \leq 1^\backprime \leq 0$. Suppose $a, b \in A$ and $a \leq b$. We have $a * b^\backprime \leq b * b^\backprime \leq 0$ and, since 0 is the minimum, $a * b^\backprime = 0$. Now, by using (r_2) and (r_3), we obtain $b^\backprime = b^\backprime * 1 \leq b^\backprime * 0^\backprime = b^\backprime * (a * b^\backprime)^\backprime \leq a^\backprime$. We can now proceed analogously to see that $a \leq b$ implies ${}'b \leq {}'a$, by using (l_1), (l_2) and (l_3).

$i) \Rightarrow iii)$: In every pseudocomplemented po-monoid the operations ${}^\backprime$ and ${}'$ are antimonotonous and the inequations (r_4), (l_4) (see Proposition 7.16), (r_5) and (l_5) (see Proposition 7.22) hold.

$iii) \Rightarrow ii)$: By using (r_4) we obtain $1^\backprime = 1 * 1^\backprime \leq 0$. Since 0 is the minimum we have $0 \leq {}'0 * 0 = {}'(1 * 0) * 0$ and, by monotonicity, $({}'(1 * 0) * 0)^\backprime \leq 0^\backprime$. Now, by (l_5), we obtain $1 \leq ({}'(1 * 0) * 0)^\backprime$ and thus $1 \leq 0^\backprime$. By using (l_5) we prove that $x \preccurlyeq ({}'x)^\backprime$ holds in $\mathcal{A}$: if $a \in A$, then $a \leq ({}'(1 * a) * 1)^\backprime = ({}'a * 1)^\backprime = ({}'a)^\backprime$. Finally, let $a, b \in A$. By (l_4) we obtain $b \leq {}'(a * (b * a)^\backprime)$, and by applying monotonicity we have $({}'(a * (b * a)^\backprime))^\backprime \leq b^\backprime$. But $a * (b * a)^\backprime \leq ({}'(a * (b * a)^\backprime))^\backprime$. Therefore, $a * (b * a)^\backprime \leq b^\backprime$. The inequalities (l_1), (l_2) i (l_3) are proved in an analogous way. □

As immediate consequences of this result we obtain the following characterizations for the class $\mathbb{PM}^{\preccurlyeq}_{w_r}$ and the varieties $\mathbb{PM}^{s\ell}_{w_r}$ and $\mathbb{PM}^{\ell}_{w_r}$.

Corollary 7.31. *An order algebra $\mathcal{A} = \langle A, *, {}^\backprime, {}', 0, 1, \leq \rangle$ with algebraic type $\langle 2, 1, 1, 0, 0 \rangle$ belongs to the class $\mathbb{PM}^{\preccurlyeq}_{w_r}$ if and only if the following conditions are satisfied:*

a) $\langle A, \leq, *, 1 \rangle$ *is a po-monoid,*

b) $\mathcal{A} \models 0 \preccurlyeq x$,

c) $(r_1), \ldots, (l_3)$ *hold in* $\mathcal{A}$.

Corollary 7.32. $\mathbb{PM}^{s\ell}_{w_r}$ $(\mathbb{PM}^{\ell}_{w_r})$ *is the equational of the algebras*

$$\mathbf{A} = \langle A, \vee, *, {}^\backprime, {}', 0, 1 \rangle \quad (\mathbf{A} = \langle A, \vee, \wedge, *, {}^\backprime, {}', 0, 1 \rangle)$$

of type $\langle 2, 2, 1, 1, 0, 0 \rangle$ $(\langle 2, 2, 2, 1, 1, 0, 0 \rangle)$ *satisfying:* [2]

1. *Any set of equations defining the class of the $s\ell$-monoids (ℓ-monoids).*
2. *The equation* $0 \preccurlyeq x$.
3. *The equations:*

$r_1)$ $1^\backprime \preccurlyeq 0$	$r_2)$ $1 \preccurlyeq 0^\backprime$	$r_3)$ $x * (y * x)^\backprime \preccurlyeq y^\backprime$
$l_1)$ ${}'1 \preccurlyeq 0$	$l_2)$ $1 \preccurlyeq {}'0$	$l_3)$ ${}'(x * y) * x \preccurlyeq {}'y$

[2] In the context of semilatticed algebras we use $t_1 \preccurlyeq t_2$ as an abbreviation for the equation $t_1 \vee t_2 \approx t_2$.

The set of equations (3) can be substituted in the axiomatitzaion of $\mathbb{PM}^{s\ell}_{w_r}$ ($\mathbb{PM}^{\ell}_{w_r}$) *by the set of equations*

$$\begin{array}{llllll} r_4) & x * x^{\backprime} \preccurlyeq 0 & r_5) & x \preccurlyeq {}'(y * (x * y)^{\backprime}) & r_6) & (x \vee y)^{\backprime} \preccurlyeq x^{\backprime} \\ l_4) & {}'x * x \preccurlyeq 0 & l_5) & x \preccurlyeq ({}'(y * x) * y)^{\backprime} & l_6) & {}'(x \vee y) \preccurlyeq {}'x \end{array}$$

The equations (r_1) *i* (l_1) *can be substituted by the equations* $1^{\backprime} \approx 0$ *i* ${}'1 \approx 0$*, respectively. The equations* (r_4) *i* (l_4) *can be substituted by* $x * x^{\backprime} \approx 0$ *and* ${}'x * x \approx 0$*, respectively.*

Proof: Immediate by using the characterization of Corollary 7.31 and the fact that the quasiequations expressing the monotonicity can be substituted by the equations (r_6) a (l_6). □

Proposition 7.33. *Let* $\mathcal{A} = \langle A, *, ^{\backprime}, ', 0, 1, \leq \rangle$ *be a pseudocomplemented po-monoid with* 0 *as the minimum element. Then* A *has a maximum element, say* $\top$*, and* ${}'0 = 0^{\backprime} = \top$ *holds.*

Proof: Since 0 is the minimum, we have that, for every $a \in A$, $0 * a \leq 0$ and $a * 0 \leq 0$. From this, by (LP), we obtain $a \leq 0^{\backprime}$ and $a \leq {}'0$. In particular, ${}'0 \leq 0^{\backprime}$ and $0^{\backprime} \leq {}'0$. Therefore, ${}'0$ and $0^{\backprime}$ are the same element and this element is the maximum. □

In the following proposition we prove some properties of the pseudocomplemented monoids with 0 as the minimum element which are integral (i.e., 1 is the maximum element).

Proposition 7.34. *In every* $\mathcal{A} \in \mathbb{PM}^{\preccurlyeq}_{w}$ *are satisfied:*

$$i)\ \ x * y \preccurlyeq x \quad ii)\ \ x * y \preccurlyeq y \quad iii)\ \ x * 0 \approx 0 * x \approx 0 \quad iv)\ \ 0^{\backprime} \approx 1 \approx {}'0$$

Proof: The reduct $\langle A, *, 0, 1, \leq \rangle$ is a po-monoid with minimum element 0 and integral and thus (see Propositions 6.10 and 6.17.) *i), ii)* and *iii)* are valid on it. Property *iv)* is a consequence of (r_2), (l_2) and the fact that 1 is the maximum. □

Corollary 7.35. *An order algebra* $\mathcal{A} = \langle A, *, ^{\backprime}, ', 0, 1, \leq \rangle$ *of algebraic type* $\langle 2, 1, 1, 0, 0 \rangle$ *is of the class* $\mathbb{PM}^{\preccurlyeq}_{w}$ *if and only if*

1. $\langle A, \leq, *, 1 \rangle$ *is a po-monoid,*
2. *in* $\mathcal{A}$ *the inequations* $0 \preccurlyeq x$ *i* $x \preccurlyeq 1$ *are satisfied,*
3. *in* $\mathcal{A}$ *are satisfied:*

$$\begin{array}{llllll} r_1) & 1^{\backprime} \approx 0 & r_2) & 1 \approx 0^{\backprime} & r_3) & x * (y * x)^{\backprime} \preccurlyeq y^{\backprime} \\ l_1) & {}'1 \approx 0 & l_2) & 1 \approx {}'0 & l_3) & {}'(x * y) * x \preccurlyeq {}'y \end{array}$$

A characterization for the class $\mathbb{PM}^{\preccurlyeq}_{ew}$ is obtained, obviously, adding the condition of commutativity of the monoidal operation and substituting in Corollary 7.35 the conditions (3) with the following ones:

$p_1)\quad \neg 1 \approx 0 \qquad p_2)\quad 1 \approx \neg 0 \qquad p_3)\quad x * \neg(y * x) \preccurlyeq \neg y$

The following result is an equational characterization of the classes $\mathbb{PM}^{s\ell}_{w}$ and $\mathbb{PM}^{\ell}_{w}$.

Corollary 7.36. $\mathbb{PM}^{s\ell}_{w}$ $(\mathbb{PM}^{\ell}_{w})$ *is the equational class of the algebras*

$$\mathbf{A} = \langle A, \vee, *, ^{\backprime}, ', 0, 1\rangle \quad (\mathbf{A} = \langle A, \vee, \wedge, *, ^{\backprime}, ', 0, 1\rangle)$$

of type $\langle 2, 2, 1, 1, 0, 0\rangle$ $(\langle 2, 2, 2, 1, 1, 0, 0\rangle)$ *satisfying:*

1. *Any set of equations defining the class of the* $s\ell$*-monoids (*ℓ*-monoids),*
2. $0 \preccurlyeq x,\ x \preccurlyeq 1$
3. $r_1)\quad 1^{\backprime} \approx 0 \qquad r_2)\quad 1 \approx 0^{\backprime} \qquad r_3)\quad (x * (y * x)^{\backprime}) \preccurlyeq y^{\backprime}.$
   $l_1)\quad {}'1 \approx 0 \qquad l_2)\quad 1 \approx {}'0 \qquad l_3)\quad ({}'(x * y) * x) \preccurlyeq {}'y.$

Proof: Immediate by using the characterization of Corollary 7.35. □

In the following statements we adapt the notation and give equational characterizations to the commutative classes $\mathbb{PM}^{s\ell}_{ew_r}$, $\mathbb{PM}^{\ell}_{ew_r}$, $\mathbb{PM}^{s\ell}_{ew}$ and $\mathbb{PM}^{\ell}_{ew}$.

Corollary 7.37. $\mathbb{PM}^{s\ell}_{ew_r}$ $(\mathbb{PM}^{\ell}_{ew_r})$ *is the equational class of the algebras*

$$\mathbf{A} = \langle A, \vee, *, \neg, 0, 1\rangle \quad (\mathbf{A} = \langle A, \vee, \wedge, *, \neg, 0, 1\rangle)$$

of type $\langle 2, 2, 1, 0, 0\rangle$ $(\langle 2, 2, 2, 1, 0, 0\rangle)$ *satisfying:*

1. *Any set of equations defining the class of the commutative* $s\ell$*-monoids (*ℓ*-monoids),*
2. $0 \preccurlyeq x,$
3. $p_1)\quad \neg 1 \approx 0 \qquad p_2)\quad 1 \preccurlyeq \neg 0 \qquad p_3)\quad (x * \neg(y * x)) \preccurlyeq \neg y.$

Corollary 7.38. $\mathbb{PM}^{s\ell}_{ew}$ $(\mathbb{PM}^{\ell}_{ew})$ *is the equational class of the algebras*

$$\mathbf{A} = \langle A, \vee, *, \neg, 0, 1\rangle \quad (\mathbf{A} = \langle A, \vee, \wedge, *, \neg, 0, 1\rangle)$$

of type $\langle 2, 2, 1, 0, 0\rangle$ $(\langle 2, 2, 2, 1, 0, 0\rangle)$ *satisfying:*

1. *Any set of equations defining the class of the commutative* $s\ell$*-monoids (*ℓ*-monoids),*
2. $0 \preccurlyeq x,\ \ x \preccurlyeq 1,$
3. $p_1)\quad \neg 1 \approx 0 \qquad p_2)\quad 1 \approx \neg 0 \qquad p_3)\quad (x * \neg(y * x)) \preccurlyeq \neg y.$

Observation 7.39. Let us stress that the variety $\mathbb{PM}^{s\ell}_{ewc}$ is, precisely, the variety of the *pseudocomplemented distributive lattices*. For an equational presentation of this class, see [BD74]. In the following statement we give an equational characterization of this variety alternative to the one of [BD74].

Corollary 7.40. *$\mathbb{PM}^{s\ell}_{ewc}$ is the equational class of the algebras*

$$\mathbf{A} = \langle A, \vee, \wedge, \neg, 0, 1 \rangle$$

of type $\langle 2, 2, 1, 0, 0 \rangle$ satisfying:

1. *Any set of equations defining the class of the distributive lattices,*
2. $0 \preccurlyeq x$, $x \preccurlyeq 1$,
3. $p_1)$ $\neg 1 \approx 0$ $\quad p_2)$ $1 \approx \neg 0$ $\quad p_3)$ $(x \wedge \neg(y \wedge x)) \preccurlyeq \neg y$.

Proof: The equations (1) and (2) in Theorem 7.38 define the class $\mathring{\mathbb{M}}^{s\ell}_{ew}$. If we add the equation $x \preccurlyeq x * x$, then the operation $*$ is equal to the operation $\wedge$ (see Proposition 6.20), the class $\mathring{\mathbb{M}}^{s\ell}_{ewc}$ is that of the distributive lattices, and the pseudocomplementation is with respect to the operation $\wedge$. □

7.5 Weakly contractive pseudocomplemented monoids

In this section we introduce the notion of weak contraction in the framework of the pseudocomplemented structures. We give some characterizations and in particular we obtain a simplification in the language concerning the presentation of the class $\mathbb{PM}^{\ell}_{w\hat{c}}$.

Definition 7.41 (Weak contraction property). *We will say that a pseudocomplemented structure has the property of* weak contraction*) or that it is* weakly contractive *if it satisfies the following set of quasi-inequations:*

$$\{x * x \preccurlyeq y^{\backprime} \supset x \preccurlyeq y^{\backprime},\ x * x \preccurlyeq {}'y \supset x \preccurlyeq {}'y\} \qquad (\hat{c} \preccurlyeq)$$

In every pseudocomplemented structure the property of weak contraction can be expressed with the inequations $(x * x)^{\backprime} \preccurlyeq x^{\backprime}$ and ${}'(x * x) \preccurlyeq {}'x$. To demonstrate this we will use the following lemma.

Lemma 7.42. *Let $\mathcal{A}$ be a pseudocomplemented po-grupoid and let u, v, t be terms of the language $\langle *, {}^{\backprime}, {}', 0 \rangle$. The following conditions are equivalent:*

i) $\mathcal{A}$ is a model of the quasi-inequations $u \preccurlyeq t^{\backprime} \supset v \preccurlyeq t^{\backprime}$ and $u \preccurlyeq {}'t \supset v \preccurlyeq {}'t$.

ii) $\mathcal{A}$ is a model of the inequations $u^{\backprime} \preccurlyeq v^{\backprime}$ and ${}'u \preccurlyeq {}'v$.

Proof: Suppose that the variables occurring in u, v and t are in $\{x_1, \ldots, x_m\}$. Let $a_1, \ldots, a_n \in A$ and let us suppose $u^{\mathcal{A}}(a_1, \ldots, a_n) = a$, $v^{\mathcal{A}}(a_1, \ldots, a_n) = b$ and $t^{\mathcal{A}}(a_1, \ldots, a_n) = c$.

$i) \Rightarrow ii)$: From $a \leq {}'(a^{\backprime})$, since the second of the quasi-inequations $i)$ is satisfied, we obtain $b \leq {}'(a^{\backprime})$ which, applying (LP), is equivalent to $b * a^{\backprime} \leq 0$ and, therefore, to

$a^{\backprime} \leq b^{\backprime}$. Thus, the first inequation of ii) is satisfied. From $a \leq ({'a})^{\backprime}$, since the first of the quasi-inequations i) is satisfied, we obtain $b \leq ({'a})^{\backprime}$ which, applying (LP), is equivalent to ${'a} * b \leq 0$ which it is equivalent to ${'a} \leq {'b}$. Therefore, the second of the inequations ii) is satisfied.

$ii) \Rightarrow i)$: Suppose $a \leq c^{\backprime}$. By antimonotonicity we obtain ${'(c^{\backprime})} \leq {'a}$ and, therefore, $c \leq {'a}$. Now, since the second inequation of i) is satisfied we have that ${'a} \leq {'b}$. Thus, $c \leq {'b}$ which, by antimonotonicity implies $({'b})^{\backprime} \leq c^{\backprime}$ and, therefore, $b \leq c^{\backprime}$. Thus, the first of the quasi-inequations i) is satisfied. It can be proved analogously that the second quasi-inequation holds by using the antimonotonicity of the pseudocomplements and the first of the inequations i). □

Proposition 7.43. *Let $\mathcal{A}$ be a pseudocomplemented po-grupoid. The following conditions are equivalent:*

i) $\mathcal{A}$ satisfies the quasi-inequations ($\hat{c} \preccurlyeq$).

ii) $\mathcal{A} \models \{(x * x)^{\backprime} \preccurlyeq x^{\backprime}, {'(x * x)} \preccurlyeq {'x}\}$.

Proof: By Lemma 7.42. □

Proposition 7.44. *Every contractive pseudocomplemented po-grupoid is weakly contractive.*

Proof: Let $\mathcal{A}$ be a contractive pseudocomplemented po-grupoid. Then $\mathcal{A}$ satisfies $x \preccurlyeq x * x$. If $a \in A$, then $a * a \leq a$ and by antimonotonicity we obtain $(a * a)^{\backprime} \leq a^{\backprime}$ and ${'(a * a)} \leq {'a}$. □

Proposition 7.45. *Let* **A** *be a $\mathbb{FL}$-algebra. The following conditions are equivalent:*

i) **A** *is weakly contractive.*

ii) $\mathbf{A} \models \{x \backslash x^{\backprime} \preccurlyeq x^{\backprime}, {'x}/x \preccurlyeq {'x}\}$.

Proof: It is a consequence of the fact that, in every $\mathbb{FL}$-algebra, the equations $(x*x)^{\backprime} \approx x \backslash x^{\backprime}$ and ${'(x * x)} \approx x \backslash {'x}$ are satisfied: if $a \in A$ we have $(a * a)^{\backprime} = (a * a) \backslash 0 = a \backslash (a \backslash 0) = a \backslash a^{\backprime}$. □

Proposition 7.46. *There are pseudocomplemented po-grupoids ($s\ell$-monoids, ℓ-monoids) and $\mathbb{FL}$-algebras which are weakly contractive but are not contractive.*

Proof: Consider the product standard algebra $[0, 1]_{\Pi}$, that is, the $\mathbb{FL}_{ew}$-algebra defined by the product t-norm and its residuum in the unit real interval. This $\mathbb{FL}$-algebra is non-contractive but is weakly contractive. The reducts in the adequate languages of this algebra are examples of pseudocomplemented po-grupoids, $s\ell$-monoids and ℓ-monoids being weakly contractive but non-contractive. □

Notation 7.47. Let $\lambda \in \{\preccurlyeq, s\ell, \ell\}$ and let σ be a subsequence, possibly empty, of ew_lw_r. We will denote by $\mathbb{PM}^{\lambda}_{\sigma\hat{c}}$ the class of structures in $\mathbb{PM}^{\lambda}_{\sigma}$ being weakly contractive and by $\mathbb{FL}_{\sigma\hat{c}}$ the class of weakly contractive algebras in $\mathbb{FL}_{\sigma}$.

Note 7.48. In Chapter 8 we will show that, for each $\sigma \leq ew_lw_r$, $\mathbb{PM}^{s\ell}_{\sigma\hat{c}}$ and $\mathbb{PM}^{\ell}_{\sigma\hat{c}}$ are the classes of all the $\langle \vee, *, ^{\backprime}, ', 0, 1\rangle$-subreducts and all the $\langle \vee, \wedge, *, ^{\backprime}, ', 0, 1\rangle$-subreducts of the class $\mathbb{FL}_{\sigma\hat{c}}$, respectively.

In the following we give characterizations for the property of weak contraction in the integral classes and in the classes where 0 is the minimum element.

Proposition 7.49. *Let $\mathcal{A} \in \mathbb{PM}^{\preccurlyeq}_{w_l}$. The following conditions are equivalent:*

i) $\mathcal{A}$ is weakly contractive,

ii) $\mathcal{A} \models \{(x * x)^{\backprime} \approx x^{\backprime},\ '(x * x) \approx {'x}\}$.

Proof: It will be sufficient to prove that in $\mathcal{A}$ the inequation $(x * x)^{\backprime} \preccurlyeq x^{\backprime}$ is equivalent to the equation $(x * x)^{\backprime} \approx x^{\backprime}$. Let $a \in A$. Since $\mathcal{A}$ is integral, we have $a * a \leq a$ which by antimonotonicity implies that $a^{\backprime} \leq (a * a)^{\backprime}$. Therefore, $(a * a)^{\backprime} \leq a^{\backprime}$ is equivalent to $(a * a)^{\backprime} = a^{\backprime}$. □

Proposition 7.50. *Let $\mathcal{A} \in \mathbb{PM}^{\preccurlyeq}_{w_r}$. The following conditions are equivalent:*

i) $\mathcal{A}$ is weakly contractive,

ii) $\mathcal{A} \models \{x * (x * x)^{\backprime} \approx 0,\ '(x * x) * x \approx 0\}$.

Proof: It will be sufficient to prove that in $\mathcal{A}$ the inequation $(x*x)^{\backprime} \preccurlyeq x^{\backprime}$ is equivalent to the equation $x * (x * x)^{\backprime} \approx 0$. Let $a \in A$. By (LP) we have that $(a*a)^{\backprime} \leq a^{\backprime}$ is equivalent to $a * (a * a)^{\backprime} \leq 0$ which, since 0 is the minimum, it is equivalent to $a * (a * a)^{\backprime} \approx 0$. □

Proposition 7.51. *Let $\mathcal{A} \in \mathbb{PM}^{\preccurlyeq}_{\hat{c}}$. For every $a \in A$, if $0 \leq a$, the infimum of the sets $\{a, {'a}\}$ and $\{a, a^{\backprime}\}$ exist and it is equal to 0.*

Proof: Let $b \leq a, a^{\backprime}$. By monotonicity we have $b * b \leq a * a^{\backprime}$. Thus $b * b \leq 0$ and from this, by the antimonotonicity of the pseudocomplements, we obtain $0^{\backprime} \leq (b * b)^{\backprime}$. Now, by the property of weak contraction, $0^{\backprime} \leq b^{\backprime}$, and using that $1 \leq 0^{\backprime}$, we have $1 \leq b^{\backprime}$ which by (LP) is equivalent to $b \leq {'1}$ and, since ${'1} \leq 0$, we obtain $b \leq 0$. We can now proceed analogously to prove that when $b \leq a, {'a}$, then $b \leq 0$. □

Proposition 7.52. *Let $\mathcal{A} \in \mathbb{PM}^{\preccurlyeq}_{w}$. The following conditions are equivalent:*

i) $\mathcal{A}$ is weakly contractive,

ii) For every $a \in A$, the infimum of the sets $\{a, {'a}\}$ and $\{a, a^{\backprime}\}$ exist and it is equal to 0.

Proof: Since 0 is the minimum, by Proposition 7.51 we have that $i) \Rightarrow iv)$. Suppose now that $iv)$ is satisfied. Let $a \in A$. On the one hand, by (p_1), we have $a * (a*a)^{\backprime} \leq a^{\backprime}$ and, on the other hand, by integrality, we have $a*(a*a)^{\backprime} \leq a$. Therefore, $a*(a*a)^{\backprime} \leq 0$. Analogously we can prove ${}'(a*a)*a \leq 0$. □

Theorem 7.53. *Let* $\mathbf{A} \in \mathbb{PM}^{\ell}_{w}$. *The following conditions are equivalent:*

i) $\mathbf{A}$ *is weakly contractive,*

ii) $\mathbf{A} \models \{x \wedge x^{\backprime} \approx 0, {}'x \wedge x \approx 0\}$,

iii) The operations $\wedge$, $'$ *and* $^{\backprime}$ *satisfy (LP).*

In these conditions both pseudocomplement coincide.

Proof: $i)$ and $ii)$ are equivalent conditions because $ii)$ is the reformulation of $iv)$ in Proposition 7.52 for the latticed case.

$ii) \Rightarrow iii)$: Let $a, b \in A$ and suppose $b \leq a^{\backprime}$. By the monotonicity of $\wedge$ we have $a \wedge b \leq a \wedge a^{\backprime} \leq 0$ and, if $a \leq {}'b$, then $a \wedge b \leq {}'b \wedge b \leq 0$. As $\mathcal{A}$ is integral, we have that $a * b \leq a \wedge b$. Thus, if $a \wedge b \leq 0$, we also have $a * b \leq 0$ and, therefore, $b \leq a^{\backprime}$ and $a \leq {}'b$.

$iii) \Rightarrow ii)$: If $\wedge$, $'$ and $^{\backprime}$ satisfy (LP), obviously, for every $a \in A$, $a \wedge a^{\backprime} \leq 0$ and ${}'a \wedge a \leq 0$ and, since 0 is the minimum, $a \wedge a^{\backprime} = {}'a \wedge a = 0$.

Now suppose that these conditions are given and let $a \in A$. Then we have that ${}'a$ is the maximum of the set $\{x \in A : x \wedge a \leq 0\}$ and $a^{\backprime}$ is the maximum of $\{x \in A : a \wedge x \leq 0\}$. By the commutativity of $\wedge$ these two sets are equal. Therefore, ${}'a = a^{\backprime}$. □

Thus, the collapse of the two pseudocomplements allows us to define the class $\mathbb{PM}^{\ell}_{w\hat{c}}$ in a signature with only one negation.

Corollary 7.54. *The variety* $\mathbb{PM}^{\ell}_{w\hat{c}}$ *is definitionally equivalent to the class of the algebras* $\mathbf{A} = \langle A, \vee, \wedge, *, \neg, 0, 1\rangle$ *of type* $\langle 2, 2, 2, 1, 0, 0\rangle$ *satisfying:*

1. *Any set of equations defining the class of the* ℓ*-monoids,*
2. $0 \vee x \approx x$, $x \vee 1 \approx 1$,
3. $p_1)$ $\neg 1 \approx 0$, $\quad p_2)$ $1 \approx \neg 0$, $\quad p_3)$ $(x * \neg(y * x)) \vee \neg y \approx \neg y$,
4. $x \wedge \neg x \approx 0$.

7.6 Involutive pseudocomplemented monoids

In this section we introduce the notion of involution in the framework of the pseudocomplemented structures. We study and characterize the involutive classes of lattices and semilatticed pseudocomplemented monoids and we show that these classes are definitonaly equivalent to the class of involutive $\mathbb{FL}$-algebras. We also prove that the order is definable in the classes of involutive pseudocomplemented po-monoids such that 0 is the minimum element.

Firstly, we will see that in a pseudocomplemented po-monoid $\mathcal{A}$, for every $a \in A$, there always exist the right and left pseudocomplements of a relative to the elements of the form $'b$ and $b^{\backprime}$, with $b \in A$.

Proposition 7.55. *In every pseudocomplemented po-monoid the following quasi-inequations are satisfied:*

i) $x * z \preccurlyeq y^{\backprime} \supset z \preccurlyeq (y * x)^{\backprime}$, $z \preccurlyeq (y * x)^{\backprime} \supset x * z \preccurlyeq y^{\backprime}$,

ii) $z * x \preccurlyeq {}'y \supset z \preccurlyeq {}'(x * y)$, $z \preccurlyeq {}'(x * y) \supset z * x \preccurlyeq {}'y$.

Proof: It is sufficient to prove that the quasi-equations *i)* are satisfied because the ones in *ii)* are their mirror images. Let $a, b, c \in A$. By applying (LP), associativity, and again (LP), we have the following equivalences:

$$a * c \leq b^{\backprime} \quad \text{iff} \quad b * (a * c) \leq 0 \quad \text{iff} \quad (b * a) * c \leq 0 \quad \text{iff} \quad c \leq (b * a)^{\backprime}.$$

□

Proposition 7.56. *Let $\mathcal{A}$ be a pseudocomplemented po-monoid. For each $a, b \in A$ there exist the right and the left pseudocomplements of a relative to $'b$ and $b^{\backprime}$, that is, there exists the maximum element of the sets*

$$\mathrm{R}^{b^{\backprime}}(a) = \{x \in A : a * x \leq b^{\backprime}\}, \qquad \mathrm{L}^{'b}(a) = \{x \in A : x * a \leq {}'b\}.$$

and it holds that

$$max\ \mathrm{R}^{b^{\backprime}}(a) = (b * a)^{\backprime} \quad and \quad max\ \mathrm{L}^{'b}(a) = {}'(a * b).$$

Proof: Let $a, b, c \in A$. We have that $a * (b * a)^{\backprime} \leq b^{\backprime}$. Thus, $(b * a)^{\backprime} \in \mathrm{R}^{b^{\backprime}}(a)$. Suppose $a * c \leq b^{\backprime}$. From this we obtain, by Proposition 7.55, $c \leq (b * a)^{\backprime}$. Therefore, $(b * a)^{\backprime}$ is the maximum of $\mathrm{R}^{b^{\backprime}}(a)$. The other identity is obtained as a mirror image of the first.□

Definition 7.57. *We will say that a pseudocomplemented structure $\mathcal{A}$ is* involutive *if, for each $a \in A$, $'(a^{\backprime}) \leq a$ and $('a)^{\backprime} \leq a$. The involutive subclasses of $\mathbb{PM}_{\sigma}^{\preccurlyeq}$, $\mathbb{PM}_{\sigma}^{s\ell}$, $\mathbb{PM}_{\sigma}^{\ell}$ and $\mathbb{FL}_{\sigma}$ wil be denoted by $\mathbb{IPM}_{\sigma}^{\preccurlyeq}$, $\mathbb{IPM}_{\sigma}^{s\ell}$, $\mathbb{IPM}_{\sigma}^{\ell}$ and $\mathbb{IFL}_{\sigma}$, respectively.*

Observe that the classes $\mathbb{IPM}_{\sigma}^{s\ell}$, $\mathbb{IPM}_{\sigma}^{\ell}$ and $\mathbb{IFL}_{\sigma}$ are the subvarieties of the varieties $\mathbb{PM}_{\sigma}^{s\ell}$, $\mathbb{PM}_{\sigma}^{\ell}$ and $\mathbb{FL}_{\sigma}$ defined by the equations $'(x^{\backprime}) \preccurlyeq x$ and $('x)^{\backprime} \preccurlyeq x$ (in the commutative cases by the equation $\neg\neg x \preccurlyeq x$), respectively.

Proposition 7.58. *In every pseudocomplemented structure $\mathcal{A}$ the following conditions are equivalent:*

i) *$\mathcal{A}$ is involutive.*

ii) *For every $a \in A$, $'(a^{\backprime}) = a = ('a)^{\backprime}$.*

iii) *For each $a, b \in A$, $a \leq b$ iff $a * b^{\backprime} \leq 0$ iff $'b * a \leq 0$.*

iv) *For each $a, b \in A$, $a \leq b$ iff $b^{\backprime} \leq a^{\backprime}$ iff $'b \leq {}'a$.*

Proof: Let $a, b \in A$.

i) $\Leftrightarrow$ *ii)*: Given that in the pseudocomplemented structures we have the validity of $x \preccurlyeq {}'(x^{\backprime})$ and $x \preccurlyeq ('x)^{\backprime}$, we have that $\mathcal{A}$ is involutive if, and only if, $'(x^{\backprime}) \approx x \approx ('x)^{\backprime}$ are valid equations in $\mathcal{A}$.

i) $\Rightarrow$ *iii)*: When $\mathcal{A}$ is involutive, $a \leq b$ is equivalent to $a \leq {}'(b^{\backprime})$ and $a \leq ('b)^{\backprime}$ and thus, by (LP), it is also equivalent to $a * b^{\backprime} \leq 0$ and $'b * a \leq 0$.

iii) $\Rightarrow$ *i)*: In every pseudocomplemented structure the inequations $x * x^{\backprime} \preccurlyeq 0$ and $'x * x \preccurlyeq 0$ hold. Thus, $'a * ('a)^{\backprime} \leq 0$ and $'(a^{\backprime}) * a^{\backprime} \leq 0$ which, when *iii)* is satisfied, are equivalent to $('a)^{\backprime} \leq a$ and $'(a^{\backprime}) \leq a$, respectively.

iii) $\Leftrightarrow$ *iv)*: This is immediate since, by (LP), $a * b^{\backprime} \leq 0$ and $'b * a \leq 0$ are equivalent to $b^{\backprime} \leq a^{\backprime}$ i $'b \leq {}'a$, respectively. $\square$

Proposition 7.59. *Let $\mathcal{A}$ be an involutive pseudocomplemented structure and let $\{a_i : i \in I\}$ be a family of elements in A. Then,*

a) *If $\bigvee_{i\in I} a_i^{\backprime}$ exists, then $\bigwedge_{i\in I} a_i$ exists and $'(\bigvee_{i\in I} a_i^{\backprime}) = \bigwedge_{i\in I} a_i$ holds.*

b) *If $\bigvee_{i\in I} {}'a_i$ exists, then $\bigwedge_{i\in I} a_i$ exists and $(\bigvee_{i\in I} {}'a_i)^{\backprime} = \bigwedge_{i\in I} a_i$ holds.*

c) *If $\bigvee_{i\in I} a_i^{\backprime}$ and $\bigvee_{i\in I} {}'a_i$ exist, then $\bigwedge_{i\in I} a_i$ exists and it holds that*

$$'(\bigvee_{i\in I} a_i^{\backprime}) = \bigwedge_{i\in I} a_i = (\bigvee_{i\in I} {}'a_i)^{\backprime}.$$

Proof: We prove only *a)*, because *b)* is a mirror image of *a)* and *c)* is an immediate consequence of *a)* and *b)*. If $\bigvee_{i\in I} a_i^{\backprime}$ exists, by Proposition 7.17, we have that there exists $\bigwedge_{i\in I} {}'(a_i^{\backprime})$ and it holds that

$$'(\bigvee_{i\in I} a_i^{\backprime}) = \bigwedge_{i\in I} {}'(a_i^{\backprime}),$$

but, $\mathcal{A}$ being involutive, $'(a_i^{\backprime}) = a_i$. $\square$

Note 7.60. The following results (from Theorem 7.61 to Theorem 7.65) are a generalization of the ones for the varieties $\mathbb{IPM}^{s\ell}_{ew}$ and $\mathbb{IPM}^{\ell}_{ew}$ included in [BGCV04, Theorems 14 and 15], where they are presented without proof.

Theorem 7.61. *Let $\mathbf{A} = \langle A, \vee, *, ^{\backprime}, ', 0, 1\rangle$ be a pseudocomplemented $s\ell$-monoid. The following conditions are equivalent:*

i) *$\mathcal{A}$ is involutive.*

ii) *For each $a, b \in A$, there exists the infimum of a and b in A and it holds that $'(a^{\backprime} \vee b^{\backprime}) = \bigwedge\{a, b\} = ('a \vee 'b)^{\backprime}$.*

iii) *For each $a, b \in A$, $a \leq b$ iff $'(a^{\backprime} \vee b^{\backprime}) = a$ iff $('a \vee 'b)^{\backprime} = a$.*

Proof: $i) \Rightarrow ii)$: By Proposition 7.59.

$ii) \Rightarrow iii)$: Trivial.

$iii) \Rightarrow i)$: Suppose $a \leq a$. By $iii)$ this is equivalent to $'(a^{\backprime} \vee a^{\backprime}) = a$ and $('a \vee 'a)^{\backprime} = a$, which by idempotency are equivalent to $'(a^{\backprime}) = a$ and $('a)^{\backprime} = a$, respectively. $\square$

Theorem 7.62. *Every involutive pseudocomplemented $s\ell$-monoid is the reduct of an involutive pseudocomplemented ℓ-monoid.*

Proof: Let $\mathbf{A} = \langle A, \vee, *, ^{\backprime}, ', 0, 1\rangle$ be an involutive pseudocomplemented $s\ell$-monoid. Now we define, for each $a, b \in A$, $a \wedge b := {'(a^{\backprime} \vee b^{\backprime})}$. Condition $ii)$ of Theorem 7.61 says that $\langle A, \vee, \wedge\rangle$ is a lattice. Therefore, $\mathbf{A}$ is the $\langle \vee, *, ^{\backprime}, ', 0, 1\rangle$-reduct of the involutive pseudocomplemented ℓ-monoid $\langle \mathbf{A}, \wedge\rangle$. $\square$

Corollary 7.63. *Let $\sigma \leq ew_l w_r c$. Every $\mathbb{IPM}^{s\ell}_{\sigma}$-algebra is the reduct of a $\mathbb{IPM}^{\ell}_{\sigma}$-algebra.*

Proof: If $\mathbf{A} \in \mathbb{IPM}^{s\ell}_{\sigma}$ it is obvious that the $\mathbb{IPM}^{\ell}$-algebra $\langle \mathbf{A}, \wedge\rangle$ built in the previous theorem also satisfies the properties codified by σ. $\square$

Corollary 7.64. *Let $\mathbf{A} = \langle A, \vee, \wedge, *, ^{\backprime}, ', 0, 1\rangle$ be a pseudocomplemented ℓ-monoid. The following conditions are equivalent:*

i) $\mathbf{A} \models {'(x^{\backprime})} \approx x \approx ('x)^{\backprime}$,

ii) $\mathbf{A} \models {'(x^{\backprime} \vee y^{\backprime})} \approx x \wedge y \approx ('x \vee 'y)^{\backprime}$.

Proof: By Theorem 7.61. $\square$

Therefore, the variety $\mathbb{IPM}^{\ell}$ is the subvariety of $\mathbb{PM}^{\ell}$. It is obtained by adding to a set of equations defining $\mathbb{PM}^{\ell}$ either the equations $i)$ or the equations $ii)$.

Theorem 7.65. *Let $\mathcal{A} = \langle A, *, {}^{\backprime}, {}', 0, 1, \leq\rangle$ be a pseudocomplemented po-monoid. The following conditions are equivalent:*

i) *$\mathcal{A}$ is involutive.*

ii) *The monoidal operation is residuated and their right and left residua are defined, for each $a, b \in A$, in the following way:*

$$a\backslash b := ({}'b * a)^{\backprime}, \qquad b/a := {}'(a * b^{\backprime}).$$

Proof: *i)* $\Rightarrow$ *ii)*: Since $\mathcal{A}$ is involutive, for each $b \in A$, $({}'b)^{\backprime} = b = {}'(b^{\backprime})$. Thus, by Proposition 7.56, we have that, for each $a, b \in A$, there exist right and left relative pseudocomplements of a with respect to b and are equal to $({}'b * a)^{\backprime}$ and ${}'(a * b^{\backprime})$, respectively. But this, by Corollary 6.27, is equivalent to saying that the operation $*$ is residuated and that their right and left residua are, respectively, the operations $\backslash$ and $/$ defined, for each $a, b \in A$, by $a\backslash b := ({}'b * a)^{\backprime}$ and $b/a := {}'(a * b^{\backprime})$.

ii) $\Rightarrow$ *i)*: Let $a \in A$. From ${}'(a^{\backprime}) \leq {}'(a^{\backprime})$ we obtain ${}'(a^{\backprime}) \leq {}'(1 * a^{\backprime})$. This expression, by the definition of left residua, is equivalent to ${}'(a^{\backprime}) \leq a/1$ but, by (LR), this is equivalent to ${}'(a^{\backprime}) * 1 \leq a$, i.e., ${}'(a^{\backprime}) \leq a$. By the law of mirror images, from $({}'a)^{\backprime} \leq ({}'a)^{\backprime}$ we obtain $({}'a)^{\backprime} \leq a$. □

Theorem 7.66. *Every involutive pseudocomplemented ℓ-monoid is the reduct of an involutive $\mathbb{FL}$-algebra.*

Proof: Let $\mathbf{A} = \langle A, \vee, \wedge, *, {}^{\backprime}, {}', 0, 1\rangle$ be an involutive pseudocomplemented ℓ-monoid. Then we define, for each $a, b \in A$, $a\backslash b := ({}'b * a)^{\backprime}$ i $b/a := {}'(a * b^{\backprime})$. Condition *ii)* in Theorem 7.65 says that $\langle \mathbf{A}, \backslash, /\rangle$ is a $\mathbb{FL}$-algebra. □

Corollary 7.67. *Let $\sigma \leq ew_l w_r c$. Every $\mathbb{PM}^{\ell}_{\sigma}$-algebra is the reduct of a $\mathbb{FL}_{\sigma}$-algebra.*

Proof: If $\mathbf{A} \in \mathbb{IPM}^{\ell}_{\sigma}$, then it is obvious that the $\mathbb{FL}$-algebra $\langle \mathbf{A}, \backslash, /\rangle$ built in the previous theorem also satisfies the properties codified by σ. □

Corollary 7.68. *Let $\sigma \leq ew_l w_r c$. Every $\mathbb{PM}^{s\ell}_{\sigma}$-algebra is the reduct of a $\mathbb{FL}_{\sigma}$-algebra.*

Proof: By Corollaries 7.63 and 7.67. □

Proposition 7.69. *The $\langle \vee, \wedge, *, {}^{\backprime}, {}', 0, 1\rangle$-reduct of a $\mathbb{FL}_{\sigma}$-algebra (involutive) is a $\mathbb{PM}^{\ell}_{\sigma}$-algebra (involutive).*

Proof: Let $\mathbf{A} = \langle A, \vee, \wedge, *, \backslash, /, {}^{\backprime}, {}', 0, 1\rangle$ be a $\mathbb{FL}$-algebra. Given that the properties codified by the sequence σ do not involve the residua, it will be sufficient to prove that the $\langle \vee, \wedge, *, {}^{\backprime}, {}', 0, 1\rangle$-reduct of $\mathbf{A}$ is a $\mathbb{PM}^{\ell}$-algebra. Indeed, $\langle A, \vee, \wedge, *, 0, 1\rangle$ is a pointed ℓ-monoid because in every residuated lattice is satisfied the distributivity of the monoidal operation with respect to the operation $\vee$. On the other hand, we have that ${}^{\backprime}$ and ${}'$ satisfy (LP) as a consequence of the law of residuation and the fact that, for every $a \in A$, ${}^{\backprime}a = a\backslash 0$ and ${}'a = 0/a$. Therefore, $\langle A, \vee, \wedge, *, {}^{\backprime}, {}', 0, 1\rangle$ is a $\mathbb{PM}^{\ell}$-algebra which obviously is involutive if $\mathbf{A}$ is involutive. □

Corollary 7.70. *Let $\sigma \leq ew_l w_r c$. Then*

a) $\mathbb{IPM}^{s\ell}_{\sigma}$ *is the class of all the* $\langle \vee, *, {}^{\backprime}, {}', 0, 1\rangle$*-reducts of* $\mathbb{IPM}^{\ell}_{\sigma}$*.*

b) $\mathbb{IPM}^{\ell}_{\sigma}$ *is the class of all the* $\langle \vee, \wedge, *, {}^{\backprime}, {}', 0, 1\rangle$*-reducts of* $\mathbb{IFL}_{\sigma}$*.*

c) $\mathbb{IPM}^{s\ell}_{\sigma}$ *is the class of all the* $\langle \vee, *, {}^{\backprime}, {}', 0, 1\rangle$*-reducts of* $\mathbb{IFL}_{\sigma}$*.*

Therefore, the classes $\mathbb{IPM}^{s\ell}_{\sigma}$, $\mathbb{IPM}^{\ell}_{\sigma}$ *and* $\mathbb{IFL}_{\sigma}$ *are definitionally equivalent.*

Proof: a): It is a consequence of Corollary 7.63 and the obvious fact that the $\langle \vee, *, {}^{\backprime}, {}', 0, 1\rangle$-reduct of a $\mathbb{PM}^{\ell}_{\sigma}$-algebra is a $\mathbb{PM}^{s\ell}_{\sigma}$-algebra.

b): A consequence of Corollary 7.63 and Proposition 7.69.

c): A consequence of a) and b). □

Corollary 7.71. *Let* $\mathbf{A} = \langle A, \vee, \wedge, *, \backslash, /, {}^{\backprime}, {}', 0, 1\rangle$ *be a* $\mathbb{FL}$*-algebra. The following conditions are equivalent:*

i) $\mathbf{A} \models {}'(x^{\backprime}) \approx x \approx ({}'x)^{\backprime}$,

ii) $\mathbf{A} \models {}'(x^{\backprime} \vee y^{\backprime}) \approx x \wedge y \approx ({}'x \vee {}'y)^{\backprime}$,

iii) $\mathbf{A} \models \{x\backslash y \approx ({}'y * x)^{\backprime},\ y/x \approx {}'(x * y^{\backprime})\}$.

Proof: By Theorem 7.65. □

Therefore, the variety $\mathbb{IFL}$ is the subvariety of $\mathbb{FL}$ which is defined by either the equations i), or the equations ii), or the equations iii).

Note 7.72. The involutive pseudocomplemented ℓ-monoids are considered by J. Lambek in [Lam95] under the name of *Grišin algebras*. In this work the author cites as a reference the paper of V. N. Grišin [Gri83]. For some examples of these structures the reader can consult the cited article of Lambek and also [Lan93]. The name *Grišin algebras* is also used by H. Ono in [Ono85] for the *latticed L^0-algebras* introduced by Grišin in [Gri82]. The *Grišin algebras* considered in [Lam95] are a generalization of the algebras with the same name considered by Ono in [Ono85].

In the following result we show the fact that in the involutive pseudocomplemented monoids the integrality is equivalent to 0 being the minimum element.

Proposition 7.73. *Let* $\mathcal{A}$ *be an involutive pseudocomplemented po-monoid. The following conditions are equivalent:*

i) 0 *is the minimum element of* $\mathcal{A}$*.*

ii) $\mathcal{A}$ *is integral.*

Proof: $i) \Rightarrow ii)$: If 0 is the minimum we have $1^\backprime \leq 0$ and so $'0 \leq '(1^\backprime)$. In consequence, by the property of involution, $'0 \leq 1$. But, since $'0$ is the maximum (Proposition 7.33), we conclude that 1 is the maximum.

$ii) \Rightarrow i)$ If 1 is the maximum we have, for each $a \in A$, $'a \leq 1$. Thus, $1^\backprime \leq ('a)^\backprime = a$. But in every pseudocomplemented po-monoid $1^\backprime = 0$. Therefore, 0 is the minimum. □

Corollary 7.74. *Let* $\lambda \in \{\preccurlyeq, s\ell, \ell\}$. *Then*

$$\mathbb{IPM}^{\lambda}_{w_r} = \mathbb{IPM}^{\lambda}_{w_l} = \mathbb{IPM}^{\lambda}_{w}, \quad \mathbb{IPM}^{\lambda}_{ew_r} = \mathbb{IPM}^{\lambda}_{ew_l} = \mathbb{IPM}^{\lambda}_{ew}.$$

Note 7.75. Given that $\mathbb{PM}^{s\ell}_{ewc}$ is the variety of pseudocomplemented distributive lattices we have obviously that $\mathbb{IPM}^{s\ell}_{ewc}$ is definitionally equivalent to the variety of Boole algebras.

Proposition 7.76. *In every involutive pseudocomplemented po-monoid where* 0 *is the minimum the order is definable by means of either the equation* $x*y^\backprime \approx 0$ *or the equation* $'y * x \approx 0$.

Proof: Let $\mathcal{A}$ be an involutive pseudocomplemented po-monoid where 0 is the minimum. By *iii*) of Proposition 7.58 we have, for each $a, b \in A$, that $a \leq b$ is equivalent to $a * b^\backprime \leq 0$ and $^\backprime b * a \leq 0$. Thus, since 0 is the minimum, $a \leq b$ is equivalent to $a * b^\backprime = 0$ and $'b * a = 0$. □

Therefore, we can use these equivalences for eliminating the order relation in the inequations and quasi-inequations appearing in the definition of the class $\mathbb{IPM}^{\preccurlyeq}_{w}$ by substituting them by equations and quasi-equations. Thus, we have the following result.

Corollary 7.77. *The classes* $\mathbb{IPM}^{\preccurlyeq}_{w}$ *i* $\mathbb{IPM}^{\preccurlyeq}_{ew}$ *are definitionally equivalent to their algebraic reducts. These reducts are quasivarieties.*

Note 7.78. The class $\mathbb{IPM}^{\preccurlyeq}_{ew}$ is definitionally equivalent to the class of L^0-*àlgebres* defined by Grišin in [Gri82], to the class $\mathbb{IBCK}$ of the *BCK-involutive algebras* studied by J. Gispert and A. Torrens in [GT07], and to the class of *involutive pocrims*. The class $\mathbb{IPM}^{\preccurlyeq}_{ew}$ is a quasivariety which is not a variety (see [Krz83, Gri85]). Let us observe that the fact that $\mathbb{IPM}^{\preccurlyeq}_{ew}$ is not a variety implies that $\mathbb{IPM}^{\preccurlyeq}_{w}$ is not either.

Chapter 8

Completions and Subreducts

In this chapter two kinds of construction of a complete $\mathbb{FL}$-algebra from any $\mathbb{FL}$-algebra are considered: the *Dedekind-MacNeille completion* (*DM*-completion, to abbreviate) and the *ideal-completion* (see [Ono93, Ono03a]), which allows the construction of a complete $\mathbb{FL}$-algebra from the monoidal reduct of a $\mathbb{FL}$-algebra in such a way that this algebra is embeddable in its completion. In Section 8.2 we present these constructions and compare them. Some well-known results and some new contributions are given. It is shown that the method of the ideal-completació also works if we start from an algebra in $\mathring{\mathbb{M}}^{s\ell}_\sigma$, $\mathring{\mathbb{M}}^{\ell}_\sigma$, $\mathbb{PM}^{s\ell}_\sigma$ or $\mathbb{PM}^{\ell}_\sigma$ and we obtain that every algebra of these classes is embeddable in a complete $\mathbb{FL}_\sigma$-algebra. These embeddings have as a consequence that the classes $\mathring{\mathbb{M}}^{s\ell}_\sigma$, $\mathring{\mathbb{M}}^{\ell}_\sigma$, $\mathbb{PM}^{s\ell}_\sigma$ and $\mathbb{PM}^{\ell}_\sigma$ are the classes of all the subreducts of the algebras in the class $\mathbb{FL}_\sigma$ (see Section 8.3). We show, however, that the *DM*-completion, which works for $\mathbb{FL}_\sigma$-algebras, does not work if we start from the monoidal reduct of an algebra of the classes $\mathring{\mathbb{M}}^{s\ell}_\sigma$, $\mathring{\mathbb{M}}^{\ell}_\sigma$, $\mathbb{PM}^{s\ell}_\sigma$ or $\mathbb{PM}^{\ell}_\sigma$ because in these cases the construction of Dedekind-MacNeille does not in general produce a $\mathbb{FL}_\sigma$-algebra. The reason is that to carry out this construction the monoidal operation has to be residuated and this is not always the case.

8.1 Basic concepts

In this section we recall some basic concepts concerning complete lattices. The notion of complete algebra for the classes $\mathring{\mathbb{M}}^{s\ell}$, $\mathring{\mathbb{M}}^{\ell}$, $\mathbb{PM}^{s\ell}$, $\mathbb{PM}^{\ell}$ and $\mathbb{FL}$ is defined; it is stated that every complete $\mathring{\mathbb{M}}^{s\ell}$-algebra is the reduct of a complete $\mathring{\mathbb{M}}^{\ell}$-algebra and that every complete $\mathbb{PM}^{s\ell}$-algebra is the reduct of a complete $\mathbb{PM}^{\ell}$-algebra.

Notation 8.1. If $\mathcal{A} = \langle A, \leq \rangle$ is an ordered set and $X \subseteq A$, we denote by $X^{\rightarrow}$ the set of all the upper bounds of X in A and by $X^{\leftarrow}$ the set of all the lower bounds of X in A. Note that the supremum of X in A, if it exists, is the minimum of the set $X^{\rightarrow}$ and the infimum of X in A, if it exists, is the maximum of the set $X^{\leftarrow}$.

Proposition 8.2. (Cf. [DP90]) *Let $\mathcal{A} = \langle A, \leq \rangle$ be an ordered set. Then*

a) The supremum of A in $\mathcal{A}$ exists if, and only if, $\mathcal{A}$ has a greatest element $\top$ and, in this case, $\bigvee_{\mathcal{A}} A = \top$.

b) The supremum of $\emptyset$ in $\mathcal{A}$ exists if, and only if, $\mathcal{A}$ has a smallest element $\perp$ and, in this case, $\bigvee_{\mathcal{A}} \emptyset = \perp$.

Proof: a): If A has a *top* $\top$, then $A^{\rightarrow} = \{\top\}$ and thus, $\bigvee_{\mathcal{A}} A = \top$. If A has no *top*, then $A^{\rightarrow} = \emptyset$ and so $\bigvee_{\mathcal{A}} A$ does not exist.

b): Each element $a \in A$ satisfies (emptily) $x \leq a$ for every $x \in \emptyset$. Hence, $\emptyset^{\rightarrow} = A$ and, consequently, $\bigvee_{\mathcal{A}} \emptyset$ there exists if, and only if, A has a *bottom* $\perp$. In this case, $\bigvee_{\mathcal{A}} \emptyset = \perp$. □

Proposition 8.3. *Every complete lattice has a minimum and a maximum element.*

Proof: It is an immediate consequence of Proposition 8.2. □

Proposition 8.4. (Cf. [DP90]) *Let $\mathcal{A} = \langle A, \leq \rangle$ be a non-empty ordered set. The following conditions are equivalent.*

a) $\mathcal{A}$ is a complete lattice.

b) $\bigvee_{\mathcal{A}} X$ exists for every subset $X \subseteq A$.

Proof: $a) \Rightarrow b)$ is trivial.

$b) \Rightarrow a)$: Suppose that every subset of A has a supremum. Let $X \subseteq A$ and $a = \bigvee_{\mathcal{A}} X^{\leftarrow}$. Since $X \subseteq (X^{\leftarrow})^{\rightarrow}$ and a is the minimum of $(X^{\leftarrow})^{\rightarrow}$, we have $a \leq x$ for each $x \in X$ and so $a \in X^{\leftarrow}$. On the other hand, since $a \in (X^{\leftarrow})^{\rightarrow}$, we have that if $z \in X^{\leftarrow}$, then $z \leq a$ and hence a is the maximum of $X^{\leftarrow}$. Therefore, $\bigwedge_{\mathcal{A}} X = \bigvee_{\mathcal{A}} X^{\leftarrow}$. □

Definition 8.5. *If* $\mathbf{A}$ *is a* $\mathring{\mathbb{M}}^{s\ell}$*-algebra or a* $\mathring{\mathbb{M}}^{\ell}$*-algebra, we will say that it is* complete *if every subset of its universe A has a supremum.*

Proposition 8.6. *Let* $\mathbf{A}$ *be a* $\mathring{\mathbb{M}}^{s\ell}$*-algebra or a* $\mathring{\mathbb{M}}^{\ell}$*-algebra.* $\mathbf{A}$ *is complete if, and only if, the ordered set associated to its semilatticed reduct is a complete lattice.*

Proof: By Proposition 8.4. □

The notion of complete algebra for the algebras in $\mathbb{PM}^{s\ell}$, $\mathbb{PM}^{\ell}$ and $\mathbb{FL}$ is defined in an analogous way.

Proposition 8.7. *Every complete algebra in* $\mathring{\mathbb{M}}^{s\ell}$, $\mathring{\mathbb{M}}^{\ell}$, $\mathbb{PM}^{s\ell}$, $\mathbb{PM}^{\ell}$ *and* $\mathbb{FL}$ *has a minimum element and a maximum element.*

Proof: By Proposition 8.3. □

Proposition 8.8. *Let $\sigma \leq ew_l w_r c$. Every complete $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$-algebra is the $\langle \vee, *, 0, 1\rangle$-reduct of a complete $\mathring{\mathbb{M}}^{\ell}_{\sigma}$-algebra.*

Proof: If $\mathbf{A}$ is a complete $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$-algebra, then the ordered set associated to its semilatticed reduct is a complete lattice. It is clear that $\mathbf{A}$ is the $\langle \vee, *, 0, 1\rangle$-reduct of the complete $\mathring{\mathbb{M}}^{\ell}_{\sigma}$-algebra of universe A where the operation $\wedge$ is defined in the following way: for each $a, b \in A$,

$$a \wedge b =: \bigvee_{\mathbf{A}} \{x \in A : x \leq a \text{ i } x \leq b\}.$$

The rest of the operations are the ones in $\mathbf{A}$. □

The following result is obtained in an analogous way.

Proposition 8.9. *Every complete $\mathbb{PM}^{s\ell}_{\sigma}$-algebra is the $\langle \vee, *, `, ', 0, 1\rangle$-reduct of a complete $\mathbb{PM}^{\ell}_{\sigma}$-algebra.*

8.2 Completions

Among $\mathbb{FL}_{\sigma}$-algebras the complete ones are particularly interesting, because of the following theorem.

Theorem 8.10. *Every $\mathbb{FL}_{\sigma}$-algebra is embeddable in a complete $\mathbb{FL}_{\sigma}$-algebra.*

There are at least two well-known methods in the literature for obtaining these completions, the *Dedekind-MacNeille completion* and the *ideal-completion.* Before explaining how these two methods work, we will recall a characterization obtained by Ono (see [Ono93, Ono03a]) in which a complete $\mathbb{FL}$-algebra is constructed from a monoid $\mathbf{M} = \langle M, *, 1\rangle$ and a closure operator on $\mathcal{P}(M)$ that satisfies a certain additional condition with respect to the monoidal operation. Here we present this result in the most general case, that is, in a language containing two residuals (left and right) and two pseudocomplements. The inclusion of the two pseudocomplements in the language allows us to see the $\mathbb{PM}^{s\ell}$-algebras and the $\mathbb{PM}^{\ell}$-algebras as subreducts of the $\mathbb{FL}$-algebras, as will be shown in Section 8.3.

Notation 8.11. Given a monoid $\mathbf{M} = \langle M, *, 1\rangle$ and two subsets $X, Y \subseteq M$, we will denote by $X * Y$ the set $\{a * b : a \in X,\ b \in Y\}$.

Proposition 8.12. *Suppose that $\mathbf{M} = \langle M, *, 1\rangle$ is a monoid and that the mapping $C : \mathcal{P}(M) \longrightarrow \mathcal{P}(M)$ is a closure operator satisfying*[1]

$$C(X) * C(Y) \subseteq C(X * Y), \tag{8.1}$$

[1]For the notions concerning closure operators, see Chapter 2, page 19.

for each $X, Y \subseteq M$. Then, for every C-closed D, the structure

$$\mathbf{C}_{\mathbf{M}}^{D} = \langle \mathcal{C}_M, \vee_C, \cap, *_C, \backslash, /, ^{\backprime}, {}^{\prime}, D, C(1) \rangle$$

is a complete $\mathbb{FL}$-algebra, where

- $\mathcal{C}_M = \{X \subseteq M : C(X) = X\}$ *(the closure system associated to C),*
- $X \vee_C Y = C(X \cup Y)$,
- $X *_C Y = C(X * Y)$,
- $X \backslash Y = \{z \in M : x * z \in Y$ *for every* $x \in X\}$,
- $Y/X = \{z \in M : z * x \in Y$ *for every* $x \in X\}$,
- $X^{\backprime} = \{z \in M : x * z \in D$ *for every* $x \in X\}$,
- ${}^{\prime}X = \{z \in M : z * x \in D$ *for every* $x \in X\}$,

and for every family $\{X_i\}_{i \in I} \subseteq \mathcal{P}(M)$,

- $\bigvee_{\mathbf{C_M}} \{X_i\}_{i \in I} = C(\bigcup_{i \in I} X_i)$,
- $\bigwedge_{\mathbf{C_M}} \{X_i\}_{i \in I} = \bigcap_{i \in I} X_i$.

Observe that from the definitions of the residuals and the pseudocomplements we have that $X^{\backprime} = X \backslash D$ and ${}^{\prime}X = D/X$.

In the next proposition we recall two well-known identities which are satisfied by all the closure operators.

Proposition 8.13. *Let M be a set, $\{X_i\}_{i \in I}$ be a family of subsets of M and C be a closure operator on $\mathcal{P}(M)$. Then*

(1) $C(\bigcup_{i \in I} X_i) = C(\bigcup_{i \in I} C(X_i))$,

(2) $\bigcap_{i \in I} C(X_i) = C(\bigcap_{i \in I} C(X_i))$.

In the following proposition we state a condition equivalent to condition (8.1).

Proposition 8.14. *Let $\mathbf{M} = \langle M, *, 1 \rangle$ be a monoid and C be a closure operator on $\mathcal{P}(M)$. For each $X, Y \subseteq M$, the following conditions are equivalent:*

(a) $C(X) * C(Y) \subseteq C(X * Y)$,

(b) $C(C(X) * C(Y)) = C(X * Y)$.

Proof: Let $X, Y \subseteq M$.

$(a) \Rightarrow (b)$: On the one hand, by (a) we have $C(X)*C(Y) \subseteq C(X*Y)$. From this, using the properties of the closure operators, we obtain $C(C(X)*C(Y)) \subseteq CC(X*Y) = C(X*Y)$. On the other hand, from $X \subseteq C(X)$ and $Y \subseteq C(Y)$ it is immediate that $X*Y \subseteq C(X)*C(Y)$ and, therefore, $C(X*Y) \subseteq C(C(X)*C(Y))$.

$(b) \Rightarrow (a)$: By (b) we have $C(C(X)*C(Y)) = C(X*Y)$ but $C(X)*C(Y) \subseteq C(C(X)*C(Y))$. Therefore, $C(X)*C(Y) \subseteq C(X*Y)$. □

8.2.1 Ideal-completion of $\mathring{\mathbb{M}}^{s\ell}$-algebras and $\mathbb{FL}$-algebras

In this section we present the construction called *ideal-completion* for $\mathring{\mathbb{M}}^{s\ell}$-algebras. The main result is Theorem 8.24, where it is shown that every $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$-algebra is embeddable in its ideal-completion in such a way that all the existing residua and all the existing meets are preserved, although this embedding in general does not preserve arbitrary joins (see Proposition 8.56). Let us stress that the mentioned embedding result, already known, was obtained by Ono (cf.[Ono03a, Theorem 7]) using a method different to the one that we will use (see Note 8.30). This theorem has as a consequence that every algebra in $\mathring{\mathbb{M}}^{\ell}_{\sigma}$, $\mathbb{PM}^{s\ell}_{\sigma}$, $\mathbb{PM}^{\ell}_{\sigma}$ and $\mathbb{FL}_{\sigma}$ is embeddable in its ideal-completion.

Definition 8.15 (Ideal of a $\vee$-semilattice). *Given a $\vee$-semilattice $\mathbf{A} = \langle A, \vee \rangle$, an* ideal *of $\mathbf{A}$ is a subset $I \subseteq A$ such that:*

i) $I \neq \emptyset$,

ii) *if $y \leq x$ i $x \in I$, then $y \in I$,*

iii) *if $x, y \in I$, then $x \vee y \in I$.*

If $a \in A$, the set $\{x \in A : x \leq a\}$, which is denoted by $(a]$, is an ideal and is called the *principal ideal generated by a*. Given $X \subseteq A$, the smallest ideal containing X is denoted by $(X]$ and it is called *ideal generated by X*. Obviously, $(a] = (\{a\}]$. Note that A is the greatest ideal and note also that when A has a minimum element $\perp$, then $\perp$ belongs to every ideal and $\{\perp\}$ is the smallest ideal. An *ideal of a lattice* $\mathbf{A} = \langle A, \vee, \wedge \rangle$ is an ideal of the semilattice $\langle A, \vee \rangle$.

Let $\mathbf{A} = \langle A, \vee, *, 0, 1 \rangle$ be a $\mathring{\mathbb{M}}^{s\ell}$-algebra and C^{Id} be an operator on $\mathcal{P}(A)$ defined by $C^{Id}(X) = (X]$ for every $X \subseteq A$. It is easy to see that C^{Id} is a closure operator. Obviously the C^{Id}-closed are the ideals of the semilattice reduct of $\mathbf{A}$. In the following we will see that C^{Id} satisfies the condition (8.1) of Proposition 8.12.

Lemma 8.16. (Cf. [Grä79]) *Let $\mathbf{A}$ be a* sup-*semilattice and let $X \subseteq A$. Then the following holds.*

$$(X] = \{c \in A : c \leq a_1 \vee \cdots \vee a_n \textit{ for some } a_i \in X\}$$

Lemma 8.17. *Let* $\mathbf{A}$ *be a* $\mathring{\mathbb{M}}^{s\ell}$*-algebra. For each* $X, Y \subseteq A$, $(X] * (Y] \subseteq (X * Y]$.

Proof: Let $c \in (X]$, $d \in (Y]$. By the previous lemma we have $c \leq a_1 \vee \cdots \vee a_n$ for some $a_i \in X$, and $d \leq b_1 \vee \cdots \vee b_m$ for some $b_j \in Y$. From this, applying monotonicity we have $c * d \leq (a_1 \vee \cdots \vee a_n) * (b_1 \vee \cdots \vee b_m)$ and so, applying the distributivity of $*$ with respect to $\vee$, we have that $c * d$ is smaller than or equal to the union of $n \times m$ elements of $X * Y$ and, therefore, $c * d \in (X * Y]$. □

Proposition 8.18. *Let* $\mathbf{A}$ *be a* $\mathring{\mathbb{M}}^{s\ell}$*-algebra. Then the structure* $\mathbf{C}^D_{\langle A,*,1\rangle}$*, built using the monoidal reduct of* $\mathbf{A}$*, the closure operator* $C = C^{Id}$ *and a* C^{Id}*-closed* D*, is a complete* $\mathbb{FL}$*-algebra.*

Proof: If $\mathbf{A}$ is a $\mathring{\mathbb{M}}^{s\ell}$-algebra, by Lemma 8.17, we have $C^{Id}(X) * C^{Id}(Y) \subseteq C^{Id}(X * Y)$. Thus, C^{DM} satisfies the condition (8.1) of Proposition 8.12 and, consequently, $\mathbf{C}^D_{\langle A,*,1\rangle}$ is a complete $\mathbb{FL}$-algebra. □

Definition 8.19 (Ideal-completion of a $\mathring{\mathbb{M}}^{s\ell}$-algebra)**.** *Let* $\mathbf{A}$ *be a* $\mathring{\mathbb{M}}^{s\ell}$*-algebra. The complete* $\mathbb{FL}$*-algebra* $\mathbf{C}^{(0]}_{\langle A,*,1\rangle}$ *built using the monoidal reduct of* $\mathbf{A}$*, the operator* C^{Id} *and the* C^{Id}*-closed* $(0]$ *by the method of Proposition 8.12 will be called* ideal-completion *of* $\mathbf{A}$ *and will be denoted by* $\mathbf{A^{Id}}$*. The set of the* C^{Id}*-closed, that is, the set of the ideals of* $\mathbf{A}$*, will be denoted by* A^{Id}*.*

In the following result we show that if the initial $\mathring{\mathbb{M}}^{s\ell}$-algebra $\mathbf{A}$ is a $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$-algebra, then its ideal-completion also satisfies the properties codified by σ, that is, $\mathbf{A^{Id}}$ is a $\mathbb{FL}_{\sigma}$-algebra.

Proposition 8.20. *When* $\mathbf{A}$ *is a* $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$*-algebra, with* $\sigma \leq ew_lw_rc$*, then* $\mathbf{A^{Id}}$ *is a complete* $\mathbb{FL}_{\sigma}$*-algebra.*

Proof: If the monoidal operation of $\mathbf{A}$ is commutative, obviously $\mathbf{A^{Id}}$ is a $\mathbb{FL}_e$-algebra. If $\mathbf{A}$ is a $\mathring{\mathbb{M}}^{s\ell}_{w_r}$-algebra, 0 is the minimum of A, then $(0]$ is the smallest ideal, that is, $(0] = (\emptyset]$ and, therefore, $\mathbf{A^{Id}}$ is a $\mathbb{FL}_{w_r}$-algebra. If $\mathbf{A}$ is a $\mathring{\mathbb{M}}^{s\ell}_{w_l}$-algebra, 1 is the maximum of A, then $(1]$ is the greatest ideal, that is, $(1] = A$ and, therefore, $\mathbf{A^{Id}}$ is a $\mathbb{FL}_{w_l}$-algebra. Thus, obviously, if $\mathbf{A}$ is a $\mathring{\mathbb{M}}^{s\ell}_{w}$-algebra, $\mathbf{A^{Id}}$ is a $\mathbb{FL}_w$-algebra. Finally, we suppose that $\mathbf{A}$ is a $\mathring{\mathbb{M}}^{s\ell}_{c}$-algebra. To see that $\mathbf{A^{Id}}$ is a $\mathbb{FL}_c$-algebra, we have to show that, for every ideal I of $\mathbf{A}$, $I \subseteq (I * I]$. Let $a \in I$. Obviously, $a^2 \in I * I$ and, therefore, $a^2 \in (I * I]$ but, as $\mathbf{A}$ is a $\mathring{\mathbb{M}}^{s\ell}_{c}$-algebra, $a \leq a^2$ and thus, as $(I * I]$ is an ideal, we obtain $a \in (I * I]$. □

The following lemmas will be used to state the embedding theorem.

Lemma 8.21. *Let* $\mathbf{A}$ *be a* $\mathring{\mathbb{M}}^{s\ell}$*-algebra and let* $I, I_1, I_2 \in A^{Id}$*. The following conditions are satisfied:*[2]

$$
\begin{array}{llcl}
1) & I_1 \vee_C I_2 & = & \{a \in A : a \leq i_1 \vee i_2 \textit{ for some } i_1 \in I_1, i_2 \in I_2\} \\
2) & I_1 \cap I_2 & = & \{a \in A : a \leq i_1 \wedge i_2 \textit{ for some } i_1 \in I_1, i_2 \in I_2\} \\
3) & I_1 *_C I_2 & = & \{a \in A : a \leq i_1 * i_2 \textit{ for some } i_1 \in I_1, i_2 \in I_2\} \\
4) & I_1 \backslash I_2 & = & \{a \in A : i_1 * a \in I_2 \textit{ for some } i_1 \in I_1\} \\
5) & I_2 / I_1 & = & \{a \in A : a * i_1 \in I_2 \textit{ for some } i_1 \in I_1\} \\
6) & I^{\backprime} & = & \{a \in A : a \leq i^{\backprime} \textit{ for some } i \in I\} \\
7) & {}'I & = & \{a \in A : a \leq {}'i \textit{ for some } i \in I\}
\end{array}
\qquad (8.2)
$$

Proof:

1): $I_1 \vee_C I_2$ is the smallest ideal containing $I_1 \cup I_2$ and every such ideal must contain the set

$$J = \{a \in A : a \leq i_1 \vee i_2 \text{ for some } i_1 \in I_1, i_2 \in I_2\}.$$

But the set J is an ideal: if $b \leq a \in J$ and $a \leq i_1 \vee i_2$ then also $b \leq i_1 \vee i_2$ and, therefore, $b \in J$. If $a, b \in J$ and $a \leq i_1 \vee i_2$, $b \leq i_1' \vee i_2'$, then $a \vee b \leq (i_1 \vee i_2) \vee (i_1' \vee i_2') = (i_1 \vee i_1') \vee (i_2 \vee i_2')$ and since $i_1 \vee i_1' \in I_1$ and $i_2 \vee i_2' \in I_2$, we have $a \vee b \in J$. In consequence, $I_1 \vee_C I_2 = J$.

2): Let us consider the set

$$K = \{a \in A : a \leq i_1 \wedge i_2 \text{ for some } i_1 \in I_1, i_2 \in I_2\}.$$

If $a \in K$ and $a \leq i_1 \wedge i_2$, then also $a \leq i_1$ i $a \leq i_2$ and hence $a \in I_1$ and $a \in I_2$, that is, $a \in I_1 \cap I_2$. Reciprocally, if $a \in I_1 \cap I_2$, then from $a \leq a \wedge a$ we can deduce $a \in K$.

3): $I_1 *_C I_2$ is the smallest ideal containing $I_1 * I_2$ and every such ideal must contain the set

$$L = \{a \in A : a \leq i_1 * i_2 \text{ for some } i_1 \in I_1, i_2 \in I_2\}.$$

But L is an ideal: if $b \leq a \in L$ and $a \leq i_1 * i_2$ then also $b \leq i_1 * i_2$ and thus, $b \in L$. If $a, b \in L$ and $a \leq i_1 * i_2$, $b \leq i_1' * i_2'$, then $i_1 * i_2, i_1' * i_2' \leq (i_1 \vee i_1') * (i_2 \vee i_2')$. Therefore, $a \vee b \leq (i_1 \vee i_1') * (i_2 \vee i_2')$ and, since $i_1 \vee i_1' \in I_1$ i $i_2 \vee i_2' \in I_2$, we have that $a \vee b \in L$.

4), 5): These are exactly the definitions of the residuals in $\mathbf{A^{Id}}$.

6), 7): By the definition we have $I^{\backprime} = \{a \in A : i * a \in (0] \text{ for each } i \in I\}$. But $i * a \in (0]$ is equivalent to $i * a \leq 0$ which by (LP) it is equivalent to $a \leq i^{\backprime}$. The proof of (7) is analogous. $\square$

Lemma 8.22. *Let* $\mathbf{A}$ *be a* $\mathring{\mathbb{M}}^{s\ell}$*-algebra.*

i) For each $a, b \in A$*, if* $a \backslash b$ *and* b/a *exists, then* $(a \backslash b] = (a] \backslash (b]$ *and* $(a/b] = (a]/(b]$*.*

ii) For each $a \in A$*, if* $a^{\backprime}$ *and* ${}'a$ *exists, then* $(a^{\backprime}] = (a]^{\backprime}$ *i* $({}'a] = {}'(a]$*.*

[2] To simplify the notation, we denote by C the operator C^{Id}.

Proof: *i*) Let $a, b \in A$ and suppose that the residual $a\backslash b$ exists. By (4) of Lemma 8.21 and (LR), we have

$$(a]\backslash(b] = \{z \in A : x * z \leq b \text{ for some } x \leq a\} = \{z \in A : z \leq x\backslash b \text{ for some } x \leq a\}.$$

If $z \in (a]\backslash(b]$ then $z \leq x\backslash b$ for every $x \leq a$ and, in particular, $z \leq a\backslash b$, i.e., $z \in (a\backslash b\,]$. Now suppose that $z \leq a\backslash b$, which is equivalent to $a * z \leq b$. If $x \leq a$, by monotonicity we have that $x * z \leq a * z$ and thus $x * z \leq b$. Therefore, $z \in (a]\backslash(b]$. It is proved analogously that if there exists a/b, then $(a]/(b] = (a/b\,]$.

ii) It is a particular case of *i*) because if $a^{\backprime}$ and $'a$ exist, then $a^{\backprime} = a\backslash 0$ and $'a = 0/a$ and so $(a]^{\backprime} = (a]\backslash(0] = (a\backslash 0] = (a^{\backprime}\,]$ and, analogously, $'(a] = ('a\,]$. $\square$

Lemma 8.23. *Let* $\mathbf{A}$ *be a* $\vee$*-semilattice and suppose* $X \subseteq A$*. Then, if there exists the infimum* $\bigwedge_{\mathbf{S}} X$ *of* X *in* $\mathbf{A}$*, then the principal ideal generated by* $\bigwedge_{\mathbf{A}} X$ *is equal to the meet of the principal ideals generated by the elements of* X*, i.e.,* $(\bigwedge_{\mathbf{A}} X] = \bigcap_{a \in X} (a]$.

Proof: $b \in \bigcap_{a \in X} (a]$ *iff* $b \leq a$ for all $a \in X$ *iff* $b \leq \bigwedge_{\mathbf{A}} X$ *iff* $b \in (\bigwedge_{\mathbf{A}} X]$. $\square$

Theorem 8.24. *For every* $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$*-algebra* $\mathbf{A} = \langle A, \vee, *, 0, 1\rangle$*, the mapping defined by* $i_{\mathbf{A}}(a) = (a]$*, for each* $a \in A$*, is an embedding (i.e., a* $\langle \vee, *, 0, 1\rangle$*-monomorphism) from* $\mathbf{A}$ *into the* $\langle \vee, *, 0, 1\rangle$*-reduct of the complete* $\mathbb{FL}_{\sigma}$*-algebra* $\mathbf{A}^{\mathbf{Id}}$*, which preserves all the residuals, pseudocomplements and existing meets.*

Proof: Obviously $i_{\mathbf{A}}$ is injective, because for each $a, b \in A$, if $a \neq b$ then $(a] \neq (b]$. Let $a, b \in A$ and $\odot \in \{\vee, *\}$. By the characterizations in Lemma 8.21 we have

$$(a] \odot_C (b] = \{x \in A : x \odot y \text{ for some } x \leq a, y \leq b\}.$$

It is obvious that $(a \odot b] \subseteq (a] \odot_C (b]$. If $z \in (a] \odot_C (b]$, then $z \leq x \odot y$ with $x \leq a$ and $y \leq b$ and, by the monotonicity of $\odot$, $z \leq a \odot b$, i.e., $z \in (a \odot b]$. Therefore, $(a] \odot_C (b] = (a \odot b]$. Moreover, the distinguished element and the unit element of $\mathbf{A}^{\mathbf{Id}}$ are, respectively, the principal ideals $(0] = i_{\mathbf{A}}(0)$ and $(1] = i_{\mathbf{A}}(1)$. Therefore, $i_{\mathbf{A}}$ is an homomorphism. By Lemma 8.22 we have that $i_{\mathbf{A}}$ preserves the existing residuals and pseudocomplements. Finally, by Lemma 8.23, we have that $i_{\mathbf{A}}$ preserves the existing meets. $\square$

Nevertheless, as we will see later, in general $i_{\mathbf{A}}$ does not preserve the existing infinite meets (see Corollary 8.51 and Proposition 8.57).

Theorem 8.24 allows us to obtain the following consequences.

Corollary 8.25. *For every algebra* $\mathbf{A}$ *in* $\mathbb{PM}^{s\ell}_{\sigma}$*, the mapping* $i_{\mathbf{A}}$ *is an embedding from* $\mathbf{A}$ *into the* $\langle \vee, *, ^{\backprime}, ', 0, 1\rangle$*-reduct of the complete* $\mathbb{FL}_{\sigma}$*-algebra* $\mathbf{A}^{\mathbf{Id}}$*. This embedding preserves the existing residuals and meets.*

Corollary 8.26. *For every algebra* $\mathbf{A}$ *in* $\mathring{\mathbb{M}}^{\ell}_{\sigma}$ *the mapping* $i_{\mathbf{A}}$ *is an embedding from* $\mathbf{A}$ *into the* $\langle \vee, \wedge, *, 0, 1\rangle$*-reduct of the complete* $\mathbb{FL}_{\sigma}$*-algebra* $\mathbf{A}^{\mathbf{Id}}$*. This embedding preserves the existing pseudocomplements, residuals and meets.*

Corollary 8.27. *For every algebra* $\mathbf{A}$ *in* $\mathbb{PM}^{\ell}_{\sigma}$*, the mapping* $i_{\mathbf{A}}$ *is an embedding from* $\mathbf{A}$ *into the* $\langle \vee, *, ^{\backprime}, ', 0, 1\rangle$*-reduct of the complete* $\mathbb{FL}_{\sigma}$*-algebra* $\mathbf{A}^{\mathbf{Id}}$*. This embedding preserves the existing residuals and meets.*

Corollary 8.28. *For every algebra* $\mathbf{A}$ *in* $\mathbb{FL}_{\sigma}$*, the mapping* $i_{\mathbf{A}}$ *is an embedding from* $\mathbf{A}$ *into the* $\langle \vee, \wedge, *, \backslash, /, ^{\backprime}, ', 0, 1\rangle$*-reduct of the complete* $\mathbb{FL}_{\sigma}$*-algebra* $\mathbf{A}^{\mathbf{Id}}$*. This embedding preserves the existing meets.*

Definition 8.29. *If* $\mathbf{A}$ *is an algebra in* $\mathring{\mathbb{M}}^{\ell}_{\sigma}$*,* $\mathbb{PM}^{s\ell}_{\sigma}$*,* $\mathbb{PM}^{\ell}_{\sigma}$ *or* $\mathbb{FL}_{\sigma}$*, the complete* $\mathbb{FL}_{\sigma}$*-algebra obtained by ideal-completion of its* $\langle \vee, *, 0, 1\rangle$*-reduct it will be denoted by* $\mathbf{A}^{\mathbf{Id}}$ *and we will call it the* ideal-completion *of* $\mathbf{A}$*.*

Note 8.30. The results in Theorem 8.24 and Corollary 8.28 are obtained by Ono in [Ono03a, Theorem 7] for $\mathring{\mathbb{M}}^{s\ell}_{ew}$-algebras and $\mathbb{FL}_{ew}$-algebras by using a different method. As Ono notes in [Ono03a, p.435], his method can be easily adapted to the cases non 0-bounded, non integral or non commutative.

8.2.2 Dedekind-MacNeille completion for $\mathbb{FL}_{\sigma}$-algebras

As is well known, the Dedekind-MacNeille completion, due to MacNeille [Mac37], is the generalization to any ordered set of Dedekind's construction of the irrational numbers by *cuts.* Let us briefly recall the construction of MacNeille. Let $\mathcal{A} = \langle A, \leq\rangle$ an ordered set. The operator defined by $C(X) = (X^{\rightarrow})^{\leftarrow}$ for every $X \subseteq A$, is a closure operator. Thus, the closure system associated with C (i.e., the set of the C-closeds), with the operations defined by $I \vee_C J = ((I \cup J)^{\rightarrow})^{\leftarrow}$ and $I \wedge_C J = I \cap J$, where I and J are C-closed sets, it is a complete lattice. If $x \in A$ we will denote by $x^{\rightarrow}$ the set $\{x\}^{\rightarrow}$. Then the mapping $x \mapsto (x^{\rightarrow})^{\leftarrow}$ is an immersion of $\mathcal{A}$ in this complete lattice that preserves all the existing meets and joins in $\mathcal{A}$ (for the details, see for example [Bir73]).[3]

In [Ono93] the author expandds this construction taking as starting point any $\mathbb{FL}$-algebra. In the previous section, we have seen that the ideal-completion of a lattice also works if we start from a $\mathring{\mathbb{M}}^{s\ell}$-algebra, and that the construction can be extended to $\mathbb{FL}$-algebras using its $\langle \vee, *, 0, 1\rangle$-reduct. At this point, a natural question arises: could we make the construction of Dedekind-MacNeille from a $\mathring{\mathbb{M}}^{s\ell}$-algebra? As we will show later, the answer is negative. The underlying reason is that residuation plays a crucial role in the realization of this construction. In what follows, we present the completion of Dedekind-MacNeille for $\mathbb{FL}$-algebras. This construction is similar to Ono's in [Ono93] but with slight differences. Ono proves the result in a language with only one residual and observes that, with some modifications, a similar construction can be obtained if we have the two residuals (cf. [Ono93, p.269]).

Definition 8.31. *Let* $\mathbf{A} = \langle A, \vee, \wedge, *, \backslash, /, ^{\backprime}, ', 0, 1\rangle$ *be a* $\mathbb{FL}$*-algebra and let* C^{DM} *be the closure operator on* $\mathcal{P}(A)$ *defined by* $C^{DM}(X) = (X^{\rightarrow})^{\leftarrow}$ *for every* $X \subseteq A$*. Observe that, for each* $a \in A$*,* $(a^{\rightarrow})^{\leftarrow} = (a]$*.*

[3] Obviously, this construction can also be carried out from a semilattice or from a lattice.

In the following we state that C^{DM} satisfies condition (8.1) of Proposition 8.12 (cf.[Ono93, page 267 and following]).[4] To state the result we will use the following lemmas dealing with residuated monoids.

Lemma 8.32. *Let $\mathcal{A} = \langle A, *, \backslash, /, 1, \leq \rangle$ be a residuated monoid and let $X \subseteq A$. The following conditions are equivalent.*

i) $a \in (X^{\rightarrow})^{\leftarrow}$, i.e., for every $z \in A$, if $x \leq z$ for every $x \in X$, then $a \leq z$.

ii) For each $u, v \in A$, if $u * x \leq v$ for every $x \in X$, then $u * a \leq v$.

Proof:

i) $\Rightarrow$ *ii)*: Suppose that $u * x \leq v$ for every $x \in X$. By the law of residuation (LR) this is equivalent to saying that $x \leq u\backslash v$ for every $x \in X$. Therefore, applying *i)*, $a \leq u\backslash v$ and, again by (LR), $u * a \leq v$.

ii) $\Rightarrow$ *i)*: Suppose that, for every $x \in X$, we have $x \leq z$. This is equivalent to saying that, for every $x \in X$, $1 * x \leq z$. Therefore, applying *ii)*, $1 * a \leq z$, that is, $a \leq z$. □

Lemma 8.33. *Let $\mathbf{A}$ be a residuated monoid. Then, for each $X, Y \subseteq A$,*

$$(X^{\rightarrow})^{\leftarrow} * (Y^{\rightarrow})^{\leftarrow} \subseteq ((X * Y)^{\rightarrow})^{\leftarrow}.$$

Proof: Let $a \in (X^{\rightarrow})^{\leftarrow}$ and $b \in (Y^{\rightarrow})^{\leftarrow}$. By the above lemma, it will be sufficient to see that for each $u, v \in A$, if $u * (x * y) \leq v$ for every $x \in X$ and $y \in Y$, then $u*(a*b) \leq v$. If $u*(x*y) \leq v$, by associativity we have $(u*x)*y \leq v$. From this, since $b \in (Y^{\rightarrow})^{\leftarrow}$, by the above lemma we obtain $(u*x)*b \leq v$ which, by (LR), is equivalent to $u*x \leq v/b$ and, hence, since $a \in (X^{\rightarrow})^{\leftarrow}$, applying again the above lemma, we have $u*a \leq v/b$ which by (LR) it is equivalent to $(u*a)*b \leq v$; finally, by associativity we obtain $u * (a * b) \leq v$. □

Proposition 8.34. *Let $\mathbf{A}$ be a $\mathbb{FL}$-algebra. The structure $\mathbf{C}^{D}_{\langle A,*,1\rangle}$, built from the monoidal reduct of $\mathbf{A}$, the closure operator $C = C^{DM}$ and a C^{DM}-closed D, is a complete $\mathbb{FL}$-algebra.*

Proof: Suppose that $\leq$ is the order associated to the lattice reduct of $\mathbf{A}$. Then the structure $\langle A, *, \backslash, /, 1, \leq \rangle$ is a residuated *po*-monoid. Thus, applying Lema 8.33, we have that $C^{DM}(X) * C^{DM}(Y) \subseteq C^{DM}(X * Y)$. Therefore, C^{DM} satisfies condition (8.1) of Proposition 8.12 and, in consequence, $\mathbf{C}^{D}_{\langle A,*,1\rangle}$ is a complete $\mathbb{FL}$-algebra. □

[4]The proof that Ono makes of this fact starts from the presence of a unique residual in the language; the author indicates, however, that the proof also works, with some modifications, if we start from a language with two residuals. Here we give a version of the proof for the case that the language has two residuals (right and left) and two pseudocomplements (right and left).

Definition 8.35 (Dedekind-MacNeille Completion)**.** *Let* $\mathbf{A}$ *be a* $\mathbb{FL}$*-algebra. We will use the term* Dedekind-MacNeille completion *of* $\mathbf{A}$ *(abbreviated* DM-completion*), and denoted by* $\mathbf{A^{DM}}$*, to refer tothe complete* $\mathbb{FL}$*-algebra* $\mathbf{C}^{(0^{\rightarrow})^{\leftarrow}}_{\langle A,*,1\rangle}$ *constructed from the monoidal reduct of* $\mathbf{A}$*, the operator* C^{DM} *and the* C^{DM}*-closed* $(0^{\rightarrow})^{\leftarrow}$ *for the procedure of Proposition 8.12. We will denote its universe by* A^{DM}*.*

Observation 8.36. He have that, for every $a \in A$, $(a^{\rightarrow})^{\leftarrow} = (a]$, the distinguished element and the unit element are, respectively, equal to the principal ideals $(0]$ i $(1]$. Thus, on the one hand, the smallest C^{DM}-closed is $\{\bot\} = (\bot]$ if $\mathbf{A}$ has a *bottom* $\bot$ and it is $\emptyset$ if $\mathbf{A}$ has not a *bottom*; and, on the other hand, the greatest C^{DM}-closed is A, which is equal to $(\top]$ if $\mathbf{A}$ has a *top* $\top$.

Now we show that, if the initial $\mathbb{FL}$-algebra $\mathbf{A}$ is a $\mathbb{FL}_\sigma$-algebra, then its DM-completion is also a $\mathbb{FL}_\sigma$-algebra.

Proposition 8.37. *Let* $\sigma \leq ew_lw_rc$*. The* DM*-completion preserves the properties codified by the sequence* σ*, that is, if* $\mathbf{A}$ *is a* $\mathbb{FL}_\sigma$*-algebra,* $\mathbf{A^{DM}}$ *is also a* $\mathbb{FL}_\sigma$*-algebra.*

Proof: If $\mathbf{A}$ is commutative, obviously $\mathbf{A^{Id}}$ is also. If 0 is the minimum of $\mathbf{A}$, then $(0]$ is the minimum of $\mathbf{A^{DM}}$. If 1 $\mathbf{A}$, then $(1]$ is the maximum of $\mathbf{A^{DM}}$. Thus, if $\mathbf{A}$ is a $\mathbb{FL}_\sigma$-algebra, with $\sigma \leq w_lw_r$, $\mathbf{A^{DM}}$ is a $\mathbb{FL}_\sigma$-algebra. Finally, suppose that $\mathbf{A}$ is a $\mathbb{FL}_c$-algebra and suppose $I \in A^{DM}$. Let $a \in I$. Obviously $a^2 \in ((I * I)^{\rightarrow})^{\leftarrow}$. Suppose $y \leq z$ for every $y \in I * I$. Then $a^2 \leq z$ but, since $\mathbf{A}$ is a $\mathbb{FL}_c$-algebra, $a \leq a^2$ and so $a \leq z$. Therefore, $a \in ((I * I)^{\rightarrow})^{\leftarrow}$. Thus we have $I \subseteq ((I * I)^{\rightarrow})^{\leftarrow}$. □

In the following result we show that the C^{DM}-closed are ideals. Following [Bir73] for the case of the completions of lattices, we call the C^{DM}-closed *closed ideals.*

Proposition 8.38. *Every* C^{DM}*-closed is an ideal. Therefore, for every* $\mathbb{FL}$*-algebra* $\mathbf{A}$*,* $A^{DM} \subseteq A^{Id}$*.*

Proof: We have to prove that if $X \subseteq A$ is such that $X = (X^{\rightarrow})^{\leftarrow}$, then X is an ideal. Indeed, if $a \in (X^{\rightarrow})^{\leftarrow}$, since $a \leq y$ for every $y \in X^{\rightarrow}$, if $b \leq a$ we have that $b \leq y$ for every $y \in X^{\rightarrow}$ and, hence, $b \in (X^{\rightarrow})^{\leftarrow}$. If $a, b \in (X^{\rightarrow})^{\leftarrow}$, since $a \leq y$ and $b \leq y$ for every $y \in X^{\rightarrow}$, we have that $a \vee b \leq y$ for every $y \in X^{\rightarrow}$ and, therefore, $a \vee b \in (X^{\rightarrow})^{\leftarrow}$.

□

In the following we show that, given a $\mathbb{FL}$-algebra $\mathbf{A}$, the injective mapping $i_{\mathbf{A}}$ that states the embedding from $\mathbf{A}$ into $\mathbf{A^{Id}}$ is also an embedding from $\mathbf{A}$ into $\mathbf{A^{DM}}$ and, moreover, if $\mathbf{A}$ is complete, then $i_{\mathbf{A}}$ is an isomorphism (Cf. [Ono93, Lemma 4.5]). To state this result we need some lemmas.

Lemma 8.39. *The set of all the principal ideals of a* $\mathbb{FL}$*-algebra* $\mathbf{A}$ *is the universe of a subalgebra of the complete* $\mathbb{FL}$*-algebra* $\mathbf{A^{Id}}$*.*

Proof: The set of all the principal ideals of $\mathbf{A}$ is equal to $i_{\mathbf{A}}[A]$, which is the isomorph copy of $\mathbf{A}$ by $i_{\mathbf{A}}$ and, consequently, a subalgebra of $\mathbf{A^{Id}}$. □

Lemma 8.40. *The set of all the principal ideals of a $\mathbb{FL}$-algebra $\mathbf{A}$ is the universe of a subalgebra of the complete $\mathbb{FL}$-algebra $\mathbf{A^{DM}}$.*

Proof: Given that, for each $a \in A$, $(a] = (\{a\}^{\rightarrow})^{\leftarrow}$, closure operators C^{Id} and C^{DM} restricted to the singletons coincide and, thus, the operations of $\mathbf{A^{DM}}$ and $\mathbf{A^{Id}}$ restricted to the principal ideals are the same. Therefore, as a consequence of the fact that the set of all the principal ideals of $\mathbf{A}$ is the universe of a subalgebra of $\mathbf{A^{Id}}$, we have that this set is also the universe of a subalgebra of $\mathbf{A^{DM}}$. □

Lemma 8.41. *Let $\mathbf{A}$ be a $\mathbb{FL}$-algebra and let $X \subseteq A$. Then, if the supremum of $\bigvee_{\mathbf{A}} X$ of X in $\mathbf{A}$ exists, then the principal ideal generated by $\bigvee_{\mathbf{A}} X$ is equal to the set $(X^{\rightarrow})^{\leftarrow}$, i.e., $(\bigvee_{\mathbf{A}} X] = (X^{\rightarrow})^{\leftarrow}$.*

Proof: We use the characterization of Lemma 8.32. Let $z \leq \bigvee_{\mathbf{A}} X$. If u, v are arbitrary elements of A, suppose $u * x \leq v$ for every $x \in X$. Then, by (LR), $x \leq u\backslash v$ for every $x \in X$ and, thus, $\bigvee_{\mathbf{A}} X \leq u\backslash v$ and from this we obtain $z \leq u\backslash v$ which is equivalent to $u * z \leq v$. Therefore, $z \in (X^{\rightarrow})^{\leftarrow}$. Reciprocally, if $z \in (X^{\rightarrow})^{\leftarrow}$ we have that, given $u, v \in A$, if $u * x \leq v$ for every $x \in X$, then $u * z \leq v$. Now taking $u = 1$ and $v = \bigvee_{\mathbf{A}} X$, since $x \leq \bigvee_{\mathbf{A}} X$ for every $x \in X$, we have that $z \leq \bigvee_{\mathbf{A}} X$. □

Lemma 8.42. (Cf. [Bir73]) *Let $\mathbf{L} = \langle L, \vee, \wedge \rangle$ be a lattice. The following conditions are equivalent:*

a) *$\mathbf{L}$ is complete.*

b) *Every closed ideal of $\mathbf{L}$ is principal.*

c) *A subset of L is a closed ideal if and only if it is a principal ideal.*

Proof: $a) \Rightarrow b)$: Let us suppose that $\mathbf{L}$ is complete. If I is a closed ideal, $(I^{\rightarrow})^{\leftarrow} = I$. Let $a = \bigvee_{\mathbf{L}} I$. We will see that $I = (a]$. If $b \in I$, then $b \leq a$ and so $b \in (a]$. Since $a = \bigvee_{\mathbf{L}} I$, then $a = \bigwedge_{\mathbf{L}} I^{\rightarrow}$. Thus, if $b \in (a]$, that is, if $b \leq a$, we have that $b \in (I^{\rightarrow})^{\leftarrow} = I$.

$b) \Rightarrow a)$: Let $X \subseteq L$. By $b)$, the closed ideal $(X^{\rightarrow})^{\leftarrow}$ is principal, and so $(X^{\rightarrow})^{\leftarrow} = (a]$ for some $a \in L$. We will see that $a = \bigvee_{\mathbf{L}} X$. Indeed, since $X \subseteq (X^{\rightarrow})^{\leftarrow}$, if $b \in X$ we have $b \leq a$. On the other hand, $a \in (X^{\rightarrow})^{\leftarrow}$ implies that $a \leq b$ for every $b \in X^{\rightarrow}$.

$b) \Leftrightarrow c)$: It is immediate since the fact that every principal ideal is a closed ideal.

□

Theorem 8.43. *Given a $\mathbb{FL}_\sigma$-algebra $\mathbf{A}$, the mapping defined by $i_{\mathbf{A}} : a \mapsto (a]$, for every $a \in A$, it is an embedding from $\mathbf{A}$ into the complete $\mathbb{FL}_\sigma$-algebra $\mathbf{A^{DM}}$ which preserves all the existing meets and joins. If $\mathbf{A}$ is complete, then $i_{\mathbf{A}}$ is an isomorphism.*

Proof: As a consequence of Lemma 8.40 we have that $i_{\mathbf{A}}$ is a homomorphism. By Lemma 8.23 we have that $i_{\mathbf{A}}$ preserves the existing (infinite) meets. The mapping

$i_{\mathbf{A}}$ also preserves the existing (infinite) joins: By Lemma 8.41 we have $i_{\mathbf{A}}(\bigvee_{\mathbf{A}} X) = (\bigvee_{\mathbf{A}} X] = (X^{\rightarrow})^{\leftarrow}$. Applying (1) of Proposition 8.13 we have

$$(X^{\rightarrow})^{\leftarrow} = C^{DM}(X) = C^{DM}(\bigcup_{x \in X} \{x\}) = C^{DM}(\bigcup_{x \in X} C^{DM}(x)) = C^{DM}(\bigcup_{x \in X} (x]).$$

But observe that $C^{DM}(\bigcup_{x \in X} (x]) = \bigvee_{\mathbf{A^{DM}}} \{i_{\mathbf{A}}(x) : x \in X\} = \bigvee_{\mathbf{A^{DM}}} i_{\mathbf{A}}[X]$. Therefore, $i_{\mathbf{A}}(\bigvee_{\mathbf{A}} X) = \bigvee_{\mathbf{A^{DM}}} i_{\mathbf{A}}[X]$.

Finally, if $\mathbf{A}$ is complete, by Lemma 8.42, we have that all the closed ideals are principal and, therefore, the mapping $i_{\mathbf{A}}$ is epijective. $\square$

Therefore we have that every $\mathbb{FL}_{\sigma}$-algebra is embeddable in a complete $\mathbb{FL}_{\sigma}$-algebra as we claimed at the begining of the present chapter (Theorem 8.10). An embedding preserving all the existing meets and joins is called a *complete embedding*, and if there is a complete embedding from a $\mathbb{FL}$-algebra $\mathbf{A}$ into a $\mathbb{FL}$-algebra $\mathbf{B}$, we will say that $\mathbf{A}$ is *completely embeddable* into $\mathbf{B}$. Therefore, as the embedding $i_{\mathbf{A}}$ is complete, we can strengthen Theorem 8.10 in the following way.

Theorem 8.44. *Every $\mathbb{FL}_{\sigma}$-algebra is completely embeddable into a complete $\mathbb{FL}_{\sigma}$-algebra.*

From Theorem 8.43 and Proposition 8.12 we obtain the following characterization for complete $\mathbb{FL}$-algebras.

Corollary 8.45. *Let $\mathbf{A}$ be a $\mathbb{FL}$-algebra. The following conditions are equivalent:*

a) $\mathbf{A}$ *is complete.*

b) $\mathbf{A}$ *is isomorphic to the algebra $\mathbf{C}^{D}_{\mathbf{M}}$ for some monoid $\mathbf{M}$, some closure operator C on $\mathcal{P}(M)$ satisfying condition (8.1), and some C-closed D.*

Proof: *a)* $\Rightarrow$ *b)*: If $\mathbf{A}$ is complete, by Theorem 8.43 we have $\mathbf{A} \cong \mathbf{A^{DM}}$ and $\mathbf{A^{DM}} = \mathbf{C}^{D}_{\mathbf{M}}$, where $\mathbf{M}$ is the monoidal reduct of $\mathbf{A}$, $C = C^{DM}$ and $D = (0]$, where 0 is the distinguished element of $\mathbf{A}$.

b) $\Rightarrow$ *a)*: If $\mathbf{A}$ is isomorphic with $\mathbf{C}^{D}_{\mathbf{M}}$ for certain M, C and D, since $\mathbf{C}^{D}_{\mathbf{M}}$ is complete, by the fact that the reticular reducts of $\mathbf{A}$ and $\mathbf{C}^{D}_{\mathbf{M}}$ are isomorphic, we have that $\mathbf{A}$ is complete. $\square$

8.2.3 The DM-completion does not work for $\mathring{\mathbb{M}}^{s\ell}$-algebras

So far we have considered two methods, the ideal-completion and the DM-completion, which allow us to construct, from any $\mathbb{FL}_{\sigma}$-algebra, a complete $\mathbb{FL}_{\sigma}$-algebra so that the first is embeddable into the second. Moreover, we have seen that the ideal-completion provides a method to built from any $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$-algebra a complete $\mathbb{FL}_{\sigma}$-algebra in such a form that the first is embeddable in the $\langle \vee, *, 0, 1 \rangle$-reduct of the second.

In this subsection the following contributions are made: a) we show that the DM-completion does not work if we start from a $\mathring{\mathbb{M}}^{s\ell}$-algebra, since in general it does not cause a $\mathbb{FL}$-algebra; and b) we show that if the monoidal operation of a complete $\mathring{\mathbb{M}}^{s\ell}$-algebra is not residuated, then this algebra cannot be submerged in any complete $\mathbb{FL}$-algebra in such a way that the infinite meets are preserved.

In the following result we prove that the closure operator C^{DM} defined by the monoidal reduct of a complete $\mathring{\mathbb{M}}^{s\ell}$-algebra satisfies the condition (8.1) of Proposition 8.12 if and only if the monoidal operation is residuated and, moreover, we show that these conditions are also equivalent to the generalized distributivity of the monoidal operation with respect to the infinite meets.

Theorem 8.46. *Let* $\mathbf{A}$ *be a complete* $\mathring{\mathbb{M}}^{s\ell}$*-algebra. The following conditions are equivalent:*

i) The monoidal operation is residuated.

ii) For each $X, Y \subseteq A$, $(X^{\rightarrow})^{\leftarrow} * (Y^{\rightarrow})^{\leftarrow} \subseteq ((X * Y)^{\rightarrow})^{\leftarrow}$.

iii) $\mathbf{A}$ *satisfies the generalized distributivity of the monoidal operation with respect to the infinite meets.*

Proof:

$i) \Rightarrow ii)$: Let $\leq$ be the order associated to the reticular reduct of $\mathbf{A}$. Then $\langle A, \leq, *, \backslash, /, 1\rangle$ is a residuated *po*-monoide and, therefore, applying Lemma 8.33, we obtain $ii)$.

$ii) \Rightarrow iii)$: Let $X, Y \subseteq A$. We have to see that if condition $ii)$ holds, then $\bigvee_{\mathbf{A}} X * \bigvee_{\mathbf{A}} Y$ is the supremum of $X * Y$. Obviously, by monotonicity, $\bigvee_{\mathbf{A}} X * \bigvee_{\mathbf{A}} Y \in (X * Y)^{\rightarrow}$ and, hence, $\bigvee_{\mathbf{A}}(X * Y) \leq \bigvee_{\mathbf{A}} X * \bigvee_{\mathbf{A}} Y$. On the other hand, by monotonicity and by $ii)$, we have $\bigvee_{\mathbf{A}} X * \bigvee_{\mathbf{A}} Y \in (X^{\rightarrow})^{\leftarrow}.(Y^{\rightarrow})^{\leftarrow} \subseteq ((X.Y)^{\rightarrow})^{\leftarrow}$. Therefore, $\bigvee_{\mathbf{A}} X * \bigvee_{\mathbf{A}} Y \leq \bigvee_{\mathbf{A}}(X * Y)$.

$iii) \Rightarrow i)$: For each $a, b \in A$ we define

$$a\backslash b := \bigvee_{z\in A}\{z : a * z \leq b\}, \qquad b/a := \bigvee_{z\in A}\{z : z * a \leq b\}.$$

To see that $a\backslash b$ is the right residual of a relative to b it will be sufficient to see that this element is the maximum of the set $\{z \in A : a * z \leq b\}$. Indeed, by applying $iii)$, we have

$$a * (a\backslash b) = a * \bigvee_{z\in A}\{z : a * z \leq b\} = \bigvee_{z\in A}\{a * z : a * z \leq b\} \leq b.$$

Taking into account the law of the mirror images, we also have that b/a is the left residual of a relative to b. □

Therefore, the complete $\mathring{\mathbb{M}}^{s\ell}$-algebras with a non residuated monoidal operation are precisely those for which the closure operator C^{DM} does not satisfy the condition (8.1) of Proposition 8.12. Next we give an example of complete $\mathring{\mathbb{M}}^{s\ell}$-algebra with a non-residuated monoidal operation.

Example 8.47. Let us consider the $\mathbb{FL}_{ew}$-algebra $\mathbf{A} = \langle A, \vee, \wedge, *, \rightarrow, \neg, 0, 1\rangle$, where A is the set $\{0,1\} \cup \{x \in \mathbb{R} : \frac{1}{4} \leq x \leq \frac{3}{4}\}$, the lattice operations correspond to the standard order over the real numbers, and the other operations are defined by the following tables (where $a, b, c \in [\frac{1}{4}, \frac{3}{4}]_{\mathbb{R}}$ and $a < c$):

$*$	0	b	1
0	0	0	0
a	0	$\frac{1}{4}$	a
1	0	b	1

$\rightarrow$	0	a	c	1
0	1	1	1	1
a	0	1	1	1
c	0	$\frac{3}{4}$	1	1
1	0	a	c	1

	$\neg$
0	1
a	0
1	0

And now we consider the algebra $\mathbf{B} = \langle B, \vee, *, 0, 1\rangle$, where $B = A \setminus \{\frac{3}{4}\}$ and the operations are the restrictions of the ones defined over $\mathbf{A}$. It is clear that $\mathbf{B}$ is a complete $\mathring{\mathbb{M}}^{s\ell}_{ew}$-algebra. However, the restriction of the operation $*$ to $\mathbf{B}$ is not residuated, because the infinitary distributivity law does not hold in $\mathbf{B}$, e.g.,

$$\left(\bigvee_{\mathbf{B}}[\tfrac{1}{4}, \tfrac{3}{4})_{\mathbb{R}}\right) * \tfrac{1}{2} = 1 * \tfrac{1}{2} = \tfrac{1}{2} \quad \text{while } \bigvee_{\mathbf{B}}\{x * \tfrac{1}{2} : x \in [\tfrac{1}{4}, \tfrac{3}{4})_{\mathbb{R}}\} = \bigvee_{\mathbf{B}}\{\tfrac{1}{4}\} = \tfrac{1}{4}. \qquad \square$$

Thus, given a complete $\mathring{\mathbb{M}}^{s\ell}$-algebra, if its monoidal operation is not residuated we cannot affirm that the structure $\mathbf{C}^{D}_{\mathbf{M}}$ is a $\mathbb{FL}$-algebra. However, could it be a $\mathbb{FL}$-algebra under determinate conditions? In the following proposition we show that the answer to this question is negative: if $\mathbf{A}$ is a complete $\mathring{\mathbb{M}}^{s\ell}$-algebra such that its monoidal operation is not residuated, then the monoidal operation of the structure $\mathbf{C}^{D}_{\mathbf{M}}$ is not residuated either.

Proposition 8.48. *Let $\mathbf{A} = \langle A, \vee, *, 0, 1\rangle$ ba a complete $\mathring{\mathbb{M}}^{s\ell}$-algebra such that the monoidal operation is not residuated with respect to the semilattice order. Then the monoidal operation of the structure $\mathbf{C}^{D}_{\mathbf{M}}$ built from the monoidal reduct of $\mathbf{A}$, the closure operator C^{DM}, and any C^{DM}-closed D, it is not residuated.*

Proof: Suppose that the monoidal operation of $\mathbf{C}^{D}_{\mathbf{M}}$, with $C = C^{DM}$, is residuated. Then we have that the $\langle \vee_C, *_C, D, C(1)\rangle$-reduct of $\mathbf{C}^{D}_{\mathbf{M}}$ is a $\mathring{\mathbb{M}}^{s\ell}$-algebra, because in this context the law of residuation implies the distributivity of the monoidal operation with respect to the join operation and, therefore, as it is a complete algebra, the fact that $*_C$ is residuated, by Theorem 8.46, is equivalent to saying that if $\mathcal{T}_1$ and $\mathcal{T}_2$ are sets of C-closed, then

$$(\mathcal{T}_1^{\rightarrow})^{\leftarrow} *_C (\mathcal{T}_2^{\rightarrow})^{\leftarrow} \subseteq ((\mathcal{T}_1 *_C \mathcal{T}_2)^{\rightarrow})^{\leftarrow}. \tag{8.3}$$

Now let $X, Y \subseteq A$. Since $\langle A^{DM}, \vee_C, \cap\rangle$ is a complete lattice, the C-closed sets coincide with the principal ideals (Lemma 8.42). Therefore, taking $\mathcal{T}_1 = \{C(X)\}$ and $\mathcal{T}_2 = \{C(Y)\}$ in (8.3) we obtain

$$(C(X)] *_C (C(Y)] \subseteq (C(X) *_C (C(Y)].$$

That is, applying the definition of $*_C$,

$$(C(X) * C(Y)] \subseteq (C(X * Y)].$$

But this implies that $C(X) * C(Y) \subseteq C(X * Y)$ and, by Theorem 8.46, this is a contradiction because we are supposing that the operation $*$ is not residuated. $\square$

In the following result we show that if the monoidal operation of a complete $\mathring{\mathbb{M}}^{s\ell}$-algebra is not residuated, then this algebra cannot be embedded in any complete $\mathbb{FL}$-algebra in such a way that all the infinite joins are preserved.

Proposition 8.49. *Let* $\mathbf{A}$ *be a complete* $\mathring{\mathbb{M}}^{s\ell}$*-algebra and let* $\mathbf{B}$ *be a complete* $\mathbb{FL}$*-algebra. Suppose that there exists an embedding from* $\mathbf{A}$ *into the* $\langle \vee, *, 0, 1\rangle$*-reduct of* $\mathbf{B}$*, preserving all the existing joins. Then the monoidal operation of* $\mathbf{A}$ *is residuated.*

Proof: We will see that if there exists such an embedding, which we denote by h, then $\mathbf{A}$ satisfies the infinite distributivity of the monoidal operation with respect to the joins and, thus, in consequence with the characterization of Theorem 8.46, we will have that the monoidal operation of $\mathbf{A}$ is residuated.

Let $X, Y \subseteq A$. On the one hand, we have that $\bigvee_{\mathbf{A}}(X * Y) \leq \bigvee_{\mathbf{A}} X * \bigvee_{\mathbf{A}} Y$. On the other hand, let z be an upper bound of $X * Y$. Then $x * y \leq z$ for every $x \in X$ and every $y \in Y$ and, since h is a $\langle \vee, *, 0, 1\rangle$-homomorphism, we have that $h(x) * h(y) \leq h(z)$ for every $x \in X$ and every $y \in Y$ and, therefore,

$$\bigvee_{\mathbf{B}}(h[X] * h[Y]) \leq h(z).$$

Thus, by Theorem 8.46, since the $\langle \vee, *, 0, 1\rangle$-reduct of $\mathbf{B}$ is a complete $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$-algebra and the monoidal operation of $\mathbf{B}$ is residuated, we have that $\mathbf{B}$ satisfies the infinite distributivity of $*$ with respect to the joins. In consequence,

$$\bigvee_{\mathbf{B}} h[X] * \bigvee_{\mathbf{B}} h[Y] \leq h(z).$$

Now, taking into account that h preserves the joins, we obtain $h(\bigvee_{\mathbf{A}} X) * h(\bigvee_{\mathbf{A}} Y) \leq h(z)$, that is,

$$h(\bigvee_{\mathbf{A}} X * \bigvee_{\mathbf{A}} Y) \leq h(z),$$

and, since h is an order monomorphism we obtain $\bigvee_{\mathbf{A}} X * \bigvee_{\mathbf{A}} Y \leq z$. Thus,

$$\bigvee_{\mathbf{A}} X * Y = \bigvee_{\mathbf{A}} X * \bigvee_{\mathbf{A}} Y$$

and, in consequence, the monoidal operation of $\mathbf{A}$ is residuated. $\square$

Corollary 8.50. *The complete* $\mathring{\mathbb{M}}^{s\ell}$*-algebras with a non residuated monoidal operation cannot be embedded into any complete* $\mathbb{FL}$*-algebra in such a way that all existing infinite joins are preserved.*

Corollary 8.51. *The embedding* $i_{\mathbf{A}}$ *of a complete non residuated* $\mathring{\mathbb{M}}^{s\ell}$*-algebra* $\mathbf{A}$ *in its ideal-completion* $\mathbf{A}^{\mathbf{Id}}$ *does not preserve all the infinite joins.*

8.2.4 Comparison between the two completions in $\mathbb{FL}$-algebras

In this section we make a comparative study of the two constructions and we obtain some new interesting negative results: in Proposition 8.55 we show with an example that there are non complete $\mathbb{FL}$-algebras such that $\mathbf{A^{DM}}$ is not a subalgebra of $\mathbf{A^{Id}}$. The same example allows us to show that in general the embedding from $\mathbf{A}$ into $\mathbf{A^{Id}}$ does not preserve the existing infinite joins (Proposition 8.56). We also show that the ideal-completion preserves all the equations that contain only the operations in $\{\vee, \wedge, *, 0, 1\}$, but it does not preserve in general the equations that contain residuals or pseudocomplements: in Proposition 8.61 we show with an example that the ideal-completació does not preserve the involutive law.

In Proposition 8.38 we have seen that the C^{DM}-closed sets of a $\mathbb{FL}$-algebra $\mathbf{A}$ are ideals of $\mathbf{A}$ and, therefore, $A^{DM} \subseteq A^{Id}$. Next we will see that, in the case that $\mathbf{A}$ is complete, $\mathbf{A^{DM}}$ is a subalgebra of $\mathbf{A^{Id}}$, but we will see that, in general, for non complete $\mathbb{FL}$-algebras, $\mathbf{A^{DM}}$ is not a subalgebra of $\mathbf{A^{Id}}$.

Proposition 8.52. *If $\mathbf{A}$ is a complete $\mathbb{FL}$-algebra, then $\mathbf{A^{DM}}$ is a subalgebra of $\mathbf{A^{Id}}$.*

Proof: If $\mathbf{A}$ is complete, then the closed and the principal ideals coincide (Lemma 8.42). In consequence, since the set of the principal ideals is the universe of a subalgebra of $\mathbf{A}^{Id}$, we have $\mathbf{A^{DM}} \subseteq \mathbf{A^{Id}}$. □

Proposition 8.53. *If $\mathbf{A}$ is a finite $\mathbb{FL}$-algebra, then all the ideals of $\mathbf{A}$ are principal.*

Proof: If A is finite and $I = \{a_1, \ldots, a_n\}$ is an ideal of $\mathbf{A}$, the supremum of I is $a_1 \vee \cdots \vee a_n$ which, by the definition of the notion of ideal, must belong to I. Therefore, if $a = \bigvee_{\mathbf{A}} I$, we have that $I = (a]$. □

Proposition 8.54. *If $\mathbf{A}$ is a finite $\mathbb{FL}$-algebra, then $\mathbf{A} \cong \mathbf{A^{DM}} = \mathbf{A^{Id}}$.*

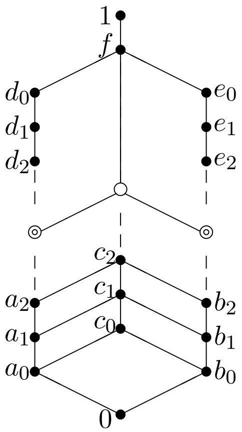
Proof: If $\mathbf{A}$ is finite, then it is complete and, hence, its closed ideals and its principal ideals coincide. Thus, since all the ideals are principal, we have that all the ideals are closed, i.e., $A^{Id} \subseteq A^{DM}$. Therefore, $A^{DM} = A^{Id}$. On the other hand, we have that the mapping $i_{\mathbf{A}} : a \mapsto [a)$ is bijective and, hence, it is an isomorphism. □

Nevertheless, as we will see in a moment, in general it is not true that $\mathbf{A^{DM}}$ is a subalgebra of $\mathbf{A^{Id}}$.

Proposition 8.55. *There are $\mathbb{FL}$-algebras $\mathbf{A}$ such that $\mathbf{A^{DM}} \not\subseteq \mathbf{A^{Id}}$.*

Proof: In order to give a counterexample we can consider the $\mathbb{FL}_{ew}$-algebra $\mathbf{A_1}$ given in Figure 8.1. In the picture we adopt the convention that the points depicted as • are the ones in the algebra $\mathbf{A_1}$, the ones depicted by ⊚ are not in the algebra but they correspond[5] to points in the Dedekind-MacNeille completion, and the points depicted

[5]Strictly speaking this means that the set of points in the algebra that are below the point ⊚ is a member of the Dedekind-MacNeille completion.



$$A_1 := \{0, f, 1\} \cup \{x_n : x \in \{a, b, c, d, e\},\ n \in \omega\}$$

$$x * y \quad := \quad \begin{cases} y, & \text{if } x = 1 \\ x, & \text{if } y = 1 \\ 0, & \text{if } x, y \neq 1 \end{cases}$$

$$x \to y \quad := \quad \begin{cases} 1, & \text{if } x \leq y \\ f, & \text{if } x \not\leq y \end{cases}$$

Figure 8.1: *A* $\mathbb{FL}$*-algebra* $\mathbf{A_1}$ *such that* $\mathbf{A_1^{DM}} \not\subseteq \mathbf{A_1^{Id}}$

by ⊚ correspond to points that are in the ideal completion but not in the Dedekind-MacNeille completion.

Let us consider the ideals $I_1 = \{0\} \cup \{a_n : n \in \omega\}$ i $I_2 = \{0\} \cup \{b_n : n \in \omega\}$, i.e., the ones corresponding to the two ⊚ points. Then, $I_1, I_2 \in A_1^{DM} \subseteq A_1^{Id}$ while

$$I_1 \vee_{\mathbf{A^{Id}}} I_2 = \{0\} \cup \{x_n : x \in \{a, b, c\}, n \in \omega\} \notin A_1^{DM}.$$

Note that in order to obtain $\mathbf{A_1^{DM}}$ from $\mathbf{A_1}$ we add two points, while in the case of $\mathbf{A_1^{Id}}$ we need to add to $\mathbf{A_1^{DM}}$ an additional point. □

Depending on our interests, one construction is more appropriate than the other. The main property of the Dedekind-MacNeille completion is that it preserves all existing infinite joins and infinite meets (Theorem 8.43), i.e., $i_{\mathbf{A}}$ is a complete (also called *regular*) embedding of $\mathbf{A}$ into $\mathbf{A^{DM}}$. On the other hand, the ideal-completion preserves all the existing infinite meets (Corollary 8.28), while it does not preserve infinite joins in general.

Proposition 8.56. *In general* $i_{\mathbf{A}}$ *is not a complete embedding of* $\mathbf{A}$ *into* $\mathbf{A^{Id}}$.

Proof: To see that $i_{\mathbf{A}}$ does not preserve the infinite joins, consider the example in Figure 8.1. We have that $i_{\mathbf{A_1}}(\bigvee_{\mathbf{A_1}}\{c_n : n \in \omega\}) = i_{\mathbf{A_1}}(f)$, while $\bigvee_{\mathbf{A_1^{Id}}}\{i_{\mathbf{A_1}}(c_n) : n \in \omega\}) = \{0\} \cup \{x_n : x \in \{a, b, c\}, n \in \omega\} \neq i_{\mathbf{A_1}}(f)$. □

In the case of $\mathring{\mathbb{M}}^{s\ell}$-algebras we have seen that the embedding $i_{\mathbf{A}}$ of a complete $\mathring{\mathbb{M}}^{s\ell}$-algebra $\mathbf{A}$ with residuated monoidal operation into its ideal-completion $\mathbf{A^{Id}}$ does not preserve the existing infinite joins (Corollary 8.51). The example considered in the above proposition shows also that there are non-complete $\mathring{\mathbb{M}}^{s\ell}$-algebras such that the ideal-completion does not preserve the existing infinite joins.

Proposition 8.57. *There are non-complete $\mathring{\mathbb{M}}^{s\ell}$-algebras such that the embedding $x \mapsto (x]$ in its ideal-completion does not preserve the existing infinite joins.*

Proof: Let $\mathbf{A}'$ be the $\langle \vee, *, 0, 1\rangle$-reduct of the algebra $\mathbf{A}$ in Figure 8.1. $\mathbf{A}'$ is a non-complete $\mathring{\mathbb{M}}^{s\ell}$-algebra. The mapping $i_{\mathbf{A}}$ is an embedding of $\mathbf{A}'$ in its ideal-completion which does not preserve the existing infinite joins. □

It is well known that the DM-completion does not preserve the lattice equations; e.g., the DM-completion of a distributive lattice is not always distributive [Fun44, Cot44, DM52, Cra62]. On the other hand, as we will see in the next, the ideal completion preserve all the equations in which only the connectives in $\langle \vee, \wedge, *, 0, 1\rangle$ appear.

Lemma 8.58. *Let $t(x_1, \ldots, x_n)$ be a term of the language of the $\mathbb{FL}$-algebras containing only connectives in $\langle \vee, \wedge, *, 0, 1\rangle$, and let $\mathbf{A}$ be a $\mathbb{FL}$-algebra. Then the mapping $t^{\mathbf{A}}$: $A^n \longrightarrow A$ defined by $\langle a_1, \ldots, a_n\rangle \mapsto t^{\mathbf{A}}(a_1, \ldots, a_n)$ is monotonous in all its arguments.*

Proof: Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ such that $a_i \leq b_i$ for $1 \leq i \leq n$. We have to see that
$$t^{\mathbf{A}}(a_1, \ldots, a_n) \leq t^{\mathbf{A}}(b_1, \ldots, b_n).$$
We use induction on the length of the term t.

- $t = x$. The we have, for every $a \in A$, $x^{\mathbf{A}}(a) = a$. If $a \leq b$, then obviously $x^{\mathbf{A}}(a) \leq x^{\mathbf{A}}(b)$.
- $t = c$, with $c \in \{0, 1\}$. Then, for every $a \in A$, $c^{\mathbf{A}}(a) = c^{\mathbf{A}}$. If $a \leq b$, then $c^{\mathbf{A}}(a) = c^{\mathbf{A}} = c^{\mathbf{A}}(b)$.
- $t = u \odot v$, with $\odot \in \{\vee, \wedge, *\}$. Suppose $a_i \leq b_i$ for $1 \leq i \leq n$. Then we have

$$(u \odot v)^{\mathbf{A}}(a_1, \ldots, a_n) = u^{\mathbf{A}}(a_1, \ldots, a_n) \odot v^{\mathbf{A}}(a_1, \ldots, a_n).$$

By the induction hypothesis we have

$$u^{\mathbf{A}}(a_1, \ldots, a_n) \leq u^{\mathbf{A}}(b_1, \ldots, b_n) \text{ i } v^{\mathbf{A}}(a_1, \ldots, a_n) \leq v^{\mathbf{A}}(b_1, \ldots, b_n).$$

Thus, since all the operations in $\{\vee, \wedge, *\}$ are monotonous, we obtain

$$(u \odot v)^{\mathbf{A}}(a_1, \ldots, a_n) \leq u^{\mathbf{A}}(b_1, \ldots, b_n) \odot v^{\mathbf{A}}(b_1, \ldots, b_n).$$

That is, $(u \odot v)^{\mathbf{A}}(a_1, \ldots, a_n) \leq (u \odot v)^{\mathbf{A}}(b_1, \ldots, b_n)$. □

Lemma 8.59. *Let $t(x_1, \ldots, x_n)$ be a term of the language of $\mathbb{FL}$-algebras and suppose that this term contain only connectives in $\langle \vee, \wedge, *, 0, 1\rangle$, Let $I_1, \ldots, I_n$ be n ideals of a $\mathbb{FL}$-algebra $\mathbf{A}$. Then the following holds:*

$$t^{\mathbf{A}^{\mathbf{Id}}}(I_1, \ldots, I_n) = \{b \in A : b \leq t^{\mathbf{A}}(a_1, \ldots, a_n) \textit{ for some } a_1 \in I_1, \ldots, a_n \in I_n\}.$$

Proof: We use induction on the length of the term.

• $t = x$. Then we have, for every $I \in A^{Id}$,

$$x^{\mathbf{A^{Id}}}(I) = I = \{b \in A : b \leq a \text{ for some } a \in I\} = \{b \in A : b \leq x^{\mathbf{A}}(a) \text{ for some } a \in I\}.$$

• $t = c$, with $c \in \{0, 1\}$. Then, for every $I \in A^{Id}$, we have

$$c^{\mathbf{A^{Id}}}(I) = (c^{\mathbf{A}}] = \{b \in A : b \leq c^{\mathbf{A}}\},$$

but, for any $a \in I$, $b \leq c^{\mathbf{A}}$ is equivalent to $b \leq c^{\mathbf{A}}(a)$.

• $t = u \odot v$, with $\odot \in \{\vee, \wedge, *\}$. Then we have

$$t^{\mathbf{A^{Id}}}(I_1, \dots, I_n) = (u \odot v)^{\mathbf{A^{Id}}}(I_1, \dots, I_n) = u^{\mathbf{A^{Id}}}(I_1, \dots, I_n) \odot v^{\mathbf{A^{Id}}}(I_1, \dots, I_n).$$

And now, applying the characterizations (8.2) of Lemma 8.21, this is equal to

$$\{b \in A : b \leq c \odot d \text{ with } c \in u^{\mathbf{A^{Id}}}(I_1, \dots, I_n), d \in v^{\mathbf{A^{Id}}}(I_1, \dots, I_n)\}$$

and, by the induction hypothesis, this is equal to

$$\{b \in A : b \leq c \odot d \text{ with } c \leq u^{\mathbf{A}}(a'_1 \dots, a'_n), d \leq v^{\mathbf{A}}(a''_1 \dots, a''_n) \text{ i } a'_i, a''_i \in I_i\}. \tag{8.4}$$

Now we see that this set is equal to

$$\{b \in A : b \leq (u \odot v)^{\mathbf{A}}(a_1, \dots, a_n) \text{ with } a_i \in I_1\}. \tag{8.5}$$

The inclusion of the set (8.5) in the set (8.4) is clear, taking $c = u^{\mathbf{A}}(a_1 \dots, a_n)$, $d = v^{\mathbf{A}}(a_1 \dots, a_n)$, $a'_1 = a''_1 = a_1$, $\dots$, i $a'_n = a''_n = a_n$. To see the other inclusion we take $a_1 = a'_1 \vee a''_1$, $\dots$, $a_n = a'_n \vee a''_n$. Since the I_i are ideals, we have $a_i \in I_i$ for every i. If b belongs to the set (8.4), by the monotonicity of $\odot$ we have

$$b \leq u^{\mathbf{A}}(a'_1 \dots, a'_n) \odot v^{\mathbf{A}}(a''_1 \dots, a''_n).$$

And now, applying Lemma 8.58 by the monotonicity of $\odot$, we have

$$b \leq u^{\mathbf{A}}(a_1 \dots, a_n) \odot v^{\mathbf{A}}(a_1 \dots, a_n).$$

That is, $b \leq (u \odot v)^{\mathbf{A}}(a_1 \dots, a_n)$. □

Theorem 8.60. *Let u and v be terms of the language of the $\mathbb{FL}$-algebras such that they contain only connectives in $\langle \vee, \wedge, *, 0, 1 \rangle$ and let $\mathbf{A}$ be a $\mathbb{FL}$-algebra such that $\mathbf{A} \models u \approx v$. Then $\mathbf{A^{Id}} \models u \approx v$.*

Proof: Suppose that the variables appearing in u and v are in $\{x_1, \dots, x_n\}$. Let $I_1, \dots, I_n$ n be arbitrary ideals of A. Let $J = u^{\mathbf{A^{Id}}}(I_1, \dots, I_n)$, $K = v^{\mathbf{A^{Id}}}(I_1, \dots, I_n)$. We want to see that $J = K$. By symmetry, it will be sufficient to see that $J \subseteq K$. Let $b \in J$. Then, by the previous lemma we have $b \leq u^{\mathbf{A}}(a_1, \dots, a_n)$ for some $a_1 \in I_1, \dots, a_n \in I_n$. But, since $\mathbf{A} \models u \approx v$, we have also $b \leq v^{\mathbf{A}}(a_1, \dots, a_n)$ and, applying again the above lemma, we can conclude $b \in K$. □

The ideal-completion does not always preserve the equations that contain the connectives of negation or implication. For example, the involutive law is not preserved in general (a $\mathbb{FL}$-algebra $\mathbf{A}$ satisfies the involutive law if $\mathbf{A} \models x \approx {}'(x^{\backprime})$ and $\mathbf{A} \models x \approx ({}'x)^{\backprime}$).

Proposition 8.61. *The involutive law is not preserved under the ideal-completion.*

Proof: As a counterexample consider the standard algebra of Łukasiewicz $[0,1]_{Ł}$, that is, the $\mathbb{FL}_{ew}$-algebra defined in the unit real interval by the t-norm of Łukasiewicz and its residuum. Now consider the subinterval $I = [0,a)$, where $0 < a < 1$. It is clear that I is an ideal of $\mathbf{A}$. We will show that $\neg\neg I = [0,a]$ and that, therefore, $I \subsetneq \neg\neg I$. Observe that if $x \in I$, that is, if $x < a$, by the monotonicity of the negation, we have $\neg a \leq \neg x$. If $b \in \neg\neg I$, since $\neg a \leq \neg x$ for every $x \in I$, we have that $b \leq \neg\neg a$ and, $\mathbf{A}$ being involutive, we can conclude $b \leq a$, i.e., $b \in [0,a]$. Reciprocally, if $b \in [0,a]$ and $b \neq a$, then $b \in I \subseteq \neg\neg I$. If $b = a$ and $y \in [0,1]$ is such that $y \leq \neg x$ for every $x \in I$, then we have, by the involutive law and the antimonotonicity of the negation, $x = \neg\neg x \leq \neg y$ for every $x \in I$ and, since a is the supremum of I, we have $a \leq \neg y$. Therefore, $a \in \neg\neg I$. □

However, the involutive law is preserved under the DM-completion. Ono [Ono93, Theorem 5.1] proves this for the commutative case and indicates that the result can be extended with slight modifications to the non commutative case. Here we will perform the proof for the non commutative case and we adapt it to the notation employed in the present work.

Theorem 8.62. (Cf. [Ono93, Theorem 5.1 ff.]) *Let $\mathbf{A}$ be an involutive $\mathbb{FL}$-algebra, i.e., satisfying the equations $x \approx {}'(x^{\backprime})$ and $x \approx ({}'x)^{\backprime}$. Then, for every $X \subseteq A$, X is a C^{DM}-closed if and only if $X = {}'(X^{\backprime})$ and $X = ({}'X)^{\backprime}$.*

Proof: We will see that, for every $X \subseteq A$, $({}'X)^{\backprime} = (X^{\rightarrow})^{\leftarrow} = {}'(X^{\backprime})$. Let $a \in A$. Then, $a \in ({}'X)^{\backprime}$ is equivalent to

a) for every $y \in A$, if $y \leq {}'x$ for every $x \in X$, then $a \leq y^{\backprime}$.

Moreover, $a \in (X^{\rightarrow})^{\leftarrow}$ is equivalent to

b) for every $z \in A$, if $x \leq z$ for every $x \in X$, then $a \leq z$.

And finally, $a \in {}'(X^{\backprime})$ is equivalent to

c) for every $y \in A$, if $y \leq x^{\backprime}$ for every $x \in X$, then $a \leq {}'y$.

These three conditions are equivalent. Indeed,

a) $\Rightarrow$ *b)*: Suppose $x \leq z$ for every $x \in X$. If $x \leq z$, since $\mathbf{A}$ is involutive we have $x \leq ({}'z)^{\backprime}$, which is equivalent to ${}'z * x \leq 0$ and, therefore, to ${}'z \leq x^{\backprime}$. Thus we have, for every $x \in X$, ${}'z \leq x^{\backprime}$ and, hence, by a), we obtain $a \leq ({}'z)^{\backprime}$ and by the involutive law, $a \leq z$.

b) $\Rightarrow$ *a)*: Suppose $y \leq {}'x$, for every $x \in X$. If $y \leq {}'x$, then $y * x \leq 0$ which is equivalent to $x \leq y^{\backprime}$. Thus we have, for every $x \in X$, $x \leq y^{\backprime}$ and hence, by b), $a \leq y^{\backprime}$.

b) $\Leftrightarrow$ *c)* is proved analogously. □

Corollary 8.63. *Let $\sigma \leq ew_l w_r c$. The DM-completion of an involutive $\mathbb{FL}_\sigma$-algebra is also an involutive $\mathbb{FL}_\sigma$-algebra.*

Proof: It is a consequence of Theorem 8.62 and the fact that the DM-completion preserves the properties codified by σ. □

Corollary 8.64. *Every involutive $\mathbb{FL}_\sigma$-algebra is completely embeddable into a complete involutive $\mathbb{FL}_\sigma$-algebra.*

Proof: It is a consequence of Theorems 8.62 and 8.43. □

In Table 8.1 we summarize the results relative to the preservation under both completions of the properties considered so far.

Preserves:	infinite meets	infinite joins	$\langle\vee,\wedge\rangle$-equa.	$\langle\vee,\wedge,*,0,1\rangle$-equa.	Involutive law
DM	YES	YES	NO	NO	YES
Ideal	YES	NO	YES	YES	NO

Table 8.1: Preservation of properties under the completions

8.2.5 On the minimality of the DM-completion

In this section we obtain some new results concerning the minimality of the DM-completion. An interesting property of the Dedekind-MacNeille completion is that it is minimal in the sense that if $\mathbf{A}$ is a complete $\mathbb{FL}$-algebra then $i_{\mathbf{A}}$ is an isomorphism between $\mathbf{A}$ and $\mathbf{A}^{\mathbf{DM}}$ (Theorem 8.43). Nevertheless, it is also possible to consider the word minimal in other senses.

(1) As a first case we can ask if for every embedding ϕ from a $\mathbb{FL}$-algebra $\mathbf{A}$ into a complete $\mathbb{FL}$-algebra $\mathbf{B}$ there is an embedding ϕ^* from $\mathbf{A}^{\mathbf{DM}}$ into $\mathbf{B}$ that extends ϕ (i.e., $\phi = \phi^* \circ i_{\mathbf{A}}$).[6] The algebra $\mathbf{A}_1$ of Figure 8.1 shows that the DM-completion $\mathbb{FL}$-algebras is not minimal in this sense. Indeed, take $\mathbf{B}$ as the ideal-completion of $\mathbf{A}_1$ and $\phi = i_{\mathbf{A}_1}$. Recall that the elements of $\mathbf{A}_1^{\mathbf{DM}}$ are the principal ideals $(a]$, with $a \in A_1$, and the ideals $I_1 = \{0\} \cup \{a_n : n \in \omega\}$ and $I_2 = \{0\} \cup \{b_n : n \in \omega\}$. Suppose that there is an embedding ϕ^* from $\mathbf{A}_1^{\mathbf{DM}}$ in $\mathbf{A}_1^{\mathbf{Id}}$ such that $\phi = \phi^* \circ \phi$. Then ϕ^* must to be the identity. Indeed, on the one hand, for each $a \in A_1$, we have $\phi^*(\phi(a)) = \phi(a)$, that is, $\phi^*((a]) = (a]$. On the other hand, for every $i \in \omega$, we have
$$(a_i] \subsetneq I_1 \subsetneq (d_i] \quad \text{i} \quad (b_i] \subsetneq I_2 \subsetneq (e_i].$$
Since the fact that ϕ^* restricted to the principal ideals is the identity and that, moreover, it preserves the order and is injective we have, for every $i \in \omega$,
$$(a_i] \subsetneq \phi^*(I_1) \subsetneq (d_i] \quad \text{i} \quad (b_i] \subsetneq \phi^*(I_2) \subsetneq (e_i].$$

[6] In this sense it is known that the Dedekind-MacNeille completions of partial orders are minimal [Rus98, pp. 72–74].

Thus, a fortiori, $\phi^*(I_1) = I_1$ and $\phi^*(I_2) = I_2$. That is, ϕ^* is the identity and hence $\phi^*(\mathbf{A_1^{DM}}) = \mathbf{A_1^{DM}}$ is a subalgebra of $\mathbf{A_1^{Id}}$, contradicting Proposition 8.55.

(2) Another possibility to consider is whether for every complete embedding ϕ from a ϕ from a $\mathbb{FL}$-algebra $\mathbf{A}$ into a complete $\mathbb{FL}$-algebra $\mathbf{B}$ there is a complete embedding ϕ^* from $\mathbf{A^{DM}}$ into $\mathbf{B}$ extending ϕ.[7] We will see that the DM-completion is not minimal in this new sense either. To give a counterexample, take $\mathbf{B}$ as the Gödel standard algebra $[0, 1]_G$, that is, the $\mathbb{FL}$-algebra defined on the unit real interval by the Minimum t-norm and its residuum. Take $\mathbf{A}$ as the subalgebra of $\mathbf{B}$ with universe $A = [0, \frac{1}{4}) \cup (\frac{3}{4}, 1]$, and, as the embedding ϕ, take the inclusion of A in B. It is easy to see that this embedding preserves the arbitrary existing meets and joins, since every existing meet and join is in fact a minimum or a maximum. Note that $\mathbf{A^{DM}}$ is the result of adding a point, say a, between $[0, \frac{1}{4})$ and $(\frac{3}{4}, 1]$. Suppose now there is a complete embedding ϕ^* from $\mathbf{A^{DM}}$ in $\mathbf{B}$ extending ϕ. Since we have that $\bigvee_{\mathbf{A^{DM}}}[0, \frac{1}{4}) = a = \bigwedge_{\mathbf{A^{DM}}}(\frac{3}{4}, 1]$, if ϕ^* preserves arbitrary joins and meets we have, on the one hand,

$$\phi^*(a) = \phi^*(\bigvee_{\mathbf{A^{DM}}}[0, \frac{1}{4})) = \bigvee_{\mathbf{B}} \phi^*([0, \frac{1}{4})) = \bigvee_{\mathbf{B}}[0, \frac{1}{4}) = \frac{1}{4}$$

and, on the other hand,

$$\phi^*(a) = \phi^*(\bigwedge_{\mathbf{A^{DM}}}(\frac{3}{4}, 1]) = \bigwedge_{\mathbf{B}} \phi^*((\frac{3}{4}, 1]) = \bigwedge_{\mathbf{B}}(\frac{3}{4}, 1] = \frac{3}{4},$$

and this is absurd.

(3) Lastly, we can consider the word *minimal* as meaning that for every complete embedding ϕ from a $\mathbb{FL}$-algebra $\mathbf{A}$ into a complete $\mathbb{FL}$-algebra $\mathbf{B}$ there is an embedding (maybe not complete) ϕ^* from $\mathbf{A^{DM}}$ into $\mathbf{B}$ that extends ϕ. The Dedekind-MacNeille completion of a $\mathbb{FL}$-algebra is not minimal in thse sense either. This easily follows from the fact that the Dedekind-MacNeille completion of a lattice (without further structure) is not minimal in this very sense. A counterexample to this last statement is given by the lattice $\mathbf{L}$ in Figure 8.2. This lattice is considered by Funayama in [Fun44]. $\mathbf{L}$ is a distributive lattice such that $\mathbf{L^{DM}}$ is not modular (this is shown by the sublattice given by the five points marked with $\checkmark$). Now take $\mathbf{A}$ as the lattice $\mathbf{L}$ and $\mathbf{B}$ as the ideal completion of $\mathbf{L}$, where we adopt the same convention as on page 139.[8]

[7] The Dedekind-MacNeille completions of Boolean algebras are minimal in this sense [Hal74, Theorem 11(Chapter 21)].

[8] Note that the fact that the DM-completion is not minimal in this sense implies that it is not minimal in the sense of (2) either.

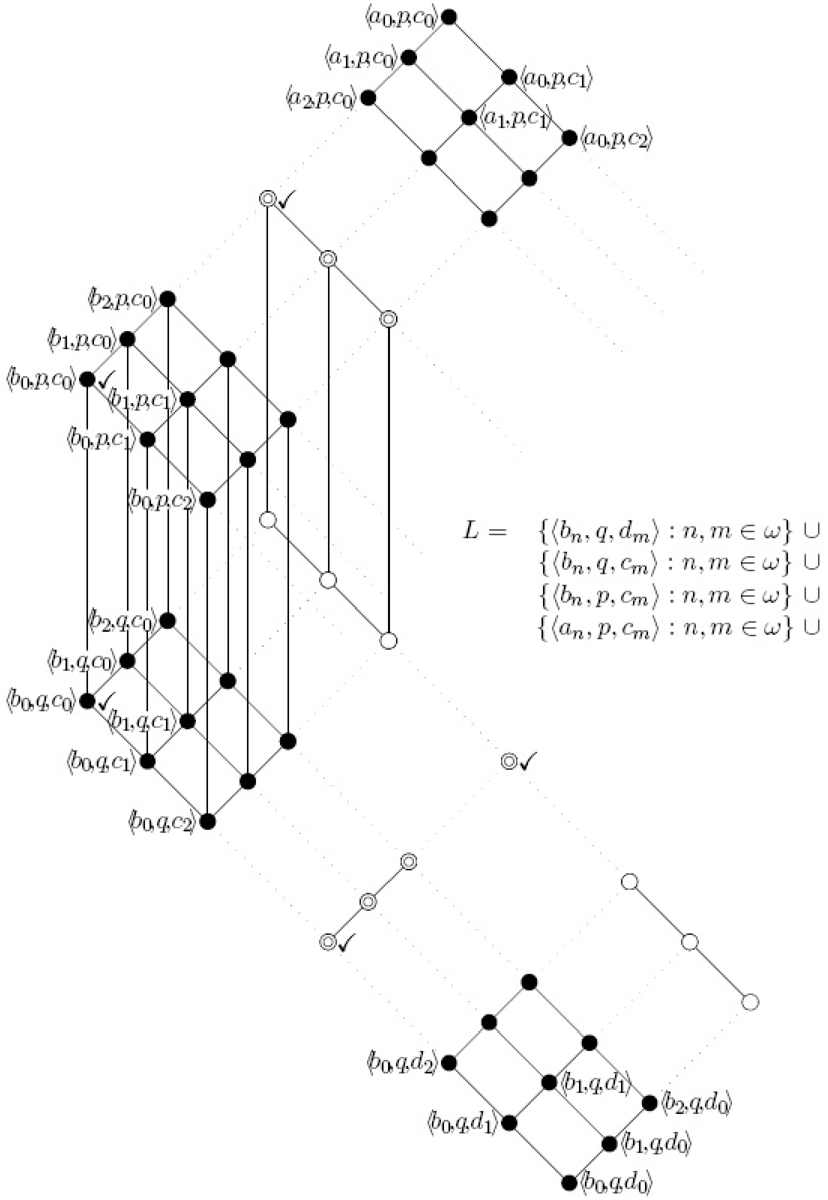


Figure 8.2: Funayama's distributive lattice $\mathbf{L}$ such that $\mathbf{L^{DM}}$ is not modular

8.3 Reducts and subreducts

Given that in every $\mathbb{FL}_\sigma$-algebra the distributivity of the monoidal operation with respect to the operation $\vee$ is satisfied, the $\langle \vee, *, 0, 1\rangle$-reduct of every $\mathbb{FL}$-algebra is in $\mathring{\mathbb{M}}^{s\ell}_\sigma$ and the $\langle \vee, \wedge, *, 0, 1\rangle$-reduct of every $\mathbb{FL}_\sigma$-algebra is in $\mathring{\mathbb{M}}^{\ell}_\sigma$. Moreover, since the operations $^{\backprime}$ and $'$ of every $\mathbb{FL}_\sigma$-algebra satisfy (LP), the $\langle \vee, *, ^{\backprime}, ', 0, 1\rangle$-reduct of every $\mathbb{FL}_\sigma$-algebra is in $\mathbb{PM}^{s\ell}_\sigma$ and the $\langle \vee, \wedge, *, ^{\backprime}, ', 0, 1\rangle$-reduct of every $\mathbb{FL}_\sigma$-algebra is in $\mathbb{PM}^{\ell}_\sigma$. However, as we will see, it is not true that every $\mathbb{PM}^{s\ell}_\sigma$-algebra is a reduct of a $\mathbb{FL}_\sigma$-algebra an the same for $\mathbb{PM}^{\ell}_\sigma$-algebras. However, it is true in the case of the complete algebras of $\mathring{\mathbb{M}}^{s\ell}_\sigma$, $\mathring{\mathbb{M}}^{\ell}_\sigma$, $\mathbb{PM}^{s\ell}_\sigma$ and $\mathbb{PM}^{\ell}_\sigma$, since they satisfy the generalized distributivity of the monoidal operation with respect to the joins and, obviously it is also true in the case of the finite algebra of the mentioned classes.

Proposition 8.65. *A complete $\mathring{\mathbb{M}}^{\ell}$-algebra is the $\langle \vee, \wedge, *, 0, 1\rangle$-reduct of a $\mathbb{FL}$-algebra if, and only if, it satisfies the generalized distributivity of the monoidal operation with respect to the joins.*

Proof: Let $\mathbf{A}$ be a complete $\mathring{\mathbb{M}}^{\ell}$-algebra. If $\mathbf{A}$ is the reduct of aa $\mathbb{FL}$-algebra $\mathbf{B}$, then $\mathbf{A}$ satisfies the generalized distributivity because $\mathbf{B}$ satisfies it. Reciprocally, if $\mathbf{A}$ satisfies the generalized distributivity, then, by Theorem 8.46, the monoidal operation is residuated and its residuals are given by the following definitions: for every $a, b \in A$,

$$a\backslash b =: \bigvee_{\mathbf{A}}\{z \in A : a * z \leq b\}, \qquad b/a =: \bigvee_{\mathbf{A}}\{z \in A : z * a \leq b\}.$$

Moreover, we define

$$a^{\backprime} =: a\backslash 0, \qquad {}'a =: 0/a.$$

Then, obviously, the structure $\langle A, \vee, \wedge, *, \backslash, /, ^{\backprime}, ', 0, 1\rangle$, where the operations in $\langle \vee, *, 0, 1\rangle$ are the ones of $\mathbf{A}$, is a $\mathbb{FL}$-algebra. □

In Section 8.1 we have seen that every complete $\mathring{\mathbb{M}}^{s\ell}$-algebra is the reduct of a complete $\mathring{\mathbb{M}}^{\ell}$-algebra. Therefore, combining this fact with the above proposition, we obtain the following.

Proposition 8.66. *A complete $\mathring{\mathbb{M}}^{s\ell}$-algebra is the $\langle \vee, *, 0, 1\rangle$-reduct of a $\mathbb{FL}$-algebra if, and only if, satisfies the generalized distributivity.*

Proposition 8.67. *Every finite member of $\mathring{\mathbb{M}}^{s\ell}$ or $\mathring{\mathbb{M}}^{\ell}$ is the reduct of a $\mathbb{FL}$-algebra.*

Proof: Every finite algebra in $\mathring{\mathbb{M}}^{s\ell}$ or $\mathring{\mathbb{M}}^{\ell}$ are complete and satisfies the generalized distributivity. This last claim is proved with an easy induction using the distributivity of the operation $*$ with respect to the operation $\vee$. Therefore, as a consequence of Propositions 8.65 and 8.66, we obtain the result. □

Proposition 8.68. *A complete $\mathbb{PM}^{\ell}$-algebra is the $\langle \vee, \wedge, *, ^{\backprime}, ', 0, 1\rangle$-reduct of a $\mathbb{FL}$-algebra if, and only if, it satisfies the generalized distributivity.*

Proof: Let $\mathbf{A}$ be a complete $\mathbb{PM}^{\ell}$-algebra. If $\mathbf{A}$ is the reduct of a $\mathbb{FL}$-algebra $\mathbf{B}$, then $\mathbf{A}$ satisfies the generalized distributivity, because $\mathbf{B}$ satisfies it. Reciprocally, suppose that $\mathbf{A}$ satisfies the generalized distributivity and let $\mathbf{A}'$ be its $\langle \vee, \wedge, *, 0, 1\rangle$-reduct. Since $\mathbf{A}'$ is a complete $\mathring{\mathbb{M}}^{\ell}$-algebra, by Proposition 8.65, it is the $\langle \vee, \wedge, *, 0, 1\rangle$-reduct of a complete $\mathbb{FL}$-algebra $\mathbf{B}$. Now we will see that the operations of pseudocomplementation in $\mathbf{B}$ coincide with the operations of pseudocomplementation in $\mathbf{A}$ and hence $\mathbf{A}$ is the $\langle \vee, \wedge, *, {}^{\backslash}, {}', 0, 1\rangle$-reduct of $\mathbf{B}$. Applying the definition of the right pseudocomplement in $\mathbf{B}$ using the operations of $\mathbf{A}'$ (see Proposition 8.65) and using the fact that $\mathbf{A}$ is a reduct of $\mathbf{B}$ we have

$$a^{\backslash \mathbf{B}} = a \backslash 0 = \bigvee_{\mathbf{B}} \{z \in A : a * z \leq 0\} = \bigvee_{\mathbf{A}} \{z \in A : a * z \leq 0\}.$$

But, since $\mathbf{A}$ is a $\mathbb{PM}^{\ell}$-algebra and it is complete, the maximum of the set $\{z \in A : a * z \leq 0\}$ exists and it is equal to its supremum. Therefore, $a^{\backslash \mathbf{B}} = a^{\backslash \mathbf{A}}$. In an analogous way it can be proved that the left pseudocomplementation in $\mathbf{A}$ and $\mathbf{B}$ are the same operation. □

Every complete $\mathbb{PM}^{s\ell}$-algebra is the $\langle \vee, *, 0, 1\rangle$-reduct of a complete $\mathbb{PM}^{\ell}$-algebra (Proposition 8.9). Therefore, by combining this fact with the previous proposition, we obtain the following.

Proposition 8.69. *A complete $\mathbb{PM}^{s\ell}$-algebra is the $\langle \vee, *, {}^{\backslash}, {}', 0, 1\rangle$-reduct of a $\mathbb{FL}$-algebra if, and only if, satisfies the generalitzed distributivity.*

Proposition 8.70. *Every finite member of $\mathbb{PM}^{s\ell}$ or $\mathbb{PM}^{\ell}$ is the reduct of a $\mathbb{FL}$-algebra.*

Proof: It is an immediate consequence of Propositions 8.68 and 8.69. □

Proposition 8.71. *There are algebras in the classes $\mathring{\mathbb{M}}^{s\ell}$, $\mathring{\mathbb{M}}^{\ell}$, $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^{\ell}$ which there are not the reduct of any $\mathbb{FL}$-algebra.*

Proof: Let us consider the algebra in Example 8.47, that is, the $\mathbb{FL}_{ew}$-algebra $\mathbf{A} = \langle A, \vee, \wedge, *, \rightarrow, \neg, 0, 1\rangle$, where $A = \{0,1\} \cup \{x \in \mathbb{R} : \frac{1}{4} \leq x \leq \frac{3}{4}\}$, the lattice operations correspond to the standard order on the real numbers and the other operations are given by the following tables (where $a, b, c \in [\frac{1}{4}, \frac{3}{4}]_{\mathbb{R}}$ and $a < c$):

$*$	0	b	1
0	0	0	0
a	0	$\frac{1}{4}$	a
1	0	b	1

$\rightarrow$	0	a	c	1
0	1	1	1	1
a	0	1	1	1
c	0	$\frac{3}{4}$	1	1
1	0	a	c	1

	$\neg$
0	1
a	0
1	0

Let $\mathbf{B} = \langle B, \vee, *, 0, 1\rangle$ i $\mathbf{C} = \langle C, \vee, *, \neg, 0, 1\rangle$, where $C = B = A \setminus \{\frac{3}{4}\}$ and the operations are the restrictions of the ones defined on $\mathbf{A}$. $\mathbf{B}$ is a complete $\mathring{\mathbb{M}}^{s\ell}_{ew}$-algebra and $\mathbf{C}$ is a complete $\mathbb{PM}^{s\ell}_{ew}$-algebra. However, the restrictions of the operation $*$ to $\mathbf{B}$ and to

C are not residuated because, as we have seen, **B** does not satisfy the generalized distributivity and **C** does not satisfy it neither. Thus **B** and **C** are, respectively, examples of $\mathring{\mathbb{M}}^{s\ell}$-algebras and $\mathbb{PM}^{s\ell}$-algebras which are not the reduct of any $\mathbb{FL}$-algebra. Obviously, $\langle \mathbf{B}, \wedge \rangle$ and $\langle \mathbf{C}, \wedge \rangle$ are, respectively, examples of $\mathring{\mathbb{M}}^{\ell}$-algebras and $\mathbb{PM}^{\ell}$-algebras which are not the reduct of any $\mathbb{FL}$-algebra. □

Proposition 8.72. *There are $\mathring{\mathbb{M}}^{s\ell}$-algebras which are not the reduct of any $\mathbb{PM}^{s\ell}$-algebra and $\mathring{\mathbb{M}}^{\ell}$-algebras which are not the reduct of any $\mathbb{PM}^{\ell}$-algebra.*

Proof: Let us consider the algebra $\mathbb{PM}^{\ell}_{ew}$-algebra $\mathbf{A} = \langle A, \vee, \wedge, *, \neg, 0, 1 \rangle$, where $A = [0, \frac{1}{2}] \cup \{1\}$, the lattice operations corresponds to the standard order on the real numbers and the other operations are given by the following tables (where $a, b \in [0, \frac{1}{2}]_{\mathbb{R}}$):

$*$	0	b	1
0	0	0	0
a	0	0	a
1	0	b	1

	$\neg$
0	1
a	$\frac{1}{2}$
1	0

Let $\mathbf{B} = \langle B, \vee, \wedge, *, 0, 1 \rangle$ on $B = A \setminus \{\frac{1}{2}\}$, where the operations are the restriction of the ones defined on $\mathbf{A}$. $\mathbf{B}$ is a $\mathring{\mathbb{M}}^{s\ell}_{ew}$-algebra but the restriction of the operation $*$ to $\mathbf{B}$ is not pseudocomplemented, because the set $\{x \in B : a * x \leq 0\}$ is the full interval $[0, \frac{1}{2})_{\mathbb{R}}$ which does not have maximum. The same example shows that there are $\mathring{\mathbb{M}}^{s\ell}$-algebras which are not the reduct of any $\mathbb{PM}^{\ell}$-algebra. □

Nevertheless, as a consequence of the embeddings obtained in Section 8.2.1 we have the following results:

Theorem 8.73. *Let $\sigma \leq ew_l w_r c$.*

i) $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$ *is the class of the $\langle \vee, *, 0, 1 \rangle$-subreducts of the algebras of the classes* $\mathbb{PM}^{s\ell}_{\sigma}$ *and* $\mathbb{FL}_{\sigma}$.

ii) $\mathring{\mathbb{M}}^{\ell}_{\sigma}$ *is the class of the $\langle \vee, \wedge, *, 0, 1 \rangle$-subreducts of the algebras of the classes* $\mathbb{PM}^{\ell}_{\sigma}$ *and* $\mathbb{FL}_{\sigma}$.

iii) $\mathbb{PM}^{s\ell}_{\sigma}$ *is the class of the $\langle \vee, *, ^{\backprime}, ^{\prime}, 0, 1 \rangle$-subreducts of the algebras of the class* $\mathbb{FL}_{\sigma}$.

iv) $\mathbb{PM}^{\ell}_{\sigma}$ *is the class of the $\langle \vee, \wedge, *, ^{\backprime}, ^{\prime}, 0, 1 \rangle$-subreducts of the algebras of the class* $\mathbb{FL}_{\sigma}$.

Proof: i) is a consequence of Theorem 8.24, ii) of Corollary 8.26, iii) of Corollary 8.25 and i) of Corollary 8.27. □

To finish the section we will show that the weakly contractive $\mathbb{PM}^{s\ell}_{\sigma}$-algebras and $\mathbb{PM}^{\ell}_{\sigma}$-algebras are the classes of the subreducts in the corresponding languages of the class of the weakly contractive $\mathbb{FL}_{\sigma}$-algebras

Proposition 8.74. *Let* $\mathbf{A}$ *be a weakly contractive* $\mathbb{PM}^{s\ell}$*-algebra or a weakly contractive* $\mathbb{PM}^{\ell}$*-algebra. Then its ideal-completion is weakly contractive.*

Proof: Let I be an ideal of $\mathbf{A}$. We must see that $(I * I]^{\backprime} \subseteq I^{\backprime}$. Therefore, using the characterization of Lemma 8.21, we have that given any $b \in I$, if $a \in (I * I]^{\backprime}$, then $a \leq b^{\backprime}$. Since $b \in I$, we have $b * b \in I * I$ and, therefore, $b * b \in (I * I]$. On the other hand, since $a \in (I * I]^{\backprime}$, we have $a \leq c^{\backprime}$ for each $c \in (I * I]$ and, in particular, $a \leq (b * b)^{\backprime}$. But, since $\mathbf{A}$ is weakly contractive, $(b * b)^{\backprime} \leq b^{\backprime}$ and, therefore, $a \leq b^{\backprime}$. By the mirror images principle we also obtain that ${}'(I * I] \subseteq {}'I$. □

Theorem 8.75. *Let* $\sigma \leq e w_l w_r$.

i) Every $\mathbb{PM}^{s\ell}_{\sigma\hat{c}}$*-algebra is embeddable in a complete* $\mathbb{FL}_{\sigma\hat{c}}$*-algebra. Therefore,* $\mathbb{PM}^{s\ell}_{\sigma\hat{c}}$ *is the class of the* $\langle \vee, *, {}^{\backprime}, {}', 0, 1\rangle$*-subreducts of the algebras of the class* $\mathbb{FL}_{\sigma\hat{c}}$*.*

ii) Every $\mathbb{PM}^{\ell}_{\sigma\hat{c}}$*-algebra is embeddable in a complete* $\mathbb{FL}_{\sigma\hat{c}}$*-algebra. Therefore,* $\mathbb{PM}^{\ell}_{\sigma\hat{c}}$ *is the class of the* $\langle \vee, \wedge, *, {}^{\backprime}, {}', 0, 1\rangle$*-subreducts of the algebras of the class* $\mathbb{FL}_{\sigma\hat{c}}$*.*

Proof: *i*) is a consequence of Corollary 8.25 and Proposition 8.74. *ii*) is a consequence of Corollary 8.27 and Proposition 8.74. □

Chapter 9

Algebraic Analysis of some Implication-free Fragments

In this chapter we study the fragments in the languages $\langle \vee, *, 0, 1\rangle$, $\langle \vee, \wedge, *, 0, 1\rangle$ and $\langle \vee, *, {}^{\backprime}, {}', 0, 1\rangle$ of the systems $\mathcal{FL}_\sigma$ and their associated external deductive systems $\mathfrak{e}\mathcal{FL}_\sigma$. In Section 9.1 we prove that the subsystems $\mathcal{FL}_\sigma[\vee, *, 0, 1]$, $\mathcal{FL}_\sigma[\vee, \wedge, *, 0, 1]$, $\mathcal{FL}_\sigma[\vee, *, {}^{\backprime}, {}', 0, 1]$ and the system $\mathcal{FL}_\sigma[\vee, \wedge, *, {}^{\backprime}, {}', 0, 1]$ are algebraizable, having as respective equivalent algebraic semantics the varieties $\mathring{\mathbb{M}}^{s\ell}_\sigma$, $\mathring{\mathbb{M}}^{\ell}_\sigma$, $\mathbb{PM}^{s\ell}_\sigma$, $\mathbb{PM}^{\ell}_\sigma$ and we also prove that the system $\mathcal{FL}_\sigma$ is algebraizable with equivalent algebraic semantics the variety $\mathbb{FL}_\sigma$. Using the algebraization results of Section 9.1 and those of in Section 8.3 where it was established that these classes are subreducts of $\mathbb{FL}_\sigma$, in Section 9.2 we obtain that the mentioned subsystems are fragments of $\mathcal{FL}_\sigma$ and that the corresponding external deductive systems are fragments of $\mathfrak{e}\mathcal{FL}_\sigma$. It is also shown that each system $\mathcal{FL}_\sigma$ is equivalent to its associated external deductive system but it is shown that the considered fragments are not equivalent to any deductive system. It is also shown that $\mathfrak{e}\mathcal{FL}_\sigma[\vee, *, 0, 1]$, $\mathfrak{e}\mathcal{FL}_\sigma[\vee, \wedge, *, 0, 1]$, $\mathfrak{e}\mathcal{FL}_\sigma[\vee, *, {}^{\backprime}, {}', 0, 1]$ and $\mathfrak{e}\mathcal{FL}_\sigma[\vee, \wedge, *, {}^{\backprime}, {}', 0, 1]$ are not protoalgebraic but have respectively the varieties $\mathring{\mathbb{M}}^{s\ell}_\sigma$, $\mathring{\mathbb{M}}^{\ell}_\sigma$, $\mathbb{PM}^{s\ell}_\sigma$, $\mathbb{PM}^{\ell}_\sigma$ as algebraic semantics with defining equation $1 \vee p \approx p$. In Section 9.3 we give some decidability results for some of the fragments considered. In Section 9.4 we define the basic substructural systems with weak contraction $\mathcal{FL}_{\sigma\hat{c}}$ and characterize the fragments in the languages $\langle \vee, *, {}^{\backprime}, {}', 0, 1\rangle$ and $\langle \vee, \wedge, *, {}^{\backprime}, {}', 0, 1\rangle$ of these systems and their associated external deductive systems.

9.1 Algebraization

We will use the letter Ψ as a generic denotation of the languages of the classes $\mathring{\mathbb{M}}^{s\ell}_\sigma$, $\mathring{\mathbb{M}}^{\ell}_\sigma$, $\mathbb{PM}^{s\ell}_\sigma$, $\mathbb{PM}^{\ell}_\sigma$ and $\mathbb{FL}_\sigma$ and we will use $\mathbb{K}_\sigma[\Psi]$ as a generic denotation for all these classes of algebras. In the following we will show that every subsystem $\mathcal{FL}_\sigma[\Psi]$ is algebraizable and that the class $\mathbb{K}_\sigma[\Psi]$ is its equivalent quasivariety semantics.

Definition 9.1. *We define the translations τ from Ψ-sequents to Ψ-equations and ρ from Ψ-equations to Ψ-sequents in the following way:*

$$\tau(\varphi_0, ..., \varphi_{m-1} \Rightarrow \varphi) := \begin{cases} \{(\varphi_0 * ... * \varphi_{m-1}) \vee \varphi \approx \varphi\}, & \text{if } m \geq 1, \\ \{1 \vee \varphi \approx \varphi\}, & \text{if } m = 0, \end{cases}$$

$$\tau(\varphi_0, ..., \varphi_{m-1} \Rightarrow \emptyset) := \begin{cases} \{\varphi_0 * ... * \varphi_{m-1} \vee 0 \approx 0\}, & \text{if } m \geq 1, \\ \{1 \vee 0 \approx 0\}, & \text{if } m = 0, \end{cases}$$

$$\rho(\varphi \approx \psi) := \{\varphi \Rightarrow \psi,\ \psi \Rightarrow \varphi\}.$$

Note that the translation τ is well defined since the languages Ψ contain all the connectives in $\{\vee, *, 0, 1\}$.

Lemma 9.2. *For every $\mathcal{FL}_\sigma[\Psi]$-theory Φ,*

$$\rho(\varphi \preccurlyeq \psi) \subseteq \Phi \quad \textit{iff} \quad \varphi \Rightarrow \psi \in \Phi.$$

Thus, in particular, the derivability of the sequents in $\rho(\varphi \preccurlyeq \psi)$ is equivalent to the derivability of the sequent $\varphi \Rightarrow \psi$.

Proof: We have that $\rho(\varphi \preccurlyeq \psi) = \rho(\varphi \vee \psi \approx \psi) = \{\varphi \vee \psi \Rightarrow \psi, \psi \Rightarrow \varphi \vee \psi\}$. The sequent $\psi \Rightarrow \varphi \vee \psi$ is derivable from $\psi \Rightarrow \psi$ using $(\Rightarrow \vee_2)$. Thus it is sufficient to prove that the sequents $\varphi \vee \psi \Rightarrow \psi$ and $\varphi \Rightarrow \psi$ are interderivable in $\mathbf{FL}[\Psi]$. Let us consider the following formal proofs:

$$\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \varphi \vee \psi}\,(\Rightarrow \vee_1) \qquad \varphi \vee \psi \Rightarrow \psi}{\varphi \Rightarrow \psi}\,(Cut) \qquad \dfrac{\varphi \Rightarrow \varphi \qquad \psi \Rightarrow \psi}{\varphi \vee \psi \Rightarrow \psi}\,(\vee \Rightarrow)$$

□

Notation 9.3. If $\bar{x} = x_0, \ldots, x_{m-1}$ is a sequence of elements in a Ψ-algebra $\mathbf{A}$, we define $\prod \bar{x} := 1$ if $\bar{x}$ is the empty sequence, $\prod \bar{x} := x_0$ if $m = 1$, and $\prod \bar{x} := x_0 * \ldots * x_{m-1}$ if $m \geq 1$. In particular, if $\mathbf{A}$ is the algebra of Ψ-formulas, for each sequence $\Gamma = \varphi_0, \ldots, \varphi_{m-1}$, we will have $\prod \Gamma := 1$ if Γ is the empty sequence, $\prod \Gamma := \varphi_0$ if $m = 1$ and $\prod \Gamma := \varphi_0 * \ldots * \varphi_{m-1}$ if $m \geq 1$.

Lemma 9.4. *If Γ is a sequence of Ψ-formulas, then the sequent $\Gamma \Rightarrow \prod \Gamma$ is derivable in $\mathbf{FL}_\sigma[\Psi]$.*

Proof: By induction on the length of the sequence Γ.

- If $m = 0$, then $\Gamma \Rightarrow \prod \Gamma$ is the sequent $\emptyset \Rightarrow 1$, that is, the axiom $(\Rightarrow 1)$.
- If $m > 0$ and $\Gamma = \varphi_0, \ldots, \varphi_{m-1}$, by the induction hypothesis we have that $\varphi_0, \ldots, \varphi_{m-2} \Rightarrow \varphi_0 * \ldots * \varphi_{m-2}$ is derivable. From this sequent and $\varphi_{m-1} \Rightarrow \varphi_{m-1}$, applying $(\Rightarrow *)$, we obtain $\Gamma \Rightarrow \prod \Gamma$. □

Lemma 9.5. *For every* $\varsigma \in Seq_{\Psi}^{\omega\times\{0,1\}}$, $\varsigma \dashv\vdash_{\mathbf{FL}_\sigma[\Psi]} \rho\tau(\varsigma)$.

Proof: We consider two cases: *a)* $\varsigma = \Gamma \Rightarrow \varphi$ and *b)* $\varsigma = \Gamma \Rightarrow \emptyset$.

a): By the definition of τ we have $\rho\tau(\varsigma) = \rho(\prod\Gamma \preccurlyeq \varphi)$. But, by Lemma 9.4, we have $\rho(\prod\Gamma \preccurlyeq \varphi) \dashv\vdash \prod\Gamma \Rightarrow \varphi$. Thus, it will be sufficient to prove $\Gamma \Rightarrow \varphi \dashv\vdash \prod\Gamma \Rightarrow \varphi$. The sequent $\Gamma \Rightarrow \varphi$ is obtained from the derivable sequent $\Gamma \Rightarrow \prod\Gamma$ (see Lemma 9.4) and the sequent $\prod\Gamma \Rightarrow \varphi$ by applying (Cut).

We will show that $\prod\Gamma \Rightarrow \varphi$ is obtained from $\Gamma \Rightarrow \varphi$ using induction on the length of the sequence Γ. If $n = 0$, we must see that $\emptyset \Rightarrow \varphi \vdash 1 \Rightarrow \varphi$ and this is clear by applying $(1 \Rightarrow)$. If $n > 0$, we have the following derivation:

$$\cfrac{\cfrac{\Gamma \Rightarrow \varphi}{\varphi_0, \ldots, \varphi_{m-2} * \varphi_{m-1} \Rightarrow \varphi}\ (* \Rightarrow)}{\prod\Gamma \Rightarrow \varphi}\ \text{(Induction hypothesis)}$$

b): In this case we have $\rho\tau(\varsigma) = \rho(\prod\Gamma \preccurlyeq 0)$. Thus, we must prove

$$\Gamma \Rightarrow \emptyset \dashv\vdash \prod\Gamma \Rightarrow 0.$$

The sequent $\Gamma \Rightarrow \emptyset$ can be obtained from $\prod\Gamma \Rightarrow 0$ in the following way:

$$\cfrac{\cfrac{\Gamma \Rightarrow \prod\Gamma \qquad \prod\Gamma \Rightarrow 0}{\Gamma \Rightarrow 0}\ (\text{Tall}) \qquad 0 \Rightarrow \emptyset}{\Gamma \Rightarrow \emptyset}\ (Cut)$$

To see that $\prod\Gamma \Rightarrow 0$ can be obtained from $\Gamma \Rightarrow \emptyset$ we use induction on the length of the sequence Γ. If $n = 0$, we must see that $\emptyset \Rightarrow \emptyset \vdash 1 \Rightarrow 0$ and this is immediate using $(1 \Rightarrow)$ and $(\Rightarrow 0)$. If $n > 0$ we have the following derivation:

$$\cfrac{\cfrac{\Gamma \Rightarrow \emptyset}{\varphi_0, \ldots, \varphi_{m-2} * \varphi_{m-1} \Rightarrow \emptyset}\ (* \Rightarrow)}{\prod\Gamma \Rightarrow 0}\ \text{(Induction hypothesis)}$$

□

Lemma 9.6. *For every* $\varphi \approx \psi \in Eq_\Psi$, $\varphi \approx \psi \mathrel{=\mid\models}_{\mathbb{K}_\sigma[\Psi]} \tau\rho(\varphi \approx \psi)$.

Proof: We must prove that $\varphi \approx \psi \mathrel{=\mid\models}_{\mathbb{K}_\sigma[\Psi]} \{\varphi \vee \psi \approx \psi, \psi \vee \varphi \approx \varphi\}$ and this is trivial.

□

Lemma 9.7. *For every* $\mathbf{A} \in \mathbb{K}_\sigma[\Psi]$ *we define* R *as the set*

$$\{\langle \bar{x}, \bar{y}\rangle \in A^m \times A^n : \langle m, n\rangle \in \omega \times \{0,1\},\ \mathbf{A} \models \tau(p_0, \ldots, p_{m-1} \Rightarrow q_0, \ldots, q_{n-1})[\![\bar{x}, \bar{y}]\!]\}.$$

Then R *is a* $\mathcal{FL}_\sigma[\Psi]$*-filter.*

Proof: Given one of the languages Ψ, we must prove that, for every σ and every $\mathbf{A} \in \mathbb{K}_\sigma[\Psi]$, the set R contains the interpretations of the axioms of $\mathbf{FL}_\sigma[\Psi]$ and is closed under the interpretation of the rules of $\mathbf{FL}_\sigma[\Psi]$. First, observe that the set R is equal to

$$\{\langle \bar{x}, a\rangle \in A^m \times A : m \in \omega, \prod \bar{x} \leq a\} \cup \{\langle \bar{x}, \emptyset\rangle \in A^m \times \{\emptyset\} : m \in \omega, \prod \bar{x} \leq 0\}.$$

For every algebra $\mathbf{A}$ in the classes considered, R contains all the pairs of the form $\langle a, a\rangle$ (where $a \in A$), $\langle 0, \emptyset\rangle$ and $\langle \emptyset, 1\rangle$, that is, R contains the interpretations of the axioms. Next we will see that in each case R is closed under the interpretation of the rules. We begin with the rules common to all the calculi under consideration.

From now on we will use the symbol δ to denote the empty set or an arbitrary element of A. Then $c_\delta \in A$ is defined as 0 if $\delta = \emptyset$ and as δ if $\delta \in A$.

- *Cut rule*:

$$\frac{\Gamma \Rightarrow \varphi \qquad \Sigma, \varphi, \Pi \Rightarrow \Delta}{\Sigma, \Gamma, \Pi \Rightarrow \Delta}\ (\mathrm{Cut})$$

Suppose $\langle \bar{x}, a\rangle \in R$ and $\langle\langle \bar{y}, a, \bar{z}\rangle, \delta\rangle \in R$. Then $\prod \bar{x} \leq a$ and $\prod \bar{y} * a * \prod \bar{z} \leq c_\delta$. By monotonicity we have $\prod \bar{y} * \prod \bar{x} * \prod \bar{z} \leq \prod \bar{y} * a * \prod \bar{z}$ and hence $\prod \bar{y} * \prod \bar{x} * \prod \bar{z} \leq c_\delta$. Therefore, $\langle\langle \bar{y}, \bar{x}, \bar{z}\rangle, \delta\rangle \in R$.

- *Rules for* $\vee$:

$$\frac{\Sigma, \varphi, \Gamma \Rightarrow \Delta \qquad \Sigma, \psi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi \vee \psi, \Gamma \Rightarrow \Delta}\ (\vee \Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi}\ (\Rightarrow \vee_1) \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi}\ (\Rightarrow \vee_2)$$

$(\vee \Rightarrow)$: If $\langle\langle \bar{x}, a, \bar{y}\rangle, \delta\rangle \in R$ and $\langle\langle \bar{x}, b, \bar{y}\rangle, \delta\rangle \in R$, then we have $\prod \bar{x} * a * \prod \bar{y} \leq c_\delta$ and $\prod \bar{x} * b * \prod \bar{y} \leq c_\delta$. Thus, $(\prod \bar{x} * a * \prod \bar{y}) \vee (\prod \bar{x} * b * \prod \bar{y}) \leq c_\delta$ and by distributivity we have $\prod \bar{x} * (a \vee b) * \prod \bar{y} \leq c_\delta$ and, in consequence, $\langle\langle \bar{x}, a \vee b, \bar{y}\rangle, \delta\rangle \in R$.

$(\Rightarrow \vee_1)$: If $\langle \bar{x}, a\rangle \in R$, then $\prod \bar{x} \leq a \leq a \vee b$. Therefore, $\langle \bar{x}, a \vee b\rangle \in R$.

$(\Rightarrow \vee_2)$: Analogous to the above case.

- *Rules for* $*$:

$$\frac{\Sigma, \varphi, \psi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi * \psi, \Gamma \Rightarrow \Delta}\ (* \Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi \qquad \Pi \Rightarrow \psi}{\Gamma, \Pi \Rightarrow \varphi * \psi}\ (\Rightarrow *)$$

$(* \Rightarrow)$: If $\langle\langle \bar{x}, a, b, \bar{y}\rangle, \delta\rangle \in R$, then $\prod \bar{x} * a * b * \prod \bar{y} \leq c_\delta$. Therefore, $\langle\langle \bar{x}, a * b, \bar{y}\rangle, \delta\rangle \in R$.

$(\Rightarrow *)$: If $\langle \bar{x}, a\rangle \in R$ and $\langle \bar{y}, b\rangle \in R$, then $\prod \bar{x} \leq a$ and $\prod \bar{y} \leq b$ and thus, by monotonicity, we have that $\prod \bar{x} * \prod \bar{y} \leq a * b$ and, therefore, $\langle\langle \bar{x}, \bar{y}\rangle, a * b\rangle \in R$.

- *Rules* $(1 \Rightarrow)$ *and* $(\Rightarrow 0)$:

$$\frac{\Sigma, \Gamma \Rightarrow \Delta}{\Sigma, 1, \Gamma \Rightarrow \Delta}\ (1 \Rightarrow) \qquad \frac{\Gamma \Rightarrow \emptyset}{\Gamma \Rightarrow 0}\ (\Rightarrow 0)$$

$(1 \Rightarrow)$: If $\langle \bar{x}, \bar{y}\rangle \in R$, then $\prod \bar{x} * \prod \bar{y} \leq c_\delta$ and hence $\prod \bar{x} * 1 * \prod \bar{y} \leq c_\delta$. Therefore, $\langle\langle \bar{x}, 1, \bar{y}\rangle, \delta\rangle \in R$.

$(\Rightarrow 0)$: If $\langle \bar{x}, \emptyset \rangle \in R$, then $\prod \bar{x} \leq 0$, that is, $\langle \bar{x}, 0 \rangle \in R$.

If $\mathbf{A}$ is a $\mathbb{PM}^{s\ell}$-algebra, we must prove moreover that the set R is closed under the interpretation of the rules for ${}^{\backprime}$ and ${}'$:

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma, \varphi^{\backprime} \Rightarrow \emptyset}\ ({}^{\backprime} \Rightarrow) \qquad \frac{\varphi, \Gamma \Rightarrow \emptyset}{\Gamma \Rightarrow \varphi^{\backprime}}\ (\Rightarrow {}^{\backprime})$$

$$\frac{\Gamma \Rightarrow \varphi}{{}'\varphi, \Gamma \Rightarrow \emptyset}\ ({}' \Rightarrow) \qquad \frac{\Gamma, \varphi \Rightarrow \emptyset}{\Gamma \Rightarrow {}'\varphi}\ (\Rightarrow {}')$$

$({}^{\backprime} \Rightarrow)$: If $\langle \bar{x}, a \rangle \in R$, then $\prod \bar{x} \leq a$. By monotonicity and the fact that $a^{\backprime}$ is the right pseudocomplement of a we have $\prod \bar{x} * a^{\backprime} \leq a * a^{\backprime} \leq 0$, that is, $\langle \langle \bar{x}, a^{\backprime} \rangle, \emptyset \rangle \in R$.

$(\Rightarrow {}^{\backprime})$: If $\langle \langle a, \bar{x} \rangle, \emptyset \rangle \in R$, then $a * \prod \bar{x} \leq 0$ and, by the pseudocomplementation law, $\prod \bar{x} \leq a^{\backprime}$, that is, $\langle \bar{x}, a^{\backprime} \rangle \in R$.

$({}' \Rightarrow), (\Rightarrow {}')$: We can proceed analogously by using the properties of the left pseudocomplement.

If $\mathbf{A}$ is one of the classes $\mathring{\mathbb{M}}^{\ell}$, $\mathbb{PM}^{\ell}$ or $\mathbb{FL}$, that is, if Ψ contains the connective $\wedge$, we must see moreover that the set R is closed under the interpretation of the introduction rules for this connective:

$$\frac{\Sigma, \varphi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ (\wedge_1 \Rightarrow) \quad \frac{\Sigma, \psi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ (\wedge_2 \Rightarrow) \quad \frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}\ (\Rightarrow \wedge)$$

$(\wedge_1 \Rightarrow)$: If $\langle \langle \bar{x}, a, \bar{y} \rangle, \delta \rangle \in R$, then $\prod \bar{x} * a * \prod \bar{y} \leq c_\delta$. From $a \wedge b \leq a$ by the monotonicity we obtain $\prod \bar{x} * (a \wedge b) * \prod \bar{y} \leq \prod \bar{x} * a * \prod \bar{y}$. Thus $\prod \bar{x} * (a \wedge b) * \prod \bar{y} \leq c_\delta$. Therefore, $\langle \langle \bar{x}, a \wedge b, \bar{y} \rangle, \delta \rangle \in R$.

$(\wedge_2 \Rightarrow)$: Analogously.

$(\Rightarrow \wedge)$: If $\langle \bar{x}, a \rangle \in R$ and $\langle \bar{x}, b \rangle \in R$, then $\prod \bar{x} \leq a$ and $\prod \bar{x} \leq b$. Thus $\prod \bar{x} \leq a \wedge b$. Therefore, $\langle \bar{x}, a \wedge b \rangle \in R$.

If $\mathbf{A} \in \mathbb{FL}$, we must show that, moreover, R is closed under the introduction rules to the connectives $\backslash$ and $/$:

$$\frac{\Gamma \Rightarrow \varphi \qquad \Sigma, \psi, \Pi \Rightarrow \Delta}{\Sigma, \Gamma, \varphi \backslash \psi, \Pi \Rightarrow \Delta}\ (\backslash \Rightarrow) \qquad \frac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \backslash \psi}\ (\Rightarrow \backslash)$$

$$\frac{\Gamma \Rightarrow \varphi \quad \Sigma, \psi, \Pi \Rightarrow \Delta}{\Sigma, \psi / \varphi, \Gamma, \Pi \Rightarrow \Delta}\ (/ \Rightarrow) \qquad \frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \psi / \varphi}\ (\Rightarrow /)$$

$(\backslash \Rightarrow)$: If $\langle \bar{x}, a \rangle \in R$ and $\langle \langle \bar{y}, b, \bar{z} \rangle, \delta \rangle \in R$, then $\prod \bar{x} \leq a$ and $\prod \bar{y} * b * \prod \bar{z} \leq c_\delta$. By monotonicity and the properties of the right residuum we have $\prod \bar{x} * (a \backslash b) \leq a * (a \backslash b) \leq b$ and, hence, $\prod \bar{y} * \prod \bar{x} * (a \backslash b) * \prod \bar{z} \leq \prod \bar{y} * b * \prod \bar{z} \leq c_\delta$. In consequence, $\langle \langle \bar{y}, \bar{x}, a \backslash b, \bar{z} \rangle, \delta \rangle \in R$.

$(\Rightarrow \backslash)$: If $\langle \langle a, \bar{x} \rangle, b \rangle \in R$, then $a * \prod \bar{x} \leq b$ and, by the law of residuation, $\prod \bar{x} \leq a \backslash b$. Therefore, $\langle \bar{x}, a \backslash b \rangle \in R$.

$(/ \Rightarrow)$, $(\Rightarrow /)$: We can proceed analogously using the properties of the left residuum.

Finally, if $\mathbf{A} \in \mathbb{K}_\sigma[\Psi]$, with σ non empty, we want to see that R is closed under all the structural rules codified by σ.

- *Exchange*:

$$\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta} \quad (e \Rightarrow)$$

If $\langle\langle \bar{x}, a, b, \bar{y}\rangle, \delta\rangle \in R$, then $\prod \bar{x} * a * b * \prod \bar{y} \leq c_\delta$. Thus, if $\mathbf{A}$ is commutative, we have $\prod \bar{x} * b * a * \prod \bar{y} \leq c_\delta$. Consequently, $\langle\langle \bar{x}, b, a, \bar{y}\rangle, \delta\rangle \in R$.

- *Weakening*:

$$\frac{\Sigma, \Gamma \Rightarrow \Delta}{\Sigma, \varphi, \Gamma \Rightarrow \Delta} \quad (w \Rightarrow) \qquad \frac{\Gamma \Rightarrow \emptyset}{\Gamma \Rightarrow \varphi} \quad (\Rightarrow w)$$

$(w \Rightarrow)$: If $\langle\langle \bar{x}, \bar{y}\rangle \delta\rangle \in R$, then $\prod \bar{x} * \prod \bar{y} \leq c_\delta$; thus, if $\mathbf{A}$ is integral, by monotonicity we have $\prod \bar{x} * a * \prod \bar{y} \leq \prod \bar{x} * 1 * \prod \bar{y} \leq \prod \bar{x} * \prod \bar{y} \leq c_\delta$. That is, $\langle\langle \bar{x}, a, \bar{y}\rangle, \delta\rangle \in R$.

$(\Rightarrow w)$: If $\langle \bar{x}, \emptyset\rangle \in R$, then $\prod \bar{x} \leq 0$. Thus, if 0 is the minimum, $\prod \bar{x} \leq a$. Therefore, $\langle \bar{x}, a\rangle \in R$.

- *Contraction*:

$$\frac{\Sigma, \varphi, \varphi, \Gamma \Rightarrow \Delta}{\Sigma, \varphi, \Gamma \Rightarrow \Delta} \quad (c \Rightarrow)$$

If $\langle\langle \bar{x}, a, a, \bar{y}\rangle, \delta\rangle \in R$, then we have $\prod \bar{x} * a * a * \prod \bar{y} \leq c_\delta$. From this, when $\mathbf{A}$ has the property of increasing idempotency, we obtain $\prod \bar{x} * a * \prod \bar{y} \leq c_\delta$. Therefore, $\langle\langle \bar{x}, a, \bar{y}\rangle, \delta\rangle \in R$. □

Lemma 9.8. *For every theory $\Phi \in Th\,\mathcal{FL}_\sigma[\Psi]$, the set*

$$\theta_\Phi := \{\langle \varphi, \psi\rangle \in Fm^2_\Psi : \rho(\varphi \approx \psi) \subseteq \Phi\}$$

is a congruence relative to the quasivariety $\mathbb{K}_\sigma[\Psi]$.

Proof: Let Φ be a $\mathcal{FL}_\sigma[\Psi]$-theory. By the definition of the translation ρ we have

$$\theta_\Phi := \{\langle \varphi, \psi\rangle \in Fm^2_\Psi : \{\varphi \Rightarrow \psi, \psi \Rightarrow \varphi\} \subseteq \Phi\}$$

but, by Corollary 4.25, this is the Leibniz congruence $\Omega\Phi$ of the theory Φ.

Let us denote $\mathbf{Q}[\Psi] := Fm_\Psi/\Omega\Phi$. We want to see that $\mathbf{Q}[\Psi]$ is a $\mathbb{K}_\sigma[\Psi]$-algebra. To see this, we will prove the following conditions:

1. If $\Psi = \langle \vee, *, 0, 1\rangle$, then $\mathbf{Q}[\Psi] \in \mathring{\mathbb{M}}^{s\ell}$.
2. If $\Psi = \langle \vee, *, \backslash, /, 0, 1\rangle$, then $\mathbf{Q}[\Psi] \in \mathbb{PM}^{s\ell}$.
3. If $\langle \wedge\rangle \leq \Psi$, then $\mathbf{Q}[\Psi]$ satisfies a set of equations defining the class of lattices.
4. If $\Psi = \mathfrak{L}$, then $\mathbf{Q}[\Psi] \in \mathbb{FL}$.

5. If $e \leq \sigma$, then $\mathbf{Q}[\Psi] \in \mathbb{K}_e[\Psi]$.
6. If $w_l \leq \sigma$, then $\mathbf{Q}[\Psi] \in \mathbb{K}_{w_l}[\Psi]$.
7. If $w_r \leq \sigma$, then $\mathbf{Q}[\Psi] \in \mathbb{K}_{w_r}[\Psi]$.
8. If $c \leq \sigma$ then, $\mathbf{Q}[\Psi] \in \mathbb{K}_c[\Psi]$.

1) If $\varphi \approx \psi$ belongs to a set of equations defining the $\mathring{\mathbb{M}}^{s\ell}$-algebras (Theorem 6.3), we will see that the sequents $\varphi \Rightarrow \psi$ and $\psi \Rightarrow \varphi$ are derivable and thus $\langle \varphi, \psi \rangle \in \Omega\Phi$.

• *Equations defining the $\langle \vee \rangle$-semilattices:*

$(x \vee y) \vee z \approx x \vee (y \vee z)$; $x \vee y \approx y \vee x$; $x \vee x \approx x$. The corresponding sequents are easily obtained by using (*Axiom*) and the introduction rules for $\vee$.

• *Equations defining the $\langle *, 1 \rangle$-monoids:*

$(x*y)*z \approx x*(y*z)$. The corresponding sequents are easily obtained by using (*Axiom*) and the rules $(\Rightarrow *)$ and $(* \Rightarrow)$.

$1*x \approx x$. The sequent $1*\varphi \Rightarrow \varphi$ is obtained by applying $(* \Rightarrow)$ to the sequent $1, \varphi \Rightarrow \varphi$ which it is obtained from $\varphi \Rightarrow \varphi$ using $(1 \Rightarrow)$. The sequent $\varphi \Rightarrow 1 * \varphi$ is obtained from the axioms $\emptyset \Rightarrow 1$ and $\varphi \Rightarrow \varphi$ using the rule $(\Rightarrow *)$.

• *Equations of distributivity of $*$ with respect to $\vee$:*

$(x \vee y) * z \approx (x * z) \vee (y * z)$. Let us consider the following derivations:

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi \qquad \gamma \Rightarrow \gamma}{\varphi, \gamma \Rightarrow \varphi * \gamma}\,(\Rightarrow *)}{\varphi, \gamma \Rightarrow (\varphi * \gamma) \vee (\psi * \gamma)}\,(\Rightarrow \vee_1) \qquad \dfrac{\dfrac{\psi \Rightarrow \psi \qquad \gamma \Rightarrow \gamma}{\psi, \gamma \Rightarrow \psi * \gamma}\,(\Rightarrow *)}{\psi, \gamma \Rightarrow (\varphi * \gamma) \vee (\psi * \gamma)}\,(\Rightarrow \vee_2)}{\varphi \vee \psi, \gamma \Rightarrow (\varphi * \gamma) \vee (\psi * \gamma)}\,(\vee \Rightarrow)}{(\varphi \vee \psi) * \gamma \Rightarrow (\varphi * \gamma) \vee (\psi * \gamma)}\,(* \Rightarrow)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \varphi \vee \psi}\,(\Rightarrow \vee_1) \qquad \gamma \Rightarrow \gamma}{\varphi, \gamma \Rightarrow (\varphi \vee \psi) * \gamma}\,(\Rightarrow *)}{\varphi * \gamma \Rightarrow (\varphi \vee \psi) * \gamma}\,(* \Rightarrow) \qquad \dfrac{\dfrac{\dfrac{\psi \Rightarrow \psi}{\psi \Rightarrow \varphi \vee \psi}\,(\Rightarrow \vee_2) \qquad \gamma \Rightarrow \gamma}{\psi, \gamma \Rightarrow (\varphi \vee \psi) * \gamma}\,(\Rightarrow *)}{\psi * \gamma \Rightarrow (\varphi \vee \psi) * \gamma}\,(* \Rightarrow)}{(\varphi * \gamma) \vee (\psi * \gamma) \Rightarrow (\varphi \vee \psi) * \gamma}\,(\vee \Rightarrow)$$

$z * (x \vee y) \approx (z * x) \vee (z * x)$. The corresponding sequents are mirror images of the previous ones and, therefore, are also derivable (Theorem 4.14).

2) If $\Psi = \langle \vee, *, {}^{\backprime}, {}', 0, 1 \rangle$, in addition we must prove that $\mathbf{Q}[\Psi]$ satisfies the equations in Theorem 7.25 relative to the pseudocomplements. Indeed:

• *Equations* $1^{\backprime} \approx 0$; $1 \preccurlyeq 0^{\backprime}$; $x * (y * x)^{\backprime} \preccurlyeq y^{\backprime}$.

$1^{\backprime} \approx 0$. The sequents ${}^{\backprime}1 \Rightarrow 0$ and $0 \Rightarrow 1^{\backprime}$ are $\mathcal{FL}_\sigma[\Psi]$-derivable:

$$\dfrac{\dfrac{\emptyset \Rightarrow 1}{1^{\backprime} \Rightarrow \emptyset}\,({}^{\backprime} \Rightarrow)}{1^{\backprime} \Rightarrow 0}\,(\Rightarrow 0) \qquad \dfrac{\dfrac{0 \Rightarrow \emptyset}{1, 0 \Rightarrow \emptyset}\,(1 \Rightarrow)}{0 \Rightarrow 1^{\backprime}}\,(\Rightarrow {}^{\backprime})$$

$1 \preccurlyeq 0^{\backprime}$. The sequent $1 \Rightarrow 0^{\backprime}$ is obtained by means of the following derivation:

$$\dfrac{\dfrac{0 \Rightarrow \emptyset}{\emptyset \Rightarrow 0^{\backprime}}\,(\Rightarrow {}^{\backprime})}{1 \Rightarrow 0^{\backprime}}\,(1 \Rightarrow)$$

$x * (y * x)^{\backprime} \preccurlyeq y^{\backprime}$. The sequent $\varphi * (\psi * \varphi)^{\backprime} \Rightarrow \psi^{\backprime}$ is obtained by means of the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\psi \Rightarrow \psi \qquad \varphi \Rightarrow \varphi}{\psi, \varphi \Rightarrow \psi * \varphi}\,(\Rightarrow *)}{\psi, \varphi, (\psi * \varphi)^{\backprime} \Rightarrow \emptyset}\,({}^{\backprime} \Rightarrow)}{\varphi, (\psi * \varphi)^{\backprime} \Rightarrow \psi^{\backprime}}\,(\Rightarrow {}^{\backprime})}{\varphi * (\psi * \varphi)^{\backprime} \Rightarrow \psi^{\backprime}}\,(* \Rightarrow)$$

• *Equations* ${}'1 \approx 0$; $1 \preccurlyeq {}'0$; $(x * {}'(y * x)) \preccurlyeq {}'y$. The sequents corresponding to these equations are the mirror images of the sequents corresponding to the equations concerning the right pseudocomplement and, by Theorem 4.14, they are derivable.

• *Equations* $(x \vee y)^{\backprime} \preccurlyeq x^{\backprime}$; ${}'(x \vee y) \preccurlyeq {}'x$. By the law of the mirror images, it is sufficient to see that the sequent $(\varphi \vee \psi)^{\backprime} \Rightarrow \varphi^{\backprime}$ is derivable. Let us consider the following derivation:

$$\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \varphi \vee \psi}\,(\Rightarrow \vee_1)}{\varphi, (\varphi \vee \psi)^{\backprime} \Rightarrow \emptyset}\,({}^{\backprime} \Rightarrow)}{(\varphi \vee \psi)^{\backprime} \Rightarrow \varphi^{\backprime}}\,(\Rightarrow {}^{\backprime})$$

3) If $\wedge \in \Psi$, we must see that $\mathbf{Q}[\Psi]$ satisfies a set of equations defining the lattices. We have already seen that using the introduction rules for $\vee$ we can prove that $\mathbf{Q}[\Psi]$ satisfies the commutativity and the associativity of the operation $\vee$. Therefore, it will be sufficient to prove that it satisfies also the commutativity and the associativity of the operation $\wedge$ and the absorption laws:

• *Equations* $x \wedge y \approx y \wedge x$; $x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$; $x \wedge y \preccurlyeq x$; $x \preccurlyeq x \vee y$.

The sequents corresponding to the three first equations are easily obtained using (*Axiom*) and the introduction rules for the connective $\wedge$. The sequent corresponding to the last equation is obtained from (*Axiom*) using $(\Rightarrow \vee_1)$.

4) If $\Psi = \langle \vee, \wedge, *, \backslash, /, {}^{\backprime}, {}', 0, 1 \rangle$, to prove that $\mathbf{Q}[\Psi]$ is a $\mathbb{FL}$-algebra, it only remains to see that it satisfies the equations involving the residuals and the pseudocomplements in the equational characterization of the class $\mathbb{FL}$ in Theorem 6.51

• *Equations* $x * ((x \backslash z) \wedge y) \preccurlyeq z$, $y \preccurlyeq x \backslash ((x * y) \vee z)$. The corresponding formal proofs are the following:

$$\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi \qquad \gamma \Rightarrow \gamma}{\varphi, \varphi \backslash \gamma \Rightarrow \gamma}\,(\backslash \Rightarrow)}{\varphi, (\varphi \backslash \gamma) \wedge \psi \Rightarrow \gamma}\,(\wedge_1 \Rightarrow)}{\varphi * ((\varphi \backslash \gamma) \wedge \psi) \Rightarrow \gamma}\,(* \Rightarrow) \qquad \dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi \qquad \psi \Rightarrow \psi}{\varphi, \psi \Rightarrow \varphi * \psi}\,(\Rightarrow *)}{\varphi, \psi \Rightarrow (\varphi * \psi) \vee \gamma}\,(\Rightarrow \vee_1)}{\psi \Rightarrow \varphi \backslash ((\varphi * \psi) \vee \gamma)}\,(\Rightarrow \backslash)$$

• *Equations* $((z/x) \wedge y) * x \preceq z$, $y \preceq ((y * x) \vee z)/x$. The sequents corresponding to these equations are the mirror images of the sequents corresponding to the equations involving the right residuum and, therefore, they are also derivable.

• *Equations defining the pseudocomplements:* $x^{\backprime} \approx x\backslash 0$; ${}'x \approx 0/x$. Let us consider the following derivations corresponding to the equation defining the right pseudocomplement:

$$\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi, \varphi^{\backprime} \Rightarrow \emptyset}\,({}^{\backprime} \Rightarrow)}{\varphi, \varphi^{\backprime} \Rightarrow 0}\,(\Rightarrow 0)}{\varphi^{\backprime} \Rightarrow \varphi\backslash 0}\,(\Rightarrow \backslash) \qquad \dfrac{\dfrac{\varphi \Rightarrow \varphi \qquad 0 \Rightarrow \emptyset}{\varphi, \varphi\backslash 0 \Rightarrow \emptyset}\,(\backslash \Rightarrow)}{\varphi\backslash 0 \Rightarrow \varphi^{\backprime}}\,(\Rightarrow {}^{\backprime})$$

The sequents corresponding to the left pseudocomplement are the mirror images of the previous ones and thus are also derivable.

5) If $e \leq \sigma$ we must prove that $\mathbf{Q}[\Psi]$ satisfies $x * y \approx y * x$. By symmetry, it is sufficient to derive the sequent $\varphi * \psi \Rightarrow \psi * \varphi$, which is easily obtained using (*Axiom*), $(\Rightarrow *)$, $(e \Rightarrow)$ and $(* \Rightarrow)$.

6) If $w_l \leq \sigma$ we must prove that $\mathbf{Q}[\Psi]$ satisfies $x \preccurlyeq 1$. It will be sufficient to prove that the sequent $\varphi \Rightarrow 1$ is derivable. Indeed, from $\emptyset \Rightarrow 1$ we obtain $\varphi \Rightarrow 1$ using $(w \Rightarrow)$.

7) If $w_r \leq \sigma$ we must prove that $\mathbf{Q}[\Psi]$ satisfies $0 \preccurlyeq x$. It will be sufficient to prove that $0 \Rightarrow \varphi$ is derivable. Indeed, from the sequent $0 \Rightarrow \emptyset$ we obtain $0 \Rightarrow \varphi$ using $(\Rightarrow w)$.

8) If $w_r \leq \sigma$ we must prove that $\mathbf{Q}[\Psi]$ satisfies $x \preccurlyeq x * x$. It will be sufficient to prove that $\varphi \Rightarrow \varphi * \varphi$ is derivable. Let us consider the following derivation:

$$\dfrac{\dfrac{\varphi \Rightarrow \varphi \qquad \varphi \Rightarrow \varphi}{\varphi, \varphi \Rightarrow \varphi * \varphi}\,(\Rightarrow *)}{\varphi \Rightarrow \varphi * \varphi}\,(c \Rightarrow)$$

□

Theorem 9.9 (Algebraization). *Every Gentzen system $\mathcal{FL}_\sigma[\Psi]$ is algebraizable, with equivalent algebraic semantics the variety $\mathbb{K}_\sigma[\Psi]$.*

Proof: We use the translations τ of ρ in Definition 9.1. With these translations, the four conditions of Lemma 3.3 are satisfied: condition 1) by Lemma 9.5, condition 2) by Lemma 9.6, condition 3) by Lemma 9.7, and condition 4) by Lemma 9.8. □

In Table 9.1 we point out the subsystems $\mathcal{FL}_\sigma[\Psi]$ and the corresponding classes of algebras that are their equivalent quasivariety semantics (e.q.s.).

The results of algebraization allow us to obtain the following consequences

Corollary 9.10. *If $\mathbf{A} \in \mathbb{K}_\sigma[\Psi]$, then the sequential Leibniz operator $\Omega_{\mathbf{A}}$ is an isomorphism between the lattice of $\mathcal{FL}_\sigma[\Psi]$-filters and the lattice of $\mathbb{K}_\sigma[\Psi]$-congruences.*

Proof: It is an immediate consequence of Theorem 3.4 and Theorem 9.9. □

Corollary 9.11. *The subdirectly irreducible algebras of one of the classes $\mathbb{K}_\sigma[\Psi]$ are exactly the algebras of the class with the smallest non trivial $\mathcal{FL}_\sigma[\Psi]$-filter.*

Gentzen system	e.q.s.
$\mathcal{FL}_\sigma[\vee, *, 0, 1]$	$\mathring{\mathbb{M}}^{s\ell}_\sigma$
$\mathcal{FL}_\sigma[\vee, \wedge, *, 0, 1]$	$\mathring{\mathbb{M}}^{\ell}_\sigma$
$\mathcal{FL}_\sigma[\vee, *, ^\backprime, ', 0, 1]$	$\mathbb{PM}^{s\ell}_\sigma$
$\mathcal{FL}_\sigma[\vee, \wedge, *, ^\backprime, ', 0, 1]$	$\mathbb{PM}^{\ell}_\sigma$
$\mathcal{FL}_\sigma$	$\mathbb{FL}_\sigma$

Table 9.1: Systems $\mathcal{FL}_\sigma[\Psi]$ and their equivalent quasivariety semantics

9.2 Fragments of $\mathcal{FL}_\sigma$ and $\mathfrak{e}\mathcal{FL}_\sigma$ in the languages with disjunction and fusion and without implications

In the following, using the results concerning subreducts of Section 8.3 and the results of algebraization of the previous section, we will state that if Ψ is one of the languages $\langle \vee, *, 0, 1\rangle$, $\langle \vee, \wedge, *, 0, 1\rangle$, $\langle \vee, *, ^\backprime, ', 0, 1\rangle$ or $\langle \vee, \wedge, *, ^\backprime, ', 0, 1\rangle$, each system $\mathcal{FL}_\sigma[\Psi]$ coincides with the Ψ-fragment of $\mathcal{FL}_\sigma$. As an immediate consequence of this fact we have that the external deductive systems associated to these fragments are fragments of the external deductive system associated to $\mathcal{FL}_\sigma$.

Theorem 9.12. *Let Ψ be one of the languages $\langle \vee, *, 0, 1\rangle$, $\langle \vee, \wedge, *, 0, 1\rangle$, $\langle \vee, *, ^\backprime, ', 0, 1\rangle$ or $\langle \vee, \wedge, *, ^\backprime, ', 0, 1\rangle$. For each σ, the system $\mathcal{FL}_\sigma[\Psi]$ is the Ψ-fragment of $\mathcal{FL}_\sigma$.*

Proof: We want to see that, for ever $\Phi \cup \{\varsigma\} \subseteq Seq_\Psi^{\omega\times\{0,1\}}$,

$$\Phi \vdash_{\mathbf{FL}_\sigma} \varsigma \quad \textit{iff} \quad \Phi \vdash_{\mathbf{FL}_\sigma[\Psi]} \varsigma.$$

Let τ be the translation of Definition 9.1. Then we have the following chain of equivalences:

$$\Phi \vdash_{\mathbf{FL}_\sigma} \varsigma \quad \textit{iff} \quad \tau(\Phi) \vDash_{\mathbb{FL}_\sigma} \tau(\varsigma) \quad \textit{iff} \quad \tau(\Phi) \vDash_{\mathbb{K}_\sigma[\Psi]} \tau(\varsigma) \quad \textit{iff} \quad \Phi \vdash_{\mathbf{FL}_\sigma[\Psi]} \varsigma.$$

The first equivalence is obtained applying Theorem 9.9; the second one is due to the fact that, in each case, the class $\mathbb{K}_\sigma[\Psi]$ is the class of all the Ψ-subreducts of the class $\mathbb{FL}_\sigma$ (Theorem 8.73); the third one is a consequence of Theorem 9.9. □

Corollary 9.13. *Let Ψ be one of the languages $\langle \vee, *, 0, 1\rangle$, $\langle \vee, \wedge, *, 0, 1\rangle$, $\langle \vee, *, ^\backprime, ', 0, 1\rangle$ or $\langle \vee, \wedge, *, ^\backprime, ', 0, 1\rangle$. For each σ, the external system $\mathfrak{e}\mathcal{FL}_\sigma[\Psi]$ associated to $\mathcal{FL}_\sigma[\Psi]$ is the Ψ-fragment of the external system $\mathfrak{e}\mathcal{FL}_\sigma$ associated to $\mathcal{FL}_\sigma$.*

Proof: Applying the definition of external system and Theorem 9.12 we have, for every $\Gamma \cup \{\varphi\} \subseteq Fm_\Psi$,

$$\Gamma \vdash_{\mathfrak{e}\mathcal{FL}_\sigma} \varphi \quad \text{iff} \quad \{\emptyset \Rightarrow \gamma : \gamma \in \Gamma\} \vdash_{\mathbf{FL}_\sigma} \emptyset \Rightarrow \varphi \quad \text{iff}$$
$$\{\emptyset \Rightarrow \gamma : \gamma \in \Gamma\} \vdash_{\mathbf{FL}_\sigma[\Psi]} \emptyset \Rightarrow \varphi \quad \text{iff} \quad \Gamma \vdash_{\mathfrak{e}\mathcal{FL}_\sigma[\Psi]} \varphi.$$ □

Next, using the algebraization results for the systems $\mathcal{FL}_\sigma$, we will show that every one of these Gentzen systems is equivalent to its associated external deductive system.[1]

Theorem 9.14. *Let $\sigma \leq ew_l w_r c$. The system $\mathcal{FL}_\sigma$ is equivalent to its associated external deductive system $\mathfrak{e}\mathcal{FL}_\sigma$.*

Proof: We define the translations τ' from $\mathfrak{L}$-sequents to $\mathfrak{L}$-formulas and ρ' from $\mathfrak{L}$-formulas to $\mathfrak{L}$-sequents in the following way:

$$\tau'(\varphi_0,...,\varphi_{m-1} \Rightarrow \varphi) := \begin{cases} \{\varphi_{m-1}\backslash(\varphi_{m-2}\backslash(\ldots\backslash(\varphi_0\backslash\varphi)\ldots))\}, & \text{if } m \geq 1, \\ \\ \{\varphi\}, & \text{if } m = 0, \end{cases}$$

$$\tau'(\varphi_0,...,\varphi_{m-1} \Rightarrow \emptyset) := \begin{cases} \{\varphi_{m-1}\backslash(\varphi_{m-2}\backslash(\ldots\backslash(\varphi_0\backslash 0)\ldots))\}, & \text{if } m \geq 1, \\ \\ \{0\}, & \text{if } m = 0, \end{cases}$$

$$\rho'(\varphi) := \{\emptyset \Rightarrow \varphi\}.$$

To see that $\mathcal{FL}_\sigma$ and $\mathfrak{e}\mathcal{FL}_\sigma$ are equivalent we will prove that the following conditions are satisfied (see Chapter 3, pg.35):

a) For every $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathfrak{L}}$, $\Gamma \vdash_{\mathfrak{e}\mathcal{FL}_\sigma} \varphi$ sii $\rho'[\Gamma] \vdash_{\mathbf{FL}_\sigma} \rho'(\varphi)$,

b) For every $\varsigma \in Seq_{\mathfrak{L}}^{\omega\times\{0,1\}}$, $\varsigma \dashv\vdash_{\mathbf{FL}_\sigma} \rho'\tau'(\varsigma)$.

a): Given that $\vdash_{\mathfrak{e}\mathcal{FL}_\sigma}$ and $\vdash_{\mathbf{FL}_\sigma}$ are finitary, we will restrict ourselves to finite sets of formulas without lost of generality. Suppose $\Gamma = \{\varphi_0,...,\varphi_{m-1}\}$, with $m \in \omega$. Then we have

$$\Gamma \vdash_{\mathfrak{e}\mathcal{FL}_\sigma} \varphi \quad \text{iff} \quad \{\emptyset \Rightarrow \varphi_0,...,\emptyset \Rightarrow \varphi_{m-1}\} \vdash_{\mathfrak{e}\mathcal{FL}_\sigma} \emptyset \Rightarrow \varphi \quad \text{iff} \quad \rho'[\Gamma] \vdash_{\mathbf{FL}_\sigma} \rho'(\varphi).$$

b): Let $\varsigma = \Gamma \Rightarrow \Delta$. Let us define δ as the formula 0 if Δ is the empty sequence and as the formula φ if Δ is constituted by the formula φ. If Γ is a sequence of m formulas $\varphi_0,\ldots,\varphi_{m-1}$ we will use the following abbreviation:

$$\Gamma\backslash\delta := \begin{cases} \varphi_{m-1}\backslash(\varphi_{m-2}\backslash(\ldots\backslash(\varphi_0\backslash\delta)\ldots)), & \text{si } m \geq 1; \\ \delta, & \text{si } m = 0. \end{cases}$$

Using the above conventions and the definition of the translations ρ' and τ we have

$$\rho'\tau'(\Gamma \Rightarrow \Delta) = \emptyset \Rightarrow \Gamma\backslash\delta.$$

From this, using the fact that, for every $\mathfrak{L}$-formula ψ, the sequents $\emptyset \Rightarrow \psi$ and $1 \Rightarrow \psi$ are interderivable, we obtain

$$\rho'\tau'(\Gamma \Rightarrow \Delta) \dashv\vdash_{\mathbf{FL}_\sigma} 1 \Rightarrow \Gamma\backslash\delta.$$

[1] In Section 4.4 we have summarized some Hilbert-style presentations for these external systems.

Let ρ be the translation considered in Definition 9.1. By Lemma 9.2 we have that the sequent $1 \Rightarrow \Gamma\backslash\delta$ and the sequents in $\rho(1 \preccurlyeq \Gamma\backslash\delta)$ are interderivable and, therefore,

$$\rho'\tau'(\Gamma \Rightarrow \Delta) \dashv\vdash_{\mathbf{FL}_\sigma} \rho(1 \preccurlyeq \Gamma\backslash\delta).$$

Now observe that in the equational system associated to $\mathbb{FL}_\sigma$ the equations $1 \preccurlyeq \Gamma\backslash\delta$ and $\prod\Gamma \preccurlyeq \delta$ are interderivable (by the law of residuation). Thus, since $\mathcal{FL}_\sigma$ and $\langle\mathfrak{L}, \vDash_{\mathbb{FL}_\sigma}\rangle$ are equivalent (Theorem 9.9), the translations by ρ of these equations are interderivable in $\mathcal{FL}_\sigma$:

$$\rho(1 \preccurlyeq \Gamma\backslash\delta) \dashv\vdash_{\mathbf{FL}_\sigma} \rho(\prod\Gamma \preccurlyeq \delta).$$

But $\rho(\prod\Gamma \preccurlyeq \delta) = \rho\tau(\Gamma \Rightarrow \Delta)$, where τ is the translation from sequents in equations of Definition 9.1 and, therefore, applying again the equivalence of $\mathcal{FL}_\sigma$ and $\langle\mathfrak{L}, \vDash_{\mathbb{FL}}\rangle$ we have $\rho\tau(\Gamma \Rightarrow \Delta) \dashv\vdash_{\mathbf{FL}_\sigma} \Gamma \Rightarrow \Delta$. Consequently,

$$\rho'\tau'(\Gamma \Rightarrow \Delta) \dashv\vdash_{\mathbf{FL}_\sigma} \Gamma \Rightarrow \Delta. \qquad \square$$

Now we give an alternative proof of the well known result concerning the algebraization of the external deductive system associated to a calculus $\mathbf{FL}_\sigma$ (see for instance [GO06]) using Theorem 9.14 and the algebraization result for $\mathcal{FL}_\sigma$

Corollary 9.15. *For every sequence σ, the deductive system $\mathfrak{e}\mathcal{FL}_\sigma$ is algebraizable with equivalent algebraic semantics the variety $\mathbb{FL}_\sigma$, with set of equivalence formulas $\varphi\Delta\psi = \{\varphi\backslash\psi, \psi\backslash\varphi\}$ and defining equation $1 \vee \varphi \approx \varphi$.*

Proof: Fix a sequence σ and take the translations τ and ρ of Theorem 9.9 and τ' and ρ' of Theorem 9.14. We define the translations $\tau'' := \tau\rho'$ from $\mathfrak{L}$-formulas to $\mathfrak{L}$-equations and $\rho'' := \tau'\rho$ from $\mathfrak{L}$-equations to $\mathfrak{L}$-formulas, that is, $\tau''(\varphi) = \{1 \preccurlyeq \varphi\}$ and $\rho''(\varphi \approx \psi) = \{\varphi\backslash\psi, \psi\backslash\varphi\}$. With these translations it is immediate that $\mathfrak{e}\mathcal{FL}_\sigma$ is algebraizable with equivalent algebraic semantics the variety $\mathbb{FL}_\sigma$. Moreover, given the definitions of τ'' and ρ'', we have that the set of equivalence formulas is$\{\varphi\backslash\psi, \psi\backslash\varphi\}$ and the defining equation is $1 \preccurlyeq \varphi$ (see Chapter 3, end of pg.37). $\square$

In the following result we show that the classes $\mathring{\mathbb{M}}^{s\ell}_\sigma$, $\mathring{\mathbb{M}}^{\ell}_\sigma$, $\mathbb{PM}^{s\ell}_\sigma$ and $\mathbb{PM}^{\ell}_\sigma$ are not equivalent to any deductive system. The proof is the same as that of [RV93, Theorem 3.1], where it is proved that the variety of pseudocomplemented distributive lattices is not the equivalent algebraic semantics to any deductive system. As a corollary we obtain that, unlike the systems $\mathcal{FL}_\sigma$ and $\mathfrak{e}\mathcal{FL}_\sigma$, the systems $\mathcal{FL}_\sigma[\Psi]$, where Ψ is one of the four considerate languages, are not equivalent to any deductive system and therefore, they are not equivalent to their associated external deductive systems.

Theorem 9.16. *The classes $\mathring{\mathbb{M}}^{s\ell}_\sigma$, $\mathring{\mathbb{M}}^{\ell}_\sigma$, $\mathbb{PM}^{s\ell}_\sigma$, $\mathbb{PM}^{\ell}_\sigma$ are not the equivalent algebraic semantics to any deductive system.*

Proof: Suppose that $\mathcal{S}$ is an algebraizable deductive system with equivalent algebraic semantics $\mathbb{PM}^{s\ell}_\sigma$. Then, for every $\mathbb{PM}^{s\ell}_\sigma$-algebra $\mathbf{A}$, the Leibniz operator $\Omega_{\mathbf{A}}$ is an isomorphism between the lattice of $\mathcal{S}$-filters and the lattice of $\mathbb{PM}^{s\ell}_\sigma$-congruences. Now consider the algebra $\mathbf{A} = \langle\{0, a, b, c, 1\}, \vee, *, \neg, 0, 1\rangle$, where $\vee$ are $*$ are the supremum and the infimum corresponding to the order $0 < a < b < c < 1$, and where $\neg$ is defined by $\neg 0 = 1$, $\neg a = \neg b = \neg c = \neg 1 = 0$. $\mathbf{A}$ is a pseudocomplemented distributive lattice and, consequently, $\mathbf{A} \in \mathbb{PM}^{s\ell}_{\sigma}$ for every sequence σ. We have that

$$Con_{\mathbb{PM}^{s\ell}_{\sigma}}(\mathbf{A}) = Con(\mathbf{A}) = \{\triangle_A, \theta_{ab}, \theta_{bc}, \theta_{abc}, \theta_{c1}, \theta_{bc1}, \theta_{abc1}, \theta_{ab} \cup \theta_{c1}, \nabla_A\},$$

where $\triangle_A$ is the identity relation, $\nabla_A = A^2$, and $\theta_{x_1 \dots x_n}$ is the congruence that identifies only $x_1 \dots x_n$. Thus, since there is a bijection between the $\mathcal{S}$-filters and the $\mathbb{PM}^{s\ell}_{\sigma}$-congruences, it must have 9 $\mathcal{S}$-filters. On the other hand, since $\mathcal{S}$ is algebraizable it has theorems (see [BP89, Theorem 4.7]) and thus the smallest $\mathcal{S}$-filter, say X, is non-empty and therefore there are 8 subsets which contain it strictly and this is possible in the considered set only if X has only one element. So, since $\Omega_{\mathbf{A}}$ is an isomorphism, we have that $\Omega_{\mathbf{A}}(X) = \triangle_A$ but this is not true in any case. Indeed, $\Omega_{\mathbf{A}}(\{0\}) = \theta_{abc1}$, $\Omega_{\mathbf{A}}(\{a\}) = \theta_{bc1}$, $\Omega_{\mathbf{A}}(\{b\}) = \theta_{c1}$, $\Omega_{\mathbf{A}}(\{c\}) = \theta_{ab}$, $\Omega_{\mathbf{A}}(\{1\}) = \theta_{abc}$.

Now, to prove that $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$ is not the equivalent algebraic semantics of any deductive system it is sufficient to consider the $\langle \vee, *, \emptyset, 1\rangle$-reduct, say $\mathbf{A}'$, of the previous algebra $\mathbf{A}$. We have $Con_{\mathring{\mathbb{M}}^{s\ell}_{\sigma}}(\mathbf{A}') = Con_{\mathbb{PM}^{s\ell}_{\sigma}}(\mathbf{A})$ and the same argument works in this case.

Finally, to prove that $\mathbb{PM}^{\ell}_{\sigma}$ and $\mathring{\mathbb{M}}^{\ell}_{\sigma}$ are not the equivalent algebraic semantics of any deductive system it is sufficient to consider the algebras $\mathbf{B} = \langle \mathbf{A}, \wedge\rangle$ and $\mathbf{B}' = \langle \mathbf{A}', \wedge\rangle$, where $\wedge$ es defined, for each $x, y \in A$, as $x \wedge y := x * y$. We have that $\mathbf{B} \in \mathbb{PM}^{\ell}$ and $\mathbf{B}' \in \mathring{\mathbb{M}}^{\ell}$ and in both cases the same argument used above applies. $\square$

Corollary 9.17. *The Gentzen systems*

$$\mathcal{FL}_{\sigma}[\vee, *, 0, 1],\ \mathcal{FL}_{\sigma}[\vee, \wedge, *, 0, 1],\ \mathcal{FL}_{\sigma}[\vee, *, \backslash, ', 0, 1] \text{ and } \mathcal{FL}_{\sigma}[\vee, \wedge, *, \backslash, ', 0, 1]$$

are not equivalent to any deductive system and, therefore, none of these Gentzen systems is equivalent to its associated external deductive system.

Proof: It is an immediate consequence of Theorem 9.9 and Theorem 9.16. $\square$

Theorem 9.18. *The deductive systems*

$$\mathfrak{e}\mathcal{FL}_{\sigma}[\vee, *, 0, 1],\ \mathfrak{e}\mathcal{FL}_{\sigma}[\vee, \wedge, *, 0, 1],\ \mathfrak{e}\mathcal{FL}_{\sigma}[\vee, *, \backslash, ', 0, 1] \text{ and } \mathfrak{e}\mathcal{FL}_{\sigma}[\vee, \wedge, *, \backslash, ', 0, 1]$$

are not protoalgebraic.

Proof: Let Ψ be one of the four considerate languages. Each system $\mathfrak{e}\mathcal{FL}_{\sigma}[\Psi]$ is a subsystem of $\mathfrak{e}\mathcal{FL}_{ewc}[\Psi]$ which, by Corollary 9.13, is the Ψ-fragment of $\mathfrak{e}\mathcal{FL}_{ewc}$, that is, of the intuitionistic logic presented in the language $\mathfrak{L}$ of **FL**. As the protoalgebraicity is monotonic, if we have that the systems $\mathfrak{e}\mathcal{FL}_{ewc}[\Psi]$ are not protoalgebraic, then we will be able to conclude that all the systems $\mathfrak{e}\mathcal{FL}_{\sigma}[\Psi]$ are not protoalgebraic.

The system $\mathfrak{e}\mathcal{FL}_{ewc}[\vee, *, 0, 1]$ is the $\langle \vee, *, 0, 1\rangle$-fragment of $\mathfrak{e}\mathcal{FL}_{ewc}$, that is, of the intuitionistic logic that, given the presence of the rules of left weakening and contraction in the corresponding calculus, it is a notational copy of the $\langle\vee,\wedge,0,1\rangle$-fragment of the classical logic and it is known that this fragment is not protoalgebraic (see [FV91]). Given that the behavior of the connectives $\wedge$ and $*$ is the same one in this context, it is easy to see it that $\mathfrak{e}\mathcal{FL}_{ewc}[\vee,\wedge,*,0,1]$ is not protoalgebraic either.

The system $\mathfrak{e}\mathcal{FL}_{ewc}[\vee,*,\backslash,',0,1]$ is definitionally equivalent to $\mathfrak{e}\mathcal{FL}_{ewc}[\vee,*,\neg,0,1]$ and this system is a notational copy of the $\langle\vee,\wedge,\neg,0,1\rangle$-fragment of the intuitionistic logic, which is known not to be protoalgebraic (see [BP89, Theorem 5.13]). It is easy to see that $\mathfrak{e}\mathcal{FL}_{ewc}[\vee,\wedge,*,\neg,0,1]$ is not protoalgebraic either. □

Although they are non-protoalgebraic, these systems have an algebraic semantics, as we show in the following result.

Theorem 9.19. *For every σ, $\mathring{\mathbb{M}}^{s\ell}_\sigma$ ($\mathring{\mathbb{M}}^{\ell}_\sigma$, $\mathbb{PM}^{s\ell}_\sigma$, $\mathbb{PM}^{\ell}_\sigma$) is an algebraic semantics for $\mathfrak{e}\mathcal{FL}_\sigma[\vee,*,0,1]$ ($\mathfrak{e}\mathcal{FL}_\sigma[\vee,\wedge,*,0,1]$, $\mathfrak{e}\mathcal{FL}_\sigma[\vee,*,\backslash,',0,1]$, $\mathfrak{e}\mathcal{FL}_\sigma[\vee,\wedge,*,\backslash,',0,1]$) with defining equation $1\preccurlyeq p$.*

Proof: As a consequence of Theorem 9.9 we have, for every $\Sigma\cup\{\varphi\}\subseteq Fm_{\langle\vee,*,0,1\rangle}$,

$$\{\emptyset\Rightarrow\psi:\psi\in\Sigma\}\vdash_{\mathbf{FL}_\sigma[\vee,*,0,1]}\emptyset\Rightarrow\varphi \quad \textit{iff} \quad \{\tau(\emptyset\Rightarrow\psi):\psi\in\Sigma\}\models_{\mathring{\mathbb{M}}^{s\ell}_\sigma}\tau(\emptyset\Rightarrow\varphi).$$

That is,

$$\Sigma\vdash_{\mathfrak{e}\mathcal{FL}_\sigma[\vee,*,0,1]}\varphi \quad \textit{iff} \quad \{1\preccurlyeq\psi:\psi\in\Sigma\}\models_{\mathring{\mathbb{M}}^{s\ell}_\sigma}1\preccurlyeq\varphi.$$

The other three cases are analogously proved. □

9.3 Some results on decidability

In this section we give decidability results for some of the fragments considered. When $\sigma\leq w_l$ it is well known that the variety $\mathbb{FL}_\sigma$ has the FEP. Using this fact and the results about subreducts in Section 8.3 we show that, when $\sigma\leq w_l$, the varieties $\mathring{\mathbb{M}}^{s\ell}_\sigma$, $\mathring{\mathbb{M}}^{\ell}_\sigma$, $\mathbb{PM}^{s\ell}_\sigma$ and $\mathbb{PM}^{\ell}_\sigma$ have the FEP. This fact allow us to prove decidability results for the corresponding systems $\mathcal{FL}_\sigma[\Psi]$ and $\mathfrak{e}\mathcal{FL}_\sigma[\Psi]$.

Theorem 9.20. (Cf.[GJKO07, Theorem 6.46]) *Let σ be such that $w_l\leq\sigma$. The variety $\mathbb{FL}_\sigma$ has the Finite Embeddability Property.*

Theorem 9.21. *Let σ be such that $w_l\leq\sigma$. The varieties $\mathring{\mathbb{M}}^{s\ell}_\sigma$, $\mathring{\mathbb{M}}^{\ell}_\sigma$, $\mathbb{PM}^{s\ell}_\sigma$ and $\mathbb{PM}^{\ell}_\sigma$ have the Finite Embeddability Property.*

Proof: It is enough to prove the first part. Let $\mathbf{A}$ be any algebra in $\mathring{\mathbb{M}}^{s\ell}_\sigma$ and let $\mathbf{B}$ be a finite partial subalgebra of $\mathbf{A}$. By Theorem 8.73, $\mathbf{A}$ is embeddable in a $\mathbb{FL}_\sigma$-algebra $\mathbf{A}'$. Let i be such an embedding. Now we have that $i[\mathbf{B}]$ is a finite partial subalgebra of $\mathbf{A}'$. By Theorem 9.20 we have that $\mathbb{FL}_\sigma$ has the FEP. Therefore, $\mathbf{A}'$ can be embedded in a finite $\mathbb{FL}_\sigma$-algebra $\mathbf{D}$. Let h be this embedding and let $\mathbf{D}'$ be the $\langle\vee,*,0,1\rangle$-reduct of $\mathbf{D}$. $\mathbf{D}'$ is a finite $\mathring{\mathbb{M}}^{s\ell}_\sigma$-algebra and the map $h\circ i$ is an embedding from $\mathbf{B}$ into $\mathbf{D}'$. A similar argument runs for $\mathring{\mathbb{M}}^{\ell}_\sigma$, $\mathbb{PM}^{s\ell}_\sigma$, and $\mathbb{PM}^{\ell}_\sigma$. □

Corollary 9.22. *Let σ be as in Theorem 9.21. The quasi-equational (and universal) theory of each one of the varieties $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$, $\mathring{\mathbb{M}}^{\ell}_{\sigma}$, $\mathbb{PM}^{s\ell}_{\sigma}$ and $\mathbb{PM}^{\ell}_{\sigma}$ is decidable.*

Note 9.23. Observe that the method used to obtain the previous result can only be applied to the case of the varieties $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$, $\mathring{\mathbb{M}}^{\ell}_{\sigma}$, $\mathbb{PM}^{s\ell}_{\sigma}$ and $\mathbb{PM}^{\ell}_{\sigma}$ falling under the scope of Theorem 9.21. For example, it is well known that the varieties $\mathbb{FL}$, $\mathbb{FL}_e$ and $\mathbb{FL}_{w_r}$ do not have the FEP (see [GJKO07, Theorem 6.56]).

Corollary 9.24. *Let σbe as in Theorem 9.21. The Gentzen systems $\mathcal{FL}_\sigma[\vee, *, 0, 1]$, $\mathcal{FL}_\sigma[\vee, \wedge, *, 0, 1]$, $\mathcal{FL}_\sigma[\vee, *, \backslash, ', 0, 1]$, and $\mathcal{FL}_\sigma[\vee, \wedge, *, \backslash, ', 0, 1]$ are decidable, i.e., their sets of entailments of the form $\{\Gamma_i \Rightarrow \Delta_i : i \in I\} \vdash \Gamma \Rightarrow \Delta$, with I finite, are decidable.*

Proof: It is an immediate consequence of the algebraization (Theorem 9.9) and Corollary 9.21. □

Corollary 9.25. *Let σ be as in Theorem 9.21. The external systems $\mathfrak{e}\mathcal{FL}_\sigma[\vee, *, 0, 1]$, $\mathfrak{e}\mathcal{FL}_\sigma[\vee, \wedge, *, 0, 1]$, $\mathfrak{e}\mathcal{FL}_\sigma[\vee, *, \backslash, ', 0, 1]$, and $\mathfrak{e}\mathcal{FL}_\sigma[\vee, \wedge, *, \backslash, ', 0, 1]$ are decidable, i.e., their entailments of the form $\Gamma \vdash \varphi$, with Γ finite, are decidable.*

Proof: By Theorem 9.19, these deductive systems have, respectively, the varieties $\mathring{\mathbb{M}}^{s\ell}_{\sigma}$, $\mathring{\mathbb{M}}^{\ell}_{\sigma}$, $\mathbb{PM}^{s\ell}_{\sigma}$ and $\mathbb{PM}^{\ell}_{\sigma}$ as algebraic semantics. The result is an immediate consequence of this last fact and Corollary 9.21. □

9.4 On some systems with weak contraction

In this section we define the substructural systems with weak contraction $\mathcal{FL}_{\sigma\hat{c}}$, where $\sigma \leq ew_lw_r$. We show that the subsystems $\mathcal{FL}_{\sigma\hat{c}}[\vee, *, \backslash, ', 0, 1]$, $\mathcal{FL}_{\sigma\hat{c}}[\vee, \wedge, *, \backslash, ', 0, 1]$ and the system $\mathcal{FL}_{\sigma\hat{c}}$ are algebraizable and have as respective equivalent algebraic semantics the varieties $\mathbb{PM}^{s\ell}_{\sigma\hat{c}}$, $\mathbb{PM}^{\ell}_{\sigma\hat{c}}$ and $\mathbb{FL}_{\sigma\hat{c}}$ (for the definition of these varieties see Section 7.5). Let Ψ be either the language $\langle \vee, *, \backslash, ', 0, 1\rangle$ or the language $\vee, \wedge, *, \backslash, ', 0, 1$. We prove that, for each $\sigma \leq ew_lw_r$, the external system $\mathfrak{e}\mathcal{FL}_{\sigma\hat{c}}[\Psi]$ associated to $\mathcal{FL}_{\sigma\hat{c}}[\Psi]$ is the Ψ-fragment of the external system $\mathfrak{e}\mathcal{FL}_{\sigma\hat{c}}$ associated to $\mathcal{FL}_{\sigma\hat{c}}$.

Definition 9.26 (Substructural calculi with weak contraction)**.** *Let $\sigma \leq ew_lw_r$. We will denote by $\mathbf{FL}_{\sigma\hat{c}}$ the calculus obtained by adding to the rules of $\mathbf{FL}_\sigma$ the structural $\omega \times \{0\}$-rule of contraction (which we also call* rule of weak contraction*):*

$$\frac{\Sigma, \varphi, \varphi, \Gamma \Rightarrow \emptyset}{\Sigma, \varphi, \Gamma \Rightarrow \emptyset} \quad (\hat{c} \Rightarrow).$$

We will denote by $\mathcal{FL}_{\sigma\hat{c}}$ the Gentzen system that this rule determines.

Theorem 9.27. *Let $\sigma \leq ew_lw_r$. Then,*

i) *$\mathcal{FL}_{\sigma\hat{c}}[\vee, *, \backslash, ']$ is algebraizable, with equivalent algebraic semantics the variety $\mathbb{PM}^{s\ell}_{\sigma\hat{c}}$.*

ii) $\mathcal{FL}_{\sigma\hat{c}}[\vee,\wedge,*,\backslash,']$ *is algebraizable, with equivalent algebraic semantics the variety* $\mathbb{PM}^{\ell}_{\sigma\hat{c}}$.

iii) $\mathcal{FL}_{\sigma\hat{c}}$ *is algebraizable, with equivalent algebraic semantics the variety* $\mathbb{FL}_{\sigma\hat{c}}$.

Proof: Let $\mathbb{K} \in \{\mathbb{PM}^{s\ell}_{\sigma}, \mathbb{PM}^{\ell}_{\sigma}, \mathbb{FL}_{\sigma}\}$ and let $\Psi_{\mathbb{K}}$ be the language of $\mathbb{K}$. Let $\mathbb{K}_{\hat{c}}$ be the class obtained by adding the equations $(x*x)^{\backslash} \preccurlyeq x^{\backslash}$ and $'(x*x) \preccurlyeq {}'x$ to any set of equations defining $\mathbb{K}$. Next we will show that $\mathcal{FL}_{\sigma\hat{c}}[\Psi_{\mathbb{K}}]$ is algebraizable with equivalent algebraic semantics the variety $\mathbb{K}_{\hat{c}}$ with the translations τ and ρ in Definition 9.1. We use Lemma 3.3. Thus, we want to prove that:

a) For every $\mathbf{A} \in \mathbb{K}_{\hat{c}}$, the set R of Lemma 9.7 is a $\mathcal{FL}_{\sigma\hat{c}}[\Psi_{\mathbb{K}}]$-filter.

b) For every $\mathcal{FL}_{\sigma\hat{c}}[\Psi_{\mathbb{K}}]$-theory Φ, if Θ_{Φ} is the set defined in Lemma 9.8, the quotient $Fm_{\Psi_{\mathbb{K}}}/\Theta_{\Phi}$ is a $\mathbb{K}_{\hat{c}}$-algebra.

a): In the proof of Lemma 9.7 we have seen that R is a $\mathcal{FL}_{\sigma}[\Psi_{\mathbb{K}}]$-filter. Thus, it will be sufficient to prove that if $\mathbf{A} \in \mathbb{K}$, then R is closed under the rule of weak contraction. Let $\langle\langle \bar{x}, a, a, \bar{y}\rangle, 0\rangle \in R$. Then $\prod \bar{x} * a * a * \prod \bar{y} \leq 0$. Now, applying (LP) and the hypothesis that $\mathbf{A}$ is weakly contractive we have:

$\prod \bar{x} * a * a * \prod \bar{y} \leq 0$ *iff* $a * a * \prod \bar{y} \leq (\prod \bar{x})^{\backslash}$ *iff* $a * a * \prod \bar{y} * {}'((\prod \bar{x})^{\backslash}) \leq 0$ *iff* $\prod \bar{y} * {}'((\prod \bar{x})^{\backslash}) \leq (a*a)^{\backslash}$ *iff* $\prod \bar{y} * {}'((\prod \bar{x})^{\backslash}) \leq a^{\backslash}$ *iff* $a * \prod \bar{y} * {}'((\prod \bar{x})^{\backslash}) \leq 0$ *iff* $a * \prod \bar{y} \leq (\prod \bar{x})^{\backslash}$ *iff* $\prod \bar{x} * a * \prod \bar{y} \leq 0$. Therefore, $\langle\langle \bar{x}, a, \bar{y}\rangle, 0\rangle \in R$.

b): By Lemma 9.8 we have that $Fm_{\Psi_{\mathbb{K}}}/\Theta_{\Phi}$ is a $\mathbb{K}$-algebra. Thus, we only need to show that the sequents $(\varphi * \varphi)^{\backslash} \Rightarrow \varphi^{\backslash}$ and $'(\varphi * \varphi) \Rightarrow {}'\varphi$ are derivable in $\mathbf{FL}_{\sigma\hat{c}}[\Psi_{\mathbb{K}}]$. By the law of mirror images it is sufficient to prove the derivability of one of the sequents. Let us consider the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi \qquad \varphi \Rightarrow \varphi}{\varphi, \varphi \Rightarrow \varphi * \varphi}\,(\Rightarrow *)}{\varphi, \varphi, (\varphi * \varphi)^{\backslash} \Rightarrow \emptyset}\,(^{\backslash} \Rightarrow)}{\varphi, (\varphi * \varphi)^{\backslash} \Rightarrow \emptyset}\,(\Rightarrow \hat{c})}{(\varphi * \varphi)^{\backslash} \Rightarrow \varphi^{\backslash}}\,(\Rightarrow {}^{\backslash})$$

□

Theorem 9.28. *Let* Ψ *be one of the languages* $\langle \vee, *, \backslash, ', 0, 1\rangle$ *or* $\langle \vee, \wedge, *, \backslash, ', 0, 1\rangle$. *For each* $\sigma \leq ew_l w_r$, *the system* $\mathcal{FL}_{\sigma\hat{c}}[\Psi]$ *is the* Ψ*-fragment of* $\mathcal{FL}_{\sigma}$.

Proof: Analogous to the proof of Theorem 9.12 but now using Theorem 9.27 and the fact that the class $\mathbb{K}_{\sigma\hat{c}}[\Psi]$ is the class of all the Ψ-subreducts of the class $\mathbb{FL}_{\sigma\hat{c}}$ (Theorem 8.75). □

Corollary 9.29. *Let* Ψ *be one of the languages* $\langle \vee, *, \backslash, ', 0, 1\rangle$ *or* $\langle \vee, \wedge, *, \backslash, ', 0, 1\rangle$. *For each* $\sigma \leq ew_l w_r$, *the external system* $\mathfrak{e}\mathcal{FL}_{\sigma\hat{c}}[\Psi]$ *associated to* $\mathcal{FL}_{\sigma\hat{c}}[\Psi]$ *is the* Ψ*-fragment of the external system* $\mathfrak{e}\mathcal{FL}_{\sigma\hat{c}}$ *associated to* $\mathcal{FL}_{\sigma\hat{c}}$.

Proof: Applying the definition of external system and Theorem 9.28. □

Chapter 10

Three Implication-free Fragments of t-Norm Based Fuzzy Logics

It is well known that the logic MTL [EG01], the most general of the t-norm based fuzzy logics, is an axiomatic extension of Monoidal Logic [Höh95]. As is pointed out in [EGGC03], Monoidal Logic (ML, for short) is equivalent to the external deductive system $\mathfrak{e}\mathcal{FL}_{ew}$ associated to the calculus $\mathbf{FL}_{ew}$. The logic ML has been considered in the literature under other names: H_{BCK} (see [OK85]), $IPC^*\backslash c$ (*Intuitionistic Propositional Calculus without contraction*; see [AV02, BGCV06]). In this chapter we analyze the fragments of ML in the languages without implication nor negation $\langle \vee, *, 0, 1\rangle$, $\langle \vee, \wedge, *, 0, 1\rangle$ and $\langle \vee, \wedge, 0, 1\rangle$. The main results are the following:

- Let Ψ be one of the languages $\langle \vee, *, 0, 1\rangle$, $\langle \vee, \wedge, *, 0, 1\rangle$ or $\langle \vee, \wedge, 0, 1\rangle$. Then, we show by using an algebraic procedure that the Ψ-fragment of ML coincides with the Ψ-fragment of the classical logic (Theorem 10.2).
- By a proof-theoretical procedure we obtain that the external deductive system $\mathfrak{e}\mathcal{FL}_{ew}[\vee, \wedge, 0, 1]$ is equal to the $\langle \vee, \wedge, 0, 1\rangle$-fragment of the classical logic (Theorem 10.5).

On the other hand, we have the following facts:

- ML is equal to the external deductive system $\mathfrak{e}\mathcal{FL}_{ew}$ (see Section 4.4)
- $\mathfrak{e}\mathcal{FL}_{ew}[\vee, *, 0, 1]$ and $\mathfrak{e}\mathcal{FL}_{ew}[\vee, \wedge, *, 0, 1]$ are the $\langle \vee, *, 0, 1\rangle$-fragment and the $\langle \vee, \wedge, *, 0, 1\rangle$-fragment of $\mathfrak{e}\mathcal{FL}_{ew}$, respectively (by Corollary 9.13 in the previous chapter).

Thus, we have that the systems $\mathfrak{e}\mathcal{FL}_{ew}[\vee, *, 0, 1]$, $\mathfrak{e}\mathcal{FL}_{ew}[\vee, \wedge, *, 0, 1]$, and $\mathfrak{e}\mathcal{FL}_{ew}[\vee, \wedge, 0, 1]$ are equal to the fragments in the corresponding languages of ML and, therefore, they are equal to the fragments in the languages $\langle \vee, *, 0, 1\rangle$, $\langle \vee, \wedge, *, 0, 1\rangle$, and $\langle \vee, \wedge, 0, 1\rangle$ of classical logic, respectively.

Since MTL is an axiomatic extension of ML, we can conclude that the fragments in these three languages of all the t-norm based fuzzy logics are equal to the corresponding fragments of classical logic. This means that the connectives of additive and multiplicative conjunction are indistinguishable in the corresponding fragments without implication and negation of ML (i.e., $\mathfrak{e}\mathcal{FL}_{ew}$). This fact has as a corollary that these two kinds of conjunction are also indistinguishable in the same fragments of all t-norm based fuzzy logics. In other words, when not dealing with implication and negation, the t-norm based fuzzy logics behave classically.

10.1 Analysis of the fragments

Next we will show that the $\langle \vee, *, 0, 1\rangle$, $\langle \vee, \wedge, *, 0, 1\rangle$ and $\langle \vee, \wedge, 0, 1\rangle$-fragments of $\mathfrak{e}\mathcal{FL}_{ew}$ are exactly the $\langle \vee, *, 0, 1\rangle$, $\langle \vee, \wedge, *, 0, 1\rangle$ and $\langle \vee, \wedge, 0, 1\rangle$-fragments of classical logic. We will use the following axiomatization of the $\langle \vee, \wedge\rangle$-fragment of classical logic.

Theorem 10.1. (Cf.[DP80, FV91]) *The $\langle \vee, \wedge\rangle$-fragment of classical propositional logic is axiomatized by the following rules:*

(R1) $\varphi \vee \varphi \vdash \varphi$
(R2) $\varphi \vdash \varphi \vee \psi$
(R3) $\varphi \vee \psi \vdash \psi \vee \varphi$
(R4) $\varphi \vee (\psi \vee \gamma) \vdash (\varphi \vee \psi) \vee \gamma$
(R5) $\varphi \wedge \psi \vdash \varphi$
(R6) $\varphi \wedge \psi \vdash \psi \wedge \varphi$
(R7) $\{\varphi, \psi\} \vdash \varphi \wedge \psi$
(R8) $\varphi \vee (\psi \wedge \gamma) \vdash (\varphi \vee \psi) \wedge (\varphi \vee \gamma)$
(R9) $(\varphi \vee \psi) \wedge (\varphi \vee \gamma) \vdash \varphi \vee (\psi \wedge \gamma)$

An easy adaptation of the proof of [DP80] allows us to conclude that the $\langle \vee, \wedge, 0, 1\rangle$-fragment of the classical propositional logic can be axiomatized by adding the axiom

(A1) 1

and the rule

(R10) $0 \vdash \varphi$

to the previous axiomatization.

Let us denote by IPL^* and CPL^*, respectively, the intuitionistic and the classical propositional logics, in the language $\langle \vee, \wedge, *, \to, 0, 1\rangle$, where the behaviour of $*$ is exactly the same as $\wedge$.

Theorem 10.2. *Let Ψ be any of the languages $\langle \vee, *, 0, 1\rangle$, $\langle \vee, \wedge, *, 0, 1\rangle$ or $\langle \vee, \wedge, 0, 1\rangle$. Then, the Ψ-fragment of $\mathfrak{e}\mathcal{FL}_{ew}$ is equal to the Ψ-fragment of CPL^*. That is,*

$$\Psi\text{-}\,\mathfrak{e}\mathcal{FL}_{ew} = \Psi\text{-}\,CPL^*.$$

Proof: Taking into account that the external deductive system $\mathfrak{e}\mathcal{FL}_{ewc}$ is equal to IPL^*, and the fact that $\Psi\text{-}IPL^* = \Psi\text{-}CPL^*$, it is obvious that $\Psi\text{-}\mathfrak{e}\mathcal{FL}_{ew} \leq \Psi\text{-}CPL^*$. Thus, to prove the theorem it is enough to check that each of the rules (R1)-(R9) in Theorem 10.1 ((A1) and (R10) are immediate) are derivable rules in each Ψ-fragment of $\mathfrak{e}\mathcal{FL}_{ew}$.

Let $\Gamma \cup \{\varphi\} \subseteq Fm_\Psi$. Let $\vdash_\Psi$ be the consequence relation of $\Psi\text{-}\mathfrak{e}\mathcal{FL}_{ew}$. By applying the definition of Ψ-fragment and the fact that the system $\mathfrak{e}\mathcal{FL}_{ew}$ is algebraizable with equivalent algebraic semantics the variety $\mathbb{FL}_{ew}$ (Corollary 9.15) we have

$$\Gamma \vdash_\Psi \varphi \quad \text{iff} \quad \Gamma \vdash_{\mathfrak{e}\mathcal{FL}_w} \varphi \quad \text{iff} \quad \{\gamma \approx 1 : \gamma \in \Gamma\} \models_{\mathbb{FL}_{ew}} \varphi \approx 1.$$

We will use this semantical characterization of the Ψ-fragments of $\mathfrak{e}\mathcal{FL}_{ew}$ to check the derivability of the rules. Obviously we have that $\emptyset \vdash_\Psi 1$ since $\models_{\mathbb{FL}_{ew}} 1 \approx 1$. If $\alpha \vdash_\Psi \beta$ is one of the rules $(R1) - (R4)$, it is easy to see that $\alpha \approx 1 \models_{\mathbb{FL}_{ew}} \beta \approx 1$. If $\odot$ is one of the connectives in $\{\wedge, *\}$, then it is also easy to see that

$\varphi \odot \psi \approx 1 \models_{\mathbb{FL}_{ew}} \varphi \approx 1$; $\varphi \wedge \psi \approx 1 \models_{\mathbb{FL}_{ew}} \psi \odot \varphi \approx 1$; and $\{\varphi \approx 1, \psi \approx 1\} \models_{\mathbb{FL}_{ew}} \varphi \odot \psi \approx 1$.

So, $(R5) - (R7)$ are also derivable rules in each Ψ-fragment. Let $\odot \in \{\wedge, *\}$. To check that (R8) and (R9) are derivable we have to show that

$$\varphi \vee (\psi \odot \gamma) \approx 1 \mathrel{=\joinrel|\joinrel\models}_{\mathbb{FL}_{ew}} (\varphi \vee \psi) \odot (\varphi \vee \gamma) \approx 1. \tag{10.1}$$

For this purpose we will apply the well-known fact that every variety is generated as a quasivariety by its subdirectly irreducible members. So, a quasiequation holds in $\mathbb{FL}_{ew}$ if, and only if it holds in every subdirectly irreducible $\mathbb{FL}_{ew}$-algebra. Thus, it will be sufficient to prove that the double inference (10.1) holds in this subclass of $\mathbb{FL}_{ew}$.

Let $\mathbf{A}$ be a subdirectly irreducible algebra of $\mathbb{FL}_{ew}$. Let us recall that the subdirectly irreducible algebras of $\mathbb{FL}_{ew}$ have the following property (see [KO01, Proposition 1.4]):

$$\text{For all } a, b \in A, \text{ if } a \vee b = 1, \text{ then } a = 1 \text{ or } b = 1. \tag{10.2}$$

Now let $a, b, c \in A$ and suppose that $a \vee (b \odot c) = 1$. So, by (10.2), we have that $a = 1$ or $b \odot c = 1$. If $a = 1$, then $(a \vee b) \odot (a \vee c) = 1 \odot 1 = 1$ (since $1 \wedge 1 = 1$ and $1 * 1 = 1$). If $b \odot c = 1$ then $b = 1$ and $c = 1$ and so, we also have $(a \vee b) \odot (a \vee c) = 1 \odot 1 = 1$.

Conversely, suppose that $(a \vee b) \odot (a \vee c) = 1$. This is equivalent to having $a \vee b = 1$ and $a \vee c = 1$. By (10.2), $a \vee b = 1$ implies that $a = 1$ or $b = 1$ and $a \vee c = 1$ implies that $a = 1$ or $c = 1$. If $a = 1$ we have $a \vee (b \odot c) = 1$. If $a < 1$ we have $b = c = 1$ and so we also obtain $a \vee (b \odot c) = 1$.

Then, we can conclude that $\Psi\text{-}CPL^* \leq \Psi\text{-}\mathfrak{e}\mathcal{FL}_{ew}$. Thus, the proof is finished. □

Corollary 10.3. *Let Ψ be any of the languages $\langle \vee, *, 0, 1\rangle$, $\langle \vee, \wedge, *, 0, 1\rangle$ or $\langle \vee, \wedge, 0, 1\rangle$. The Ψ-fragment of every t-norm based fuzzy logic is equal to the Ψ-fragment of classical logic.*

Proof: It is an immediate consequence of Theorem 10.2 and the fact that t-norm based fuzzy logics are axiomatic extensions of Monoidal Logic ML, which is equal to the system $\mathfrak{e}\mathcal{FL}_{ew}$. □

Corollary 10.4. *The $*$-fragment and the $\wedge$-fragment of every t-norm based fuzzy logic are equal to the $\wedge$-fragment of classical logic.*

Proof: It is an immediate consequence of Corollary 10.3. □

In Corollary 9.13 we have characterized the fragments of $\mathfrak{e}\mathcal{FL}_{ew}$ in the languages $\langle \vee, \wedge, *, 0, 1\rangle$ and $\langle \vee, *, 0, 1\rangle$ as the external deductive systems $\mathfrak{e}\mathcal{FL}_{ew}[\vee, *, 0, 1]$ and $\mathfrak{e}\mathcal{FL}_{ew}[\vee, *, 0, 1]$ associated to the subsystems $\mathcal{FL}_{ew}[\vee, *, 0, 1]$ and $\mathcal{FL}_{ew}[\vee, \wedge, *, 0, 1]$, respectively. In the following, the external system $\mathfrak{e}\mathcal{FL}_{ew}[\vee, \wedge, 0, 1]$ will be characterized by means of proof-theoretical methods.

Theorem 10.5. *The external system $\mathfrak{e}\mathcal{FL}_{ew}[\vee, \wedge, 0, 1]$ is equal to the $\langle \vee, \wedge, 0, 1\rangle$-fragment of classical logic.*

Proof: Obviously, we have $\mathfrak{e}\mathcal{FL}_{ew}[\vee, \wedge, 0, 1] \leq \langle \vee, \wedge, 0, 1, \rangle$-$CPL^*$. To prove the converse it will be sufficient to show that the rules (R8) and (R9) in Theorem 10.2 are derivable rules of $\mathfrak{e}\mathcal{FL}_{ew}[\vee, \wedge, \emptyset, 1]$ (the rules (R1), ..., (R7) and (R10), and the axiom (A1) are immediate). It is easy to see that the sequent $\varphi \vee (\psi \wedge \gamma) \Rightarrow (\varphi \vee \psi) \wedge (\varphi \vee \gamma)$ is derivable in $\mathbf{FL}_{ew}[\vee, \wedge, 0, 1]$. So, by applying (Cut), we have that

$$\emptyset \Rightarrow \varphi \vee (\psi \wedge \gamma) \vdash_{\mathbf{FL}_{ew}[\vee,\wedge,0,1]} \emptyset \Rightarrow (\varphi \vee \psi) \wedge (\varphi \vee \gamma)$$

and so,

$$\varphi \vee (\psi \wedge \gamma) \vdash_{\mathfrak{e}\mathcal{FL}_{ew}[\vee,\wedge,0,1]} (\varphi \vee \psi) \wedge (\varphi \vee \gamma).$$

Now we are going to prove that

$$(\varphi \vee \psi) \wedge (\varphi \vee \gamma) \vdash_{\mathfrak{e}\mathcal{FL}_{ew}[\vee,\wedge,\emptyset,1]} \varphi \vee (\psi \wedge \gamma).$$

It is easy to check that the sequent $(\varphi \vee \psi) \wedge (\varphi \vee \gamma) \Rightarrow (\varphi \vee (\psi \wedge \gamma))$ (distributivity law) is not derivable in $\mathbf{FL}_{ew}[\vee, \wedge, 0, 1]$.[1] However, we will see that if we repeat the formula $(\varphi \vee \psi) \wedge (\varphi \vee \gamma)$, then the sequent which is obtained is derivable, that is, the following sequent

$$(\varphi \vee \psi) \wedge (\varphi \vee \gamma), (\varphi \vee \psi) \wedge (\varphi \vee \gamma) \Rightarrow \varphi \vee (\psi \wedge \gamma) \tag{10.3}$$

is derivable in $\mathbf{FL}_{ew}[\vee, \wedge, 0, 1]$. Then, by using this derivable sequent and two applications of (Cut), we obtain that $\emptyset \Rightarrow (\varphi \vee \psi) \wedge (\varphi \vee \gamma) \vdash_{\mathbf{FL}_{ew}[\vee,\wedge,0,1]} \emptyset \Rightarrow \varphi \vee (\psi \wedge \gamma)$, i.e.,

$$(\varphi \vee \psi) \wedge (\varphi \vee \gamma) \vdash_{\mathfrak{e}\mathcal{FL}_{ew}[\vee,\wedge,0,1]} \varphi \vee (\psi \wedge \gamma).$$

Now, let us check the derivability of the sequent (10.3). Firstly, we will derive the sequents

[1] A simple semantical proof of this statement is that if this sequent is derivable in $\mathbf{FL}_{ew}[\vee, \wedge, 0, 1]$, then it would be derivable in $\mathbf{FL}_{ew}$ and then, by the algebraization, $\mathbb{FL}_{ew}$-algebras would be distributive, a contradiction.

$$\varphi, \varphi \vee \gamma \Rightarrow \varphi \vee (\psi \wedge \gamma) \text{ and } \psi, \varphi \vee \gamma \Rightarrow \varphi \vee (\psi \wedge \gamma)$$

(in these proofs we implicitly use the *exchange* rule):

$$\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi, \varphi \vee \gamma \Rightarrow \varphi}\,(w \Rightarrow)}{\varphi, \varphi \vee \gamma \Rightarrow \varphi \vee (\psi \wedge \gamma)}\,(\Rightarrow \vee_1)$$

$$\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\psi, \varphi \Rightarrow \varphi}\,(w \Rightarrow)}{\psi, \varphi \Rightarrow \varphi \vee (\psi \wedge \gamma)}\,(\Rightarrow \vee_1) \qquad \dfrac{\dfrac{\dfrac{\psi \Rightarrow \psi}{\psi, \gamma \Rightarrow \psi}\,(w \Rightarrow) \qquad \dfrac{\gamma \Rightarrow \gamma}{\psi, \gamma \Rightarrow \gamma}\,(w \Rightarrow)}{\psi, \gamma \Rightarrow \psi \wedge \gamma}\,(\Rightarrow \wedge)}{\psi, \gamma \Rightarrow \varphi \vee (\psi \wedge \gamma)}\,(\Rightarrow \vee_2)}{\psi, \varphi \vee \gamma \Rightarrow \varphi \vee (\psi \wedge \gamma)}\,(\vee \Rightarrow)$$

And now we have:

$$\dfrac{\dfrac{\dfrac{\varphi, \varphi \vee \gamma \Rightarrow \varphi \vee (\psi \wedge \gamma) \qquad \psi, \varphi \vee \gamma \Rightarrow \varphi \vee (\psi \wedge \gamma)}{\varphi \vee \psi, \varphi \vee \gamma \Rightarrow \varphi \vee (\psi \wedge \gamma)}\,(\vee \Rightarrow)}{\varphi \vee \psi, (\varphi \vee \psi) \wedge (\varphi \vee \gamma) \Rightarrow \varphi \vee (\psi \wedge \gamma)}\,(\wedge_2 \Rightarrow)}{(\varphi \vee \psi) \wedge (\varphi \vee \gamma), (\varphi \vee \psi) \wedge (\varphi \vee \gamma) \Rightarrow \varphi \vee (\psi \wedge \gamma)}\,(\wedge_1 \Rightarrow)$$

□

Corollary 10.6. *The following conditions hold:*

a) $\mathfrak{e}\mathcal{FL}_{ew}[\vee, *, 0, 1] = \langle \vee, *, 0, 1 \rangle\text{-}\mathfrak{e}\mathcal{FL}_{ew} = \langle \vee, *, 0, 1 \rangle\text{-}CPL^*$.

b) $\mathfrak{e}\mathcal{FL}_{ew}[\vee, \wedge, *, 0, 1] = \langle \vee, \wedge, *, 0, 1 \rangle\text{-}\mathfrak{e}\mathcal{FL}_{ew} = \langle \vee, \wedge, *, 0, 1 \rangle\text{-}CPL^*$.

c) $\mathfrak{e}\mathcal{FL}_{ew}[\vee, \wedge, 0, 1] = \langle \vee, \wedge, 0, 1 \rangle\text{-}\mathfrak{e}\mathcal{FL}_{ew} = \langle \vee, \wedge, 0, 1 \rangle\text{-}CPL^*$.

Proof: *a*), *b*) and *c*) are consequences of Theorem 10.2 and Corollary 9.15 and 10.5, respectively. □

It is well known that $\langle \vee, \wedge, 0, 1 \rangle$-fragment of classical logic can be obtained from derivable sequents in the well known sequent calculus for Intuitionistic logic. The equalities *a*) and *c*) allow us to obtain two alternative sequent axiomatizations of the $\langle \vee, \wedge, 0, 1 \rangle$-fragment of classical logic by means of two sequent calculi *without contraction* (that is, sequent axiomatizations that allow us to obtain this deductive system as an external system of a Gentzen system), as we summarize in the following results.

Notice that this calculus is equal, up to notation, to $\mathbf{FL}_{ew}[\vee, \wedge, 0, 1]$.

Corollary 10.7. *The $\langle \vee, \wedge, 0, 1 \rangle$-fragment of classical logic is equal to the external deductive system associated to the Gentzen system defined by the following axioms and rules:*[2]

[2]We denote by $(\wedge \Rightarrow)_m$ and $(\Rightarrow \wedge)_m$ the rules of introduction of the connective $\wedge$. The subindex m stresses the *multiplicative* character of this rules and state a difference in notation with respect to the *additive* rules $(\wedge_1 \Rightarrow)$, $(\wedge_2 \Rightarrow)$ and $(\Rightarrow \wedge)$.

Axioms:

$$\varphi \Rightarrow \varphi \quad \textit{(Axiom 1)} \qquad 0 \Rightarrow \varphi \quad \textit{(Axiom 2)} \qquad \emptyset \Rightarrow 1 \quad \textit{(Axiom 3)}$$

Structural rules:

$$\frac{\Gamma \Rightarrow \varphi \qquad \varphi, \Pi \Rightarrow \xi}{\Gamma, \Pi \Rightarrow \xi} \ \textit{(Cut)}$$

$$\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \xi}{\Gamma, \psi, \varphi, \Pi \Rightarrow \xi} \ (e \Rightarrow) \qquad \frac{\Gamma \Rightarrow \xi}{\varphi, \Gamma \Rightarrow \xi} \ (w \Rightarrow)$$

Rules of introduction of connectives:

$$\frac{\varphi, \Gamma \Rightarrow \xi \qquad \psi, \Gamma \Rightarrow \xi}{\varphi \vee \psi, \Gamma \Rightarrow \xi} \ (\vee \Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi} \ (\Rightarrow \vee_1) \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi} \ (\Rightarrow \vee_2)$$

$$\frac{\varphi, \psi, \Gamma \Rightarrow \xi}{\varphi \wedge \psi, \Gamma \Rightarrow \xi} \ (\wedge \Rightarrow)_m \qquad \frac{\Gamma \Rightarrow \varphi \qquad \Pi \Rightarrow \psi}{\Gamma, \Pi \Rightarrow \varphi \wedge \psi} \ (\Rightarrow \wedge)_m$$

Corollary 10.8. *The $\langle \vee, \wedge, 0, 1 \rangle$-fragment of classical logic is equal to the external deductive system associated to the Gentzen system defined by the following axioms and rules:*

Axioms:

$$\varphi \Rightarrow \varphi \quad \textit{(Axiom 1)} \qquad 0 \Rightarrow \varphi \quad \textit{(Axiom 2)} \qquad \emptyset \Rightarrow 1 \quad \textit{(Axiom 3)}$$

Structural rules:

$$\frac{\Gamma \Rightarrow \varphi \qquad \varphi, \Pi \Rightarrow \xi}{\Gamma, \Pi \Rightarrow \xi} \ \textit{(Cut)}$$

$$\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \xi}{\Gamma, \psi, \varphi, \Pi \Rightarrow \xi} \ (e \Rightarrow) \qquad \frac{\Gamma \Rightarrow \xi}{\varphi, \Gamma \Rightarrow \xi} \ (w \Rightarrow)$$

Rules of introduction of connectives:

$$\frac{\varphi, \Gamma \Rightarrow \xi \qquad \psi, \Gamma \Rightarrow \xi}{\varphi \vee \psi, \Gamma \Rightarrow \xi} \ (\vee \Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi} \ (\Rightarrow \vee_1) \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi} \ (\Rightarrow \vee_2)$$

$$\frac{\varphi, \Gamma \Rightarrow \xi}{\varphi \wedge \psi, \Gamma \Rightarrow \xi} \ (\wedge_1 \Rightarrow) \qquad \frac{\psi, \Gamma \Rightarrow \xi}{\varphi \wedge \psi, \Gamma \Rightarrow \xi} \ (\wedge_2 \Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi} \ (\Rightarrow \wedge)$$

Note 10.9. As is well known, the $\langle \vee, \wedge, 0, 1\rangle$-fragment of Intuitionistic logic (and as a consequence the $\langle \vee, \wedge, 0, 1\rangle$-fragment of Gödel logic) is the same as the $\langle \vee, \wedge, 0, 1\rangle$-fragment of classical logic. In this chapter we have seen that an analogous situation happens between the $\langle \vee, \wedge, *, 0, 1\rangle$-fragment of the intuitionistic logic without contraction (i.e., Monoidal Logic) and the $\langle \vee, \wedge, *, 0, 1\rangle$-fragment of classical logic (and as a consequence the same situation happens for every t-norm based logic). In other words, when not dealing with implication and negation, the t-norm based fuzzy logics behave classically.

Chapter 11

Conclusions and Future Work

The work presented in this monograph can be seen as a first step towards the analysis of all the fragments without implication of the basic intuitionistic substructural systems $\mathcal{FL}_\sigma$, and their associated external deductive systems $\mathfrak{e}\mathcal{FL}_\sigma$. We have analyzed the fragments in the four languages (without implications) that contains the connectives in $\{\vee, *, 0, 1\}$. Let $\Psi \in \{\langle\vee, *, 0, 1\rangle, \langle\vee, \wedge, *, 0, 1\rangle, \langle\vee, *, ^{\backprime}, ^{\prime}, 0, 1\rangle, \langle\vee, \wedge, *, ^{\backprime}, ^{\prime}, 0, 1\rangle\}$. For every sequence σ, we have characterized the Ψ-fragments of $\mathcal{FL}_\sigma$ and $\mathfrak{e}\mathcal{FL}_\sigma$ in the following way:

a) We have shown that an axiomatization for the Ψ-fragment of $\mathcal{FL}_\sigma$ is the calculus $\mathbf{FL}_\sigma[\Psi]$ obtained by dropping from the calculus $\mathbf{FL}_\sigma$ the rules for the connectives that there are not in Ψ.

b) We have that the Ψ-fragment of the external system $\mathfrak{e}\mathcal{FL}_\sigma$ associated to $\mathcal{FL}_\sigma$ is characterized as the external system $\mathfrak{e}\mathcal{FL}_\sigma[\Psi]$ associated to $\mathcal{FL}_\sigma[\Psi]$.

c) We have characterized algebraically the Ψ-fragments of $\mathcal{FL}_\sigma$, by providing an equivalent quasivariety semantics (e.q.s.) for each one of them, such as it is shown in the following table.

Fragment	**e.q.s.**
$\langle\vee, *, 0, 1\rangle$-$\mathcal{FL}_\sigma$	$\mathring{\mathbb{M}}^{s\ell}_\sigma$
$\langle\vee, \wedge, *, 0, 1\rangle$-$\mathcal{FL}_\sigma$	$\mathring{\mathbb{M}}^{\ell}_\sigma$
$\langle\vee, \wedge, *, ^{\backprime}, ^{\prime}, 0, 1\rangle$-$\mathcal{FL}_\sigma$	$\mathbb{PM}^{s\ell}_\sigma$
$\langle\vee, \wedge, *, ^{\backprime}, ^{\prime}, 0, 1\rangle$-$\mathcal{FL}_\sigma$	$\mathbb{PM}^{\ell}_\sigma$

d) We have proved that these fragments are not equivalent to any deductive system. Thus, a fortiori, each system $\mathcal{FL}_\sigma[\Psi]$ is not equivalent to its associated external system $\mathfrak{e}\mathcal{FL}_\sigma[\Psi]$. In fact, we have proved that these external systems are not even protoalgebraic. Nevertheless, there is a weaker connection among the systems $\mathfrak{e}\mathcal{FL}_\sigma[\Psi]$

and the considered varieties because, as we have proved, the systems $\mathfrak{e}\mathcal{FL}_\sigma[\vee, *, 0, 1]$, $\mathfrak{e}\mathcal{FL}_\sigma[\vee, \wedge, *, 0, 1]$, $\mathfrak{e}\mathcal{FL}_\sigma[\vee, \wedge, *, \backprime, \prime, 0, 1]$, and $\mathfrak{e}\mathcal{FL}_\sigma[\vee, \wedge, *, \backprime, \prime, 0, 1]$ have, respectively, the varieties $\mathring{\mathbb{M}}^{s\ell}_\sigma$, $\mathring{\mathbb{M}}^{\ell}_\sigma$, $\mathbb{PM}^{s\ell}_\sigma$ and $\mathbb{PM}^{\ell}_\sigma$, as an algebraic semantics with defining equation $1 \preccurlyeq p$.

For the particular case $\sigma = ew$ we have shown that the fragments in the languages $\langle \vee, *, 0, 1\rangle$, $\langle \vee, \wedge, *, 0, 1\rangle$ and $\langle \vee, \wedge, 0, 1\rangle\}$ coincide with the fragments of Classical Logic in the very languages.

Notice that for $\sigma = ewc$, the systems $\mathfrak{e}\mathcal{FL}_{ewc}[\vee, *, 0, 1]$ and $\mathfrak{e}\mathcal{FL}_{ewc}[\vee, \wedge, *, 0, 1]$ are definitionally equivalent to the $\langle \vee, \wedge\rangle$-fragment of the Intuitionistic (and Classical) Logic, and the systems $\mathfrak{e}\mathcal{FL}_{ewc}[\vee, \wedge, *, \backprime, \prime, 0, 1]$ and $\mathfrak{e}\mathcal{FL}_{ewc}[\vee, \wedge, *, \backprime, \prime, 0, 1]$ are definitionally equivalent to the $\langle \vee, \wedge, \neg\rangle$-fragment of the Intuitionistic Logic. These fragments are characterized by means of Hilbert-style axiomatizations (see [DP80, FV91] for the $\langle \vee, \wedge\rangle$-fragment, and [RV94] for the $\langle \vee, \wedge, \neg\rangle$-fragment). With the exception of these cases, providing Hilbert-style axiomatizations for the systems $\mathfrak{e}\mathcal{FL}_\sigma[\Psi]$ remains open.

As future work our goal is to extend our analysis to the implication-free fragments of $\mathcal{FL}_\sigma$ and $\mathfrak{e}\mathcal{FL}_\sigma$ for the languages containing the connectives in $\{*, 0, 1\}$ that have not been studied in this monograph, that is, $\langle \wedge, *, \backprime, \prime, 0, 1\rangle$, $\langle *, \backprime, \prime, 0, 1\rangle$, $\langle \wedge, *, 0, 1\rangle$ and $\langle *, 0, 1\rangle$.

Another research lines where we are nowadays working are the following:

a) The study of the internal system associated with each substructural Gentzen system $\mathcal{FL}_\sigma$, and its fragments.

b) The analysis of possible axiomatizations by means of Tarski-style conditions for both the external and the internal system associated to the Gentzen systems $\mathcal{FL}_\sigma$.

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Symbols Index

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