On modal expansions of t-norm based logics with rational constants

Amanda Vidal Wandelmer

Supervised by Félix Bou, Francesc Esteva and Lluís Godo
Tutor: Joan Gispert

2015
Consejo Superior de Investigaciones Científicas (CSIC)
Instituto de Investigación en Inteligencia Artificial.

Universitat de Barcelona
Programa de Doctorado en Lógica Pura y Aplicada

A Tommaso

Abstract

According to Zadeh, the term “fuzzy logic” has two different meanings: wide and narrow. In a narrow sense it is a logical system which aims a formalization of approximate reasoning, and so it can be considered an extension of many-valued logic. However, Zadeh also says that the agenda of fuzzy logic is quite different from that of traditional many-valued logic, as it addresses concepts like linguistic variable, fuzzy if-then rule, linguistic quantifiers etc. Hájek, in the preface of his foundational book Metamathematics of Fuzzy Logic, agrees with Zadeh’s distinction, but stressing that formal calculi of many-valued logics are the kernel of the fuzzy logic in the narrow sense. Hájek’s book undertakes the study of the so-called Basic Fuzzy logic (BL), having continuous triangular norms (t-norm) and their residua as semantics for the conjunction and implication respectively, and of its most prominent extensions, namely Łukasiewicz, Gödel and Product fuzzy logics. Taking advantage of the fact that a t-norm has residuum if, and only if, it is left-continuous, the logic of the left-continuous t-norms, called MTL, was soon after introduced.

On the other hand, classical modal logic is an active field of mathematical logic, originally introduced at the beginning of the XXth century for philosophical purposes, that more recently has shown to be very successful in many other areas, specially in computer science. This success is mainly due to the huge number of real-world scenarios that can be modelled using Kripke relational structures, that are the most well-known semantics for classical modal logics.

Modal expansions of non-classical logics, in particular of many-valued logics, have also been studied in the literature. In this thesis we focus on the study of some modal logics over MTL, using natural generalizations of the classical Kripke relational structures where propositions at possible worlds can be many-valued, but keeping classical accessibility relations.

In more detail, the main goal of this thesis has been to study modal expansions of the logic of a left-continuous t-norm, defined over the language of MTL expanded with rational truth-constants and the Monteiro-Baaz Δ operator, whose intended (standard) semantics is given by Kripke models with crisp accessibility relations and taking the unit real interval $[0, 1]$ as set of truth-values. To get complete axiomatizations, already known techniques based on the canonical model construction are used, but this requires to ensure that the underlying (propositional) fuzzy logic is strongly standard complete. This constraint leads us to consider axiomatic systems with infinitary inference rules, already at the propositional level. A second goal of the thesis has been to also develop an automated reasoning software tool to solve satisfiability and logical consequence

problems for some of the fuzzy modal logics considered.

This dissertation is structured in four parts. After a gentle introduction, Part I contains the needed preliminaries for the thesis be as self-contained as possible. Most of the theoretical results are developed in Parts II and III. Part II focuses on solving some problems concerning the strong standard completeness of underlying non-modal propositional logics, as a necessary step previous to the study of the modal expansions. We first present an axiomatic system for the non-modal propositional logic of a left-continuous t-norm that makes use of a unique infinitary inference rule, the *density rule*, that solves several problems pointed out in the literature. We further expand this axiomatic system in order to also characterize arbitrary operations over $[0, 1]$ satisfying certain regularity conditions. However, since this axiomatic system turns out to be not well-behaved for the modal expansion, we search for alternative axiomatizations with some particular kind of inference rules (that will be called conjunctive). Unfortunately, this kind of axiomatization does not necessarily exist for all left-continuous t-norms (in particular, it does not exist for the Gödel logic case), but we identify a wide class of t-norms for which it works. This “well-behaved” t-norms include all ordinal sums of Łukasiewiczand Product t-norms.

Part III focuses on the modal expansion of the logics presented before. We propose axiomatic systems (which are, as expected, modal expansions of the ones given in the previous part) respectively strongly complete with respect to local and global Kripke semantics defined over frames with crisp accessibility relations and worlds evaluated over a “well-behaved” left-continuous t-norm. We also study some properties and extensions of these logics and also show how to use it for axiomatizing the possibilistic logic over the very same t-norm. Later on, we characterize the algebraic companion of these modal logics, provide some algebraic completeness results and study the relation between their Kripke and algebraic semantics.

Finally, Part IV of the thesis is devoted to a software application, mNiBLoS, that uses Satisfiability Modulo Theories in order to build an automated reasoning system to reason over modal logics evaluated over *BL* algebras. The acronym of this applications stands for a modal Nice BL-logics Solver. The use of *BL* logics along this part is motivated by the fact that continuous t-norms can be represented as ordinal sums of three particular t-norms: Gödel, Łukasiewicz and Product ones. It is then possible to show that these t-norms have alternative characterizations that, although equivalent from the point of view of the logic, have strong differences for what concerns the design, implementation and efficiency of the application. For practical reasons, the modal structures included in the solver are limited to the finite ones (with no bound on the cardinality).

Contents

Abstract vii

Introduction 1

I Preliminaries 13

1 Universal algebra and abstract algebraic logic 15
- 1.1 Universal algebra . . . 15
- 1.2 Algebraic logic . . . 20

2 Many-valued logics: MTL and BL 27
- 2.1 *MTL*: the logic of left-continuous t-norms . . . 27
- 2.2 *BL*: the logic of continuous t-norms . . . 29
- 2.3 Notable extensions of *BL* and *MTL* . . . 31
- 2.4 Construction of continuous t-norms . . . 33
- 2.5 *MTL* expansions . . . 36
  - Logics with Δ . . . 36
  - Logics with truth constants: Pavelka completeness . . . 37

3 Modal Logics 41
- 3.1 Classical modal logic . . . 42
  - Kripke semantics . . . 42
  - Syntactic level: the axiomatization problem . . . 44
  - Algebraic semantics . . . 46
- 3.2 Fuzzy modal logics . . . 47
  - *MTL*-based Kripke semantics . . . 48
  - Remarkable differences with the classical modal logic . . . 51
  - State of the art . . . 53

II Strongly complete axiomatizations of the logic of a t-norm 61

4 Strongly standard complete systems 63

4.1 Why a new approach towards a logic of $[0,1]_*$? 64
4.2 Axiomatizations with the density rule 66
4.3 A general result on semilineary 77
4.4 Expansions by operations from $[0,1]$ 80
The general problem: representable operations 80
Strong standard completeness 85

5 Modally well-behaved axiomatizations 91
5.1 Conjunctive inference rules 91
5.2 Algebraic models of the density rule 95
5.3 T-norms accepting a conjunctive axiomatization 98

III Modal expansions of some t-norm based logics 103

6 Many-valued modal logics: $\mathbf{K}_*$ and $\mathbf{K}_*^{\mathbf{g}}$ 105
6.1 The semantics: crisp local and global modal logics over $[\mathbf{0},\mathbf{1}]_*^{\mathrm{Q}}$ . . 106
6.2 Axiomatization and completeness 107
Completeness: the Canonical Model 114
The global modal logic 116
6.3 Extensions and further properties 118
Interdefinability of $\Box$ and $\Diamond$ 118
On the Finite Model Property 120
Some usual axiomatic extensions 122
6.4 Possibilistic many-valued logic 123

7 Many-valued modal algebras 127
7.1 Algebraic semantics of K_* and GK_* 127
Algebraic completeness of the global modal logic 128
Algebraic completeness of the local modal logic 129
7.2 Kripke semantics and algebraic semantics 133
From Kripke models to modal algebras: complex algebras 134
From modal algebras to Kripke models: canonical models 136
Correspondence results 139
7.3 Order preserving logics 140

IV Aplications 143

8 Satisfiability Modulo Theories 145
8.1 The satisfiability problem 145
8.2 Formal definitions 147
SAT encodings 147
SMT-solvers approach 148
8.3 Standardisation: language and solvers 150

9 The theory behind the solver **151**
9.1 The propositional level . . . 152
Nice *BL* logics . . . 152
Efficiency issues: the product components case . . . 154
Rational constants and Δ operator . . . 158
9.2 Some modal expansions . . . 160

10 mNiBLoS: modal Nice BL Logics Solver **165**
10.1 What does mNiBLoS do? . . . 166
10.2 Design of the application . . . 167
10.3 Use of mNiBLoS . . . 168
Pre execution . . . 169
Inputs . . . 169
The generated SMT-code . . . 171
Output . . . 182
Examples . . . 183

11 mNiBLoS Statistics **187**
11.1 Tests of *BL* theorems . . . 187
11.2 Randomly generated formulas: number of variables vs. length of the formula . . . 188

12 Conclusions and open problems **197**
12.1 Main contributions . . . 197
12.2 Publications and Comunications . . . 199
12.3 Future work . . . 200

References **202**

Acknowledgements

"Begin at the beginning"
the King said, very gravely,
"and go on till you come to the end:
then stop."

Lewis Carroll

Quiero agradecer como soportes indispensables en el desarrollo de esta tesis a mis directores, Lluís Godo, Francesc Esteva y Félix Bou, los cuales me han brindado un constante apoyo y guía a través del proceso investigador. Su atención para conmigo, el tiempo y la energía que me han dedicado y su comprensión y ayuda en los momentos más necesarios han hecho posible el desarrollo y la escritura de este trabajo. También quiero agradecer su acogida a los investigadores del Institut d'Investigació en Intel·ligència Artificial, y muy especialmente a Carles Sierra, por haberme aceptado inicialmente como su estudiante; a Felip Manyá, por haberme mostrado las posibilidades e intereses de la aplicación de los resultados teóricos; y finalmente, a todo el equipo administrativo y técnico del IIIA, por haberme siempre ayudado con las necesidades burocráticas y electrónicas. También deseo agracer el soporte económico recibido por parte de los proyectos Agreement Technologies (CONSOLIDER CSD2007-0022), EdeTRI (MINECO, TIN2012-39348-C02-01) y GEAR (CSIC ref.201350E112), así como del CSIC a través de la Beca JAE-predoc.

Agradezco también a los profesores del Máster de Lógica Pura y Aplicada de la Universitat de Barcelona el haberme introducido a la materia y haberme interesado en ella. Muy especialmente, también quiero agraceder a mis amigos y compañeros en el trance del doctorado, por todos los ratos pasados juntos y las risas, preocupaciones, trabajos y ocios compartidos: Eduardo Hermo, Roberto Palomares, Hugo Albuquerque y Pedro Esteban.

I want to thank Mehdi Dastani, Stefano Aguzzoli, Carles Noguera and Petr Cintula for their hospitality in, respectively, Utrech, Milano and Prague, and Carlos Chesñevar and Ana Casali for their kind treat when I visited them in Argentina. I would also like to thank all the researchers that showed interest and gave me feedback on my work in the conferences and symposiums I have attended to in these past years.

Quiero agradecerles a mis padres, Julián y Alicia, su muy importante papel en lo que al desarrollo de esta tesis se refiere: me han apoyado, empujado y soportado siempre que ha sido necesario, y su confianza en mi y su tesón para animarme a empezar y terminar las cosas que me he propuesto son factores dificiles de cuantificar, e imprescindibles tanto para esta tesis como para mi misma. Igualmente, he de agradecerle a mi primo Pepe su modelo como investigador y persona, por ser el primero que me ilusionó con las matemáticas y la informática, y que, a pesar de sus consejos, me ha mostrado en si mismo aciertos y errores que han contribuido enormemente no sólo a este trabajo sino a toda mi formación.

Asimismo, deseo agradecerles a mis amigos, y en particular a la gente de la partida semanal de rol, el haberme dado un importante componente de ocio a lo largo de estos años de estudio.

Finalmente, he de darle las gracias enormemente a Tommaso Moraschini, colega, amigo y amante, por toda su compañía, cariño y ayuda sin la cual esta tesis no habría, muy probablemente, existido. Por su ánimo en los periodos de bajo optimismo, su siempre atenta ayuda en temas de álgebra y su apoyo, alegría, aguante y cuidado en todo momento. Por cómo guía mi vida con la suya, por haberme mostrado qué cosas son importantes y, en suma, por todo.

Introduction

Logic is known as the science that studies valid reasoning. As such it has been a prominent branch, since its origins more than 2000 years ago, of mathematics, philosophy and, more recently, computer science. It formally approaches the notion of truth and studies its flow across an argumentation, in the sense that a logically valid argumentation transmits truth from the premises of discourse to its conclusions.

A logic is characterized by a language and a deductive system. The former one delimits the concepts and propositions that can be referred to and the latter characterizes the deductions or argumentations within the elements of the language that are considered correct under the logic.

Even though logic has been applied in different formal fields, its origins are linked to the representation and analysis of natural reasoning. In this setting, the language is the one used in the human communication, that is the natural language and the deductive system consists on the "philosophically correct" argumentations. Since the beginning, it was generally considered that natural languages should be replaced by formal languages, given their vagueness and ambiguity. These formal languages should represent an idealization of the concepts expressed in natural language and entailment among the elements of the formal language represents a paradigm of the correct reasoning in natural language.

This leads to the establishment of the so-called **classical mathematical logic**. Nowadays, this term refers to the largest and most widely studied and used formal logic, whose language makes use of conjunction, disjunction and negation (with intended meanings of *and*, *or* and *not* respectively). It is characterized by the following properties:

1. The law of excluded middle: for every statement, either it or its negation is true.

2. The double negation elimination: a statement and its two-times negation are equivalent.

3. The law of non-contradiction: a statement cannot be both true and not true at the same time and in the same sense.

4. The principle of explosion: from a contradiction it can be deduced any proposition.

5. Monotonicity and idempotency of the deduction: the same consequences can be derived from many instances of the same hypothesis as from just one instance and any derived fact may be freely extended with additional assumptions.

6. Commutativity of conjunction.

7. De Morgan laws: disjunction and conjunction are mutually definable via the negation.

Along the XXth century, the logical tradition, that regarded vague and uncertain concepts as imperfect statements that should be avoided or translated to exact notions, develops the idea of formalizing such non-precise propositions. This creates an intermediate level of formalisms, in between the very restrictive classical logics and the too informal natural reasoning. What is called truth begins to be also seen as a concept relative to a state of knowledge and not necessarily with respect to an objective, completely and precisely known state of the world.

For this reason, the previous set of properties began to change, giving birth to **non-classical logics**. This is a wide class that goes from logics that do not fulfil some of the above laws to those that are defined over an expanded language and fulfil a larger set of characteristics. Non-classical logics address the problem of the replacement of natural language by formal languages in a more flexible way that the classical one, allowing a more accurate characterization of real-world statements.

In this work we are mainly interested in two classes of non-classical logics, which were born at the same time but that have followed quite separated and different paths. We are talking about **modal logics**, which appear as an expansion of classical logic with non-truth-functional (modal) operators and **fuzzy logics** (and in particular, **many-valued logics**), that disregard the law of the excluded middle (and motivated from this fact, generally also the non-contradiction and the double negation elimination) and allow for propositions to take more than just the truth values true or false.

The origins of both modal and fuzzy logics go back to around the 20s of the XX century. Logics with a higher expressivity level than classical logics appear or are formalized (from intuitive previously considered notions) at this time, with different objectives and notations. C. I. Lewis publishes in 1918 [88] ideas concerning the addition of certain symbols (in modern notation, $\Box$ or $\Diamond$) as prefixes for a formula φ to get expressions like "the proposition φ is necessary" or "the proposition φ is possible" respectively, defining new modes of truth. On the other hand, historical examples from Aristotle challenging the bivalence principle, based on propositions referring to future events inspired Łukasiewicz to define a new kind of logic. He came up with a three-valued logic whose third value had the intended meaning of *possibly*.

These two formalisms quickly follow, nevertheless their near origins, independent paths. Already in the 60s, when both are being intensively studied and reach the status of independent fields within the area of logics, very different techniques, characteristics and applications are linked to each one of them.

Fuzzy logics are born with the foundational 1965 paper by Lotfi Zadeh entitled “Fuzzy Sets” [128]. Here Zadeh proposes the idea of fuzzy sets as those whose characteristic function is no longer evaluated in $\{0,1\}$, but can rather take different values in an partially ordered set (usually in $[0,1]$, but also other universes have been considered, e.g. complete lattices). The set-theoretic operations on membership functions are defined using the operations on the evaluation set: the intersection is defined using the minimum and the union with the maximum. Fuzzy logic in its narrow/technical sense, as presented by Zadeh, refers to many-valued logics that handle gradual properties (as opposed to fuzzy logic in its wide sense, which is a generic expression that mostly refers to that part of soft computing where fuzzy sets and rules are used).

Many-valued logics consist of formal systems that have a set of truth values containing $\{0,1\}$ and whose truth functions are defined satisfying some natural monotonicity conditions on the other values of the set. A formalism that has been commonly adopted in order to uniformly approach the study of many-valued logics is that of taking as truth values subsets of the real unit interval. The fuzzy connectives originate from the definition and algebraic study of the set-theoretical operations arising from the fuzzy set theory and are essentially developed in the eighties, when this field lives a significant development. It is then when researchers recognise that an appropriate definition for the intersections and unions of fuzzy sets is a class of associative monotonic connectives known as triangular norms (t-norms for short), together with their De Morgan dual triangular co-norms. From here, related implication and negation functions are also studied, giving place to a whole algebraic description of logics. The syntactical issues of fuzzy logic followed these semantical definitions.

The first many-valued logic is introduced, as we commented before, by Jan Łukasiewicz, in 1920 [90], when the concept of many-valued logic as defined above was still not developed. The logic presented there is a three-valued logic. It includes the usual $\top$ (true) and $\perp$ (false) truth values and a the third value P is introduced in order to deal with contingent futures, meaning *possible*. The semantics for this logic is given through the truth tables of its operations:

$\vee$	$\perp$	P	$\top$
$\perp$	$\perp$	P	$\top$
P	P	$\top$	$\top$
$\top$	$\top$	$\top$	$\top$

$\wedge$	$\perp$	P	$\top$
$\perp$	$\perp$	$\perp$	$\perp$
P	$\perp$	P	P
$\top$	$\perp$	P	$\top$

$\rightarrow$	$\perp$	P	$\top$
$\perp$	$\top$	$\top$	$\top$
P	P	$\top$	$\top$
$\top$	$\perp$	P	$\top$

$\neg$	
$\perp$	$\top$
P	P
$\top$	$\perp$

Assigning a numeric interpretation to the previous set of values, taking into account the behaviour of the minimum and maximum operations, leads to the more usual presentation of 3-valued Łukasiewicz logic as the logic over the set $\{0, \frac{1}{2}, 1\}$. Moreover, the definition of the implication and negation are the key points of the further extensions of this approach towards the many-valued interpretation of logics. Indeed, first Łukasiewicz himself and later in collaboration with Tarski, studied the generalization of the above 3-valued logic to n-valued and infinitely-valued logics (taking this values in the $[0, 1]$ interval). This is done defining the new implication and negation operation as follows

$$\begin{aligned} x \to y &:= \min\{1, 1 - x + y\} \\ \neg x &:= 1 - x \end{aligned}$$

Łukasiewicz also proposes a Hilbert-style axiomatic system for this logic, and conjectures that the theorems of this system coincide with the tautologies over the algebra over [0,1] with the usual $\vee$ and $\wedge$ operations and the $\to$ and $\neg$ operations defined as above. However, it was not until much later that this fact was proven.

In 1930, Heyting formalizes the intuitionistic logic [81], whose principles were given by Brouwer during the early years of the XXth century in his proposal of intuitionistic mathematics. It rejects the law of the excluded middle and the double negation elimination but maintains the law of non-contradiction and the principle of explosion. Its formalization is given syntactically, i.e., proposing axioms and inference rules from which the deductive system is inductively defined. In 1932, Gödel proved that the intuitionistic logic is not a finitely valued logic [65] and defined a family logics intermediate between classical and intuitionistic logic. These turn to have semantics over bounded linearly ordered algebras, with the usual interpretation of $\vee$ and $\wedge$ respectively as maximum and minimum operations of the algebra, but a different definition of implication and negation. Letting 1 and 0 represent the top and bottom elements of the algebras,

$$\begin{aligned} x \to y &:= \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases} \\ \neg x &:= \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Later on, following the truth-functional setting of Zadeh, other formalisms are proposed to model the operations between fuzzy sets and indirectly, of many-valued logics operations. Some conditions are needed, like associativity and commutativity of the conjunction and disjunction operations. Alsina, Trillas and Valverde suggest, with the objective of modelling the intersection (conjunction) of fuzzy sets, a very general class of functions used in other mathematical fields ([4]), the triangular t-norms (t-norms). From these, their duals and the so-called weak negation functions arise, allowing the addressing of the union (disjunction) and complement (negation) of fuzzy sets. T-norms are binary operations on $[0, 1]$ that are associative, commutative, monotonically increasing and have 1 as the neutral element. Regarding implication, mainly two kind of operations are proposed. One is defined as in the classical case but using the weak negation instead of the usual one, i.e., $x \to y \coloneqq \neg x \vee y$. The other satisfies the residuation law over the conjunction $*$: $x \to y \coloneqq \max\{z : x * z \leq y\}$. Since only the second kind of implication makes true in general the usual form of the Modus Ponens rule (from x and $x \to y$ infer y), this is the one that has been generally adopted in the many-valued logics field.

It turns out that the implication operations of Łukasiewicz and Gödel logics are, respectively, the residua of two continuous t-norms:

$$x *_{Ł} y \coloneqq \max\{0, x + y - 1\} \qquad x *_G y \coloneqq \min\{x, y\}$$

Later, a third continuous t-norm, the product of real numbers, appears as a basic one together with the Łukasiewicz and Gödel ones: any other continuous t-norm can be defined from these [101]. An axiomatic system for the logic of the product t-norm is finally proposed by Hájek, Godo and Esteva in [75]. Once it is shown that the logics defined by the three main continuous t-norms and their residua enjoy a syntactical calculus (finitely given), Hájek proposes in [70, 71] a new logical system aiming to capture their common characteristics. He names it Basic Fuzzy Logic (BL) and conjectures it corresponds to the logic (in the sense of valid equations) given by the class of all continuous t-norms, as finally proved in [30].

However, continuity is not a necessary condition for a t-norm to have a residuum: it is only necessary for the t-norm to be left-continuous in order to enjoy this property. Motivated by this, Esteva and Godo propose in [49] the Monoidal t-norm Logic (MTL), an axiomatic system aiming to characterize the logic arising from these operations. Jenei and Montagna prove in [84] that MTL is complete with respect to the family of standard algebras arising from left-continuous t-norms. MTL becomes then the weaker axiomatic system arising from t-norms having a unique residuum.

On the other hand, modal logics are deeply studied branches of mathematical logic. As we said before, they are formally introduced in the pioneering work of C.I.Lewis of 1918 as a way to talk about modal notions (possibility and necessity). In 1933, due to concerns in the foundations of mathematics, Gödel introduces modal operators as a way to formalise the notion of mathematical provability ($\Box\varphi$ means "φ is provable" and $\Diamond\varphi$ means "φ is consistent"). In the following decades, many other modal operators are introduced and investigated, formalizing truth from very different spaces. For instance, temporal logics include modalities of the kind of "eventually", "always", "never"...; deontic logics models concepts like "it is permitted" and "it is mandatory"; epistemic logics, use modalities talking about knowledge, belief or trust; dynamic logics are multi-modal logics with modalities of the form $\langle\alpha\rangle\varphi$ meaning "after the execution of program α, φ is possible" and $[\alpha]\varphi$ meaning "after the execution of α, φ necessarily holds" etc. Many well-known researchers interested in non-classical logics work on these areas, like G.H. von Wright, A. Prior, J. Hintikka and D. Lewis.

Up to the 50s, modal logic is understood as just a logic formalizing modal notions. However, when the relational semantics, based on graphs, is developed in the late 50s and early 60s (by Kripke, Joyal and others), it is shown that standard modal logics can be seen as fragments of first or second order predicate logics. Indeed, modalities quantify, in some sense, over the worlds or states of these relational structures: $\Box$ quantifies universally while $\Diamond$ does so existentially. The mathematical study of modal logics in this vein has proves that modal logics enjoy a good balance between expressive power and computational complexity of logical systems. While the satisfiability problem on first order predicate logic is undecidable, this is not the case for the more general classical modal logic K.

Later on, other disciplines become interested in using modalities for modelling and working with their particular problems. In particular, temporal, dynamic and epistemic logics are deeply studied and used in Artificial Intelligence, economic game theory and computer science. Epistemic and temporal operators are crucial in the formalization of knowledge-based programming and belief-based agent systems, temporal and dynamic logics are used in industry for automated verification of hardware and software, dynamic and epistemic logics are used in game theory...

Modal logic today can be seen as a large family of studies concerning modal-like notions, with the philosophical and mathematical basis above commented but in constant partnership with other fields. The applications of modal logics are living a particularly fruitful period and formalizations using modal-like operators seem to happen in all kind of environments. Moreover, the developments done at the applied side of modal logics are also influencing the theoretical studies in a circular way that enriches the whole theory.

The interaction of modal and fuzzy logics is a matter of study that has received attention from different research groups and specialists within the logic field. With this, we refer to the addition of modal operators, which mimic the classical modal operators behaviour, to fuzzy logics. Significant differences between the classical modal logic and the new systems can be easily detected, since the formal definition of a Kripke structure can now include many-valued notions. Thus, the idea of "related states" is brought to a fuzzy dimension and understood as "related states to some degree". Moreover, the fact that propositions have a many-valued interpretation leads to interesting challenges concerning the axiomatization and study of these logics. For instance, the fact that the necessity and possibility operators are not in general mutually derivable diversifies the modal expansions that can be considered, and problems solved at the classical level, like decidability and complexity questions, do not have a direct generalization in the new context.

Several works on fuzzy modal logics have created a setting for understanding the possible ways to generalize the modal notions to a fuzzy context. In the last years, there has been a growing number of papers dealing with the combination of modal and many-valued logics. Some approaches do not fit in our framework for considering different notions of many-valued modalities, like [29, 97], but others stay as particular cases of our framework. Among the ones that fit in our framework we can cite [57, 58], [18], [24, 21, 23],[77, 117, 78]. We will comment here briefly the known axiomatizations in the literature of logics arising from classes of Kripke frames valued over a non-Boolean algebra.

The case when the algebra of evaluation is a finite Heyting algebra is considered by M. Fitting in the early 90s [57, 58]. He includes a truth constant for every element of the algebra (i.e., for every truth value), which simplifies the proofs and allows to give a unified presentation of the calculus to axiomatize the logic of the class of Kripke frames evaluated over these algebras. The uniformity refers to the fact that all these calculi share the same schemes without constants. Other papers that study these cases are [86, 87, 85]. Moreover, concerning crisp Kripke frames evaluated over arbitrary finite algebras have been presented in [55].

Concerning the more particular case of modal expansions of MTL logics, three main research lines have been followed. Łukasiewicz logic expanded with modal operators has been mainly studied by Teheux and Hansoul (see [116, 117, 78]). They focus on the study of the logic arising from Kripke models with propositions evaluated over finite subalgebras of the standard Łukasiewicz algebra and also over the whole algebra, but considering crisp accessibility relations. Given the involutive character of the Łukasiewicz negation, the modal operators are, as in the classical case, interdefinable. They present an axiomatic system for the logic, and focus afterwards on the study of the duality between the modal algebras that form the algebraic counterpart of their system and the original Kripke models that gave place to the logic. It is remarkable that, in the case of the modal logic built over the infinitely valued Łukasiewicz standard algebra, the axiomatic system incorporates an infinitary inference rule (with an infinite amount of premises).

On the other hand, studies of Kripke frames evaluated over residuated lattices, considering a non-crisp accessibility relation were developed in [18]. Within that work, the authors extend finitely-valued MTL logics with the $\Box$ operator, and present axiomatic systems for the modal logics arising from Kripke models where both the accessibility relation of the structures and the evaluation of the propositions is done over finitely valued MTL-algebras endowed with canonical constants (that is, a constant for each element of the universe). The authors also study some other modal expansions, considering the classes of Kripke models where the accessibility relation is an idempotent element of the algebra, and those whose accessibility relation is crisp.

To the best of our knowledge, other works treating the modal logics arising from Kripke frames valued on non-Boolean algebras focus on issues related with Gödel logic. Caicedo and Rodrigez present several results (see [24, 23]) concerning the addition of modal operators to the Gödel logic. They study first the expansion of Gödel logic with each one of the modal operators separately, and propose an axiomatic system for the logic defined over the Kripke models evaluated over the standard Gödel algebra, both in terms of formulas and accessibility relations. Later on, also the expansion with both $\Box$ and $\Diamond$ is considered, obtaining an axiomatic system complete with respect to the corresponding class of Kripke models. Moreover, together with Metcalfe and Rogger, they have faced the decidability problem over the modal Gödel logics in [21]. It was known from

[24] that these logics do not enjoy in general the finite model property and decidability of the satisfiability and validity problems were not known. It is proven that these logics are decidable via the definition of an alternative equivalent semantics that does enjoy the finite model property.

Concerning the modal expansion of *MTL* logics, several important open problems remain and our objective is to solve some of them. On the one hand, modal expansions of Product logic have not been studied, which we think is an important question because Product logic is, together with Łukasiewicz and Gödel logics, the main continuous t-norms from which any other can be built. More in general, studies concerning the axiomatization of modal logics arising from arbitrary left-continuous t-norms without restricting the universe to a finite one (as done in [18]) also lack in the literature. On the other hand, the practical uses of many-valued modal logics have been long theoretically developed, but there are no software applications that implement reasoners over them. We find this lack of automated tools to be a big drawback in the application of the modal many-valued logics.

These are the two main problems we develop along this dissertation. In particular, we first study the problem of the axiomatization of modal expansions -considering a crisp accesibility relation- of the logics arising from left-continuous t-norms. We then develop a software for reasoning over some of these modal logics.

In the path to do so, several interesting results have also been proved, concerning different related fields. We consider of particular interest several results from Chapters 4 and 5 dealing with strong completeness problems, studies of the relation between modal algebras and Kripke models showed in Section 7.2 from Chapter 7 and the characterization results of some of the studied logics oriented to efficiency issues presented in Chapter 9. The Doctoral dissertation is structured in the following main parts.

Part I of the dissertation introduces the necessary preliminaries to make this thesis as self-contained as possible. We describe the main elements of the three topics that will be used and developed along the thesis. The first chapter is dedicated to the more general ones, introducing some basic concepts from universal algebra and notions coming from the so-called abstract algebraic logic, which studies how logics are associated to algebraic structures. In this chapter, we present the notation issues, basic definitions and some remarkable results that will be referred to later on. The second chapter from this part shows the definitions and results concerning the many-valued logic *MTL* commented above, paying attention to its algebraic semantics and to its more common axiomatic extensions and expansions (considering additional operations in the language). Finally, the third chapter presents a brief survey on the aspects more relevant to our dissertation of modal logics. In order to see the philosophical motivations behind the work later done, we begin by defining the semantical framework that gives place to the classical modal logic and state some of the most remarkable results concerning it. We then focus on the state of the art on many-valued modal logics and present a brief resume of the works dealing with modal expansions of families of *MTL* logics.

Part II of the thesis presents the theoretical results we have developed for what concerns the strong standard completeness problem (i.e., finding an axiomatic system complete with respect to a given algebra on $[0,1]$, even when an infinite set of formulas is involved) of the standard algebras of left-continuous t-norms with truth constants. From our studies on modal expansions of many-valued logics and also analysing the literature on the topic, strong completeness and truth constants seem to be very important features of the non-modal logic, in order to be able to expand it with modalities (in a recursively enumerable way).

In the first chapter, we solve the above problem for the whole family of left-continuous t-norms further expanded with the Δ operator. The proposed axiomatic system associated with the standard canonical algebra (where the constants are interpreted by its name) expanded with Δ has an infinite set of book-keeping axioms that are needed to specify the value of the operations over the constants, but it is defined using a unique infinitary inference rule, the *density rule*. Axiomatic systems strongly complete with respect to many left-continuous t-norms (including Gödel or any ordinal sum with more than one component) in the above sense were unknown and our proposal solves a lot of this cases. We also study how it is possible to use the density rule to also build axiomatic systems that expand the logic with new operations (defined in $[0,1]$), that only need to follow some regularity conditions. An interesting result whose scope not limited to the logics we are studying here is Theorem 4.16, which proves the *Prime Theory extension property* for extensions of MTL_Δ using up to a countable set of infinitary inference rules with certain characteristics. It turns out that the density rule from Chapter 4 solves the strong standard completeness, but it is not a rule well-behaved when the logic is expanded with modalities. For this reason, in Chapter 5 we study the previous logics from the algebraic point of view and search for alternative axiomatizations using only infinitary inference rules with a determined schemata, named *conjunctive inference rules.* We prove that this kind of axiomatic system does not exist for some left-continuous t-norms (for instance, in the Gödel case) so we restrict the rest of our research to the t-norms that can be axiomatized in this more controlled way. We prove that the class of left-continuous t-norms that accept such an axiomatization is nevertheless quite large, since it contains for instance all the continuous t-norms that are ordinal sums of Łukasiewicz and Product t-norms.

Part III of this dissertation faces the study of the local and global modal logics (with Δ and truth constants) arising from the Kripke models with a crisp accessibility relation but with the formulas evaluated on the standard algebra of a t-norm admitting an axiomatization using (as infinitary rules) conjunctive inference rules only. In Chapter 6 it is addressed the problem of axiomatizing the above modal logics. We begin formally defining the (local and global) semantics of a modal logic over Kripke models as the above ones. We then propose axiomatizations for these logics using the results from Part II of the dissertation and prove they are respectively strongly complete with respect to the intended relational semantics. Observe that, in particular, this solves the problem of finding a suitable axiomatization for the modal expansions of logics arising from arbitrary ordinal sums of Łukasiewicz and Product components, which was a question not solved in the literature. Moreover, some characteristics and applications of these logics are presented, like some issues concerning partial interdefinability of $\Box$ and $\Diamond$ operators, canonical extensions of the modal logics presented before and the axiomatization of the possibilistic logic based on left-continuous t-norms that accept an axiomatization with conjunctive inference rules.

The second chapter of this part focuses on the algebraic study of the many-valued modal logics presented in Chapter 6. We begin by classifying them within the Leibniz hierarchy and presenting some algebraic completeness results for them. Next we focus on the relation between the modal MTL algebras and the class Kripke models evaluated over MTL algebras. We prove a strong relation between certain subclasses of these models and algebras and we get a new completeness result for the local modal logic that relates it to the order-preserving logic defined over the class of modal algebras.

Part IV of this dissertation is devoted to move the logics studied along Parts II and III closer to possible practical applications.

On this point, we think it is important to remark that the idea behind this part is not that of picking a particular problem and treating it with the logics previously studied, which we consider to be out of the scope of this work. Rather, what we do in this part of the dissertation is presenting a software application (*mNiBLoS, a modal Nice BL logics Solver*) that allows to, automatically and in a reasonably efficient way, work over a large family of the logics studied in the previous parts of the thesis. Precisely, it is a solver for a large family of BL logics and for some modal many-valued logics.

We begin devoting a preliminaries chapter to the software tools we are using for implementing mNiBLos. It is not included in Part I of the dissertation for being rather technical and not necessary for the comprehension of the results. Nevertheless, it is included here because the understanding of SMT solvers and the certainty on their correction play an important role in the design and development of the software application we are presenting later. The study of solver applications (that is, software oriented to check consistency of sets of equations and mathematical conditions) is a very important area within computer science, largely researched, developed and optimized and mNiBLoS is implemented using a previously existing solver of the so-called Satisfiability Modulo Theories (SMT for short). We begin by describing what SMT are, why they are interesting to build a many-valued logics solver over them and explain some of the technical results concerning the particular SMT-solver we are using, namely `z3` [42].

In the next chapter, we define the class of logics that are going to be treated with mNiBLos. For technical motivations the propositional level of these logics (called *Nice BL logics*) is settled within the BL logics, while the modal expansion considered is limited to finite Kripke structures (that is, models with a finite number of states) with a crisp accessibility relation. We present a way of computing more efficiently the previous logics than considering the operations as originally defined in the literature. This comes as a result of an alternative characterization of non-linear operations arising in the so-called Product components of *BL*-chains. We also develop an algorithm to pre-process the modal fragment, that given a modal formula constructs a Kripke structure that can be evaluated to a counter-model for the formula when it is not a theorem of the logic.

In Chapter 10 a detailed description of the main result of this part of our research is presented, namely a complete specification of mNiBLoS. Details concerning the theoretical design and the technical implementation of the solver are given and we also present some easy examples that could help an interested reader to become a user of mNiBLoS.

Finally, we present a chapter with empirical results from several performed tests. First, we show several tests run over two benchmarks already used in these kind of applications and compare the results with the ones existing in the literature. Then, we present some more general results, presenting a time chart of mNiBLoS running over randomly-generated formulas of increasing complexity and comment the results.

We close this dissertation presenting some conclusions about the work done and listing several open problems that are still open after this research.

Part I

Preliminaries

Chapter 1

Universal algebra and abstract algebraic logic

We devote this chapter to present a coherent and general environment over which the rest of our work will be settled. We begin with some basic but important notions from Universal algebra, which provide us with a uniform notation and methodology that will be used along the rest of this work. Then, we will present an overview on the issues from abstract algebraic logic that will be of use in the following chapters.

1.1 Universal algebra

With the aim of being as self-contained as possible, we will begin by recalling in this section some general definitions and results from universal algebra. Our aim is not presenting a survey on this matter (for the interested reader, we refer to a classical text-book on the matter, [20]), but rather creating a coherent environment over which it is possible to develop our studies. We will use these basic notions of universal algebra as a language in order to unify the most theoretical part of this dissertation and for this reason we will focus on presenting some logically oriented objects that will be used latter.

As commented in the introduction, one of the two defining elements of a logic is its language. Formally, a ***language*** or ***type*** is a pair $\mathfrak{L} = \langle L, \Lambda\rangle$ where $L \neq \emptyset$ is a set of symbols and $\Lambda\colon L \to \omega$ is a function that assigns an arity to each symbol in L. Symbols with arity equal to 0 are called ***constants***, while the other symbols will be referred to by ***operations*** or ***connectives***.

A language $\mathfrak{L}$ can be interpreted on algebraic structures that have the same type. These are the ***$\mathfrak{L}$-algebras***, structures of the form $\mathbf{A} = \langle A, \{f^{\mathbf{A}}\}_{f\in L}\rangle$ where $A \neq \emptyset$ is the universe of the algebra and $f^{\mathbf{A}}\colon A^{\Lambda(f)} \to A$ are the operations on A. When the language is clear from the context we will often say "algebra" instead of "$\mathfrak{L}$-algebra". Moreover, if $\mathfrak{L}$ has a finite number of symbols, we will sometimes denote it by a tuple of finite lenght with the elements $f/\Lambda(f)$ for each

$f \in \mathfrak{L}$. Moreover, we will denote its type just by the tuple of natural numbers given by $\Lambda(f)$.

We will define here some particular class of algebras that will appear along this dissertation.

Definition 1.1. Consider a language $\mathfrak{L}$ extending $\langle \vee/2, \wedge/2, \overline{0}/0, \overline{1}/0 \rangle$. A $\mathfrak{L}$-algebra $\mathbf{A}$ is a ***bounded lattice*** when $\vee^{\mathbf{A}}$ and $\wedge^{\mathbf{A}}$ are associative, commutative and idempotent operations in $\mathbf{A}$ and for all $a, b \in A$ the following hold:

- $a \wedge^{\mathbf{A}} (a \vee^{\mathbf{A}} b) = a \vee^{\mathbf{A}} (a \wedge^{\mathbf{A}} b) = a$,
- $a \wedge^{\mathbf{A}} \overline{0}^{\mathbf{A}} = \overline{0}^{\mathbf{A}}$,
- $a \vee^{\mathbf{A}} \overline{1}^{\mathbf{A}} = \overline{1}^{\mathbf{A}}$.

$\mathbf{A}$ is ***distributive*** if, moreover, for all $a, b, c \in A$,

- $a \wedge^{\mathbf{A}} (b \vee^{\mathbf{A}} c) = (a \wedge^{\mathbf{A}} b) \vee^{\mathbf{A}} (a \wedge^{\mathbf{A}} c)$,
- $a \vee^{\mathbf{A}} (b \wedge^{\mathbf{A}} c) = (a \vee^{\mathbf{A}} b) \wedge^{\mathbf{A}} (a \vee^{\mathbf{A}} c)$.

A particular algebra that is very relevant in the field of logic is the algebra generated from the language itself over a set of propositional variables. When we talk about ***formulas of a language***, we mean constructions from a countable set of elements (the ***variables***) using the symbols of the language. Formally, given a language $\mathfrak{L}$ and a countable set of variables Var, the set of formulas Fm[1] is built inductively with the following steps:

- $Var \subseteq Fm$,
- $f \in Fm$ for all $f \in L$ such that $\Lambda(f) = 0$,
- $f(\varphi_1, \ldots, \varphi_n) \in Fm$ for $f \in L$ with $\Lambda(f) = n \geq 1$ and $\varphi_1, \ldots, \varphi_n \in Fm$,
- No other elements belong to Fm.

Usually, when the maximum arity of the operations in $\mathfrak{L}$ is 2, we will write the formulas using the infix notation instead the above presented prefix one.

The set of formulas can be turned into an algebra of type $\mathfrak{L}$, denoted by $\mathbf{Fm}_L$ and called the ***formula algebra***.

The universe of $\mathbf{Fm}$ is Fm and the operations of the algebra are implicitly given in the construction of the set of formulas detailed above, i.e., for each $f \in L$,

$$
\begin{array}{rcl}
f^{\mathbf{Fm}} & = & f \text{ if } \Lambda(f) = 0 \\
f^{\mathbf{Fm}}(\varphi_1, \ldots, \varphi_n) & = & f(\varphi_1, \ldots, \varphi_n) \text{ if } \Lambda(f) = n \geq 1
\end{array}
$$

From now on we will assume that all algebras are in a fixed type $\mathfrak{L}$ and use this fact without notice.

[1] If it is needed to emphasize the language and/or the variables set, we will write $Fm_{\mathfrak{L}}(Var)$.

Given two algebras $\mathbf{A}, \mathbf{B}$ we say that $\mathbf{A}$ is a ***subalgebra*** of $\mathbf{B}$ and write $\mathbf{A} \preceq \mathbf{B}$, whenever $A \subseteq B$, $f^{\mathbf{A}} = f^{\mathbf{B}}$ for each 0-ary symbol f in $\mathfrak{L}$ and, for each n-ary operation $f \in \mathfrak{L}$ (with $n \geq 1$) $f^{\mathbf{A}}(a_1, \ldots, a_n) = f^{\mathbf{B}}(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n \in A$. Given a class of algebras K, we will write $\mathbb{S}\mathsf{K}$ to denote the class of subalgebras of the algebras in K.

We denote by $Hom(\mathbf{A}, \mathbf{B})$ the set of ***homomorphisms*** from $\mathbf{A}$ to $\mathbf{B}$, that are the mappings h from A to B such that

$$\begin{aligned} h(f^{\mathbf{A}}) &= f^{\mathbf{B}} \text{ for } f \in \mathfrak{L} \text{ s.t } \Lambda(f) = 0 \\ h(f^{\mathbf{A}}(a_1, \ldots, a_n)) &= f^{\mathbf{B}}(h(a_1), \ldots, h(a_n)) \text{ for } f \in \mathfrak{L} \text{ s.t } \Lambda(f) = n \geq 1 \end{aligned}$$

for all $a_1, \ldots, a_n \in A$. Accordingly, given $h \in Hom(\mathbf{A}, \mathbf{B})$, we will denote by $h(\mathbf{A})$ the algebra $\langle h(A), \{f^{\mathbf{B}} \restriction h(A)\rangle$ (which is a subalgebra of $\mathbf{B}$).

Injective homomorphisms are called ***embeddings***, while bijective ones are called ***isomorphisms***. We will write $\mathbf{A} \hookrightarrow \mathbf{B}$ and $\mathbf{A} \cong \mathbf{B}$ to denote respectively that there is an embedding and an isomorphism from $\mathbf{A}$ to $\mathbf{B}$. Given a class of algebras K, we denote by $\mathbb{H}\mathsf{K}$ the class of ***homomorphic images*** of the algebras in K i.e., the class $\{\mathbf{A} : \exists \mathbf{B} \in \mathsf{K} \text{ and } h \in Hom(\mathbf{B}, \mathbf{A}) \text{ surjective}\}$. We will write $\mathbb{I}\mathsf{K}$ to denote the class of ***isomorphic images*** of K, $\{\mathbf{A} : \exists \mathbf{B} \in \mathsf{K} \text{ with } \mathbf{A} \cong \mathbf{B}\}$.

A particular family of homomorphisms deeply related with logic that will be intensively used all along this thesis are the so-called evaluations. An ***evaluation*** e into an algebra $\mathbf{A}$ is an homomorphism $e : \mathbf{Fm} \to \mathbf{A}$. Observe that these homomorphisms are determined by their values on the set *Var*: indeed, for a function $h\colon Var \to A$, there is a unique homomorphism $h' \in Hom(\mathbf{Fm}, \mathbf{A})$ extending h. Given an evaluation e and a set of formulas Γ, we will write $e[\Gamma]$ instead of $\{h(\gamma) : \gamma \in \Gamma\}$. The homomorphims from the class $Hom(\mathbf{Fm}, \mathbf{Fm})$ are called ***substitutions***.

In our analysis we will make use of another kind of syntactic objects besides formulas. Formally speaking an ***equation*** is a pair of formulas $\langle \varphi, \psi \rangle \in Fm \times Fm$. Nevertheless we will make use of the more suggestive notation $\varphi \approx \psi$. We will denote the set of equations (in the language $\mathfrak{L}$) by $Eq_{\mathfrak{L}}$. Given a class of algebras K, the ***equational consequence relative to*** K is the relation $\models_{\mathsf{K}} \subseteq \mathcal{P}(Eq_{\mathfrak{L}}) \times Eq_{\mathfrak{L}}$ defined as

$$\begin{aligned} \Theta \models_{\mathsf{K}} \varphi \approx \psi \iff & \text{for every } \mathbf{A} \in \mathsf{K} \text{ and every evaluation } h \text{ in } \mathbf{A} \\ & \text{if } h\alpha = h\beta \text{ for every } \alpha \approx \beta \in \Theta, \text{ then } h\varphi = h\psi \end{aligned}$$

for every $\Theta \cup \{\varphi \approx \psi\} \subseteq Eq_{\mathfrak{L}}$. Formally speaking a ***quasi-equation*** is an ordered pair made up by a finite set of equations followed by a single equation, usually denoted by expressions of the following kind[2]

$$\varphi_1 \approx \psi_1 \wedge \cdots \wedge \varphi_n \approx \psi_n \implies \varphi \approx \psi.$$

By removing the limitation to a finite set of equations in the previous definition we obtain the so-called ***generalized quasi-equations***, i.e. expressions of the

[2]Observe $\wedge$ stands here for the semantical *and* notion, so $\varphi_1 \approx \psi_1 \wedge \varphi_2 \approx \psi_2$ stands for "equations $\varphi_1 \approx \psi_1$ and $\varphi_2 \approx \psi_2$ hold.

following kind below (where I is an arbitrary set) [3]:

$$\bigwedge_{i\in I}\varphi_i \approx \psi_i \implies \varphi \approx \psi.$$

It is clear that quasi-equations are just generalized quasi-equations with a finite set of premises and that equations are quasi-equations with empty premises. Therefore the following definition applies to these three kinds of objects. A generalized quasi-equation $\bigwedge_{i\in I}\varphi_i \approx \psi_i \implies \varphi \approx \psi$ is ***valid*** in a class of algebras K if

$$\{\varphi_i \approx \psi_i : i \in I\} \vDash_{\mathsf{K}} \varphi \approx \psi.$$

A ***congruence*** θ on an algebra $\mathbf{A}$ is an equivalence relation on A that respects the operations. We denote by $Con\mathbf{A}$ the set of congruences on $\mathbf{A}$. It is worth to remark that it is possible to order $Con\mathbf{A}$ under set-theoretic inclusion and that the resulting structure is a complete lattice whose infima coincide with intersections. Observe that the top and the bottom element of $Con\mathbf{A}$ are respectively

$$A \times A \text{ and } \mathrm{Id}_{\mathbf{A}} \coloneqq \{\langle a, a\rangle : a \in A\}.$$

Accordingly, we say that $\mathbf{A}$ is ***simple*** whenever $Con\mathbf{A} = \{A \times A, \mathrm{Id}_{\mathbf{A}}\}$.

We will denote by $\mathbf{A}/\theta$ the ***quotient algebra*** of $\mathbf{A}$, that the algebra with universe A/θ whose operations are defined by means of representatives of equivalence classes. Observe that this process is sound, since congruences preserve operations. Sometimes we will be interested in classes of algebras that are not closed under the formation of quotient algebras. In these cases the following device will be useful: given a class of algebras K, the K***-congruences*** of $\mathbf{A}$ (also called ***congruences of*** $\mathbf{A}$ ***relative to*** K) are the congruences of $\mathbf{A}$ that yield a quotient in K. We will denote their collection by $Con_{\mathsf{K}}\mathbf{A}$.

Observe that $Con\mathbf{A} = Con_{\mathbb{H}\{\mathbf{A}\}}\mathbf{A}$. It is worth to remark that in general the poset $\langle Con_{\mathsf{K}}\mathbf{A}, \subseteq\rangle$ may fail to be a complete lattice (and to be a lattice as well).

We will make use of several kinds of algebraic product-like constructions. The ***direct product*** of a family of algebras $\{\mathbf{A_i}\}_{i\in I}$ is the algebra

$$\prod_{i\in I}\mathbf{A_i} \coloneqq \langle \prod_{i\in I} A_i, \{f^{\prod_{i\in I}\mathbf{A_i}}\}_f \in \mathfrak{L}\rangle$$

where $\prod_{i\in I} A_i$ is the usual Cartesian product and the operations are defined component-wise. If $I = \emptyset$, the direct product $\prod_{i\in I}\mathbf{A_i}$ will be the one-element algebra, that is unique up to isomorphism. Finally, given a class of algebras K, we will denote by $\mathbb{P}\mathsf{K}$ the class of direct products of families of algebras in K. In order to introduce another kind of product-like structures, we will need some more concepts. A ***filter*** F on a set I is an upwards closed subset of $\mathcal{P}(I)$ that is closed under intersection and $I \in F$. We will say that F is ***proper*** whenever $F \neq \mathcal{P}(I)$, that is when $\emptyset \notin F$. Then, given a family of algebras $\{\mathbf{A_i}\}_{i\in I}$ and

[3]The fact that the set of variables is countable is important for this definition: generalized quasi-equations, as considered here, cannot have more than countable many different variables.

a proper filter F on I, it is not difficult to see that the relation $\sim_F$ on $\prod_{i\in I} \mathbf{A_i}$ defined as

$$a \sim_F b \iff \{j \in I : a[j] = b[j]\} \in F$$

is a congruence on $\prod_{i\in I} \mathbf{A_i}$. The quotient of $\prod_{i\in I} \mathbf{A_i}$ under $\sim_F$ is called the ***reduced product*** of $\{\mathbf{A_i}\}_{i\in I}$ ***with respect to*** F. We will denote it by $\prod_{i\in I} \mathbf{A_i}/F$. Finally, given a class of algebras K, we will denote by $\mathbb{P}_{\mathbb{R}}\mathsf{K}$ the class of reduced products of families of algebras in K. That is,

$$\{\prod_{i\in I} \mathbf{A_i}/F : \{\mathbf{A_i}\}_{i\in I} \subseteq \mathsf{K},\ I \text{ set and } F \text{ proper filter on } I\}.$$

A proper filter is called an ***ultrafilter*** when it enjoys one of the following equivalent properties:

- For all $X \subseteq I$, $X \in F$ if and only if $I \setminus X \notin F$,
- For all $X, Y \subseteq I$ such that $X \cup Y \in F$, either $X \in F$ or $Y \in F$,
- F is maximal in the set of proper filters of I (ordered by the inclusion).

Reduced products with respect to ultrafilters are called ***ultraproducts***. Given a class of algebras K, we denote by $\mathbb{P}_{\mathbb{U}}\mathsf{K}$ the class of ultraproducts of families of algebras is K.

The last product-like condition we shall consider is the following: an algebra $\mathbf{A}$ is a ***subdirect product*** of a family $\{\mathbf{A_i}\}_{i\in I}$ whenever it is a subalgebra of the direct product $\prod_{i\in I} \mathbf{A_i}$ and $A[j] \coloneqq \{a[j] : a \in A\}$ equals A_j for each $j \in I$. Given a class of algebras K, we denote by $\mathbb{P}_{\mathbb{S}}\mathsf{K}$ the class of subdirect products of families of algebras in K. It is worth to remark that if K is closed under the formation of subdirect products, then for every algebra $\mathbf{A}$ of the type, the poset $\langle Con_{\mathsf{K}}\mathbf{A}, \subseteq\rangle$ is a complete lattice (but in general not a sublattice of $Con\mathbf{A}$).

A class of algebras K is a ***variety*** if there exists a set of equations Υ such that $\mathsf{K} = \{\mathbf{A} : \emptyset \models_{\mathbf{A}} \Upsilon\}$. A celebrated result of Birkhoff characterizes varieties as the classes of algebras closed under the formation of homomorphic images, subalgebras and direct products. It is not difficult to see that, given a class of algebras K, there exist the smallest variety that contains K, namely the class of models of the equations that are valid in K. This variety is called the ***the variety generated by*** K and is denoted by $\mathbb{V}\mathsf{K}$. Tarski characterized it in terms of class-operators as $\mathbb{V}\mathsf{K} = \mathbb{HSP}\mathsf{K}$.

Analogously, a class of algebras K is a ***quasi-variety*** if it is axiomatized by quasi-equations. Mal'cev showed that a class of algebras is a quasi-variety if and only if is is closed under the formation of isomorphisms, subalgebras and reduced products. Moreover, given a class of algebras K, there exists the smallest quasi-variety that contains K. This quasi-variety is called the ***quasi-variety generated by*** K and is denoted by $\mathbb{Q}\mathsf{K}$. It is possible to prove that $\mathbb{Q}\mathsf{K} = \mathbb{ISP}_{\mathbb{R}}\mathsf{K}$.

Finally, a class of algebras K is a ***generalized quasi-variety*** if it is axiomatized by generalized quasi-equations. Blok and Jónsson introduced a new class-operator

$$\begin{aligned}\mathbb{U}\mathsf{K} :=\{\mathbf{A} :\ & \text{if } \mathbf{B} \preceq \mathbf{A} \text{ and } \mathbf{B} \text{ is generated by a set}\\ & \text{of cardinality } \leq |\mathit{Var}|, \text{ then } \mathbf{B} \in \mathsf{K}\}\end{aligned}$$

and proved that a class of algebras is a generalized quasi-variety if and only if it is closed under the formation of isomorphic images, subalgebras, direct products and under the $\mathbb{U}$ operator (see [13, Th. 8.1]). Moreover it is easy to see that, given a class of algebras K, there exists the smallest generalized quasi-variety including K. This generalized quasi-variety is called the ***generalized quasi-variety generated by*** K and is denoted by $\mathbb{GQ}\mathsf{K}$. Blok and Jónsson proved that $\mathbb{GQ}\mathsf{K} = \mathbb{UISP}\mathsf{K}$ (see [13, Cor. 8.2]).

It is worth to remark that generalized quasi-equations are preserved under direct products and subalgebras and so generalized quasi-varieties are closed under subdirect products. In particular this implies that if K is a generalized quasi-variety, then for every algebra $\mathbf{A}$ of the type, the poset $\langle Con_{\mathsf{K}}\mathbf{A}, \subseteq\rangle$ is a complete lattice. Moreover it is evident that quasi-varieties are special generalized quasi-varieties and, again, that varieties are special quasi-varieties.

1.2 Algebraic logic

Abstract algebraic logic is a theory that aims to provide several methods to uniformly study propositional logics using algebraic tools. We are particularly interested in this field since, over the last 30 years, a large set of powerful results has been developed for studying the different levels of the Leibniz hierarchy. This consists on a classification of logics depending on some of their properties, which can be defined equivalently from different perspectives (included syntactical and algebraic characterizations). Classifying a logic into a level of this hierarchy allows to use all the results existing in the literature concerning the corresponding level in order to study it.

This classification is the reason behind our interest on this general approach concerning the definitions and studies of logics. Even though the main work of this thesis dissertation is that of studying a particular family of logics, defining and working with them in the adequate terms will allow us to classify these logics within the Leibniz hierarchy and so gain access to this large set of results.

The definitions and results existing in the abstract algebraic logic field can be seen, in a lot of cases, as generalizations of the ones presented historically for each one of the particular logics studied before. For this reason we see it here as a theory unifying different mathematical logics and as such, is natural to present it in the first chapter of the dissertation. In this way we will be able to uniformly talk about particular cases of the notions presented along this section and make the comprehension of the two first parts of the thesis easier to the reader.

Most of the definitions and results recalled here are folklore and have been established after several works on the matter by groups of researchers, so in general, we will not include references for the main definitions and results presented in this section. Instead, for a systematic exposition of abstract algebraic logic, we refer to the main classical publications on the topic: [14, 15, 36, 60, 59].

We will begin by giving the formal definition of a sentential logic.

Definition 1.2. Let $\mathfrak{L}$ be a language. A ***(sentential) logic*** of type $\mathfrak{L}$ is a pair $L = \langle \mathfrak{L}, \vdash_L \rangle$ where $\mathfrak{L}$ is a language and $\vdash_L \subseteq \mathcal{P}(Fm_{\mathfrak{L}}) \times Fm_{\mathfrak{L}}$ is a relation (called the ***consequence*** or ***derivability relation*** of L) satisfying the following conditions for all $\Gamma \cup \Theta \cup \{\varphi\} \subseteq Fm$ the following properties hold:

1. It is a closure operator, i.e.,
   - If $\varphi \in \Gamma$ then $\langle \Gamma, \varphi \rangle \in \vdash_L$,
   - If $\langle \Gamma, \varphi \rangle \in \vdash_L$ and $\Gamma \subseteq \Theta$ then $\langle \Theta, \varphi \rangle \in \vdash_L$,
   - If $\langle \Gamma, \varphi \rangle \in \vdash_L$ and $\langle \Theta, \gamma \rangle \in \vdash_L$ for all $\gamma \in \Gamma$ then $\langle \Theta, \varphi \rangle \in \vdash_L$,
2. It is structural, i.e., for any substitution $\sigma \in Hom(\mathbf{Fm}_{\mathfrak{L}}, \mathbf{Fm}_{\mathfrak{L}})$, if $\langle \Gamma, \varphi \rangle \in \vdash_L$ then $\langle \sigma(\Gamma), \sigma(\varphi) \rangle \in \vdash_L$.

We will write $\Gamma \vdash_L \varphi$ instead $\langle \Gamma, \varphi \rangle \in \vdash_L$ and in such case we will say that φ ***follows from*** (or ***is deducible from***) Γ in L. We denote by $Th(L)$ the ***theorems*** of the logic L, which are the formulas φ such that $\emptyset \vdash_L \varphi$ (and we will write instead $\vdash_L \varphi$). A logic L is ***finitary*** whenever for all $\Gamma \cup \{\varphi\}$ such that $\Gamma \vdash_L \varphi$ there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_L \varphi$. We will sometimes abuse notation and write Fm_L (or $\mathbf{Fm}_L$) instead of $Fm_{\mathfrak{L}}$ (or $\mathbf{Fm}_{\mathfrak{L}}$).

Logics can be defined both in syntactic and in semantic terms. One of the most common ways to syntactically define a logic is to present it by means of an Hilbert-style calculus. More precisely an ***Hilbert style calculus*** is a triple $H = \langle \mathfrak{L}, Ax, IR \rangle$ where $\mathfrak{L}$ is a propositional language, $Ax \subseteq Fm_{\mathfrak{L}}$ is a set of formulas called ***axioms*** and $IR \subseteq (\mathcal{P}(Fm_{\mathfrak{L}}) \setminus \{\emptyset\}) \times Fm_{\mathfrak{L}}$ is a set of the so called ***inference rules***. H induces a consequence relation $\vdash_H$ (and thus, a logic) as follows:

- For any substitution $\sigma \in Hom(\mathbf{Fm}_{\mathfrak{L}}, \mathbf{Fm}_{\mathfrak{L}})$ and any $\varphi \in Ax$, $\emptyset \vdash_H \sigma(\varphi)$,
- For any substitution σ and $\langle \Gamma, \varphi \rangle \in IR$, $\sigma\Gamma \vdash_H \sigma\varphi$,
- If there is an inference rule $\langle \Theta, \varphi \rangle \in IR$, a substitution σ and a set $\Sigma \subseteq Fm_L$ such that $\Sigma \vdash_H \sigma\theta$ for each $\theta \in \Theta$ then $\Sigma \vdash_H \sigma\varphi$.

$\vdash_H$ is the smallest logic that satisfies all the deductions expressed by the rules and axioms of H. Another way of defining the consequence relation $\vdash_H$ relies on the following notion of ***proof***.[4]

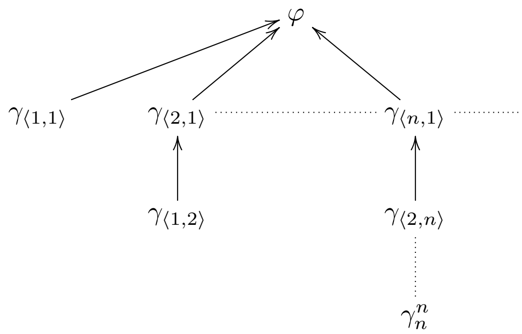
Definition 1.3. Let $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathfrak{L}}$, and H a Hilbert style calculus (on $\mathfrak{L}$). A ***proof*** of φ from Γ in H is a well-founded tree[5] labelled by formulas such that

[4] Observe that this is a generalization of the usual finite proof. We allow infinitary inference rules in our Hilbert calculus (i.e., with infinitely many premises) and so the definition of proof is more general.

[5] A well-founded tree has possibly infinite width and depth, but there are no branches of infinite length.

- The root is labelled by φ and the leaves are elements from Γ or substitutions of the axioms of H.
- For each node with label ψ of the tree, with Σ_ψ being its predecessors, there is an inference rule in H, $\langle\Theta,\theta\rangle \in IR$ and a substitution σ such that $\sigma\Theta = \Sigma_\psi$ and $\sigma\theta = \psi$.

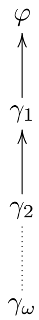
We assume the reader is familiar with the finitary proofs in Hilbert-style calculus. In order to get an idea on the meaning of an infinite proof given by the above definition, the following tree



illustrates a proof structure for H such that:[6]

- H has finitary rules $\{R_{\langle i,j\rangle}\}_{1\leq j<=i,j,i\in\mathbb{N}}$ such that there exist substitutions $\sigma_{\langle i,j\rangle}$ with $\sigma_{\langle i,j+1\rangle}(prem(R_{\langle i,j+1\rangle})) = \gamma_{\langle i,j+1\rangle}$ and $\sigma_i(con(R_{\langle i,j+1\rangle})) = \gamma_{\langle i,j\rangle}$,
- H has an infinitary inference rule iR such that there exists a substitution σ with $\sigma(prem(R)) = \{\gamma_{\langle i,1\rangle} : 1 \leq i \in \mathbb{N}\}$ and $\sigma(con(R)) = \varphi$,
- γ_n^n is an axiom of H for all $n \in \mathbb{N}$, $n \geq 1$.

On the other hand, a tree like the one depicted at the right side is not considered a deduction, even if the requirements concerning the application of rules at each step hold in the Hilbert-style axiomatic system and γ_ω is an axiom.



Keeping this in mind one can define $\Gamma \vdash_H \varphi$ if and only if there is a proof of φ from Γ in H. It is easy to show that this definition of $\vdash_H$ matches the previous

[6]For an inference rule R, we denote by $prem(R)$ the set of premises of R and by $con(R)$ the consequence of R.

one, i.e., it is the smallest logic that satisfies all the deductions expressed by the rules and axioms of H.

On the other hand, logics can be defined semantically by means of certain mathematical objects that allow to interpret their consequence relation. The most general way of doing this, at least in the propositional case, is given by the so-called ***(logical) matrices***, i.e. pairs of the form $\langle \mathbf{A}, F\rangle$ where $\mathbf{A}$ is an algebra of the type and $F \subseteq A$ is a set that represents truth among the set of possible values A. According to this intuition, any class of matrices M determined a logic as follows:

$$\begin{array}{rcl} \Gamma \models_{\mathsf{M}} \varphi & \iff & \text{for every } \langle \mathbf{A}, F\rangle \in \mathsf{M} \text{ and evaluation } h \text{ in } \mathbf{A} \\ & \text{if} & h[\Gamma] \subseteq F, \text{ then } h(\varphi) \in F \end{array}$$

for every $\Gamma \cup \{\varphi\} \subseteq Fm$.

It is well known that every logic can be determined both by a Hilbert calculus and by a class of matrices. Nevertheless it is in principle not evident how to associate to a given logic L a class of matrices M that reflects the meta-logical properties of L (and clearly such that $\models_{\mathsf{M}}$ coincides with $\vdash_L$). In order to explain how this process can be carried on in general, we need to introduce some new concepts.

Definition 1.4. Let $\langle \mathbf{A}, F\rangle$ be a matrix. The ***Leibniz congruence*** $\boldsymbol{\Omega}^{\mathbf{A}}F$ of $\langle \mathbf{A}, F\rangle$ is the largest congruence θ of $\mathbf{A}$ compatible with F in the sense that

$$\text{if } a \in F \text{ and } \langle a, b\rangle \in \theta \text{ then } b \in F.$$

In the case where $\mathbf{A}$ is the formula algebra we will omit the superscript and simply write $\boldsymbol{\Omega}F$. The Leibniz congruence naturally gives rise to a map

$$\boldsymbol{\Omega}^{\mathbf{A}} : X \to Con\mathbf{A},$$

called the ***Leibniz operator***, that sends a set $F \subseteq A$ to its Leibniz congruence $\boldsymbol{\Omega}^{\mathbf{A}}F$. The other fundamental concept that we need in order to construct algebraic semantics for a given logic is the one of filter. Given a logic L and an algebra $\mathbf{A}$, a set $F \subseteq A$ is called a ***filter*** of L (or a ***L-filter***) on $\mathbf{A}$ when it is closed under the interpretation of the rules of L, that is

$$\begin{array}{rl} \Gamma \vdash_L \varphi \Longrightarrow & \text{for every evaluation } h \text{ in } \mathbf{A} \\ & \text{if } h[\Gamma] \subseteq F, \text{ then } h(\varphi) \in F \end{array}$$

for every $\Gamma \cup \{\varphi\} \subseteq Fm$. In case L is presented by a Hilbert calculus, the filters of L on $\mathbf{A}$ are exactly the subsets of A that contains all interpretations of the axioms and that are closed under the interpretations of the rules of the calculus. In this sense Hilbert calculi may help to describe the structure of logical filters. We will denote by $\mathcal{F}i_L\mathbf{A}$ the complete lattice of L-filters on $\mathbf{A}$ ordered under the inclusion relation. We are now ready to define the class of ***models*** of L as

$$Mod_L := \{\langle \mathbf{A}, F\rangle : F \in \mathcal{F}i_L\mathbf{A}\}$$

It has long been known that any logic L is strongly complete with respect to its class of models Mod_L, that is L is the logic determined by the class of matrices Mod_L. Filters of a logic over the formula algebra are called ***theories*** of the logic.

As we already mentioned, it is not the case that every class of matrices, that yields a completeness result with respect to L, reflects the meta-logical properties of L. An example of this phenomenon is the construction of the algebraic semantics Mod_L that is by no means the intended semantics of the logic L.[7] Nevertheless, thanks to the Leibniz congruence, it turns out that the artificial semantics Mod_L can be refined in a way that (at least in well-behaved cases) it becomes the intended semantics of L. More precisely, the ***reduced models*** and the ***reduced algebras*** of L are the following classes:

$$Mod^*L := \{\langle \mathbf{A}, F\rangle : F \in \mathcal{F}i_L\mathbf{A} \text{ and } \boldsymbol{\Omega}^{\mathbf{A}} = \mathrm{Id}_{\mathbf{A}}\}$$
$$Alg^*L := \{\mathbf{A} : \langle \mathbf{A}, F\rangle \in Mod^*L \text{ for some } F \subseteq A\}.$$

It is easy to prove that L is (strongly) complete with respect to Mod^*L. Moreover in well-behaved cases the reduced models of a logic coincide with the class of matrices that was traditionally associated to logic, e.g. the reduced models of classical (intuitionistic) logic are Boolean (Heyting) algebras equipped with the singleton of the top element.

The general study of the relation that holds between a given logic L and its algebraic semantics Mod^*L gives rise to the so-called ***Leibniz hierarchy***, in which logics are classified by means of two basic features: the fact that logical equivalence (which is identified with the Leibniz congruence) is definable by means of formulas and the fact that the filters of the matrices in Mod^*L are definable by means of equations. This general classification can be useful in the study of new logics since, once a given logic is recognized to belong to a certain level of the Leibniz hierarchy, several useful results about it can be directly deduced from the general theory. For this reason we will briefly sketch the definitions of some (but not all) classes of logics in the Leibniz hierarchy.

The first class of logics we will consider was introduced by Rawiowa [108]:

Definition 1.5. A logic L in a language $\mathfrak{L}$ is ***implicative*** when there is a binary term $\to$ in the language that satisfies the following conditions:

1. $\vdash_L \varphi \to \varphi$,
2. $\varphi \to \psi, \psi \to \chi \vdash_L \varphi \to \chi$,
3. For each $f \in \mathfrak{L}$ of arity $n \geq 1$,

$$\{\varphi_i \to \psi_i, \psi_i \to \varphi_i : i \leq n\} \vdash_L f(\varphi_1, \ldots, \varphi_n) \to f(\psi_1, \ldots, \psi_n),$$

[7] For example, it is not difficult to see that the algebraic reducts of Mod_L coincide with the class of all algebras of the type.

4. $\varphi, \varphi \to \psi \vdash_L \psi$,

5. $\varphi \vdash_L \psi \to \varphi$

It is not difficult to see that if L is an implicative logic, then for every algebra $\mathbf{A} \in Alg^* L$ we have that $a \to a = b \to b$ for every $a, b \in A$. In other words the term $1 \coloneqq x \to x$ defines a constant in $Alg^* L$.

Theorem 1.6. *If L is implicative, then it is (strongly) complete with respect to the following class of matrices*

$$\{\langle \mathbf{A}, \{1\}\rangle : \mathbf{A} \in Alg^* L\}.$$

Moreover the above class coincides with $Mod^* L$.

Another class of logics that properly include the implicative ones, was introduced by Blok and Pigozzi [15]:

Definition 1.7. A logic L is ***algebraizable*** when there is a class K of algebras, a set of formulas $\mathbf{\Delta}(x, y)$ in at most two variables (called ***congruence formulas***), and a set of equations $\mathbf{E}(x)$ in at most one variable (called ***defining equations***) such that for all $\Gamma \cup \{\varphi\} \subseteq Fm_L$ the following conditions hold:

- $\Gamma \vdash_L \varphi$ if and only if $\mathbf{E}(\Gamma) \models_{\mathsf{K}} \mathbf{E}(\varphi)$;
- $x \approx y \models_{\mathsf{K}} \mathbf{E}(\mathbf{\Delta}(x, y))$ and $\mathbf{E}(\mathbf{\Delta}(x, y)) \models_{\mathsf{K}} x \approx y$.

Here $\mathbf{E}(\Gamma)$ is an abbreviation for $\{\alpha(\gamma) \approx \beta(\gamma) : \alpha \approx \beta \in \mathbf{E}(x) \text{ and } \gamma \in \Gamma\}$ for every set of formulas Γ. If L and K are related as above, then $\mathbb{GQ}\mathsf{K}$ is called ***equivalent algebraic semantics***. It should be noticed that the definition of equivalent algebraic semantics does not depend on the choice of K in the sense that if two classes K and K' are related to L as above, then $\mathbb{GQ}\mathsf{K} = \mathbb{GQ}\mathsf{K}'$. It can be proven that this class of logics strictly contains the implicative ones. One of the main achievements of the theory of algebraizable logics is the following characterization result in terms of an isomorphism between logical filters and relative congruences:

Theorem 1.8 (Isomorphism Theorem)**.** *Let L be a logic and K a generalized quasi-variety. $\mathcal{L}$ is algebraizable with equivalent algebraic semantics K if and only if, in every algebra $\mathbf{A}$ of the type, the Leibniz operator is an isomorphism between the complete lattices $\mathcal{F}i_L\mathbf{A}$ and $Con_{\mathsf{K}}\mathbf{A}$ that, moreover, commutes with inverse endomorphisms in the sense that $\mathbf{\Omega}^{\mathbf{A}}\sigma^{-1}F = \sigma^{-1}\mathbf{\Omega}^{\mathbf{A}}F$ for every endomorphism σ of $\mathbf{A}$ and filter $F \in \mathcal{F}i_L\mathbf{A}$.*

Relying on the above theorem, when L is an algebraizable logic with equivalent algebraic semantics K we will simplify the notation by writing $\mathbf{A}/F$ instead of $\mathbf{A}/\mathbf{\Omega}^{\mathbf{A}}F$ for every algebra $\mathbf{A}$ and $F \in \mathcal{F}i_L\mathbf{A}$. The last class of logics in the Leibniz hierarchy we will make use of is the following one.

Definition 1.9. A logic L in a language $\mathfrak{L}$ is ***equivalential*** when there is a set of formulas $\mathbf{\Delta}(x, y)$[8] in at most two variables for which the following equivalent conditions hold:

1. The following deductions hold:

   (R) $\vdash_L \mathbf{\Delta}(\varphi, \varphi)$

   (MP) $\varphi, \mathbf{\Delta}(\varphi, \psi) \vdash_L y$

   (Re) $\bigcup_{i \leq n} \mathbf{\Delta}(\varphi_i, \psi_i) \vdash_L \mathbf{\Delta}(f(\varphi_1 ... \varphi_n), f(\psi_1 ... \psi_n))$ for each $f \in \mathfrak{L}$ of arity $n \geq 1$

2. For all $\langle \mathbf{A}, F \rangle \in ModL$ and $a, b \in A$

$$\langle a, b \rangle \in \mathbf{\Omega}^{\mathbf{A}} F \Longleftrightarrow \mathbf{\Delta}^{\mathbf{A}}(a, b) \subseteq F.$$

In case 1 (or, equivalently, 2) holds, we will say that $\mathbf{\Delta}$ is a set of ***congruence formulas*** for L.

Remarkably, also equivalential logics can be characterized by a property of the Leibniz operator restricted to logical filters. More precisely, we have the following result.

Theorem 1.10. *A logic L is equivalential if and only if, for every algebra $\mathbf{A}$ of the type, the Leibniz operator $\mathbf{\Omega}^{\mathbf{A}}$ is monotonic over $\mathcal{F}i_L\mathbf{A}$ and commutes with inverse endomorphisms in the sense that $\mathbf{\Omega}^{\mathbf{A}}\sigma^{-1}F = \sigma^{-1}\mathbf{\Omega}^{\mathbf{A}}F$ for every endomorphism σ of $\mathbf{A}$ and filter $F \in \mathcal{F}i_L\mathbf{A}$.*

The Leibniz hierarchy comprehends other classes of logics, but these will be all the ones that are referred to along this thesis dissertation.

[8] Do not confuse this $\mathbf{\Delta}$ binary function that provides a set of equations with the Δ Baaz-Monterio unary operator of the logic, used later in this dissertation.

Chapter 2

Many-valued logics: MTL and BL

MTL logic arises as a generalization of a previously defined logic, *BL*, which is in turn a way of bringing together historically studied many-valued logics that share several characteristics.

Along this chapter we do a brief revision on these logics and their more common axiomatic extensions, going from the more basic *MTL* to more specific many-valued logics like Gödel, Łukasiewicz and Product logics. For a deeper and complete study on many-valued logics, we refer the reader to [35].

2.1 *MTL*: the logic of left-continuous t-norms

One of the most studied many-valued systems are those corresponding to logical calculi defined over the real interval $[0,1]$ and, in particular, the so-called ***t-norm based fuzzy logics***. These are logics defined over a language $\mathfrak{L} = \langle \&/2, \rightarrow /2, \wedge/2, \overline{0}/0 \rangle$, interpreted in $[0,1]$ respectively by a (left-continuous) t-norm $*$, a residuated operation $\Rightarrow_*$ for $*$, the minimum operation in $[0,1]$ and the 0 element.

A ***t-norm*** $*$ is a binary operation on $[0,1]$ that is commutative, non-decreasing in both components, associative and 1 neutral element. It is a natural generalization of the notion of minimum from lattices. Moreover, if $*$ is a left-continuous t-norm it has associated a unique residuum operation associated ($\Rightarrow_*$), defined by

$$x \Rightarrow_* y \coloneqq \max\{z : z * x \leq y\}.$$

For a left-continuous t-norm $*$, we let $[\mathbf{0},\mathbf{1}]_*$ (and call it the standard $*$-algebra) be the algebra of type $\mathfrak{L}$ defined as $\langle [0,1], *, \Rightarrow_*, \wedge, 0 \rangle$ (where $\wedge$ stands for the minimum).

Each one of these algebras determines a unique logic over formulas in the language $\mathfrak{L}$ with a countable set of variables, by considering the logical matrix

$\langle [\mathbf{0},\mathbf{1}]_*, \{1\}\rangle$.

We simply write $\Gamma \models_{[\mathbf{0},\mathbf{1}]_*} \varphi$ when φ follows from Γ in the above semantics, i.e., when for any evaluation e in $[\mathbf{0},\mathbf{1}]_*$, if $e[\Gamma] = \{1\}$ then $e(\varphi) = 1$.

An axiomatization for the logic of all left-continuous t-norm based standard algebras was proposed by Esteva and Godo in [49].

Definition 2.1. *MTL* (for monoidal t-norm logic) is the logic given by the Hilbert style calculus with *Modus Ponens* (i.e., MP : $p, p \to q \vdash q$) as its only inference rule and the following axioms:

(MTL1) $(p \to q) \to ((q \to r) \to (p \to r))$

(MTL2) $(p \,\&\, q) \to p$

(MTL3) $(p \,\&\, q) \to (q \,\&\, p)$

(MTL4a) $(p \wedge q) \to p$

(MTL4b) $(p \wedge q) \to (q \wedge p)$

(MTL4c) $(p \,\&\, (p \to q)) \to (p \wedge q)$

(MTL5a) $(p \to (q \to \chi)) \to ((p \,\&\, q) \to r)$

(MTL5b) $((p \,\&\, q) \to r) \to (p \to (q \to r))$

(MTL6) $((p \to q) \to r) \to (((q \to p) \to \chi) \to r)$

(MTL7) $\overline{0} \to p$

Other connectives can be defined from $\&, \wedge$ and $\to$ as follows.

$$
\begin{aligned}
\overline{1} &:= p \to p \\
\neg p &:= p \to \overline{0} \\
p \leftrightarrow q &:= (p \to q) \,\&\, (q \to p) \\
p \vee q &:= ((p \to q) \to q) \wedge ((q \to p) \to p)
\end{aligned}
$$

The algebraic semantics of *MTL* is the class of the so-called *MTL-algebras*. The class of *MTL*-algebras coincides with the variety of prelinear residuated lattices (that is, residuated lattices where $(x \to y) \vee (y \to x) = 1$), understanding residuated lattices as commutative, integral, bounded residuated monoids. The algebras of this variety are subdirect products of the linearly ordered algebras of the class. As expected, the operations of the *MTL*-algebras with universe $[0,1]$ are given by left-continuous t-norms and their residua. Jenei and Montagna proved in [84] that *MTL* is strongly complete with respect to the class of *MTL*-algebras defined on the real unit interval, i.e,:

Theorem 2.2. *Let $\Gamma \cup \{\varphi\}$ be a set of formulas. Then*

$$\Gamma \vdash_{MTL} \varphi \iff \Gamma \models_{[\mathbf{0},\mathbf{1}]_*} \varphi \text{ for all left-continuous t-norm } * .$$

One common property of all axiomatic extensions of MTL is that they enjoy a *local* form of the *deduction theorem*, namely, for any MTL axiomatic extension L it holds that

$$\Gamma \cup \{\varphi\} \vdash_L \psi \iff \text{there exists } n \in \mathbb{N} \text{ such that } \Gamma \vdash_L \varphi^n \to \psi,$$

where φ^n stands for $\varphi \& \overset{n}{\ldots} \& \varphi$. It is local in the sense that the value n depends of the particular formulas involved Γ, φ and ψ.

2.2 BL: the logic of continuous t-norms

If we restrict to the semantics $[\mathbf{0}, \mathbf{1}]_*$ when $*$ is a continuous t-norm a new axiomatic system can be defined. It is remarkable that the main difference with respect to MTL lies on the divisibility property (algebraically the equality $x \wedge y := x * (x \Rightarrow_* y)$), which characterizes the continuity of the t-norm. This means that the min-conjunction $\wedge$, which was not definable in MTL can be defined in this new logic from the & and $\to$ operations.

Hájek presented an axiomatic system in [71], later proven to coincide with the logic (for what concerns validity) of the continuous t-norms. This axiomatic system, named BL or Hajek's Basic Logic, coincides with that of MTL except for what concerns the already commented divisibility. BL has again MP as only inference rule and the following axioms:[1]

(BL1) $(p \to q) \to ((q \to r) \to (p \to r))$

(BL2) $(p \,\&\, q) \to p$

(BL3) $(p \,\&\, q) \to (q \,\&\, p)$

(BL4) $(p \,\&\, (p \to q)) \to (q \& (q \to p))$

(BL5a) $(p \to (q \to r)) \to ((p \,\&\, q) \to r)$

(BL5b) $((p \,\&\, q) \to r) \to (p \to (q \to r))$

(BL6) $((p \to q) \to r) \to (((q \to p) \to r) \to r)$

(BL7) $\overline{0} \to p$

In this case, also the $\wedge$ connective can be defined from & and $\to$ by letting $p \wedge q := p \,\&\, (p \to q)$.

The class of BL-algebras, i.e., the variety generated by all standard algebras based on a continuous t-norm, is formed by algebras of the form $\mathbf{A} = \langle A, \odot/2, \Rightarrow /2, 0/0 \rangle$, where the $\wedge$ operation is defined by $x \wedge y := x \odot (x \Rightarrow y)$ (and the $\vee$, $\Leftrightarrow$ and $\overline{1}$ operations are defined from the other ones as in the MTL case) and such that:

[1] These are the original set of axioms proposed by Hájek in [71]. Later Cintula showed in [33] that (BL3) is redundant.

(i) $(A, \wedge, \vee, 0, 1)$ is a lattice with the largest element 1 and the least element 0 (with respect to the lattice ordering $\leq$),

(ii) $(A, \odot, 1)$ is a commutative semigroup with the unit element 1, i.e. $\odot$ is commutative, associative and $1 \odot x = x$ for all x,

(iii) the following conditions hold for all x,y,z:

(1) $z \leq (x \Rightarrow y)$ iff $x * z \leq y$, (residuation)

(2) $x \wedge y = x \odot (x \Rightarrow y)$, (divisibility)

(3) $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$. (pre-linearity)

Thus, in other words, a BL-algebra is a bounded, integral commutative residuated lattice further satisfying (2) and (3). It is known that each BL-algebra is a subdirect product of linearly ordered BL-algebras, which has as a consequence the strong completeness of BL with respect to the linearly ordered algebras of this class. BL-algebras defined on the real unit interval $[0, 1]$, called ***standard BL-algebras***, turn to be, as intended, the ones associated to continuous t-norms.

Hájek proved in [71] that the following conditions are equivalent:

(i) $\Gamma \vdash_{BL} \varphi$,

(ii) for each BL-algebra $\mathbf{A}$ and any $\mathbf{A}$-evaluation e such that $e([\Gamma]) = \{\bar{1}^{\mathbf{A}}\}$ then $e(\varphi) = \bar{1}^{\mathbf{A}}$,

(iii) for each linearly ordered BL-algebra $\mathbf{C}$ and any $\mathbf{C}$-evaluation e such that $e([\Gamma]) = \{\bar{1}^{\mathbf{C}}\}$ then $e(\varphi) = \bar{1}^{\mathbf{C}}$.

Hájek's conjecture was that BL captured the 1-tautologies (that is, the formulas that are evaluated to 1 under any evaluation) common to all many-valued calculi defined by a continuous t-norm. In fact this was proved to be the case soon after, in [30]. That is to say, letting $TAUT(\mathbf{A})$ denote the formulas φ such that, for any evaluation $e : \mathbf{Fm} \to \mathbf{A}$ it holds that $e(\varphi) = 1$, it holds that

$$\varphi \text{ is provable in } BL \iff \varphi \in \bigcap\{TAUT([\mathbf{0}, \mathbf{1}]_*) : * \text{ is a continuous t-norm}\}$$

More than that, a stronger completeness property holds: if Γ is a *finite* set of formulas, then $\Gamma \vdash_{BL} \varphi$ if and only if for each standard BL-algebra $\mathbf{A}$ (i.e., of the form $[\mathbf{0}, \mathbf{1}]_*$ for $*$ a continuous t-norm) and any evaluation e in $\mathbf{A}$, if $e[\Gamma] = \{1\}$ then $e(\varphi) = 1$. This result is usually referred as *finite strong* standard completeness of BL. However, this property does not hold for infinite sets of formulas, i.e., BL is not strongly complete with respect to the class of standard algebras based on a continuous t-norm.

Concerned with the problem of the axiomatization of the valid equations in the standard algebra of a particular t-norm, in [52] it is addressed that problem for continuous t-norms. The authors provide there a general method to get a finite axiomatization L_*, extending BL, of the set of tautologies arising from a continuous t-norm, studying the subvarieties of the class of BL-algebras. For each L_* logic, one has that a formula φ is provable in L_* if and only if it is a tautology in $[\mathbf{0},\mathbf{1}]_*$.

2.3 Notable extensions of BL and MTL

Three outstanding examples of continuous t-norms and thus, of standard BL algebras are the following:

Gödel: $[\mathbf{0},\mathbf{1}]_{\mathbf{G}} = \langle [0,1], *_G, \Rightarrow_G, \wedge, 0\rangle$ where

$$\begin{array}{rcl} x *_G y & = & \min(x,y) \\ x \Rightarrow_G y & = & \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise.} \end{cases} \end{array}$$

Łukasiewicz: $[\mathbf{0},\mathbf{1}]_{\mathbf{Ł}} = \langle [0,1], *_{\mathbf{Ł}}, \Rightarrow_{\mathbf{Ł}}, \wedge, 0\rangle$ where

$$\begin{array}{rcl} x *_{\mathbf{Ł}} y & = & \max(x+y-1,0) \\ x \Rightarrow_{\mathbf{Ł}} y & = & \begin{cases} 1, & \text{if } x \leq y \\ 1-x+y, & \text{otherwise.} \end{cases} \end{array}$$

Product: $[\mathbf{0},\mathbf{1}]_{\mathbf{\Pi}} = \langle [0,1], *_\Pi, \Rightarrow_\Pi, \wedge, 0\rangle$ where

$$\begin{array}{rcl} x *_\Pi y & = & x \cdot y \quad \text{(product of reals)} \\ x \Rightarrow_\Pi y & = & \begin{cases} 1, & \text{if } x \leq y \\ y/x, & \text{otherwise.} \end{cases} \end{array}$$

For each one of the previous t-norms there is a corresponding formula that allows to extend BL in such a way that the resulting logic corresponds finitarily (that is, for what concerns deductions from a finite amount of premises) to the one determined by the t-norm. Namely, Łukasiewicz, Gödel and product logics can be axiomatized as extensions of BL. Indeed, it is shown in [71] that Łukasiewicz logic is the extension of BL by the axiom

(Ł) $\neg\neg p \to p$,

forcing the negation to be involutive and that Gödel logic is the extension of BL by the axiom

(G) $p \to (p\&p)$.

forcing the conjunction to be idempotent. Finally, product logic is just the extension of BL by the following two axioms:

(Π1) $\neg\neg r \to (((p\&r) \to (q\&r)) \to (p \to q))$,
(Π2) $p \wedge \neg p \to \bar{0}$.

The first axiom indicates that if $c \neq 0$, the cancellation of c on both sides of the inequality $a \cdot c \leq b \cdot c$ can be done, getting thus hence the strict monotony of the conjunction on $(0, 1]$. The last axiom is due to the fact that negation in product logic behaves such that $\neg a = a \to 0 = 0$ whenever $a > 0$.

It is known that Łukasiewicz, Gödel and Product logic are *finitely strong standard complete*. That is to say, for L being one of the previous logics and $[\mathbf{0},\mathbf{1}]_L$ its corresponding standard algebra and for any $\Gamma \cup \{\varphi\}$ a *finite* set of formulas it holds that

$$\Gamma \vdash_L \varphi \quad \Longleftrightarrow \quad \Gamma \vdash_{[\mathbf{0},\mathbf{1}]_L} \varphi$$

The case of the Gödel logic is a bit special: it is he only axiomatic extension of *BL* that enjoys the *strong standard completeness*, because it is finitary in its semantic definition (which is not the case for Ł and Π logics, for instance). Not only that: for any set $\Gamma \cup \{\varphi\}$ of formulas the following items are equivalent

- $\Gamma \vdash_G \varphi$,
- $\Gamma_0 \vdash_G \varphi$ for some finite $\Gamma_0 \subseteq \Gamma$,
- $\Gamma \models_{[\mathbf{0},\mathbf{1}]_\mathbf{G}} \varphi$,
- $\Gamma \models^{\leq}_{[\mathbf{0},\mathbf{1}]_\mathbf{G}} \varphi$, i.e, for any evaluation e in $[\mathbf{0},\mathbf{1}]_\mathbf{G}$, $\inf\{e[\Gamma]\} \leq e(\varphi)$

By defining $\neg x := x \Rightarrow 0$, it turns out that the algebraic semantics of Łukasiewicz logic, given by the class of *MV-algebras* (or Wajsberg algebras), coincides with the subvariety of *BL*-algebras further satisfying $\neg\neg x = x$. Similarly, the class of *Gödel-algebras* corresponds to the subvariety of *BL*-algebras satisfying $x * x = x$ and the class of *Product algebras*, which characterize the algebraic semantics for Product logic, coincides with the subvariety of *BL* satisfying the equations

$$\begin{aligned} x \wedge \neg x &\approx \bar{0} \\ \neg\neg z \to ((x * z \leftrightarrow y * z) \to (x \leftrightarrow y)) &\approx \bar{1} \end{aligned}$$

The importance of the Gödel, Łukasiewicz and Product t-norms (and thus, of their corresponding logics) with respect to all the other continuous t-norms will become clearer in the next section, where we present how any other continuous t-norm can be obtained from these ones. Two of these logics correspond to many-valued systems historically studied before fuzzy logics were developed. These are the well-known Łukasiewicz [89] and Gödel [65] logics[2] which are the logical systems corresponding to the so-called Łukasiewicz and minimum t-norms and their residuated implications respectively. Later, already motivated by research

[2] Gödel logic is also known as Dummett logic, referring to the scholar who axiomatized it.

on fuzzy logic, in [75] the many-valued logic corresponding to Product t-norm and its residuum was also axiomatized.

Clearly, not only Łukasiewicz logic, Gödel logic, Product logic and Hájek's BL logic—as well as the Classical Propositional Calculus[3]—can be presented as axiomatic extensions of MTL. Tables 2.1 and 2.2 collect several important axiom schemata and some of the most prominent axiomatic extensions of MTL.[4] Notice that in extensions of MTL with the divisibility axiom (Div) the additive conjunction $\wedge$ is definable and therefore we could not consider it as a primitive connective. However, for the sake of homogeneity we keep $\mathfrak{L} = \{\&, \rightarrow, \wedge, \overline{0}\}$ as the common language for all MTL extensions.

Axiom schema	**Name**
$\neg\neg p \rightarrow p$	Involution (Inv)
$\neg p \vee ((p \rightarrow p \,\&\, q) \rightarrow q)$	Cancellation (C)
$\neg(p \,\&\, q) \vee ((q \rightarrow p \,\&\, q) \rightarrow p)$	Weak Cancellation (WC)
$p \rightarrow p \,\&\, p$	Contraction (Con)
$(p \wedge q) \rightarrow p \,\&\, (p \rightarrow q)$	Divisibility (Div)
$(p \wedge \neg p) \rightarrow \overline{0}$	Pseudo-complementation (PC)
$p \vee \neg p$	Excluded Middle (EM)
$(p \,\&\, q \rightarrow \overline{0}) \vee (p \wedge q \rightarrow p\&q)$	Weak Nilpotent Minimum (WNM)
$p^{n-1} \rightarrow p^n$	n-Contraction (C_n)

Table 2.1: Some usual axiom schemata in fuzzy logics.

We conclude this section placing the logics presented in this chapter within the context of other studied logics. In the tradition of substructural logics, both *BL* and *MTL* are logics without contraction. The weakest residuated logic without contraction is Höhle's Monoidal Logic *ML* [82], equivalent to FL_{ew} (Full Lambek calculus with exchange and weakening) introduced by Kowalski and Ono [62] as well as to Adillon and Verdú's $\mathrm{IPC}^*\backslash c$ (Intuitionistic Propositional Calculus without contraction) [1], which is the logic corresponding to the variety of bounded, integral and commutative residuated lattices. From this logic, *MTL* can be obtained by adding the prelinearity axiom and from there, a hierarchy of all t-norm-based fuzzy logics can be considered as different schematic extensions [62]. Figure 2.1 shows a diagram of this hierarchy with the main logics presented.

2.4 Construction of continuous t-norms

It seems natural to wonder whether it is possible to give a way of constructing continuous t-norms with a recursive procedure, in order to be able to simplify

[3]Indeed, Classical Propositional Calculus can be presented as the extension of MTL (and of any of its axiomatic extensions) with the excluded-middle axiom (EM).

[4]Some of these logics were known well before MTL was introduced and our objective is just that of presenting them here in a uniform way.

Logic	Additional axiom schemata	References
SMTL	(PC)	[72]
ΠMTL	(C)	[72]
WCMTL	(WC)	[100]
IMTL	(Inv)	[49]
WNM	(WNM)	[49]
NM	(Inv) and (WNM)	[49]
C_nMTL	(C_n)	[28]
C_nIMTL	(Inv) and (C_n)	[28]
BL	(Div)	[71]
SBL	(Div) and (PC)	[51]
Ł	(Div) and (Inv)	[71]
Π	(Div) and (C)	[75]
G	(Con)	[71]

Table 2.2: Some axiomatic extensions of MTL obtained by adding the corresponing axiom schemata and the references (from the fuzzy logics literatura) where they were introduced.

the study of these operations (and thus, of the axiomatic extensions of *BL*). The most common method for doing so is that of the ordinal sum operation: it allows to build a t-norm from a family of t-norms, by shrinking them into disjoint subintervals of the interval $[0, 1]$.

It is based on the following well known result:

Lemma 2.3. *Let $\{*_i\}_{i\in I}$ for I a be a set of continuous t-norms and $\{(b_i, t_i)\}_{i\in I}$ a family of pairwise disjoint open intervals of $[0, 1]$ such that $\bigcup_{i\in I}[b_i, t_i] = [0, 1]$. Then, the function $*\colon [0, 1] \times [0, 1] \to [0, 1]$ defined as*

$$x * y := \begin{cases} b_i + (t_i - b_i) \cdot \left(\frac{x-b_i}{t_i-b_i} *_i \frac{y-b_i}{t_i-b_i}\right) & \textit{if } x, y \in [b_i, t_i] \\ min\{x, y\} & \textit{otherwise} \end{cases}$$

is a continuous t-norm.

The t-norm resulting from the previous construction is called the ***ordinal sum*** of $\{\langle *_i, (b_i, t_i)\rangle\}_{i\in I}$ and is denoted

$$* = \bigoplus_{i\in I} \langle *_i, (b_i, t_i)\rangle$$

When I is finite with cardinal n, we refer to it also by

$$* = \langle *_1, (b_1, t_1)\rangle \oplus \ldots \oplus \langle *_n, (b_n, t_n)\rangle$$

Intuitively, the construction of an ordinal sum is just "piling" different t-norms and considering the structure order generated with this union.

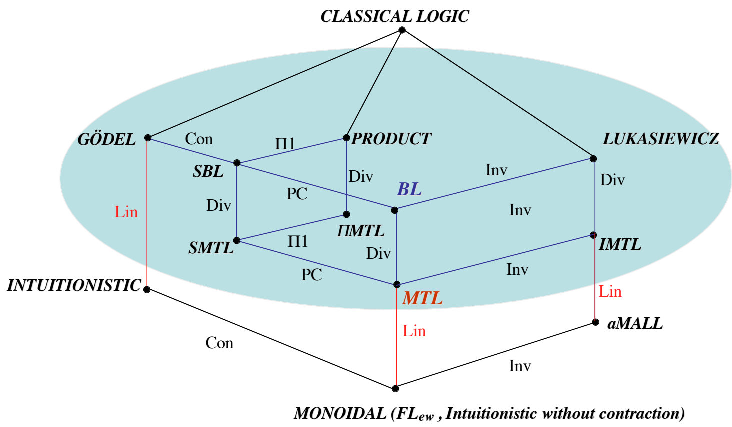


Figure 2.1: Hierarchy of some substructural and many-valued logics [50]

In can be checked that the residuum of the previous construction also has a nice characterization in terms of $\{\langle *_i, (b_i, t_i)\rangle\}_{i\in I}$. If $* = \bigoplus_{i\in I}\langle *_i, (b_i, t_i)\rangle$, its residuum $\Rightarrow_*$ is given by:

$$x \Rightarrow_* y = \begin{cases} 1 & \text{if } x \leq y \\ b_i + (t_i - b_i)\cdot\left(\frac{x-b_i}{t_i-b_i} \to_{*_i} \frac{y-b_i}{t_i-b_i}\right) & \text{if } b_i \leq y < x \leq t_i \\ y & \text{otherwise} \end{cases}$$

To work with an arbitrary axiomatic extension of the BL logic, or, equivalently, with an arbitrary standard BL-algebra (and thus, an arbitrary continuous t-norm), we know, thanks to the following well known theorem, that it is enough to deal with ordinal sums of the three particular ones commented before: $*_{\text{Ł}}$, $*_G$ and $*_\Pi$.

Theorem 2.4. *(Mostert and Shields [101], cf. [71]). Any continuous t-norm $*$ can be expressed as a countable ordinal sum of Łukasiewicz, Gödel and Product t-norms.*
That is to say, there exists I with $|I| \leq \aleph_0$, $\{(b_i, a_i)\}_{i\in I}$ family of pairwise disjoint (non-empty) subintervals of $[0,1]$ and $\{*_i\}_{i\in I}$ with $*_i \in \{*_L, *_G, *_\Pi\}$ for each $i \in I$ such that

$$* = \bigoplus_{i\in I}\langle *_i, (b_i, t_i)\rangle$$

It can be proven that the variety of BL-algebras coincides with the one generated by a particular continuous t-norm: the ordinal sum of infinitely many Łukasiewicz t-norms.

Theorem 2.5. *([2, 98])*
The variety of BL*-algebras is generated as a quasivariety by the class of all algebras of the form* $\bigoplus_{i\in I}[\mathbf{0},\mathbf{1}]_{\L}$ *for any finite* I*.*

Moreover the variety of BL*-algebras is generated as a quasivariety by the algebra* $\bigoplus_{i\in\mathbb{N}}[\mathbf{0},\mathbf{1}]_{\L}$.[5]

A corollary of this theorem is that BL is finite standard complete with respect to the standard algebra arising from the previous continuous t-norm.

2.5 MTL expansions

In the literature of t-norm based logics, one can find not only a number of axiomatic extensions of MTL but also expansions by means of introducing new connectives in the language. We address here two particular cases that have been previously studied in the literature and that play an important role along this doctoral dissertation: the expansions with truth constants and the expansions with the unary Monteiro-Baaz Δ operation.

Logics with Δ

The Δ unary connective, introduced in [9], has as intended semantics that of capturing the crisp part within fuzzy propositions. This is done fixing the interpretation of the Δ operator in any standard MTL-algebra $[\mathbf{0},\mathbf{1}]_*$ to

$$\delta x := \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

At the syntactical level, an axiomatization for Δ in the context of MTL is presented in [71].

Definition 2.6. Let L be an extension of MTL. The logic L_Δ is the extension of the L axiomatic system by the following axiom schemata and rules:

(Δ1) $\Delta p \vee \neg\Delta p$,

(Δ2) $\Delta(p \vee q) \to (\Delta p \vee \Delta q)$,

(Δ3) $\Delta p \to p$,

(Δ4) $\Delta p \to \Delta\Delta p$,

[5] We write $\bigoplus_{i\in\mathbb{N}}[\mathbf{0},\mathbf{1}]_{\L}$ to denote the algebras isomorphic to $\bigoplus_{i\in\mathbb{N}}\langle *_{\L}, (\frac{i}{i+1}, \frac{i+1}{i+2})\rangle$.

(Δ5) $\Delta(p \to q) \to (\Delta p \to \Delta q)$.

G_Δ Generalization rule for Δ: $p \vdash \Delta p$

The algebraic companion of L_Δ is given by the class of L_Δ-algebras, i.e. L-algebras expanded with a unary operator δ that satisfies the equations and quasi-equations arising from L_Δ. That is to say, for each x, y in the algebra, the following are valid equations and quasi-equations.

(Δ1) $\delta(x) \vee \neg\delta(x) \approx 1$
(Δ2) $\delta(x \vee y) \leq (\delta(x) \vee \delta(y))$
(Δ3) $\delta(x) \leq x$
(Δ4) $\delta(x) \leq \delta(\delta(x))$
(Δ5) $\delta(x \Rightarrow y) \leq (\delta(x) \Rightarrow \delta(y))$
(Δ6) $\delta(\overline{1}) \approx \overline{1}$

Observe that in any linearly ordered L_Δ-algebra, the semantics of Δ coincides with the intended semantics over the standard L-algebra, i.e., $\delta(x) = 1$ if $x = 1$, and $\delta(x) = 0$ otherwise.

It is proven that the logics L_Δ are strongly complete with respect to linearly ordered algebras of the algebraic companion. That is to say, for each set of formulas $\Gamma \cup \{\varphi\}$ the following are equivalent:

1. $\Gamma \vdash_{L_\Delta} \varphi$,
2. for each L_Δ-chain $\mathbf{A}$ and each $\mathbf{A}$-evaluation such that $e([\Gamma]) = \{\overline{1}^{\mathbf{A}}\}$ it holds that $e(\varphi) = \overline{1}^{\mathbf{A}}$,
3. for each L_Δ-algebra $\mathbf{A}$ and each $\mathbf{A}$-evaluation such that $e([\Gamma]) = \{\overline{1}^{\mathbf{A}}\}$ it holds that

Standard completeness results for L_Δ logics have been proven in the literature whenever the logic L has been shown to be standard complete. In particular, MTL_Δ is strongly complete with respect to the class of left-continuous t-norm standard algebras expanded with the Δ operation, while BL_Δ is finitely complete with respect to the class of standard algebras based on continuous t-norms with Δ. Moreover, $Ł_\Delta$, G_Δ and Π_Δ are finitely complete with respect to, correspondingly, $[\mathbf{0},\mathbf{1}]_{\mathbf{Ł}}$, $[\mathbf{0},\mathbf{1}]_{\mathbf{G}}$ and $[\mathbf{0},\mathbf{1}]_{\mathbf{\Pi}}$ expanded with the δ operation.

Logics with truth constants: Pavelka completeness

While the notion of deduction in t-norm based fuzzy logics is, in general, crisp (in the sense that a formula follows from a set of premises if it preserves the distinguished value 1), a more general notion of fuzzy deduction can be defined and thus we can consider t-norm based fuzzy logics as logics of *comparative truth*. The idea arises naturally from the fact that the residuum $\Rightarrow_*$ of a (left-continuous) t-norm $*$ satisfies the condition $x \Rightarrow_* y = 1$ if and only if $x \leq y$ for all $x, y \in [0, 1]$. This means that a formula $\varphi \to \psi$ is a logical consequence of a set Γ, i.e. $\Gamma \vdash_L \varphi \to \psi$, if the truth degree of φ is at most as high as the truth degree of ψ in any interpretation which is a model of Γ. Therefore, implications naturally capture a notion of comparative truth. In some situations it might be also interesting to explicitly represent and reason with partial degrees of truth. For instance, in any logic L_* of a left-continuous t-norm $*$, any truth-evaluation e satisfying $e(\varphi \to \psi) \geq \alpha$ and $e(\varphi) \geq \beta$, necessarily satisfies $e(\psi) \geq \alpha * \beta$ as well.

One simple and elegant way to allow for an explicit treatment of degrees of truth is by introducing truth-constants into the language. In fact, if one adds to the language new constant symbols $\overline{\alpha}$ for suitable values $\alpha \in [0, 1]$ and stipulates that $e(\overline{\alpha}) = \alpha$ for all truth-evalutations, then a formula of the kind $\overline{\alpha} \to \varphi$ becomes 1-true under an evaluation e when $\alpha \leq e(\varphi)$.

This approach was presented by Pavelka [105, 106, 107] who built a propositional many-valued logical system PL which turned out to be equivalent to the expansion of Łukasiewicz logic with a truth-constant $\overline{c}$ for each real number $r \in [0, 1]$, together with certain additional axioms. The semantics is the same as Łukasiewicz logic, just expanding the Łukasiewicz standard algebra with this set of constants and evaluating each of them by its value, i.e., expanding the evaluations e of propositional variables in $[0, 1]$ to truth-constants by requiring $e(\overline{c}) = c$ for all $c \in [0, 1]$. Although the resulting logic is not strong standard complete (SSC in the sense defined in Section 2.2) with respect to that intended semantics, Pavelka proved that his logic enjoys a different kid of semantic completeness. Namely, he defined the ***truth degree*** of a formula φ in a theory Γ as

$$\|\varphi\|_\Gamma = \inf\{e(\varphi) \mid e \text{ is a } PL\text{-evaluation model of } \Gamma\},$$

and the ***provability degree*** of φ in T as

$$|\varphi|_\Gamma = \sup\{r \in [0, 1] \mid \Gamma \vdash_{PL} \overline{r} \to \varphi\}$$

and proved that these two degrees coincide. This kind of completeness is usually known as ***Pavelka-style completeness*** and this case strongly relies on the continuity of Łukasiewicz operations. Observe that $\|\varphi\|_\Gamma = 1$ is not equivalent to $\Gamma \vdash_{PL} \varphi$, but only to $\Gamma \vdash_{PL} \overline{r} \to \varphi$ for all $r < 1$.

Later, Hájek [71] showed that Pavelka's logic PL could be simplified (while keeping the previous completeness result) limiting the expansion of the Łukasiewicz language to only a countable set of truth-constants, one for each rational number in $[0, 1]$ and the two following so-called ***book-keeping axioms***:

$$(\overline{c} \,\&\, \overline{d}) \leftrightarrow \overline{c *_{\text{Ł}} d} \qquad (\overline{c} \to \overline{d}) \leftrightarrow \overline{c \Rightarrow_{\text{Ł}} d}$$

for all $c, d \in [0, 1]_{\mathbb{Q}}$. He called this new system Rational Pavelka Logic, RPL for short. Moreover, he proved that RPL is standard complete for finite theories.

Similar rational expansions for other t-norm based fuzzy logics can be analogously defined, but unfortunately Pavelka-style completeness cannot be proven in general (for the logics extended by just the previous axioms). Łukasiewicz logic has continuous operations and an involutive negation that contribute to the Pavelka completeness, but those are not general characteristics of the continuous t-norms.

Nevertheless, expansions with truth-constants of fuzzy logics different from Łukasiewicz have been studied, mainly related to the other two outstanding continuous t-norm based logics, namely Gödel and Product logic. However, the results are not so uniform as in the Łukasiewicz case. In [71] it is studied an expansion of G_Δ (the expansion of Gödel logic G with Baaz's projection connective Δ) with a *finite* number of rational truth-constants. In [51] the authors define logical systems obtained by adding (rational) truth-constants to Gödel and Product logics with an additional involutive negation.

More in general, Cintula gives in [32] a definition of what he calls ***Pavelka-style extension*** of a particular fuzzy logic. He considers the Pavelka-style extensions of the some common continuous t-norms based fuzzy logics and for these he defines an axiomatic system with infinitary rules to overcome the discontinuities of the operations. In particular, he presents an axiomatic system with two infinitary rules Pavelka-complete with respect to the standard product algebra with Δ. In a more recent work [34], the same author approaches this problem in general for expansions of MTL. There, he proposes a method for axiomatizing a logic Pavelka complete with respect to the standard algebra of a left-continuous t-norm expanded with operations that shall follow certain restrictions, extending the logic with book-keeping axioms and with an infinitary rule for each discontinuity point of the operations. This particular work is further commented in Chapter 4, where we also present several results on this topic.

On the other hand, a systematic approach based on traditional algebraic semantics has been also considered to study completeness results (in the usual sense) for expansions of t-norm based logics with truth-constants. Indeed, as already mentioned, only the case of Łukasiewicz logic was known according to [71]. In [111] it was studied the addition of rational constants to Product logic and an axiomatic system for the theorems of this logic was given. Moreover, in [48], [53] and [54] it is addressed the problem of the addition of rational constants to other BL logics. The main idea presented in these works is that if L_* is a logic based on a left-continuous t-norm and $\mathcal{C} = \langle C, \odot, \Rightarrow, \wedge, \vee 0, 1\rangle$ is a countable subalgebra of the standard L_*-algebra $[\mathbf{0}, \mathbf{1}]_*$, then extending the logic with the book-keeping axioms for the elements in C results in a logic $L_*^{\mathcal{C}}$ strongly complete with respect to linearly ordered $L_*^{\mathcal{C}}$-algebras (the L_* algebras with an extended set of 0-ary operation symbols $\{\overline{c}\}_{c\in C}$ that behave following the book-keeping axioms). That is to say, for any set of formulas $\Gamma \cup \{\varphi\}$ the following are equivalent:

- $\Gamma \vdash_{L_*^{\mathcal{C}}} \varphi$,
- for each $L_*^{\mathcal{C}}$-chain $\mathbf{A}$, $e(\varphi) = \overline{1}^{\mathbf{A}}$ for all evaluation e in $\mathbf{A}$ which is a model of Γ.

A $L_*^{\mathcal{C}}$-chain defined over the real unit interval $[0, 1]$ is called standard. Observe that, for a logic $L_*^{\mathcal{C}}$, multiple standard chains can exist depending on the different ways of interpreting the truth-constants on $[0,1]$ (respecting the book-keeping axioms). For instance, considering the expansion of the Gödel logic with constants from $[0,1]_{\mathbb{Q}}$ (i.e., the rational numbers from $[0,1]$), the algebra $\mathbf{A} = \langle [0,1], \wedge, \rightarrow, \{\overline{c}^{\mathbf{A}}\}_{c\in[0,1]_{\mathbb{Q}}}\rangle$ where

$$\overline{c}^{\mathbf{A}} = \begin{cases} 1 & \text{if } c \geq \alpha \\ 0 & \text{otherwise} \end{cases}$$

is a $L_*^{\mathcal{C}}$-algebra with universe $[0,1]$ for any $\alpha > 0$. Among the standard chains there is one which reflects the intended semantics, the so-called ***canonical standard*** $L_*^{\mathcal{C}}$-chain

$$[\mathbf{0},\mathbf{1}]_{L_*}^{\mathcal{C}} = \langle [0,1], *, \Rightarrow, \wedge, \vee, \{c : c \in C\}\rangle,$$

i. e. the one where the truth-constants are interpreted by their names.[6]

Studying whether a logic $L_*^{\mathcal{C}}$ defined as above is complete with respect to the class of standard $L_*^{\mathcal{C}}$-chains or with respect to the canonical $L_*^{\mathcal{C}}$-chain is also a problem treated in the literature for some particular L_* logics. In [71] it was proven the canonical completeness of the expansion of Łukasiewicz logic with rational truth-constants for finite theories. On the other hand, the expansions of Gödel (and of some t-norm based logic related to the nilpotent minimum t-norm) and of Product logic with countable sets of truth-constants have been proved to be canonical complete for theorems in [53] and in [111] respectively.

One negative result for many of these logics (with the exception of Łukasiewicz logic) is that they are not (strongly nor finitely) complete with respect to their corresponding canonical standard algebras for deductions from non-empty theories (although they are finitely canonical complete if the language is restricted to formulas of the $\overline{c} \rightarrow \varphi$). We will see in Chapter 4 how it is possible to prove not only finite canonical completeness, but also its infinitary version, extending the previous logics with some infinitary rules and the Δ operator from the previous section.

[6] For the sake of a simpler notation, in Part II and Part III of this dissertation, we will omit the canonical term, and define the above kind of algebras as the standard ones. We will see that, in our case, this is the only standard algebra.

Chapter 3

Modal Logics

Modal logic is a branch of mathematical logic that focuses on reasoning with *qualification* of sentences. They have been studied from the 30's and given its versatility, it is one of the fields from mathematical logic that most has been developed and applied. One of its more remarkable applications is done by Gödel, who interprets modalities in the context of proof theory and gives place to the so-called provability logic. More recently, there has been a high number on works concerning modal logics from a quite wide range of fields. For instance, within the computer science context we can find dynamic logic [79, 80](that has two modal operators for each process, the "it is executable" and the "after it is executed" ones) and the temporal logic [109, 120] (that allows to address the "always", "sometimes" and other temporal concepts). Moreover, linked with cognitive sciences, modal logics have been used to model beliefs, trust and preferences [118, 119, 96]..., and in this context the so-called description logics have appeared. Within philosophy, modal logics have been used to study different categories of necessity, contingency, causality and so on. Modal logics have also been exploited in studies of proof theory, consistency and also complexity and decidability.

The necessity and possibility-like modal operators of the previously commented applications have a common core, around which we will center this chapter. Moreover, the high versatility of modal logics is one of the principal motivations behind the main topic of this doctoral dissertation. The addition of fuzzy characteristics to modal logics appears to be filled with several possible applications, allowing to study problems from the previously commented contexts without limiting the information to a classical setting. However, it is not a research line fully developed and many related problems are still unsolved.

In this chapter, we will overview the topics of modal logic that are relevant to the development of our work, in order to provide a solid framework over which we base following parts of the thesis. For a deeper presentation of this topic, we refer to some classical books on the matter, for instance [26, 27, 12]. We will begin by presenting the definitions and results for classical modal logic, with the two usual operators of necessity and possibility and then present an overview of the state of the art for what concerns modal expansions of fuzzy logics.

3.1 Classical modal logic

The main difference of modal logics with respect to the logics explained in the previous chapters is that the natural semantics of the former is not the algebraic one, but the so-called relational or Kripke semantics. Even though the historical development of modal logics begins with the syntax of these logics and later the semantics are discovered and studied we will present these ideas here the other way around. We think this gives a more natural idea of the motivations and behaviour of the modal logics and allows us to show the contents from the more general ones to the more specific ones. We think this approach allows the reader a clearer and more uniform comprehension on the topic.

The language of the modal logics considered in this work consists on a propositional language $\mathfrak{L}$ that adds two particular unary operations, $\Box$ and $\Diamond$ called modal operators, whose intuitive meaning is that of necessarily and possibly respectively. In the classical setting, we will consider the modal language given by $\mathfrak{L}_{\mathfrak{CM}} = \langle \vee/2, \neg/1, \bot\ /0, \Box/1\rangle$ with other usual propositional connectives defined from these ($\top \doteq x \vee \neg x$, $x \wedge y \doteq \neg(\neg x \vee \neg y)$, $x \to y \doteq \neg x \vee y$) and, in particular, $\Diamond x \doteq \neg\Box\neg x$. As usual, we will denote by Var the set of propositional variables (denumerable) used to build the set of formulas of the logic. As it is well known, classical propositional logic is complete with respect to the Boolean algebra of two elements. We will denote the (propositional) operations in this algebra by their respective symbols in the language, since this is intensively done in the literature and will clarify the notation.

Kripke semantics

Kripke semantics are developed in the 50s and allow to address formally the notion of local truth. They are based on graph-like structures that are independent from the language.

Definition 3.1. A ***Kripke or relational frame*** (usually called just Kripke frame) is a pair $\mathfrak{F} = \langle W, R\rangle$ where $W \neq \emptyset$ is a set, whose elements are called ***states*** of ***worlds*** of the frame and R is a binary relation on W called the ***accessibility relation*** of the frame. For $v, w \in W$ such that $\langle v, w\rangle \in R$, we write Rvw.

We will denote by KF the class of all Kripke frames. The use of the previous notion as a structure over which we define a logic is done by assigning to each state an evaluation that determines what propositions are true in it.

Definition 3.2. A ***Kripke model*** $\mathfrak{M}$ is a triple $\langle W, R, V\rangle$ such that $\langle W, R\rangle$ is a Kripke frame $\mathfrak{F}$ and V is a mapping from Var to W[1] called the ***evaluation of the model***. In this case, we say that $\mathfrak{M}$ is a model based on $\mathfrak{F}$.

[1]Observe this is equivalent to say that $V : Var \times W \to \{0, 1\}$, which gives a more natural idea on how to generalize this notion to a many-valued context.

We inductively (and uniquely) define the notion of a formula φ being ***satisfied*** or ***true*** in a Kripke model $\mathfrak{M}$ at state w, in symbols $\mathfrak{M}, w \Vdash \varphi$, as follows:

$$\begin{array}{rcll}
\mathfrak{M}, w \Vdash p & \text{iff} & w \in V(p), & \text{for all } x \in Var, \\
\mathfrak{M}, w \Vdash \perp & \text{iff} & never, & \\
\mathfrak{M}, w \Vdash \neg\varphi & \text{iff} & \text{not } \mathfrak{M}, w \Vdash \varphi, & \\
\mathfrak{M}, w \Vdash \varphi \vee \psi & \text{iff} & \mathfrak{M}, w \Vdash \varphi \text{ or } \mathfrak{M}, w \Vdash \psi, & \\
\mathfrak{M}, w \Vdash \Box\varphi & \text{iff} & \mathfrak{M}, v \Vdash \varphi, \text{ for all } v \in W \text{ such that } Rwv &
\end{array}$$

The class of all Kripke models will be denoted by KM.

For a set of formulas Γ and a state w of a model $\mathfrak{M}$, we write $\mathfrak{M}, w \Vdash \Gamma$ whenever $\mathfrak{M}, w \Vdash \gamma$ for all $\gamma \in \Gamma$. Moreover, we say that a Kripke model $\mathfrak{M}$ satisfies a formula φ (or a set of formulas Γ) and write $\mathfrak{M} \Vdash \varphi$ (resp. $\mathfrak{M} \Vdash \Gamma$) whenever it does so at any of its states. We say that a class of models satisfies a formula when each model in the class does so.

It is natural to see Kripke model as a frame and an evaluation with contingent information. It is interesting to define the notion of truth independently of this evaluation, as it is done in the algebraic setting and so focus on the more fundamental level of the frames. This is done through the notion of validity in a frame.

A formula φ is ***valid in a frame*** $\mathfrak{F}$ and we write $\mathfrak{F} \Vdash \varphi$ if φ is satisfied at every model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$.

The previous notion allows to semantically define two types of logical deduction. The natural idea of logical deduction implies that truth must be preserved from the premises to the consequence and this can be interpreted in two different ways. One is locally, that is, understanding truth is preserved at states of a model or frame (or a class of these). On the other hand, it is also possible to think of truth globally, being maintained in the whole model.

Formally, the two intuitive notions of deduction give rise to the following consequence relations, defined semantically.

Definition 3.3. Let C be a class of models and $\Gamma \cup \{\varphi\}$ be a set of formulas. Then we say that:

- φ is ***local consequence*** of Γ over C and write $\Gamma \Vdash_{\mathsf{C}} \varphi$ whenever for any model $\mathfrak{M}$ from C and for any state w in $\mathfrak{M}$, if $\mathfrak{M}, w \Vdash \Gamma$ then $\mathfrak{M}, w \Vdash \varphi$.
- φ is ***global consequence*** of Γ over C and write $\Gamma \Vdash^{g}_{\mathsf{C}} \varphi$ whenever for any model $\mathfrak{M}$ from C, if $\mathfrak{M} \Vdash \Gamma$ then $\mathfrak{M} \Vdash \varphi$.

It is easy to see that these two consequence relations are different. Indeed, consider the formulas φ and $\Box\varphi$. Clearly, $\varphi \Vdash^{g}_{KF} \Box\varphi$, while this is not true under the local consequence. This is in fact, the very essence behind the global deduction. On the other hand, for any class of Kripke models C, the theorems of the local and the global modal logics over C coincide. For this reason, sometimes modal logics are defined as this common set of theorems, but we will stick to the whole consequence relation definition for considering it more natural from a general point of view.

Syntactic level: the axiomatization problem

The previous notes comprehend the semantic approach to classical modal logics. However, it exists an important syntactic dimension for what concerns these logics, first from an historical point of view (modal logics arose in general in the syntactic side) and also for what concerns their possible finitary characterizations. A natural question that appears when modal logics are studied is determining which classes of frames have a corresponding logic that can be finitely axiomatizable, and the other way around, i.e., which axiomatic extensions have a relational semantics based on a class of frames whose structure has a good characterization.

We will refer to this problem as the *axiomatization problem*. It has been deeply studied and we will present here only some results that will be useful later on this thesis. Again, for a deeper study on this issues, we refer the interested reader to [26, 27, 12].

A first approach is that of fixing the axioms that hold in all the Kripke frames KF and the rules that are sound, respectively, in the local and global deductions over KF. Logics containing these are called ***normal modal logics***.

Definition 3.4.

- Let K be the logic defined by the extension of the classical propositional calculus (CPC) with the axiom $\mathsf{K}: \ \Box(p \to q) \to (\Box p \to \Box q)$ and the necessitation rule $\mathsf{N}_\Box$: from $\vdash \varphi$ infer $\vdash \Box\varphi$ for any formula φ.

- Let K^g be the logic defined from K by substituting the necessitation rule by $\mathsf{N}^{\mathsf{g}}_\Box : p \vdash \Box p$.

We will call ***normal modal logics*** those extending K (and validating the $\mathsf{N}_\Box$ rule) and ***normal global modal logics*** those extending K^g.

It is now of great interest to know how, given a normal modal logic N, to characterize -if it exists- the class of frames with respect to which this logic is complete.

Concerning K the usual approach is the definition of a special model $\mathfrak{M}^N$, called the ***canonical model of*** N, with respect to which the completeness always holds. In a certain sense, this model will play a similar role to that of the Lindembaum-Tarki algebra defined in Chapter 1. However, this process does not necessary provide a frame with the previous property.

The keystone that allows the use of the canonical model is the following lemma.

Lemma 3.5. *For any set of formulas* $\Gamma \cup \{\varphi\}$

$$\Gamma \vdash_K \varphi \iff \Gamma \cup Th(K) \vdash_{CPC} \varphi$$

where it must be understood that the set of propositional variables used in the right side is the extension of the one at the left with modal formulas that begin by $\Box$.

It is not hard to see how the previous result helps to prove completeness of K with respect to the intended semantics. If $\Gamma \nvdash_K \varphi$, it is enough to see if there exists a classical evaluation (i.e., in $\{0, 1\}$), satisfying the modal theorems (and the premises in Γ) but sending φ to 0.

For that, the canonical model has a very particular definition. It relies on the notion of maximally consistent theories of a logic L, that is, the sets of formulas that are L-consistent but such that any set of formulas *properly* containing it is L-inconsistent.

Definition 3.6. Let N be a normal modal logic. Then, the ***canonical model of*** N is the structure $\mathfrak{M}^N = \langle W^N, R^N, V^N \rangle$ defined by:

- W^N is the set of maximal theories of N,
- For $v, w \in W^N$, $R^N vw$ if and only if for any formula φ, if $\Box\varphi \in v$ then $\varphi \in w$,
- For any $p \in \mathit{Var}$, $V^N(p) = \{w \in W^N : \ p \in w\}$.

The so called Truth Lemma states that satisfaction at a state of this model is equivalent to membership (not only for variables, bot for all formulas). That is, for any formula φ and any $w \in W^N$, it holds that

$$\mathfrak{M}^N, w \Vdash \varphi \iff \varphi \in w.$$

From this result, it is not hard to prove that any normal modal logic is complete with respect to its canonical model.

For the case of the logic K, since it is sound with respect to the whole class of Kripke frames, it is easy ton conclude the completeness result: for any set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash_K \varphi \iff \Gamma \Vdash_{\mathsf{KF}} \varphi$$

For other cases, completeness with respect to the canonical model gives a semantical characterization of the logic (in terms of just one Kripke model), but it is not immediate to know with respect to which class of frames is the logic sound (so indirectly, how to characterize a class of frames with respect to which the logic is complete). In fact, there exist normal modal logics that are not complete with respect to any class of frames whatsoever. We will say that formula φ ***is canonical for a frame property*** P whenever the canonical model of any normal modal logic containing φ (in the sense that φ is a theorem of the logic) has property P and φ is valid in any class of frames with property P.

There are canonical modal formulas for some famous frame properties (see Table 3.1 below for a list of the most common ones). However, the general problem (i.e., given a particular class of frames C, generate the axioms that extend K in such a way that the resulting logic is complete with respect to C) is not always solvable.

There is, nevertheless, a partial solution to this problem that we will very briefly comment here: formulas belonging to a particular class named ***Sahlqvist***

Name	Axiom	Frame property
D	$\Box p \to \Diamond p$	Seriality ($\mathcal{D}$): $\forall v \exists w \; Rvw$
T	$\Box p \to p$	Reflexivity ($\mathcal{T}$): $\forall v \; Rvv$
B	$\Diamond \Box p \to p$	Symmetry ($\mathcal{B}$): $\forall v, w \; (Rvw \;\Rightarrow\; Rwv)$
5	$\Diamond \Box p \to \Box p$	Euclidean ($\mathit{5}$): $\forall v, w, u \; (Rvw \,\&\, Rvu \;\Rightarrow\; Rwu)$
4	$\Box p \to \Box\Box p$	Transitivity ($\mathit{4}$): $\forall v, w, u \; (Rvw \,\&\, Rwu \;\Rightarrow\; Rvu)$

Table 3.1: Common modal axioms

have a corresponding frame property expressible by a first-order condition on the states and relation, with respect to which they are canonical. The most well known modal extensions of K are done with axioms belonging to this class. Table 3.1 provides a list of some of these and their corresponding frame condition.

An interesting property of the modal logics obtained with the axiom schematas from table 3.1 is that they enjoy the ***finite model property***: they are complete with respect to classes of finite Kripke frames (that is, with finite universes). This is a consequence of the fact that any Kripke model $\mathfrak{M}$ from the associated semantics is ***witnessed***: for any formula φ and any world $w \in W$ such that $\mathfrak{M}, w \nVdash \Box\varphi$ there is $v \in W$ such that Rwv and $\mathfrak{M}, v \nVdash \varphi$. The finite model property allows, among other things, to state that the decidability of the satisfiability problem of K.

Algebraic semantics

Modal logics have also been studied from the algebraic semantics point of view (see for instance [93, 63, 83],...), and the relations obtained between the so called modal algebras and the relational semantics presented in this chapter has been called duality theory. ***Modal algebras*** arise as the algebraic semantics associated to the modal logics syntactically defined as extensions of the logic K. They turn out to be Boolean algebras endowed with an extra unary operation τ (which is the interpretation of $\Box$ in the algebra) that satisfies the modal axioms of the logic. In particular, for the minimum modal logic K, the operation τ just must enjoy two particular properties: it preserves the top element (i.e., $\tau 1 = 1$) and it distributes over the minimum (i.e., $\tau(a \wedge b) = \tau a \wedge \tau b$). It has been proved using Boolean algebrasIn particular using finite Boolean algebras, whose calculus can be done ad-hoc. that the local modal logic K is not algebraizable in the sense of Blok and Pigozzi, but nevertheless a characterization of the reduced matrices of the logic is given by Malinowski in [92].

Another algebraic study of modal logics that has been proved very fruitful is the relation existing between Kripke frames and some modal algebras. From a modal algebra $\mathbf{A}$, it is possible to build a Kripke frame $\mathbf{A}_+$, called the ***ultrafilter frame of*** $\mathbf{A}$, whose universe is the set of ultrafilters of $\mathbf{A}$ and where two states v, w are related if and only if for any $a \in A$, if $\tau a \in v$ then $a \in w$.

On the other hand, it is also possible to study the algebraic semantics starting not from a syntactically defined modal logic, but from the class of frames that define the logic semantically. Given a Kripke frame $\mathfrak{F} = \langle W, R\rangle$, we call the ***complex algebra*** of $\mathfrak{F}$ to the algebra $\mathfrak{F}^+$ with universe $\mathcal{P}(W)$ and operations $\cup$ (interpreting $\vee$), the complement operation for subsets of W ^{c} (i.e., $X^c = W \setminus X$, that interprets the $\neg$ operation of the language), W as the top element and the unary operation τ, that interprets the $\Box$ symbol, defined by

$$\tau X = \{y \in W : \ \forall x, \text{ if } xRy \text{ then } y \in X\}$$

It is proven that these algebras are modal algebras in the sense that they belong to the algebraic semantics of K and not only that: the ultrafilter frame and the complex algebra constructions preserve several properties from one side to the other. Moreover, strong results can be obtained using classical theorems like the Stone Representation Theorem (that states that the variety of Boolean algebras is generated by $\mathbf{2}$) or the Stone duality. The Jónsson-Tarski Theorem states that any modal algebra $\mathbf{A}$ can be embedded within the complex algebra $(\mathbf{A}_+)^+$ (the complex algebra built from the ultrafilter frame of $\mathbf{A}$). On the other hand, it is possible to check that given a Kripke frame $\mathfrak{F} = \langle W, R\rangle$, the ultrafilter frame built from its complex algebra, that is to say, $(\mathfrak{F}^+)_+$, coincides with the ultrafilter extension of $\mathfrak{F}$, i.e., the filter whose universe is the set of ultrafiters of W and where the relation is such that Rvw if and only if the union of the sets of states accessible (under R) from each state (from W) belonging to v is a subset of w.

All this machinery has been largely studied and exploited, also paying attention to the similarities of these structure with certain topological spaces. An important result is the Goldblatt-Thomason theorem [69], which states that a class of Kripke frames closed under ultrafilter extensions is modally definable if and only if it reflects ultrafilter extensions and is closed under generated subframes, homomorphic images and disjoint unions. Thanks to these kind of equivalence results, the axiomatization problem has undergone important advances and other topics on modal logic have also gained tools that simplify their study.

It was proven by Goldblatt that there is a duality[2] between modal algebras and Kripke frames and these has allowed to prove very important theorems of modal logic, some concerning the axiomatization problem (that is, of understanding which classes of frames give place to axiomatizable modal logics) and others referring to more abstract questions out of the scope of this work but of great importance to the development of modal logics.

3.2 Fuzzy modal logics

Researchers coming both from modal logic, algebraic logic and many-valued logics fields have studied the possibilities of expanding many-valued logics with modal-like operators (or endowing modal logics with many-valued characteristics). There exist several works on this matter, considering different kind of

[2] It is not on the scope of this work enter in discussion of this issues, but we just want to remark on this paragraph the great importance of the relations between the modal algebras and the Kripke Frames. For the interested reader we refer to [67, 68, 61].

modal operators and many-valued frames and models. Some approaches that do not fit the work presented in this dissertation are for instance [97] and [29], where the interpretation of the modalities or the underlying logics are out of our considerations.

The seminal works by Fitting [57, 58] are the first ones to interpret many-valued modal logics as logics arising from Kripke frames where the propositions and the accesibility relation are evaluated over many-valued algebras and the modal operators are a generalization of the classical ones.

Following this interpretation, we will focus here on the modal extensions of the *MTL* logics. For what concerns modal expansions of *MTL*-logics, the usual approach has been first understanding the semantic structures and from this knowledge, try to provide nice axiomatic systems for the corresponding logics.

The Kripke semantics built over fuzzy structures aims to be a natural generalization of the classical Kripke models presented in the previous section. The most general definitions at this respect, in the sense that they cope with *MTL*-algebras (even though only up to finite cardinality) were presented in [18], defining Kripke frames and models over bounded commutative residuated lattices. However, in that work the authors consider a language (the usual language of bounded residuated lattices) with a constant for each element of the algebra, expanded only by the modal $\Box$ operator. One could think that this is enough to address the whole set of modal notions (of necessity and possiblity), as it was in the classical case. However, we will see that in the many-valued context, in general, the two modal operators are no longer inter-definable. For this reason, since our objective is presenting the most general possible structures, we will consider the language expanded with both modal operators, as it is done for instance in [23].

The language of the many-valued modal logic that we will be using in the rest of this chapter is given by

$$\langle \&/2, \rightarrow/2, \wedge/2, \Box/1, \Diamond/1, \overline{0}/0\rangle.$$

Some extra (usual) operations are defined from these ones as usual (namely $\overline{1} \doteq \varphi \rightarrow \varphi$, $\neg\varphi \doteq \varphi \rightarrow \overline{0}$ and $\varphi \vee \psi \doteq ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi))$. Fm denotes the set of formulas constructed using this language.

MTL-based Kripke semantics

The natural intuition, after studying the semantics of the classical modal logics, is that both the relation and the evaluations of (classical) Kripke models are defined in terms of subsets because of reasons coming from a version of the Stone representation theorem for Boolean algebras: every Boolean algebra is isomorphic to a field of sets. This leads to thinking that in fact, the general definition of Kripke models is based on interpreting accesibility relations and evaluations into Boolean algebras, even though is equivalent to the one we defined in the previous section because of such theorem.

This seems a good enough motivation to generalize the notion of Kripke structures to the following ones.

Definition 3.7. Let $\mathbf{A}$ be a *MTL*-algebra. An **A-*Kripke frame*** (and in general, called ***many-valued Kripke frame***) is a pair $\mathfrak{F} = \langle W, R \rangle$ where $W \neq \emptyset$ is a set whose elements are called ***states*** of the frame and R is a binary relation on W evaluated in A, that is, $R\colon W \times W \to A$, called the ***accessibility relation***. We denote by $\mathsf{KF}(\mathbf{A})$ the class of $\mathbf{A}$-Kripke frames.

If $R[W \times W] \subseteq \{0, 1\}$, the frame is said to be ***crisp***. If $R[W \times W] \subseteq Idp(\mathbf{A}) \coloneqq \{a \in A : a \odot a = a\}$, the frame is said to be ***idempotent***. We denote these two classes respectively by $\mathsf{KF}_C(\mathbf{A})$ and $\mathsf{KF}_I(\mathbf{A})$.

The last considerations from the previous definition allow us to address more clearly different properties of these kind of frames. We denote by $\mathsf{KF}(\mathsf{C})$ the class of frames based on a class of *MTL*-algebras C and respectively by $\mathsf{KF}_C(\mathsf{C})$ and $\mathsf{KF}_I(\mathsf{C})$ to the crisp and idempotent frames based on the algebras from C. When C is the whole class of *MTL*-algebras, we refer to this class simply as KF. It is clear that, for an arbitrary class of *MTL*-algebras C, $\mathsf{KF}_C(\mathsf{C}) \subseteq \mathsf{KF}_I(\mathsf{C}) \subseteq \mathsf{KF}(\mathsf{C})$ and that these three cases differ when there is at least one algebra $\mathbf{A} \in \mathsf{C}$ where $\{0, 1\} \subsetneq \{a \in A : a \odot a = a\} \subsetneq A$.

The meaning of endowing a many-valued Kripke frame with an evaluation, that is, defining a many-valued Kripke model arises intuitively as a generalization of the classical Kripke-models.The only consideration to take into account is to choose the same algebra to evaluate both the accessibility relation and the formulas at each world.

Definition 3.8. Let $\mathbf{A}$ be a *MTL*-algebra. An **A-*Kripke model*** (and in general, called ***many-valued Kripke model***) is a triple $\langle W, R, e \rangle$ such that $\langle W, R \rangle$ is a $\mathbf{A}$-Kripke frame $\mathfrak{F}$ and e is a mapping from $W \times Var$ to A called the ***evaluation of the model***. In this case, we say that $\mathfrak{M}$ is a model based on $\mathfrak{F}$.

In the classical case it makes sense to talk about satisfiability of a formula in a world. Now, this notion must be adapted to the many-valued setting and so what is natural is to consider *the degree of truth* taken by a formula in a world in a model. A natural way to do this is to inductively extend the evaluation e, whose universe is just the set of variables, to a mapping $e' : W \times Fm \to A$. The inductive steps for what concerns the non-modal operators arise naturally, but the modal ones could be defined in different ways.

The definition we give here has been generally accepted by the community, mainly for two reasons. On the one hand, it is clearly a generalization of the classical case: it coincides with the usual classical definitions when the frame is crisp and the algebra of evaluation is $\mathbf{2}$. On the other hand, it behaves as a restriction of the most usual definitions of the predicate models of first order many-valued logics. We will not give details concerning this relation, but it is well known that modal logics can be seen as a fragment of predicate logics and this is not an exception in the many-valued logics case. For this reason, defining modal logics according to the generally-accepted first order definitions seems the natural way to proceed.

Then, let $\mathbf{A}$ be a *MTL*-algebra, $\mathfrak{M}$ a $\mathbf{A}$-Kripke model and w a state from $\mathfrak{M}$. We inductively define $e(w, \varphi)$, the truth-degree of a formula φ in $\mathfrak{M}$ at state w

as follows:

$$
\begin{aligned}
e(w,p) &= e(w,p), \text{ for all } p \in \mathit{Var} \\
e(w,\overline{0}) &= \overline{0}^{\mathbf{A}} \\
e(w,\varphi \star \psi) &= e(w,\varphi) \star^{\mathbf{A}} e(w,\psi), \text{ for } \star \text{ being } \&, \rightarrow \text{ and } \wedge \\
e(w,\Box\varphi) &= \begin{cases} \inf_{v\in W}\{Rwv \Rightarrow e(v,\varphi)\}, & \text{if this value exists in } A \\ \text{undefined} & \text{otherwise} \end{cases} \\
e(w,\Diamond\varphi) &= \begin{cases} \sup_{v\in W}\{Rwv \odot e(v,\varphi)\} & \text{if this value exists in } A \\ \text{undefined} & \text{otherwise} \end{cases}
\end{aligned}
$$

This definition of truth-degree is in some sense disturbing: there exist models where the evaluation of certain formulas at some worlds is left undefined. This is a well-known problem coming from the predicate fuzzy logics and the usual (but not for this reason any more elegant) method to fix it is to consider only the so called ***safe*** models: those for which all the truth values for all the formulas at all the worlds are defined. Clearly, complete algebras (that is, those for which the infimum and the supremum of any subset of its universe always exist) generate safe models, but it is not clear how to characterize other safe models, making this concept uncomfortable to work with. Nevertheless, to the best of our knowledge, this approach is the only one that has been followed up to now and it does not seem clear how to avoid the reference to safe models.

Thus, if we do not specify the contrary, whenever we talk about a many-valued Kripke model it shall be assumed it is a safe model. Given a MTL-algebra $\mathbf{A}$, we denote by $\mathsf{KM}(\mathbf{A})$ the class of safe $\mathbf{A}$-Kripke models. Similarly, for a class C of MTL-algebras $\mathsf{KM}(\mathsf{C})$ denotes the union of all the $\mathsf{KM}(\mathbf{A})$ classes for each $\mathbf{A} \in \mathsf{C}$ and if this class consists of all the MTL-algebras, we will simply write KM.

For what concerns the notions of (top) satisfiability, validity and deduction, the definitions coincide with those from the classical modal logic setting, only taking into account that the considered models must be safe.

The notion of (top) satisfiability at a state is naturally inherited from the degree of truth at that state. We write $\mathfrak{M}, w \Vdash \varphi$ if and only if $e(w,\varphi) = \overline{1}^{\mathbf{A}}$ and for a set of formulas Γ, $\mathfrak{M}, w \Vdash \Gamma$ if and only if $\mathfrak{M}, w \Vdash \gamma$ for each $\gamma \in \Gamma$. Similarly, the notion of validity in frames is very similar to the classical one. A formula φ is ***valid at state w in a many-valued Kripke frame*** $\mathfrak{F}$, and we write $\mathfrak{F}, w \Vdash \varphi$, when φ is true at state w in any (safe) Kripke model based on $\mathfrak{F}$. A formula φ is valid in a frame $\mathfrak{F}$ ($\mathfrak{F} \Vdash \varphi$) if it is valid at every state of $\mathfrak{F}$. Finally, a formula φ is valid in a class of frames F and we write $\mathsf{F} \Vdash \varphi$ if it is valid in very frame of the class.

We define the notion of logical consequence for a modal logic arising from a class of models in a very similar way to the one presented in the previous section.

Definition 3.9. Let C be a class of many-valued models and $\Gamma \cup \varphi$ be a set of formulas. Then we say that:

- φ is ***local consequence*** of Γ over C and write $\Gamma \Vdash_{\mathsf{C}} \varphi$ whenever for all model $\mathfrak{M}$ from C and for all state w in $\mathfrak{M}$, if $\mathfrak{M}, w \Vdash \Gamma$ then $\mathfrak{M}, w \Vdash \varphi$.

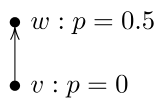
- φ is ***global consequence*** of Γ over C and write $\Gamma \Vdash^{g}_{\mathsf{C}} \varphi$ whenever for all models $\mathfrak{M}$ from C, if $\mathfrak{M} \Vdash \Gamma$ then $\mathfrak{M} \Vdash \varphi$.

Remarkable differences with the classical modal logic

The logics with deduction relations $\Vdash_{\mathsf{C}}$ and $\Vdash^{g}_{\mathsf{C}}$ for a class C of many-valued Kripke models have important differences with respect to their classical counterparts, that is, when C is a class of classical Kripke models. We proceed to detail the most remarkable ones, illustrating in some sense why the study of many-valued modal logics shall be done, in many cases, very differently to the one done in the classical modal logic case. Later, we present a list with some formulas and deductions that are (are not) valid in the logics arising respectively from the crisp, idempotent and the whole class of many-valued Kripke frames, namely $\mathsf{KF}_C(\mathsf{MTL})$, $\mathsf{KF}_I(\mathsf{MTL})$ and $\mathsf{KF}(\mathsf{MTL})$.

The fist observation, which we already remarked when defining the language of the many-valued modal logics, is that the $\Diamond$ and $\Box$ operators are no longer interdefinable with the same relation we had in the classical case. Indeed, it suffices to think in a *MTL*-algebra with a non-involutive negation, for instance, the Gödel standard algebra $[\mathbf{0},\mathbf{1}]_{\mathbf{G}}$ defined in the previous chapter.

It is not hard to define a $[\mathbf{0},\mathbf{1}]_{\mathbf{G}}$-Kripke model where, at some world v, $e(v, \Diamond\varphi) \neq e(v, \neg\Box\neg\varphi)$ (and similarly for what concerns $\Box\varphi$ and $\neg\Diamond\neg\varphi$). Observe that in the model from the right side, where $W = \{v, w\}$ and $R = \{\langle v, w\rangle\}$, and $e(w, p) = 0.5$. Then, $e(v, \Diamond p) = 0.5$, but clearly $e(v, \neg\Box\neg p) = 1$.



There exist some cases where still the $\Box$ and the $\Diamond$ operator are interdefinable, for instance, the modal expansions of logics with an involutive negation (an example of which is the Łukasiewicz logic), but as we have remarked above, this is not true in general. Even if some of the differences between many-valued modal logics and classical ones seem to be inherent to the fuzziness of the accessibility relation, the interdefinability of $\Diamond$ and $\Box$ keeps non being valid over the restricted class of many-valued models build over Kripke frames with crisp accessibility relation. A consequence of this is that modal expansions of many-valued logics comprehend now, at least three logics that do not necessarily coincide: the $\Box$ expansion, the $\Diamond$ expansion and the expansion with both $\Box$ and $\Diamond$ operations at the same time.

Another notable difference is that the well known axiom $\mathsf{K} : \Box(p \to q) \to (\Box p \to \Box q)$ is no longer valid in the whole class of many-valued frames, that is, in the minimum many-valued modal logic.

As an example consider the Kripke model from the right, defined over an arbitrary standard *MTL*-algebra (i.e., with $[0,1]$ as universe), where the universe is $W = \{v, w\}$, the accessibility relation is given by $R(v,w) = 0.5$ and $R(v,v) = R(w,w) = R(w,v) = 0$ and evaluated with $e(w,p) = 0.2$ and $e(w,q) = 0.15$ (and $e(v,p) = e(v,q) = 0$). In this model, $e(v, \Box(p \to q)) = 1$, but $e(v, \Box p) = 0.4$ and $e(w, \Box q) = 0.3$ (so clearly, $e(v, \Box p \to \Box q) < 1$).

$$\begin{tikzcd} \bullet\, w : p = 0.2, q = 0.1 \\ \bullet\, v : p = 0, q = 0 \arrow[u, "0"] \end{tikzcd}$$

The K validity is directly linked to the kind of frames that defines the logic: it was proven in [18] that for a class C of idempotent Kripke structures (that is, where each accessibility relation is evaluated on the idempotent elements of the respective algebra), K is a valid formula. For what concerns non idempotent frames, a result that still holds in the local deduction (and also in the global one) is that if $\varphi \to \psi$ is a theorem then so is $\Box\varphi \to \Box\psi$. In the global logic, it is easy to see that $\varphi \to \psi \Vdash^g_{\mathsf{C}} \Box\varphi \to \Box\psi$.

The third point we want to turn our attention to is the fact that the logics arising from $\Vdash_{\mathsf{KF}}$ and $\Vdash^g_{\mathsf{KF}}$ do no enjoy in general the finite model property. Indeed, it is seen already in [17] that $\Box\neg\neg\varphi \to \neg\neg\Box\varphi$ is valid in any many-valued structure with a finite universe evaluated over the over the standard Gödel algebra, while it is easy to check that it is not a theorem of the modal expansion of the Gödel logic. For instance, let $\mathfrak{M}$ be the $[\mathbf{0},\mathbf{1}]_{\mathbf{\Pi}}$-Kripke model with universe $\mathbb{N}$ and $R = \{\langle 0, n\rangle : n \geq 1\}$, and with evaluation $e(n{+}1, p) = \frac{1}{n+1}$. Clearly $e(0, \neg\Box p) = 1$ (because $e(0, \Box p) = 0$), but since for all $n \geq 1$ it holds that $e(n,p) > 0$, then $e(n, \neg p) = 0$ and so $e(0, \Diamond\neg x) = 0$ (and then $e(0, \neg\neg\Diamond\neg p) = 0$ too). Observe that this model belongs in fact to the class of crisp Kripke models, so as a consequence we immediately know that the finite model property does not hold either for the modal logics based on the classes KF_C and KF_I. This makes difficult knowing whether many-valued modal logics are decidable or not. In what follows we will see some results on this issue that have been developed in the recent literature, but the general problem remains open.

The previous differences between classical and many-valued modal logics witness the deep structural and even philosophical changes when adding many-valuedness to modal logics. As it is expected, many other characteristics of the two families of logics also differ. We now present some formulas and deductions that are valid in the many-valued modal logics, some of them maintained from the classical modal logic and others which are clearly restrictions of more general characteristics from the classical setting. We will include here some results that use a language expanded with constant symbols, denoted by $\overline{c}, \overline{d}, \ldots$, since our main research work is done over this kind of extended language too. The interpretation of these symbols on a $\mathbf{A}$-Kripke model for $\mathbf{A}$ a MTL-algebra with constant operations is the natural one: $e(v, \overline{c}) = \overline{c}^{\mathbf{A}}$ for all $v \in W$ and each constant symbol considered.

All the following assertions are proved in [18].

- The following formula schemata are valid in all the many-valued structures:
  - $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$,
  - $\neg\neg\Box p \to \Box\neg\neg p$,
  - $\Box(\overline{c} \to p) \leftrightarrow (\overline{c} \to \Box p)$,
- The following formula schemata are valid in all idempotent structures:
  - $\mathsf{K} : \Box(p \to q) \to (\Box p \to \Box q)$,
  - $(\Box p \,\&\, \Box q) \to \Box(p \,\&\, q)$,
- The following formula schemata are valid in all crisp structures:
  - $\Box\overline{0} \vee \neg\Box\overline{0}$,
  - $\Box\overline{0} \vee (\Box\overline{c} \leftrightarrow \overline{c})$.
- The following deductions are valid for C being all the many-valued structures :
  - N : If φ is a theorem of $\Vdash_{\mathsf{C}}$, then so is $\Box\varphi$,
  - $\mathsf{N}^g : p \Vdash^g_{\mathsf{C}} \Box p$,
  - If $\varphi \to \psi$ is a theorem of $\Vdash_{\mathsf{C}}$, then so is $\Box\varphi \to \Box\psi$,
  - $\varphi \to \psi \Vdash^g_{\mathsf{C}} \Box\varphi \to \Box\psi$,
  - If $p \to \Gamma \Vdash_{\mathsf{C}} p \to \varphi$ for p propositional variable not appearing in $\Gamma \cup \{\varphi\}$ then $\Box\Gamma \Vdash_{\mathsf{C}} \Box\varphi$,
- The following deductions are valid for C being the class of crisp structures:
  - $\Gamma \Vdash_{\mathsf{C}} \varphi$ implies that $\Box\Gamma \Vdash_{\mathsf{C}} \Box\varphi$,
  - $\Gamma \Vdash^g_{\mathsf{C}} \varphi$ implies that $\Box\Gamma \Vdash^g_{\mathsf{C}} \Box\varphi$,

State of the art

We will recall here the most important studies and results obtained in the field of modal expansions of *MTL*-logics. The sketch on the state of the art is presented mainly following the thematic development, in order to give an idea of the results that have been obtained the different lines of work concerning modal expansions of *MTL* logics.

Along this chapter, we are not going to detail the methods used in each one of the presented works to prove the completeness results we are listing for the sake of brevity. Nevertheless, we will remark that in the works we are about to comment, the approach followed for proving completeness of their respective axiomatizations with respect to the corresponding modal logics follows, in more or less direct ways, the canonical model construction. Clearly each case has its own particularities and techniques (some of which we will remark in order to show the common points and the differences of the work we present in Chapter 6), but some ideas can be seen in common and, in fact, as generalizations of points commented in the proof of completeness of the classical modal logic. In the first place, corresponding versions of the Lemma 3.5 are valid in the new contexts, considering the many-valued propositional logic that is being modally expanded instead of the classical propositional calculus. This allows to use the canonical model notion, taking into account that the states are now the evaluations in this new algebra (or class of algebras), instead that the original maximal theories (that can be seen as evaluations in $\{0, 1\}$). The core of the problem is turned to prove the truth lemma for the new canonical model where the different techniques, corresponding to each particular logic, are exploited.

For what concerns each one of the particular cases studied, we present below a list with the main results found in the literature.

Łukasiewicz Logics

The main theoretical results concerning modal expansion of Łukasiewicz logics have been done by Hansoul and Teheux ([116, 117, 78]), and focus on the study of the logic arising from crisp Kripke frames evaluated over both finite subalgebras of $[\mathbf{0}, \mathbf{1}]_{\text{Ł}}$ and the standard Łukasiewicz algebra itself. It is remarkable that this is the first case where infinitely-valued algebras are considered. They study the $\Box$ fragment, but given that the Łukasiewicz negation is involutive, the three modal expansions of Ł_n and Ł logics, that is, the expansions with $\Box$ or $\Diamond$ separatedly, or with both, are mutually definable.

Apart from studying the usual axiomatization problem they also present a new kind of Kripke models, based on the interpretation over a finite subalgebra of $[\mathbf{0}, \mathbf{1}]_{\text{Ł}}$, but also allowing the limitation of the evaluations on each world to possibly different subalgebras of the original one. Moreover, they perform a deep study concerning the duality relation between the complex algebras (those arising from the Kripke models evaluated over finite MV-chains) and the canonical models, obtained applying the canonical-model construction to the Lindembau-Tarski algebras of formulas. This is done aiming at solving the axiomatization problem (that of deciding which classes of frames have a corresponding axiomatic system) and in their works the authors present a version of the Sahlqvist results in the Łukasiewicz context.

Some of the results presented in Hansoul and Teheux works, studying the duality relation between Łukasiewicz Kripke models and modal algebras, are out of the scope of this dissertation so we will not detail them here. Concerning the topics related to our research, the main results from their works are the axiomatizations of the local modal logics built over crisp Kripke frames evaluated over the finite subalgebras of $[\mathbf{0}, \mathbf{1}]_{\text{Ł}}$ and also the local modal logic of the same frames evaluated over $[\mathbf{0}, \mathbf{1}]_{\text{Ł}}$.

It is remarkable that in the standard Łukasiewicz algebra, for each $q \in [0, 1]_{\mathbb{Q}}$ there exists a propositional formula in one variable τ_q that characterizes the interval $[q, 1]$, i.e., such that $\tau_q(a) = 1$ if $a \in [q, 1]$ and $\tau_q(a) < 1$ otherwise (see for instance [104]). Moreover, if the algebra has a finite universe, this formula strongly characterizes the interval, i.e., it holds that the formula $\tau_q(a) \vee \neg\tau_q(a)$ is valid for each a in $\mathbf{A}$. This fact is one of the keystones used by the authors to prove their results concerning the modal expansions Łukasiewicz logics. It is remarkable that, whereas the case of Kripke models evaluated over finite algebras relies on the fact that the logics $Ł_n$ are semantically finitary (and so naturally strongly standard complete), in the the infinite-valued case an infinitary rule is added in order to get strong completeness at propositional level. The authors also present a class of algebras named many-valued modal algebras, that are MV-algebras expanded with one unary operator that satisfies the equations coming from the axiomatic system.

The axiomatic systems presented in the previously commented works are the following ones.

In [117, 78], the authors study also the local modal expansions, considering a crisp accesibility relation, of the Łukasiewicz logics. Let $Ł_n$ be the usual Hilbert style calculus of the n-valued Łukasiewciz logic and $\mathbf{Ł_n}$ its corresponding algebra. Define $MŁ_n$ by extending $Ł_n$ with

$$
\begin{array}{rcl}
\mathsf{K} & : & \Box(p \to q) \to (\Box p \to \Box q) \\
\text{axioms:} & & \Box(p \oplus p) \leftrightarrow (\Box p \oplus \Box p) \\
 & & \Box(p \odot p) \leftrightarrow (\Box p \odot \Box p) \\
 & & \Box(p \oplus p^m) \leftrightarrow (\Box p \oplus (\Box p)^m) \text{ for each positive integer } m \\
\mathsf{N}_\Box & : & \text{for all formula}\varphi,\ \text{from } \vdash \varphi \text{ infer } \vdash \Box\varphi
\end{array}
$$

Then, for any set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash_{MŁ_n} \varphi \iff \Gamma \Vdash_{\mathsf{KF}_C(\mathbf{Ł_n})} \varphi$$

On the other hand, the axiomatization of the local modal logic arising from the class of crisp Kripke frames evaluated over the standard Łukasiewicz algebra $[\mathbf{0},\mathbf{1}]_{\mathbf{Ł}}$ is given by the extension of the Hilbert style calculus of the Łukasiewciz logic, Ł, with the same modal axioms and rules used in the finite case plus the infinitary rule

$$\frac{p \oplus p^m \text{ for all } m \text{ possitive integer}}{p}$$

<u>Logics based on finite bounded commutative RLs with canonical constants</u>

In [18], the authors study the problem of finding a Hilbert-style axiomatization for the global and local modal logics with only $\Box$ that arise, respectively, from the classes of crisp, idempotent and unrestricted $\mathbf{A}$-Kripke frames, for $\mathbf{A}$ being a finite bounded commutative residuated lattice endowed with a canonical constants (i.e., a constant symbol for each element of the universe interpreted by its value, that is for each $a \in A$ there is $\overline{a}$ in the language with $\overline{a}^{\mathbf{A}} = a$). The finite MTL-algebras with canonical constants fall in this class. In this article, finite algebras are used mainly because this ensures that the non-modal propositional logic is finitary, which plays an important role. Moreover, this allows to have canonical constant symbols without resorting to a non-denumerable language, which can bring complications for what respects some classical and often used results that are only proved for denumerable languages. Moreover, the use of these constants is said to be difficult to overcome and are shown several results illustrating the expressive power of this expanded language.

The main results from this publication are the successful axiomatizations of the local logics arising, respectively, from $\mathsf{KF}_I(A)$ and $\mathsf{KF}(A)$, for $\mathbf{A}$ residuated lattice as above. For some particular algebras (those that have a unique coatom), they also present an axiomatization for the global and local logics arising from $\mathsf{KF}_C(A)$. Their approach is based on translating the deductions into theorems, and then prove completeness of this set with respect to the theorems in the semantic counterpart. On the other hand, the authors also present an axiomatization for both the local and global modal logics arising from the classes of frames $\mathsf{KF}_C(A)$ and $\mathsf{KF}(A)$ for $\mathbf{A}$ being a finite MV-chain, in this case *without* considering constants in the language.

A remarkable observation concerning constants is shown in Proposition 4.3 from [18]. It states that given a residuated lattice $\mathbf{A}$ as above (with canonical constants!) it is possible to prove that for two $\mathbf{A}$-Kripke models $\mathfrak{M} = \langle W, R, e\rangle$ and $\mathfrak{M}' = \langle W', R', e'\rangle$ and two states $w \in W, w' \in W'$, the mappings $e(w)\colon Fm \to A$ and $e'(w')\colon Fm \to A$ (i.e., $e(w)(\varphi) = e(w, \varphi)$ and the same for e') coincide (that is, $e(w)(\varphi) = e'(w')(\varphi)$ for each formula φ) if and only if they evaluate to $\overline{1}^{\mathbf{A}}$ exactly the same formulas. It is very interesting that the previous statement holds *without using the canonical constants* if $\mathbf{A}$ is a subalgebra of the standard Łukasiewicz algebra, as we remarked in the previous point, which allows to study the modal expansions of these logic without adding truth constants.

In Table 3.2 we present the axiomatic systems and the corresponding relational semantics definition of the logics studied in [18] (recall that the language it that of the bounded commutative residuated lattices with the $\Box$ operator). We let $H^c_{\mathbf{A}}$ denote an axiomatic calculus (that is, axiom schematas plus deduction rules) for the non-modal logic defined over the finite bounded commutative residuated lattice $\mathbf{A}^c$ with canonical constants.

For what concerns finite MV-chains, let $Ł_n$ be the usual Hilbert style calculus of the n-valued Łukasiewciz logic, $\mathbf{Ł_n}$ its corresponding algebra and $\mathsf{R_a}$, for each $a > 0$ in $Ł_n$ (that is, $\frac{i}{n-1}$ for $1 \leq i \leq n-1$) the following rule:

$$\mathsf{R_a}: \ \frac{(\tau_{a_2 \odot b}(p_2) \wedge \cdots \wedge \tau_{a_n \odot b}(p_n)) \to \tau_{a \odot b}(p) \text{ for each } b \in Ł_n, b > \neg a}{(\tau_{a_2}(\Box p_2) \wedge \cdots \wedge \tau_{a_n}(\Box p_n)) \to \tau_a(\Box p)}$$

where $a_i = \frac{i-1}{n-1}$ (and τ_a denotes the characteristic formula of $[a, 1]$). Then, the results obtained about axiomatizations of the modal expansions of L_n are shown in Table 3.3.

Gödel Logic

Works studying the modal expansions of Gödel logoc are mainly done by Caicedo and Rodriguez. In [24] the authors address the axiomatization of the

$H^c_{\mathbf{A}}$ **axiomatic calculus plus:**	**Complete with respect to:**
$\Box\overline{1}$ $(\Box p \wedge \Box q) \to \Box(p \wedge q)$ $\Box(\overline{a} \to p) \leftrightarrow (\overline{a} \to \Box p)$ from $\vdash \varphi \to \psi$ infer $\vdash \Box\varphi \to \Box\psi$	local modal logic arising from $\mathbf{A}^c$-Kripke models
$\Box\overline{1}$ $(\Box p \wedge \Box q) \to \Box(p \ \wedge q)$ $\Box(\overline{a} \to p) \leftrightarrow (\overline{a} \to \Box p)$ from $\vdash \varphi \to psi$ infer $\vdash \Box\varphi \to \Box\psi$ $(\Box p \,\&\, \Box p) \to \Box(p \,\&\, p)$	local modal logic arising from $\mathbf{A}^c$-Kripke models with an involutive accessibility relation
$\Box\overline{1}$ $(\Box p \wedge \Box q) \to \Box(p \wedge q)$ $\Box(\overline{a} \to p) \leftrightarrow (\overline{a} \to \Box p)$ $\Box(\overline{k} \vee p) \to \overline{k} \vee \Box p$ (k the coatom in $\mathbf{A}$) from $\vdash \varphi \to \psi$ infer $\vdash \Box\varphi \to \Box\psi$	local modal logic arising from $\mathbf{A}^c$-Kripke models with a crisp accessibility relation
$\Box\overline{1}$ $(\Box p \wedge \Box q) \to \Box(p \wedge q)$ $\Box(\overline{a} \to p) \leftrightarrow (\overline{a} \to \Box p$ $\Box(\overline{k} \vee p) \to \overline{k} \vee \Box p$ (k is the coatom in $\mathbf{A}$) $p \to q \vdash \Box p \to \Box q$	global modal logic arising from $\mathbf{A}^c$-Kripke models with a crisp accessibility relation

Table 3.2: Finite bounded commutative RL modal expansions ([18])

$Ł_n$ **axiomatic calculus plus:**	**Complete with respect to:**
$\Box\overline{1}$ $(\Box p \wedge \Box q) \to \Box(p \ \wedge q)$ from $\varphi \to \psi$ infer $\vdash \Box\varphi \to \Box\psi$ $\mathsf{R_a}$ for each $a > 0$ in $Ł_n$	local modal logic arising from $Ł_n$-Kripke models
$\Box\overline{1}$ $(\Box p \wedge \Box q) \to \Box(p \ \wedge q)$ from $\vdash \varphi \to \psi$ infer $\vdash \Box\varphi \to \Box\psi$ $\mathsf{R_a}$ for each $a \in Ł_n \setminus \{0\}$ $\tau_a(\Box p) \to \Box\tau_a(p)$ for each $a \in Ł_n \setminus \{0\}$	local modal logic arising from $Ł_n$-Kripke models with a crisp accessibility relation
$\Box\overline{1}$ $(\Box p \wedge \Box q) \to \Box(p \ \wedge q)$ $p \to q \vdash \Box p \to \Box q$ $\mathsf{R_a}$ for each $a \in Ł_n \setminus \{0\}$ $\tau_a(\Box p) \to \Box\tau_a(p)$ for each $a \in Ł_n \setminus \{0\}$	global modal logic arising from $Ł_n$-Kripke models with a crisp accessibility relation

Table 3.3: Finite MV-chains modal expansions ([18])

logics arising from the class of Kripke frames where both propositions at each world and the accessibility relation of the models are infinitely valued in the standard Gödel algebra $[\mathbf{0},\mathbf{1}]_{\mathbf{G}}$. They propose strongly complete axiomatizations for the $\Box$-fragment and the $\Diamond$-fragment of the resulting minimal logic, proving also that the $\Box$-fragment arising from the classes of crisp, idempotent and unrestricted $[\mathbf{0},\mathbf{1}]_{\mathbf{G}}$-valued Kripke models coincides. On the other hand, they prove that the $\Box$-fragment does not enjoy the finite model property, while the $\Diamond$-fragment does so.

Their studies show a common point that seems to appear along the different works concerning completeness proofs for the modal expansions of other many-valued logics. This is the strong standard completeness of the underling propositional logic. The approach for proving completeness passes, up to now,through the use of many-valued versions of the Lemma 3.5. Then, if the propositional logic is not strongly complete, it is no possible to go further after using this kind of lemma. In the particular case of the Gödel logic, not only the logic is strongly complete with respect to the standard algebra, but it also preserves the order relation (in the sense that $\Gamma \vdash_G \varphi$ if and only if $\Gamma \models^{\leq}_{[\mathbf{0},\mathbf{1}]_{\mathbf{G}}} \varphi$), and this fact is intensively used in the completeness proof. In particular, in the proof of the Truth Lemma of the corresponding canonical model, it is possible to present a constructive solution thanks to the order preserving character of the logic and to the nice behaviour of the endomorphisms in the standard Gödel algebra. That is, resorting to those tools, for each state and each modal formula $\mathsf{M}\varphi$ evaluated to a value α and each $\epsilon > 0$ the authors construct a a successor state in the canonical model that sends φ to a distance below ϵ from α. Sadly, it seems unlikely this reasoning can work for other logics, that in general do not enjoy any of these two properties.

The axiomatic systems commented are shown in the Table 3.4.

On the other hand, in [23], the same authors study the expansion of Gödel logic with the two modal operators $\Box$ and $\Diamond$ at the same time, and prove that this logic is equivalent to the Fischer-Servi intuitionistic modal logic plus the prelinearity axiom (that comes from the axiomatic system of Gödel logic). They also briefly develop the relation between modal algebras arising from the so-called bi-modal Gödel Kripke models and the canonical models coming from the Lindembaum-Tarski algebras. Once again, the main ideas used in the completeness proof resort to the methods used in the "mono"-modal expansions of Gödel logic presented in [24], which are hardly translatable to other frameworks. The axiomatic system presented in [95] is shown in Table 3.4.

A more recent work concerning the modal expansion of the Gödel logic has been done by again Caicedo and Rodriguez in collaboration with Metcalfe and Roggers, see [21] and [95]. The main contribution of this paper is to establish the decidability of validity (and this means deciding whether a certain formula is or is not a theorem of the logic) on in the class of Kripke models evaluated over the standard Gödel algebra (with also fuzzy relations) and in the crisp models from the previous class. For doing so they provide an alternative Kripke semantics for these logics that have the same valid formulas as the original semantics, but also

G **Hilbert style calculus plus:**	**Complete with respect to:**
$\mathsf{K}: \ \Box(p \to q) \to (\Box p \to \Box q)$ $\mathsf{Z}_\Box: \ \neg\neg\Box p \to \Box\neg\neg p$ $\mathsf{N}_\Box$: from $\vdash \varphi$ infer $\vdash \Box\varphi$	(on the $\Box$-language) local modal logic arising from $[\mathbf{0},\mathbf{1}]_{\mathbf{G}}$-Kripke models **which coincides with** the one from $[\mathbf{0},\mathbf{1}]_{\mathbf{G}}$-Kripke models with crisp accessibility relation
$\mathsf{D}_\Diamond: \ \Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$ $\mathsf{Z}_\Diamond: \ \Diamond\neg\neg p \to \neg\neg p$ $\mathsf{F}_\Diamond: \ \neg\Diamond\overline{0}$ $\mathsf{N}_\Diamond$: from $\vdash \varphi \to \psi$ infer $\vdash \Diamond\varphi \to \Diamond\psi$	(on the $\Diamond$-language) local modal logic arising from $[\mathbf{0},\mathbf{1}]_{\mathbf{G}}$-Kripke models
$\mathsf{K}: \ \Box(p \to q) \to (\Box p \to \Box q)$ $\mathsf{D}_\Diamond: \ \Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$ $\mathsf{F}_\Diamond: \ \neg\Diamond\overline{0}$ $\mathsf{FS1}: \ \Diamond(p \to q) \to (\Box p \to \Diamond q)$ $\mathsf{FS2}: \ (\Diamond p \to \Box q) \to \Box(p \to q)$ $\mathsf{N}_\Box$: from $\vdash \varphi$ infer $\vdash \Box\varphi$ $\mathsf{N}_\Diamond$: from $\vdash \varphi \to \psi$ infer $\vdash \Diamond\varphi \to \Diamond\psi$	(on the language with $\Box$ and $\Diamond$) local modal logic arising from $[\mathbf{0},\mathbf{1}]_{\mathbf{G}}$-Kripke models

Table 3.4: Gödel modal expansions ([24] and [95])

admit some kind of finite model property. The key idea of the new semantics is to restrict evaluations of modal formulas at a given world to a particular finite set of truth values.

Even though it does not seem clear if the ideas appearing at these papers can be translated to other many-valued logics (once again, the nice behaviour of Gödel endomorphisms is used), the importance of the results presented in [21] and [95] and the idea of working with an alternative semantics to approach the problem of decidability seems to us very interesting. It is the main article concerning decidability on some many-valued modal logic (the modal expansion of Gödel logic), together with some few works concerning decidability on some particular expansions [3] and approaches the problem with new methods that could bring some light to the more general context.

[3] For instance, in [25] it is proven the decidability of the problem of positive-satisfiability or 1-validity in a description product logic and in [74] it is studied the problem of decidability of the modal logics over BL considering frames with a total accessibility relation

Part II

Strongly complete axiomatizations of the logic of a t-norm

Chapter 4

Strongly standard complete systems

The main theoretical objective of this doctoral dissertation is to study and axiomatize the minimum modal expansions of logics arising from a left-continuous t-norm. In doing so we have realized it was important to solve first some issues concerning the non-modal (propositional) axiomatic system. A very desirable property of the underlying non-modal axiomatic system is that it enjoys strong completeness with respect to an standard algebra that, moreover, has a set of truth constants interpreted densely.

Within this chapter, we focus on presenting and studying, given a left-continuous t-norm $*$, an axiomatic system that enjoys strong completeness with respect to the standard algebra of $*$ with rational truth constants and the Δ operator. We begin by doing a brief review of the state of the art. We present the main existing results about strong completeness of *MTL* logics and the studies on the expansion of these logics with rational constants (for a deeper explanation of these issues, see Chapter 2) and justify why the solutions found in the literature do not satisfactorily fulfil our needs. To solve the problem, we propose in Section 4.2 a first axiomatic calculus for each left-continuous t-norm, and we show some general results concerning it. We then prove that adding a particular infinitary inference rule (the *density rule*) to this systems results in a calculus strongly complete with respect to the standard algebra of the t-norm, expanded with Δ and dense constants.

The last section of this chapter comprehends a study on how to extend the previous system in order to axiomatize the standard algebra of a left-continuous t-norm expanded with additional operations that satisfy certain regularity conditions. A general result that follows from this is the proof of the so-called *prime theory extension property* for a large family of logics that use up to a countable set of infinitary inference rules.

Before continuing, we want to point out that during this chapter $*$ stands for an arbitrary left-continuous t-norm.

4.1 Why a new approach towards a logic of $[0, 1]_*$?

As we remarked before, we are interested in an axiomatic system with certain characteristics. First, taking into account we need constant symbols interpreted canonically in the standard algebra, it is natural that our objective is axiomatizing the logics arising from *only one* left-continuous t-norm (since otherwise, the behaviour of the constant symbols cannot be fixed). Thus, we are interested in the axiomatization of the logic arising from one t-norm expanded with constant symbols. Another desired characteristic is the strong standard completeness of the axiomatic system, so it is natural to look at the existing results dealing with this matter, in order to search for useful results and/or methods.

Within the existing works, we could not find the particular results we needed to study later the modal expansion of logics arising from a left-continuous t-norm. We do here a brief review of these works, remarking what they lack for being applicable for our purposes.

In [52] it is shown that the variety of BL-algebras generated by a single BL-chain on $[0, 1]$ is finitely axiomatizable and an algorithm to obtain this axiomatization is given. Unfortunately, since our aim is to work with a logic that enjoys strong standard completeness, we cannot resort to many of the results and methods presented in these works. The proof of finite completeness of the axiomatic system presented with respect to the variety generated by the corresponding standard algebra exploits the fact that for each continuous t-norm there exists a continuous t-norm with a certain structure that generates the same variety as the original one. These particular t-norms determine the axiomatization of the logic arising from the variety. In our case, however, this is not strong enough: the fact that two t-norms generate the same variety does not imply that their generated generalized quasi-varieties coincide. Thus we cannot assume the logic (in the sense of all valid deductions) over these two standard algebras is the same.

On the other hand, there exist several works concerned with *BL* and *MTL* logics expanded with truth constants, but they still lack some desirable results that we aim to get. On the one hand, Esteva et. al. present in [48] results concerning the expansion of the logic of a continuous t-norm with constant symbols forming a set isomorphic to the rational numbers in $[0, 1]$. The axiomatic systems studied there are the ones resulting from adding the book-keeping axioms to the underlying axiomatic systems of certain *BL*-logics, which in general are not complete with respect to the corresponding standard algebra with constants interpreted as intended (i.e., by its name). The completeness results presented in [48] are with respect to all standard algebras: that is to say, not only those where the constants are "properly" interpreted, but also non-isomorphic ones, where the constants have different behaviours (which is, for our purposes, not functional). Moreover, except for the Gödel case (where it is proven strong completeness with respect to the class of standard algebras), no strong completeness results are presented.

On the other hand, a study more adapted to our needs has been done by Cintula in [32] and [34], following the Pavelka-style completeness approach. He remarks the need of including infinitary inference rules in an axiomatic system in order to be able to prove it is Pavelka-complete whenever there is some non-continuous operation in the algebra (see [34, Prop. 17]). He specifies a family of infinitary rules for each discontinuity point of each operation that serve as candidates to axiomatize Pavelka-complete MTL logics. Concerning only the t-norm and the residuum operation, these families of rules are the following ones:

For each $\langle x_1, x_2\rangle$ discontinuity point of the $*$ operation, the set of rules

$$\frac{\{\overline{r_1} \to p, \overline{r_2} \to q : r_1 \in [0, x_1)_\mathbb{Q}, r_2 \in [0, x_2)_\mathbb{Q}\}}{\overline{r} \to (p \,\&\, q)} \text{ for each rational } r \leq x_1 * x_2 \tag{4.1}$$

For each $\langle x_1, x_2\rangle$ discontinuity point of the $\Rightarrow_*$ operation, the set of rules

$$\frac{\{p \to \overline{r_1}, \overline{r_2} \to q : r_1 \in (x_1, 1]_\mathbb{Q}, r_2 \in [0, x_2)_\mathbb{Q}\}}{\overline{r} \to (p \to q)} \text{ for each rational } r \leq x_1 \Rightarrow_* x_2 \tag{4.2}$$

A remarkable result is [34, Cor. 23]: it states that given a left-continuous t-norm $*$ and an axiomatic system AS such that

- AS extends the finitary logic of the standard algebra of $*$,
- AS validates all book-keeping axioms and the rule $\overline{r} \vdash \overline{0}$ for all $r < 1$,
- AS validates all the above infinitary rules,
- AS validates the infinitary rule

$$\{\overline{c} \to p : c \in [0,1)_\mathbb{Q}\} \vdash p$$

- AS is semilinear (i.e., strongly complete with respect to the linearly ordered algebras of the algebraic companion),

then AS is Pavelka-complete.

Some cases like Product and Łukasiewicz logics with Δ have a finite amount of discontinuity points which are, moreover, over rational elements over $[0,1]$, so it is only necessary to consider a finite quantity of the previous infinitary inference rules. The addition of these finite sets of infinitary inference rules (and the book-keeping axioms) to the usual Hilbert calculus of Łukasiewicz and Product logics with Δ [1] allows to prove Pavelka style completeness of these logics. However, this construction does not work in general. It is not proven if the system resulting of the addition of all the infinitary rules associated to the discontinuity points of the operations to the (finitary) calculus of the t-norm results in a semilinear logic. In particular, for cases like the Gödel logic

[1]The Δ operator is used to prove semilinearity of the axiomatic system.

and t-norms arising from ordinal sums (that have a non-countable amount of discontinuity points in the diagonal of the residuum), an axiomatization does not directly follow from [34] since there it is not proven the semilinearity of these systems.

Concerning strong completeness of *BL* logics, a remarkable work is the one presented by Montagna in [99]. Instead using Δ, he resorts in his work the storage operator $\circ$, i.e., an unary operator that determines the greatest idempotent element below the attribute. He defines an infinitary rule that added to the axiomatic system of *BL* with $\circ$ makes it strongly complete with respect to the class of standard *BL* algebras with $\circ$:

$$\frac{\{p \vee (q \to r^n): \quad \text{for all } n \in \omega\}}{p \vee (q \to r^\circ)}$$

Moreover, this rule also works particularly well for the cases of Product and Łukasiewicz logics, where the storage operator coincides with the Δ operator. That is to say, adding the previous infinitary rule to the Π_Δ and $Ł_\Delta$ axiomatic systems results into axiomatic systems that are strongly complete with respect to $[\mathbf{0},\mathbf{1}]_{\mathbf{\Pi_\Delta}}$ and $[\mathbf{0},\mathbf{1}]_{\mathbf{Ł\Delta}}$ correspondingly. Even though we are in a different context (*MTL* algebras with constant symbols and Δ) some of the ideas and methods developed along [99] have been source of great inspiration to us.

Our aim is that of filling the open issues left in the literature reviewed above and providing axiomatizations that are strongly complete with respect to the logics of the left-continuous t-norms with rational constants and for reasons that will become clearer below, with the Monteiro-Baaz Δ operator. In Section 4.4, we will also pay some attention to the generalization of this study to the case of the logics arising from standard left-continuous t-norms algebras with rational constant symbols expanded with arbitrary operations from $[0,1]$ that respect some regularity conditions, complementing in some senses the studies from [34].

4.2 Axiomatizations with the density rule

For the motivations explained above, instead of trying to extend the axiomatizations from [52] or [48], we follow a different approach. The intuition is that, having (at least) rational constants, and knowing that the operations are either left or right continuous, it should be possible to determine the exact behaviour of the operations on an arbitrary algebra of the class knowing the values taken in the constant symbols. We detail how to proceed with this reasoning.

The language of the logics we will define and study within this chapter is inherited from the MTL one, extended with the Monteiro-Baaz Δ unary operator. Moreover, for each left-continuous t-norm $*$, we consider a set of countable constants $\mathcal{C}_*$ given by the countable subalgebra of $[\mathbf{0},\mathbf{1}]_*$ generated by the rationals on $[0,1]$. This consideration implies that the set of constants considered is closed under the algebra operations, a characteristic necessary in order write down the book-keeping axioms. Formally, we let the language along this one and the next chapters to be

$$\mathfrak{L} = \langle \&/2, \to/2, \wedge/2, \Delta/1, \{\overline{c}/0\}_{c\in\mathcal{C}_*}\rangle.$$

Towards a definition of a logic of $*$, we will begin by specifying the semantics we aim to axiomatize. Since in all our work the algebras used are extended with at least the rational constant symbols and the Δ operation, we will abuse notation for the sake of readability when naming the main algebras and logics.

Definition 4.1. We call ***(expanded) standard algebra of*** $*$ to the algebra

$$[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}} := \langle [0,1], *, \Rightarrow_*, \wedge, \delta, \{c\}_{c\in\mathcal{C}_*}\rangle$$

where $\Rightarrow_*$ is the residuum of the t-norm, for each constant symbol $\overline{c}$ of the language (that is to say, with $c \in \mathcal{C}_*$), $c = \overline{c}^{[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}}$ and δ is the interpretation of the Monteiro-Baaz operator

$$\delta(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

Observe that the universe of $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$ is the real unit interval, and the $\mathbb{Q}$ stands for the expansion with constants containing the rational elements from $[0,1]$.

An initial calculus, that will be later extended in order to truly become a logic for $*$ is the following one. $MTL_*^{\mathbb{Q}}$ is the expansion of MTL (Definition 2.1) in the language $\mathfrak{L}$ using the following axioms and rules:

- Δ axioms and (G_Δ) rule (see Definition 2.6),
- Book-keping axioms (see [71]), i.e.:
  - (C1) $(\overline{c} \,\&\, \overline{d}) \leftrightarrow \overline{c * d}$ for each $\overline{c}, \overline{d} \in \mathcal{C}_*$
  - (C2) $(\overline{c} \to \overline{d}) \leftrightarrow \overline{c \Rightarrow_* d}$ for each $\overline{c}, \overline{d} \in \mathcal{C}_*$
  - (C3) $\neg\Delta\overline{c}$ for each $\overline{c} \in \mathcal{C}_* \setminus \{\overline{1}\}$.

The notion of proof and of provable formula in $MTL_*^{\mathbb{Q}}$ are the usual (finitary) ones, which allow us to use without any problem all the finite deductions existing in the MTL logic with Δ. The following is a list with some theorems and (finitary) valid deduction of MTL^{Δ} (and so of $\mathrm{MTL}_*^{\mathbb{Q}}$ too) that will be of use later.

Remark 4.2. The following formulas and deductions schemata are valid in $\mathrm{MTL}_*^{\mathbb{Q}}$.

1. $p \leftrightarrow p \vee p$
2. $p \vee (q \to r) \vdash ((\neg\Delta p \wedge q) \to r)$,
   $p \vee (q \to r) \vdash (q \to (\Delta p \vee r))$
3. $(p \wedge q) \to r \vdash (p \to r) \vee (q \to r)$,
   $p \to (q \vee r) \vdash (p \to q) \vee (p \to r)$

4. $\neg\Delta p \to q \vdash p \vee q$

5. $p \to \Delta q \vdash \neg p \vee q$

6. $\neg p \vdash p \to q$

7. $(p \to q) \wedge (r \to s) \vdash (p \,\&\, r) \to (q \,\&\, s)$

8. $(p \to q) \wedge (r \to s) \vdash (q \to r) \to (p \to s)$

9. $\Delta(p \to \Delta(q \to r)) \to \Delta((q \wedge \neg\Delta\neg p) \to r)$,
   $\Delta(p \to \Delta(q \to r)) \to \Delta(q \to (r \vee \Delta\neg p))$

10. $(\Delta(p \to q) \vee \Delta\neg r) \to \Delta(r \to (p \to q))$

11. $(\Delta p \to q) \leftrightarrow (\neg\Delta p \vee q)$

Our aim is to extend the previous system $\mathrm{MTL}_*^{\mathbb{Q}}$ to obtain an axiomatic system strongly complete with respect to $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$.

A remarkable result from [34] (see Prop.17) states that it is not possible to get such an axiomatic system without adding infinitary inference rules whenever there is some non-continuous operation in $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$. Therefore the use of inference rules with an infinite set of premises is unavoidable. On the other hand, even though what we intuitively get from the previous result is that the problems arise from the discontinuity points of the operations, we considered that the solution could pass not through addressing each one of these points separately, but rather by finding a way to treat all the discontinuity points *at the same time*.

We were inspired by the behaviour of the deduction rule

$$\frac{(A \to p) \vee (p \to B) \vee C}{(A \to B) \vee C}$$

where p is a propositional variable not occurring in A, B or C.

This is well known from the area of first-order non-classical logics, firstly presented by Takeuti and Titani in [115] to axiomatize the Intuitionistic predicate logic. It exploits the concept of free variable from first order logics and its validity in a given algebra forces its universe to be dense (in the sense that between two different elements there is always a third one in between).

In our framework, it is possible to propose a similar rule with an infinite number of premises (that depend on a "free" constant symbol) in order to enforce the density of the constants within the elements of the algebra.

Definition 4.3. L_*^∞ is the extension of $MTL_*^{\mathbb{Q}}$ with the ***density rule***

$$\mathsf{D}^\infty : \frac{\{(p \to \overline{c}) \vee (\overline{c} \to q)\}_{c \in \mathcal{C}_*}}{p \to q}$$

Recall that the notion of deduction within L_*^∞ is the one given in Definition 1.3.

It is easy to prove the soundness of L_*^∞ with respect to $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$.

Lemma 4.4. *Let* $\Gamma \cup \{\varphi\} \in Fm$. *Then* $\Gamma \vdash_{L_*^\infty} \varphi$ *implies that* $\Gamma \models_{[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}} \varphi$.

Proof. It is just needed to prove the soundness of the new infinitary rule. We can prove it by contraposition. Since $\mathcal{C}_*$ is dense in $[0,1]$, it is clear that for any $a, b \in [0,1]$ such that $b < a$ there exists $c_0 \in \mathcal{C}_*$ such that $b < c_0 < a$. Thus, $(a \Rightarrow_* c_0) \vee (c_0 \Rightarrow_* b) = \overline{1}^{\mathbf{A}}$ does not hold, proving the soundness of D^∞. ⊠

We know that any extension of $MTL_*^{\mathbb{Q}}$ by inference rules is Rasiowa-implicative and so in particular L_*^∞ is implicative too and so, algebraizable. The characterization of the algebraic companion of L_*^∞ is given by the class L_*^∞ of L_*^∞***-algebras***. These are algebras of the form $\mathbf{A} = \langle A, \odot, \Rightarrow, \wedge, \delta^{\mathbf{A}}, \{\overline{c}^{\mathbf{A}}\}_{c \in \mathcal{C}_*}\rangle$ where $\langle A, \odot, \Rightarrow, \wedge, \delta^{\mathbf{A}}\rangle$ is a MTL_Δ-algebra, $\mathbf{A}$ validates the equations arising from the book-keeping axioms of $\mathcal{C}_*$ and so does the generalized quasi-equation

$$\mathcal{Q} : \bigwedge_{c \in \mathcal{C}_*} [(x \to \overline{c}) \vee (\overline{c} \to y) \approx \overline{1}] \quad \Longrightarrow \quad [x \to y \approx \overline{1}]$$

At this point, it is a natural question whether this class is a proper generalized quasi-variety. It can be proven to be not a quasi-variety by means of an example.

Example 4.5. Clearly, $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$ belongs to L_*^∞. We can construct an ultrapower of $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$ for which the generalized quasi-equation $\mathcal{Q}$ does not hold, and so L_*^∞ is not a quasi-variety, since it is not closed under ultraproducts.

Let S be the Fréchet filter, that is, the set of cofinite subsets of $\mathbb{N}$. From the well known ultrafilter lemma there is an ultrafilter U that extends S. Then, consider $\prod_{n \in \mathbb{N}} [\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}/U$. Recall that this means that for $a, b \in \prod_{n \in \mathbb{N}} [\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$, the equivalence relation $\sim_U$ is given by $a \sim_U b \Leftrightarrow \{n \in \mathbb{N} : \ a[n] = b[n]\} \in U$.

Let p be the element in $\prod_{n \in \mathbb{N}} [\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$ defined component-wise by $p[n] = \frac{2^n - 1}{2^n}$ for each $n \in \mathbb{N}$. It is clear that for each $c \in \mathcal{C}_*$ there is some $s \in \mathbb{N}$ such that $p[s] \geq c$ (pick $s \geq log_2 \frac{1}{1-c}$, so naturally $p[s] \geq c$).

Denote by $c^{\mathbb{N}}$ the element such that $c^{\mathbb{N}}[n] = c$ for each $n \in \mathbb{N}$. Then for all $c \in \mathcal{C}_*$ it holds that $(c^{\mathbb{N}} \Rightarrow p)[i] = 1$ for all $i \geq s$ and thus since U extends S and $\{n \in \mathbb{N} : \ c^{\mathbb{N}} \Rightarrow p)[i] = 1\} \in S$, it holds that $c^{\mathbb{N}} \Rightarrow p \sim_U 1$. Given that the interpretation of the constants in this ultraproduct is by definition (inherited from $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$) the equivalence classes of $c^{\mathbb{N}}$ for each $c \in \mathcal{C}_*$, we are verifying the antecedent of the generalized quasi-equations $\mathcal{Q}$.

However, $p[n] < 1$ for each $n \in \mathbb{N}$ and thus it is not true that $p \sim_U 1$, i.e., $p \neq 1$ in $\prod_{n \in \omega} [\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}/U$.

With this we have build an example where the premises of $\mathcal{Q}$ hold but not its consequence. This implies that $\prod_{n \in \omega} [\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}/U$ is not an element of the class L_*^∞, which concludes the example. ⊠

Towards the proof of strong standard completeness of L_*^∞, we begin by checking that this logic is strong complete with respect to the linearly ordered algebras from its corresponding algebraic companions.

We first prove the some technical results concerning the syntactical behaviour of L_*^∞. It is in the following results where the importance of the Δ operator is remarked: if not using Δ, it is unclear whether similar theorems can be proved, up to not knowing whether if the the same axiomatic system without Δ is strongly complete with respect to linearly ordered algebras.

Lemma 4.6. *The following is a derived rule in L_*^∞,*

$$\frac{\{r \vee (p \to \overline{c}) \vee (\overline{c} \to q)\}_{c \in \mathcal{C}_*}}{r \vee (p \to q)}.$$

Proof. We can check $r \vee (p \to \overline{c}) \vee (\overline{c} \to q) \vdash_{L_*^\infty} ((p \wedge (\neg \Delta r)) \to \overline{c}) \vee (\overline{c} \to (q \vee \Delta r)$. It follows from the fact that finitary deductions of L_*^∞ coincide with those of $\mathrm{MTL}_*^{\mathbb{Q}}$ and using 2. of Remark 4.2. Since this is true for each $\overline{c}$, we can apply the infinitary inference rule D^∞ and so get $(p \wedge (\neg \Delta r)) \to (q \vee \Delta r)$. Using 3. of Remark 4.2, we have that $(p \to q) \vee (p \to \Delta r) \vee ((\neg \Delta r) \to q) \vee ((\neg \Delta r) \to \Delta r)$. From here, since $p \to \Delta r \vdash_{\mathrm{MTL}_*^{\mathbb{Q}}} \Delta r \vee \neg p$, $\neg \Delta r \to q \vdash_{\mathrm{MTL}_*^{\mathbb{Q}}} \Delta r \vee q$ and $\neg \Delta r \to \Delta r \vdash_{\mathrm{MTL}_*^{\mathbb{Q}}} \Delta r$ we can conclude that $(p \to q) \vee \Delta r$. ⊠

From this, the Δ Deduction theorem is a direct consequence.

Lemma 4.7 (Deduction theorem)**.** *For any set of formulas $\Gamma \cup \{\varphi, \psi\}$,*

$$\Gamma, \varphi \vdash_{L_*^\infty} \psi \iff \Gamma \vdash_{L_*^\infty} \Delta\varphi \to \psi.$$

Proof. Working by induction on the proof, it is enough to see it holds for any possible last rule.

The only non-direct case is D^∞, for which we proceed as follows.

For the left-to-right direction, assume that ψ is $\chi \to \delta$ and that $\Gamma, \varphi \vdash_{L_*^\infty} (\chi \to \overline{c}) \vee (\overline{c} \to \delta)$ for eah $c \in \mathcal{C}_*$. By Induction hypothesis, $\Gamma \vdash_{L_*^\infty} \Delta\varphi \to ((\chi \to \overline{c}) \vee (\overline{c} \to \delta))$. From 11. of Remark 4.2 it follows that $\Gamma \vdash_{L_*^\infty} \neg\Delta\varphi \vee (\chi \to \overline{c}) \vee (\overline{c} \to \delta)$. Since this happens for each $c \in \mathcal{C}_*$ and using the previous lemma, we have that $\Gamma \vdash_{L_*^\infty} \neg\Delta\varphi \vee (\chi \to \delta)$. From the same theorem of $\mathrm{MTL}_*^{\mathbb{Q}}$ used above we can deduce that $\Gamma \vdash_{L_*^\infty} \Delta\varphi \to (\chi \to \delta)$, which concludes the proof.

The other direction is more direct. For ψ, χ and δ as above, assume $\Gamma \vdash_{L_*^\infty} \Delta\varphi \to ((\chi \to \overline{c}) \vee (\overline{c} \to \delta))$ for each $c \in \mathcal{C}_*$. By Induction hypothesis, $\Gamma, \varphi \vdash_{L_*^\infty} (\chi \to \overline{c}) \vee (\overline{c} \to \delta)$. Thus, using rule D^∞ we have $\Gamma, \varphi \vdash_{L_*^\infty} \chi \to \delta$. ⊠

With these two results, is now easy to check syntactically a notion that naturally arises from the semantics.

Lemma 4.8. *Let $\Gamma \cup \{\varphi, \psi, \alpha\}$ be a set of formulas such that*

- $\Gamma \nvdash_{L_*^\infty} \alpha$ *and*
- $\Gamma, \varphi \to \psi \vdash_{L_*^\infty} \alpha$.

Then there is $c \in \mathcal{C}_*$ such that $\Gamma, \neg\Delta((\varphi \to \overline{c}) \vee (\overline{c} \to \psi)) \nvdash_{L_*^\infty} \alpha$.

Proof. Suppose towards a contradiction that for all $c \in \mathcal{C}_*$, it holds that

$$\Gamma, \neg\Delta((\varphi \to \overline{c}) \vee (\overline{c} \to \psi)) \vdash_{L_*^\infty} \alpha.$$

By the Deduction Theorem (and knowing that $\neg\neg\Delta\varphi \leftrightarrow \Delta\varphi$ is a theorem of the logic), this is equivalent to have that $\Gamma \vdash_{L_*^\infty} \neg\Delta((\varphi \to \overline{c}) \vee (\overline{c} \to \psi)) \to \alpha$ Then, from 11. of Remark 4.2 this implies that $\Gamma \vdash_{L_*^\infty} \neg\neg\Delta((\varphi \to \overline{c}) \vee (\overline{c} \to \psi)) \vee \alpha$. Since a double negation over a Δx element is the identity function and given that $\Delta\varphi \to \varphi$ is an axiom of the logic it follows that $\Gamma \vdash_{L_*^\infty} (\varphi \to \overline{c}) \vee (\overline{c} \to \psi) \vee \alpha$ From Lemma 4.6, we have that we can apply rule D^∞ and thus obtain $\Gamma \vdash_{L_*^\infty} (\varphi \to \psi) \vee \alpha$. Applying 11. of Remark 4.2 again we get that $\Gamma \vdash_{L_*^\infty} \neg\Delta(\varphi \to \psi) \to \alpha$

From second assumption of the lemma, we know that $\Gamma \vdash_{L_*^\infty} \Delta(\varphi \to \psi) \to \alpha$, so the both deductions together lead to $\Gamma \vdash_{L_*^\infty} (\Delta(\varphi \to \psi) \vee \neg\Delta(\varphi \to \psi)) \to \alpha$. However, we know that $\Gamma \vdash_{L_*^\infty} \Delta(\varphi \to \psi) \vee \neg\Delta(\varphi \to \psi)$, since this formula is an axiom of *MTL*. Then, by MP, we have that $\Gamma \vdash_{L_*^\infty} \alpha$, which contradicts the first assumption of the lemma. ⊠

With these tools it is now not difficult to check the so called *prime theory extension property* for L_*^∞. Recall that prime theory stands for a theory such that for each pair of formulas φ, ψ, either $\varphi \to \psi$ or $\psi \to \varphi$ belongs to it.

Theorem 4.9. *Let $\Gamma \cup \{\alpha\}$ be a set of formulas such that $\Gamma \nvdash_{L_*^\infty} \alpha$. Then, there is a prime theory Γ' expanding Γ such that $\Gamma' \nvdash_{L_*^\infty} \alpha$.*

Proof. As Montagna noticed in [99], it is interesting to observe that by the presence of infinitary rules, many standard constructions do not work. For instance, one might be tempted to use Zorn's lemma to obtain a maximal theory closer under D^∞, Γ' extending Γ such that $\varphi \notin \Gamma'$. But in this case Zorn's lemma does not apply, as the union of a chain of L_*^∞-theories may fail to be a theory closer under D^∞. Thus we shall proceed in a slightly different way.

Along the following prove, we abuse notation and for a set of formulas T we write $Cn(T)$ to denote the set of formulas derivable from T in L_*^∞, i.e., $Cn(T) = \{\varphi \in Fm : \ T \vdash_{L_*^\infty} \varphi\}$.

Let $\langle \varphi_n, \psi_n \rangle$ be an enumeration of all pairs of formulas from Fm. We can build a chain $\Gamma \subseteq \Gamma_0 \subseteq ... \subseteq \Gamma_n \subseteq ...$ such that for each $n \in \omega$,

- $\Gamma_n \nvdash \alpha$ and
- either $\varphi_n \to \psi_n$ or $\psi_n \to \varphi_n$ belong to Γ_{2n+1} and
- if $\varphi_n \to \psi_n \notin \Gamma_{2n+1}$ (resp. if $\psi_n \to \varphi_n \notin \Gamma_{2n+1}$) there is $c \in \mathcal{C}_*$ such that $\neg\Delta((\varphi_n \to \overline{c}) \vee (\overline{c} \to \psi_n)) \in \Gamma_{2n+2}$ (resp. $\neg\Delta((\psi_n \to \overline{c}) \vee (\overline{c} \to \varphi_n)) \in \Gamma_{2n+2}$.
- **step 0:** Put $T_0 = Cn(\Gamma)$.

- **step 2n+1:** If $\alpha \notin Cn(\Gamma_{2n}, \varphi_n \to \psi_n))$ put $\Gamma_{2n+1} = Cn(\Gamma_{2n}, \varphi_n \to \psi_n))$. Otherwise, put $\Gamma_{2n+1} = Cn(\Gamma_{2n}, \psi_n \to \varphi_n))$. Observe that in any case, $\alpha \notin \Gamma_{2n+1}$: if both $\alpha \in Cn(\Gamma_{2n}, \varphi_n \to \psi_n))$ and $\alpha \in Cn(\Gamma_{2n}, \psi_n \to \varphi_n))$, by the deduction theorem and the semilinearity axiom of MTL, we would have that $\Gamma_{2n} \vdash \alpha$, which is not possible.

- **step 2n+2:** Suppose $\Gamma_{2n+1} = Cn(\Gamma_{2n}, \varphi_n \to \psi_n))$ (the case where $\Gamma_{2n+1} = Cn(\Gamma_{2n}, \psi_n \to \varphi_n))$ is proved analogously).

  If $\alpha \notin Cn(\Gamma_{2n+1}, \psi_n \to \varphi_n)$ then put $\Gamma_{2n+2} = Cn(\Gamma_{2n+1}, \psi_n \to \varphi_n)$.

  Otherwise, observe that we have that $\Gamma_{2n+1} \nvdash_{L_*^\infty} \alpha$ and also that $\Gamma_{2n+1}, \psi_n \to \varphi_n \vdash_{L_*^\infty} \alpha$. Then, from Lemma 4.8 there is c such that $\alpha \notin Cn(\Gamma_{2n+1}, \neg\Delta((\varphi \to \overline{c}) \vee (\overline{c} \to \psi)))$, so put $\Gamma_{2n+2} = Cn(\Gamma_{2n+1}, \neg\Delta((\varphi \to \overline{c}) \vee (\overline{c} \to \psi)))$.

  It is easy to check that each one of the Γ_n from before holds all the characteristics we aimed to. Let then $\Gamma' = \bigcup_{i \in \mathbb{N}} \Gamma_i$. We can prove that Γ' is a L_*^∞-theory by seeing it is closed under the derivation rules. The only non-direct case is the infinitary rule D^∞ (Γ' is closed under all the finitary rules since this happens for each Γ_n).

  Suppose towards a contradiction that $(\varphi \to \overline{c}) \vee (\overline{c} \to \psi) \in \Gamma'$ for each $c \in \mathcal{C}_*$ and that $\varphi \to \psi \notin \Gamma'$.

  The formula $\psi \to \varphi$ was added at step $2k+1$ for some k, so consider then step $2k+2$. Since $\varphi \to \psi$ was not added, necessarily $\Gamma_{2k+2} = Cn(\Gamma_{2k+1}, \neg\Delta((\varphi \to \overline{c_0}) \vee (\overline{c_0} \to \psi)))$ for some $c_0 \in \mathcal{C}_*$. But then, since $(\varphi \to \overline{c}) \vee (\overline{c} \to \psi) \in \Gamma'$ for each $c \in \mathcal{C}_*$, in particular $(\varphi \to \overline{c_0}) \vee (\overline{c_0} \to \psi) \in \Gamma_r$ for some $r \in \mathbb{N}$. Then, taking the maximum of this two indexes, i.e., $m = max\{r, 2k+2\}$, it turns out that $\varphi \to \overline{c_0}) \vee (\overline{c_0} \to \psi) \in \Gamma_m$ and $\neg\Delta((\varphi \to \overline{c_0}) \vee (\overline{c_0} \to \psi)) \in \Gamma_m$. Then, $\bot \in \Gamma_m$, which is a contradiction (since we know that $\Gamma_m \nvdash_{L_*^\infty} \alpha$).

  It is then easy to conclude that $\Gamma' \nvdash_{L_*^\infty} \alpha$. Otherwise, since Γ' is closed under derivations, $\alpha \in \Gamma' = \bigcup_{i \in \mathbb{N}} \Gamma_i$ and so $\alpha \in \Gamma_i$ for some i, which is not true. ⊠

A corollary of the previous result will be of use later on.

Corollary 4.10. *Let* $\mathbf{A}$ *be a countable* L_*^∞*-algebra and* F *a* L_*^∞*-filter on it. Then* F *coincides with the intersection of all prime filters of* $\mathbf{A}$ *containing* F*.*

Proof. Being $\mathbf{A}$ countable, it is direct that $\mathbf{A}$ is isomorphic to $\mathbf{Fm}/\mathbf{\Omega}\Gamma$ for some set of formulas Γ. We let $i : \mathbf{A} \to \mathbf{Fm}/\mathbf{\Omega}\Gamma$ denote this isomorphism. Moreover, a filter F on it is isomorphic (using i) to $g^-1(1)$ where $g \in Hom(\mathbf{Fm}, \mathbf{Fm}/\mathbf{\Omega}\Sigma)$ for some L_*^∞-theory Σ with $\Gamma \subseteq \Sigma$. Having an element a from A not in F is now equivalent to say, for $i(a) = \varphi/\mathbf{\Omega}\Gamma$, that $\varphi/\mathbf{\Omega}\Gamma$ with $\varphi \notin \Sigma$ (so $g(\varphi/\mathbf{\Omega}\Gamma) < 1$). Then, we have that $\Sigma \nvdash_{L_*^\infty} \varphi$ and so from the previous Theorem we have that there is Σ' prime theory extending Σ with $\Sigma' \nvdash_{L_*^\infty} \varphi$.

Since $\Gamma \subseteq \Sigma \subseteq \Sigma'$ and given that the logic is equivalential (it is algebraizable) we know that $\boldsymbol{\Omega}\Gamma \subseteq \boldsymbol{\Omega}\Sigma \subseteq \boldsymbol{\Omega}\Sigma'$. Let then $h = h_1 \circ g$, where h_1 is the natural projection from $\mathbf{Fm}/\boldsymbol{\Omega}\Sigma$ to $\mathbf{Fm}/\boldsymbol{\Omega}\Sigma'$ and g is the natural projection from $\mathbf{Fm}/\boldsymbol{\Omega}\Gamma$ to $\mathbf{Fm}/\boldsymbol{\Omega}\Sigma$. Let $P = h^{-1}(1)$. Then we can prove the following list of characteristics.

- $i^{-1}(P)$ **is a L_*^∞-filter on $\mathbf{A}$**. Follows from the fact that $h^{-1}(1)$ is a filter of the logic in $\mathbf{Fm}/\boldsymbol{\Omega}\Gamma$ and this is isomorphic via i^{-1} to $\mathbf{A}$.
- $i^{-1}(F) \subseteq i^{-1}(P)$. It is immediate from the fact that $g^{-1}(1) \subseteq h^{-1}(1)$.
- $i^{-1}(P)$ **is prime**. We know that for any φ, ψ, either $\varphi \to \psi \in \Sigma'$ or $\psi \to \varphi \in \Sigma'$, so $h((\varphi \to \psi)/\boldsymbol{\Omega}\Gamma) = 1$ or $h((\psi \to \varphi)/\boldsymbol{\Omega}\Gamma) = 1$. Then for every $\varphi/\boldsymbol{\Omega}\Gamma$, $\psi/\boldsymbol{\Omega}\Gamma$, either $\varphi/\boldsymbol{\Omega}\Gamma \Rightarrow \psi/\boldsymbol{\Omega}\Gamma \in h^{-1}(1)$ or $\psi/\boldsymbol{\Omega}\Gamma \Rightarrow \varphi/\boldsymbol{\Omega}\Gamma \in h^{-1}(1)$. Then, since i is an isomorphism, we can conclude that $i^{-1}(P)$ is prime too.
- $a \notin i^{-}1(P)$. It is equivalent to see that $i(a) = \varphi/\boldsymbol{\Omega}\Gamma \notin P$. This is immediate since $\varphi \notin \Sigma'$, so $h(\varphi/\boldsymbol{\Omega}\Gamma) < 1$.

It is not difficult to prove now that $F = \bigcap\{P : P \text{ prime } L_*^\infty\text{-filter on } \mathbf{A}\}$, as we proceed to do. The $\subseteq$ direction is easy: for any element $x \in F$ and any filter P extending F, then $x \in P$ too. Concerning the $\supseteq$ direction, let $a \notin F$. From the reasoning above, we know there is some prime filter P extending F with $a \notin P$. Then, clearly $a \notin \bigcap\{P : P \text{ prime } L_*^\infty\text{-filter on } \mathbf{A}\}$ either, which concludes the proof. ⊠

Relying either on Theorem 4.9 or in this last Corollary, it is not difficult to see that L_*^∞ is strongly complete with respect to the linearly ordered algebras from its algebraic companion.

Theorem 4.11. *Let $\Gamma \cup \varphi \subseteq Fm$. Then the following are equivalent.*

1. $\Gamma \vdash_{L_*^\infty} \varphi$
2. $\Gamma \models_{\mathbf{C}} \varphi$ *for all* $\mathbf{C} \in \mathsf{L}_*^\infty$ *such that* $\mathbf{C}$ *is linearly ordered.*

Proof. We just need to check 2. $\Rightarrow$ 1., since the other direction follows from the definition of the class of L_*^∞-algebras.

The general completeness result states that $\Gamma \nvdash_{L_*^\infty} \varphi$ implies that there exists $\mathbf{A} \in \mathsf{L}_*^\infty$, $F \in Fi_{L_*^\infty}\mathbf{A}$ and $h \in Hom(\mathbf{Fm}, \mathbf{A})$ such that $h([\Gamma]) \subseteq F$ and $h(\varphi) \notin F$. We can restrict ourselves to a countable subalgebra of $\mathbf{A}$ with universe $A' = h([Fm])$ and the corresponding L_*^∞ filter on $\mathbf{A}'$ given by $F' = F \cap A'$. By the previous lemma, there is a prime L_*^∞-filter P of $\mathbf{A}'$ that contains F' and such that $h([\Gamma]) \subseteq P$ and $h(\varphi) \notin P$. It is an exercise to see that $\mathbf{A}'/P$ (i.e., $\mathbf{A}'/\boldsymbol{\Omega}^{\mathbf{A}}P$) is a linearly ordered algebra from L_*^∞.[2] To conclude, recall that $\overline{h} = \pi_P \circ h$ is an homomorphism from $\mathbf{Fm}$ to $\mathbf{A}'/P$, where $\pi_P : \mathbf{A}' \to \mathbf{A}'/P$ is the projection

[2] First, it is clear that $\mathbf{A}'/\boldsymbol{\Omega}^{\mathbf{A}'}P$ belongs to L_*^∞, since $\boldsymbol{\Omega}^{\mathbf{A}'}P$ is a congruence of $\mathbf{A}'$ relative to the class L_*^∞ (and so by definition, $\mathbf{A}'/\boldsymbol{\Omega}^{\mathbf{A}'}P \in \mathsf{L}_*^\infty$). To check that it is linearly ordered,

on the quotient algebra. Since for any $\psi \in P$ it holds that $\pi_P(\psi) = \overline{1}^{\mathbf{A}'/P}$ and for any $\psi \notin P$ we have that $\pi_P(\psi) < \overline{1}^{\mathbf{A}'/P}$ it follows that $\overline{h}([\Gamma]) \subseteq \{\overline{1}^{\mathbf{A}'/P}\}$ and $\overline{h}(\varphi) < \overline{1}^{\mathbf{A}'/P}$. ⊠

For future reference, it is worth to introduce here a notion concerning the second point of the previous lemma. We say that a logic L that is strongly complete with respect to the linearly ordered algebras of its class in the above sense is ***semilinear***.

In order to conclude that L_*^∞ is strongly complete with respect to $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$ we resort here to a natural method. We define a mapping from any (countable) linearly ordered L_*^∞-algebra into $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$ and check this mapping is an embedding. Having this in mind, the reason behind the addition of the rule D^∞ is now clearer: we can check that, over the linearly ordered L_*^∞-algebras, the constants are dense with respect to the the elements of the algebra, which is of great use for proving the embedding property of our mapping.

Lemma 4.12. *Let $\mathbf{A} \in \mathsf{L}_*^\infty$ be linearly ordered, and $a < b$ in A. Then there is $c \in \mathcal{C}_*$ such that $a < \overline{c}^{\mathbf{A}} < b$.*

Proof. Towards a contradiction, suppose that there is no such c. Then, since $\mathbf{A}$ is linearly ordered we have that for all $c \in \mathcal{C}_*$, either $b \leq \overline{c}^{\mathbf{A}}$ or $\overline{c}^{\mathbf{A}} \leq a$. Then we have that, for all $c \in \mathcal{C}_*$, $[(b \Rightarrow \overline{c}^{\mathbf{A}}) \vee (\overline{c}^{\mathbf{A}} \Rightarrow a)] = 1$, which means that the premises of the generalized quasiequation $\mathcal{Q}$ are true and thus it can be applied. The consequence of this instantiation of $\mathcal{Q}$ is that $b \leq a$, which contradicts the assumptions of the lemma. ⊠

With this, a natural natural mapping from any numerable linearly ordered L_*^∞-algebra into $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$ can be defined. In order to express this map it in the clearest possible way, let us previously define the following sets.

Given a linearly ordered L_*^∞-algebra $\mathbf{A}$ and an element $a \in A$, we consider the following subsets of $[0,1]$:

$$\begin{aligned} \mathcal{C}_a^+ &:= \{c \in \mathcal{C}_* : a \leq_{\mathbf{A}} \overline{c}^{\mathbf{A}}\} \\ \mathcal{C}_a^- &:= \{c \in \mathcal{C}_* : \overline{c}^{\mathbf{A}} \leq_{\mathbf{A}} a\} \end{aligned}$$

Clearly, for each $a \in A$, $\mathcal{C}_a^-$ is downward closed and $\mathcal{C}_a^+$ is upward closed. Moreover, it also holds that $\sup \mathcal{C}_a^- = \inf \mathcal{C}_a^+$ for any $a \in A$. Indeed, these two values cannot be different, since if that was the case the previous Lemma would imply the existence of a constant d between them (i.e., $d \in \mathcal{C}_*$ such that $d \notin \mathcal{C}_a^-$ and $d \notin \mathcal{C}_a^+$). However, $\mathbf{A}$ is linearly ordered so we have that either $a \leq \overline{d}^{\mathbf{A}}$ or $\overline{d}^{\mathbf{A}} \leq a$, which contradicts the previous statement.

given that L_*^∞ is implicative, its set of congruence formulas can be defined by $\mathbf{\Delta}(\varphi,\psi) = \{\varphi \to \psi, \psi \to \varphi\}$. From the definition of equivalential logic 1.9 this leads to have that if $a \in P$, then $\langle a, \overline{1}^{\mathbf{A}'}\rangle \in \mathbf{\Omega}^{\mathbf{A}'}P$ and so $a/\mathbf{\Omega}^{\mathbf{A}'}P = \overline{1}^{\mathbf{A}'}/\mathbf{\Omega}^{\mathbf{A}'}P$. Since the projection is an homomorphism, for $a, b \in A'$, if $a \to b \in P$ then $a//\mathbf{\Omega}^{\mathbf{A}'}P \Rightarrow b/\mathbf{\Omega}^{\mathbf{A}'}P = (a \Rightarrow b)/\mathbf{\Omega}^{\mathbf{A}'}P = \overline{1}^{\mathbf{A}'}/\mathbf{\Omega}^{\mathbf{A}'}P$ (so $a \leq b$ in $\mathbf{A}'/P$) and otherwise, since P is prime, $b/\mathbf{\Omega}^{\mathbf{A}'}P \Rightarrow a/\mathbf{\Omega}^{\mathbf{A}'}P = (b \Rightarrow a)/\mathbf{\Omega}^{\mathbf{A}'}P = \overline{1}^{\mathbf{A}'}/\mathbf{\Omega}^{\mathbf{A}'}P$ (so $b \leq a$ in $\mathbf{A}'/P$).

Lemma 4.13. *Let* $\mathbf{A}$ *be a* L_*^∞*-chain. Then, the function* $\rho : A \to [0,1]$ *such that* $\rho(a) = \sup \mathcal{C}_a^- = \inf \mathcal{C}_a^+$ *is an embedding from* $\mathbf{A}$ *into* $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$*.*

Proof. First note that for any constant $\overline{d}$, $d = \min \mathcal{C}^+_{\overline{c}^{\mathbf{A}}} = max\mathcal{C}^-_{\overline{c}^{\mathbf{A}}}$ and so $\rho(\overline{d}^{\mathbf{A}}) = d = \overline{d}^{[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}}$.

For what concerns the operations, we can resort to the density of the constants in $\mathbf{A}$ and in $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$. This means that in order to check that two elements $a, b \in [0,1]$ coincide, it is enough to check that for each constant c, if $a < c$ then $b \leq c$ and that if $c < a$ then $c \leq b$.

We can prove the homomorphism conditions just for the $\odot$, $\Rightarrow, \wedge$ and $\delta^{\mathbf{A}}$ operations, since the rest are definable from these ones.

- First of all, the case of δ is trivial, since $\delta^{\mathbf{A}} x = 1$ if and only if $x = 1$ in the algebra. Then $\rho(\delta^{\mathbf{A}} a) = \inf \mathcal{C}^+_{\delta a} = 1$ if and only if $\delta^{\mathbf{A}} a = 1$ in A, i.e., if and only if $a = 1$ in A (by the definition of Δ over a chain). Then, this happens if and only if $\delta\rho(a) = 1$. On the other hand, if $a < 1$ in A then $\delta^{\mathbf{A}} a = 0$ and thus, $\rho(\delta^{\mathbf{A}} a) = 0$. Since $a < 1$, there is c such that $a < \overline{c}^{\mathbf{A}} < 1$, so $\rho(a) < 1$ and thus, $\delta\rho(a) = 0$ too.

- Concerning $\odot$, observe that for any $a, b \in A$, since $\odot$ is an increasing function in both components (4.2[(7.)]) it holds that $c \in \mathcal{C}_a^-$ and $d \in \mathcal{C}_b^-$ implies that $c*d \in \mathcal{C}^-_{a\odot b}$, and similarly, $c \in \mathcal{C}_a^+$ and $d \in \mathcal{C}_b^+$ implies that $c*d \in \mathcal{C}^+_{a\odot b}$.

  Let $c \in \mathcal{C}_*$ be such that $c < \rho(a) * \rho(b) = \sup \mathcal{C}_a^- * \sup \mathcal{C}_b^-$. Using that $*$ has a residuum that coincides with the order operation in the algebra it follows that there exist $d_1 \in \mathcal{C}_a^-$ and $d_2 \in \mathcal{C}_b^-$ such that $c < d_1 * d_2$. [3] Then, from the previous remark and given that $\mathcal{C}^-_{a\odot b}$ is downward closed, $c \in \mathcal{C}^-_{a\odot b}$ too and thus, $c \leq \sup \mathcal{C}^-_{a\odot b} = \rho(a \odot b)$.

  For the other direction, we firs prove an auxiliary claim:

  CLAIM: Let $\mathbf{A} \in \mathrm{L}_*^\infty$ be linearly ordered. Then

  $$a \odot b = \sup\{\overline{c*d}^{\mathbf{A}} : \overline{c}^{\mathbf{A}} \leq a, \overline{d}^{\mathbf{A}} \leq b\}$$

  First we check that $a \odot b \geq \sup\{\overline{c*d}^{\mathbf{A}} : \overline{c}^{\mathbf{A}} \leq a, \overline{d}^{\mathbf{A}} \leq b\}$. For any $\overline{r}^{\mathbf{A}} \leq \sup\{\overline{c*d}^{\mathbf{A}} : \overline{c}^{\mathbf{A}} \leq a, \overline{d}^{\mathbf{A}} \leq b\}$, by definition there exist c, d with $\overline{c}^{\mathbf{A}} \leq a, \overline{d}^{\mathbf{A}} \leq b$ such that $\overline{r}^{\mathbf{A}} \leq \overline{c*d}^{\mathbf{A}}$. By the monotonicity of $\odot$, $\overline{c}^{\mathbf{A}} \odot \overline{d}^{\mathbf{A}} \leq a \odot b$ and from the book-keeping axioms, $\overline{r}^{\mathbf{A}} \leq a \odot b$.

  To see that $a \odot b \leq \sup\{\overline{c*d}^{\mathbf{A}} : \overline{c}^{\mathbf{A}} \leq a, \overline{d}^{\mathbf{A}} \leq b\}$ observe that for any $\overline{r}^{\mathbf{A}} \geq \sup\{\overline{c*d}^{\mathbf{A}} : \overline{c}^{\mathbf{A}} \leq a, \overline{d}^{\mathbf{A}} \leq b\}$, by definition it holds

[3] If for all two constants like above $d_1 * 2 \leq c$, applying residuation $d_1 \leq d2 \to c$ and so the supremum can be taken in the left side. Similarly, we get that $\sup \mathcal{C}_a^- * \sup \mathcal{C}_b^- \leq c$ which contradicts the assumptions.

that $\overline{r}^{\mathbf{A}} \geq \overline{c * d}^{\mathbf{A}}$ for any c, d like in the formula. Then, by the book-keeping axioms, $\overline{r}^{\mathbf{A}} \geq \overline{c}^{\mathbf{A}} \odot \overline{d}^{\mathbf{A}}$ for such c, d. Applying that $\odot$ is a residuated operation we have that $\overline{c}^{\mathbf{A}} \leq \overline{d}^{\mathbf{A}} \Rightarrow \overline{r}^{\mathbf{A}}$. We can now take the supremum at the left side, so $a = \sup\{\overline{c}^{\mathbf{A}} : \overline{c}^{\mathbf{A}} \leq a\} \leq \overline{d}^{\mathbf{A}} \Rightarrow^{\mathbf{A}} \overline{r}^{\mathbf{A}}$. Proceeding similarly for the other component, we get that $a *^{\mathbf{A}} b = \sup\{\overline{c}^{\mathbf{A}} : \overline{c}^{\mathbf{A}} \leq a\} \odot \sup\{\overline{d}^{\mathbf{A}} : \overline{d}^{\mathbf{A}} \leq b\}$, concluding the proof of the claim.

Now, let $c \in \mathcal{C}_*$ be such that $c \leq \rho(a \odot b)$. By definition, $\overline{c}^{\mathbf{A}} \leq a \odot b$ and by the previous claim,
$\overline{c}^{\mathbf{A}} \leq \sup\{\overline{c * d}^{\mathbf{A}} : \overline{c}^{\mathbf{A}} \leq a, \overline{d}^{\mathbf{A}} \leq b\}$. Then, there exist $c_0, c_1 \in \mathcal{C}_*$ with $\overline{c_0}^{\mathbf{A}} \leq a$ and $\overline{c_1}^{\mathbf{A}} \leq b$ such that $\overline{c}^{\mathbf{A}} \leq \overline{c_0 * c_1}^{\mathbf{A}} = \overline{c_0}^{\mathbf{A}} \odot \overline{c_1}^{\mathbf{A}}$. Then, $c \leq c_0 * c_1$. Given that $c_0 \in \mathcal{C}_a^-$ and $c_1 \in \mathcal{C}_b^-$, then from the first remark we have that $c_0 * c_1 \in \mathcal{C}_{a\odot b}^-$ and given that this set is downwards closed, $c \in \mathcal{C}_{a\odot b}^-$ too, concluding the proof.

- The reasoning for $\wedge$ is exactly the same done for $\odot$, since it is also true that $\wedge$ is an increasing function in both components, which moreover, is continuous.
- The approach to the $\Rightarrow$ connective can be simplified using that it is the residuum of $\odot$. Indeed, first consider $c \in \mathcal{C}_*$ such that $c \leq e(a) \to e(b)$. By residuation (on $[0, 1]$), $c * e(a) \leq e(b)$. By definition of e over the constants, this is the same that $e(\overline{c}^{\mathbf{A}}) * e(a) \leq e(b)$. Then, from the previous point of the proof, $e(\overline{c}^{\mathbf{A}} \odot a) \leq e(b)$. Given that from he definition of e it is immediate that it is order-preserving, we have that $\overline{c}^{\mathbf{A}} \odot a \leq b$. Applying now residuation of $\odot$, $\overline{c}^{\mathbf{A}} \leq a \Rightarrow b$. Then, again by the definition of e, $c = e(\overline{c}^{\mathbf{A}}) \leq e(a \Rightarrow b)$.

  For the other direction, let $c \in \mathcal{C}_*$ such that $c \leq e(a \Rightarrow b)$. By definition, $\overline{c}^{\mathbf{A}} \leq a \Rightarrow b$. By residuation of $\odot$it follows that $\overline{c}^{\mathbf{A}} \odot \leq b$. Then, $e(\overline{c}^{\mathbf{A}} \odot a) \leq e(b)$ and from the previous point of the proof, $c * e(a) = e(\overline{c}^{\mathbf{A}}) * e(a)) = e(\overline{c}^{\mathbf{A}} \odot a) \leq e(b)$. By residuation of $*$ in $[0, 1]$, $c \leq e(a) \to e(b)$, concluding the proof.

On the other hand, we know that for any two elements a, b of a linearly ordered L_*^∞-algebra,

$$\neg\delta^{\mathbf{A}}(a \Rightarrow b) = \begin{cases} 1 & \text{if } b < a \\ 0 & \text{if } a \leq b \end{cases}$$

From here, it is immediate to see that any homomorphism between two different L_*^∞-chains $\mathbf{A}$ and $\mathbf{A}'$ is injective. Indeed, if $b < a$ in A, then under an homomorphism h we have that $h(\neg\delta^{\mathbf{A}}(a \Rightarrow b)) = 1$ and thus, being an homomorphism, that $\neg\delta^{\mathbf{A}'}(h(a) \Rightarrow' h(b)) = 1$, so $h(b) < h(a)$ in $\mathbf{A}'$.[4] This concludes the proof. ⊠

[4]This can be also seen as a direct consequence of the fact that all L_*^∞-chains are relatively simple.

An immediate corollary of the previous result is the following characterization of the L_*^∞-algebras.

Corollary 4.14. *Any L_*^∞-algebra is a subalgebra of a direct product of $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$.*

Strong standard completeness of L_*^∞ follows now easily.

Theorem 4.15 (Strong Standard Completeness of L_*^∞)**.** *Let $\Gamma \cup \{\varphi\} \subseteq Fm$. Then the following are equivalent:*

1. $\Gamma \vdash_{L_*^\infty} \varphi$
2. $\Gamma \models_{[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}} \varphi$.

Proof. 1. implies 2. comes from the soundness of L_*^∞. For what respects the other implication, suppose that $\Gamma \not\vdash_{L_*^\infty} \varphi$. Then, by Lemma 4.11 there is a linearly ordered L_*^∞-algebra $\mathbf{A}$ and an homomorphism $h : \mathbf{Fm} \to \mathbf{A}$ such that $h([\Gamma]) \subseteq \{1\}$ and $h(\varphi) < 1$. It is immediate that $h([\mathbf{Fm}])$ is a countable subalgebra of $\mathbf{A}$ (thus linearly ordered) and so it can be embedded into $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$ by the embedding e built in the previous lemma. Then, it is clear that $e \circ h : \mathbf{Fm} \to [\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$ is an homomorphism such that $e \circ h(\Gamma) \subseteq \{1\}$ and $e \circ h(\varphi) < 1$. ⊠

In other words, we have proved that the density rule is always enough to provide a strongly complete axiomatization for the standard algebra of a left-continuous t-norm.

4.3 A general result on semilineary

The section is mainly motivated by the wish of knowing when does an infinitary logic expanding $MTL_\triangle$ enjoy strong completeness with respect to linearly ordered algebras of its algebraic counterpart. When developing the results previously presented and in particular, Theorem 4.9, we realized the main ideas used there did not depend on the particular logic we were studying. In fact, we can prove a more general version of that theorem, that is of use in the next section and also in Chapter 5.

This is a quite interesting result concerning the semilinearity on extensions of $MTL_\triangle$ with infinitary inference rules. It is also of great importance for the development of this dissertation, since we will resort to it along the next section, and, more importantly, in the next Chapter (see Lemmas 5.11 and 5.11).

Theorem 4.16. *Let L be an expansion of $MTL_\triangle$ such that*

- *L is implicative for $\to$,*
- *L is axiomatized by a countable amount of inference deduction rules, each one closed under $\vee$ and with a finite number of variables.*

Let $\Gamma \cup \{\alpha\} \subseteq Fm$ be such that $\Gamma \not\vdash_L \alpha$. Then there is a prime theory Γ' extending Γ such that $\Gamma' \not\vdash_L \alpha$.

Proof. First, it is easy to check that the Deduction Lemma with Δ keeps holding, by the same reasoning that in Lemma 4.7. Moreover, for each rules R denote by $prem(R)$ the premises of rule and by $con(R)$ its consequence. We can first prove that the result analogous to Lemma 4.8 holds also here naturally.

> CLAIM: If $T \not\vdash_L \alpha$ and $T \vdash_L \sigma(con(R_i)) \to \alpha$ for some rule R_i and some substitution σ, then there is $\rho_i \in \sigma([prem(R_i)])$ such that $T, \neg\Delta\rho_i \not\vdash_L \alpha$.
>
> Suppose that $T, \neg\Delta\rho_i \vdash_L \alpha$ for all $\rho_i \in \sigma([prem(R_i)])$. Then, by the Deduction theorem, $T \vdash_L \Delta\neg\Delta\rho_i \to \alpha$. Using the $\Delta\neg\Delta\varphi \leftrightarrow \neg\Delta\varphi$ is a theorem of the logic and point 11. from Remark 4.2 we get that $T \vdash_L \neg\neg\Delta\rho_i \vee \alpha$ for all $\rho_i \in \sigma([prem(R_i)])$. Since $\neg\neg\Delta\varphi \leftrightarrow \Delta\varphi$ is a theorem of the logic this is equivalent to have $T \vdash_L \Delta\rho_i \vee \alpha$ for all $\rho_i \in \sigma([prem(R_i)])$. Now, given that the inference rules are closed under $\vee$, we can apply R_i and get that $T \vdash_L \sigma(con(R_i)) \vee \alpha$. By rule G_Δ, point 11. from Remark 4.2 it follows that $T \vdash_L \sigma(con(R_i))\neg\Delta \to \alpha$. Finally, since $\Delta\varphi \vee \neg\Delta\varphi$ is a theorem of the logic, this implies that $T \vdash_L \alpha$, which is a contradiction.

For this reason, if the number of rules is finite, the proof can be done exactly as the one from Theorem 4.9 (instead of splitting in even and odd steps, we can simply split in $mod(k+1)$ steps for k being the number of rules).

Consider the case where there are infinite deduction rules, $\mathcal{R} = \bigcup_{i\in\mathbb{N}} R_i$. Let $\langle\varphi_n, \psi_n\rangle$ be an enumeration of all formulas in the language and as in Theorem 4.9, let $Cn(T)$ stand for the consequences of T in the logic L.

Initialize a variable $CL = \emptyset$ (where CL stands for Conditions List); CL must be understood as a set that will store the changes that are stored in order to be done at each step. Intuitively, the idea behind this proof is ordering the deduction rules and manage, for each pair of formulas, all the rules indexed below the current step at one time (which is a finite set) and the rest of the rules, when the step corresponding to its index arrives. Consider the following construction of theories:

- **step 0** $\Gamma_0 = Cn(\Gamma)$,
- **step 2n+1**
    1. If $\alpha \notin Cn(\Gamma_{2n}, \varphi_n \to \psi_n)$ put $\Gamma^0_{2n+1} = Cn(\Gamma_{2n}, \varphi_n \to \psi_n)$. Otherwise, put $\Gamma^0_{2n+1} = Cn(\Gamma_{2n}, \psi_n \to \varphi_n)$
    2. For each $\chi \to \delta$ in CL (a finite amount so we consider them indexed by i going from 0 to s), consider all substitutions σ_k, with $1 \leq k \leq K$,[5] such that $\sigma(con(R_n)) = \chi \to \delta$, or if $\chi = \overline{1}$, the substitutions such that $\sigma(con(R_n)) = \delta$ and let $S_k = \sigma_k[prem(R_n)]$.

[5] There are at most a finite amount of such substitutions because of the restriction on the variables of the rules.

From the observation from the beginning of this proof, we know there is $\rho_1 \in S1$ such that $\Gamma^{i-1}_{2n+1} \nvdash_L \neg\Delta\rho_1 \to \alpha$. Similarly, then there is $\rho_2 \in S_2$ such that $\Gamma^{i-1}_{2n+1}, \neg\Delta\rho_1 \nvdash_L \neg\Delta\rho_2 \to \alpha$ and this for all numbers below K. Thus, there are $\rho_1 \in S_1, ..., \rho_K \in S_K$ such that $\Gamma^{i-1}_{2n+1}, \neg\Delta\rho_1, ..., \neg\Delta\rho_K \nvdash_L \alpha$. Put $\Gamma^{i}_{2n+1} = Cn(\Gamma^{i-1}_{2n+1}, \neg\Delta\rho_1, ..., \neg\Delta\rho_K)$. Repeat this process for $1 \leq i \leq |CL|$.

Then, let $\Gamma_{2n+1} = \Gamma^{i}_{2n+1}$ for $i = |CL|$.

- **step 2n+2** Suppose $\Gamma^0_{2n+1} = Cn(\Gamma_{2n}, \psi_n \to \varphi_n)$. If $\alpha \notin Cn(\Gamma_{2n+1}, \varphi_n \to \psi_n)$ then let $\Gamma_{2n+2} = Cn(\Gamma_{2n+1}, \varphi_n \to \psi_n)$. Otherwise, add $\varphi_n \to \psi_n$ to CL and do the following process.

  Put $\Gamma^0_{2n+2} = \Gamma_{2n+1}$. For each $1 \leq i \leq n$, consider all substitutions σ_k (with $1 \leq k \leq K$) such that $\sigma_k(con(R_i)) = \varphi_n \to \psi_n$ or, if $\varphi_n = 1$, the substitutions such that $\sigma_k(con(R_i)) = \psi_n$ and also the sets $S_k = \sigma_k[prem(R_i)]$. As before, there is $\rho_1 \in S1$ such that $\Gamma^{i-1}_{2n+2} \nvdash_L \neg\Delta\rho_1 \to \alpha$. Similarly, then there is $\rho_2 \in S_2$ such that $\Gamma^{i-1}_{2n+2}, \neg\Delta\rho_1 \nvdash_L \neg\Delta\rho_2 \to \alpha$ and this for all numbers below K. Thus, there are $\rho_1 \in S_1, ..., \rho_K \in S_K$ such that $\Gamma^{i-1}_{2n+2}, \neg\Delta\rho_1, ..., \neg\Delta\rho_K \nvdash_L \alpha$. Put $\Gamma^{i}_{2n+2} = Cn(\Gamma^{i-1}_{2n+1}, \neg\Delta\rho_1, ..., \neg\Delta\rho_K)$.

  Then, let $\Gamma_{2n+2} = \Gamma^{n}_{2n+2}$.

It is easy to see that for each i, $\Gamma_i \subseteq \Gamma_{i+1}$ and that $\Gamma_i \nvdash_L \alpha$. To check that $\Gamma' = \bigcup_{i\in\mathbb{N}} \Gamma_i$ is closed under deduction rules, consider some rule R_n and suppose that there is a substitution σ such that $\varphi = \sigma(con(R_n)) \notin \Gamma'$ and $\sigma(prem(R_n)) \subseteq \Gamma'$. Then, assume the pair of formulas $\langle \overline{1}, \varphi\rangle$ is indexed by k. If $n \leq k$, then we have that $\neg\Delta\rho_i \in \Gamma^{n}_{2k+2}$ for some $\rho_i \in \sigma(prem(R_n))$ (recall that all possible substitutions where considered). Then $\neg\Delta\rho_i \in \Gamma_{2k+2}$. Otherwise, we have that $\neg\Delta\rho_i \in \Gamma^{s}_{2n+1}$ (where s is some finite number below n) for some $\rho_i \in \sigma(prem(R_n))$, so $\neg\Delta\rho_i \in \Gamma_{2n+2}$.

Since $\sigma(prem(R_n)) \subseteq \Gamma'$, then in particular $\rho_i \subseteq \Gamma'$ and thus, there is some k_0 where $\rho_i \in \Gamma_k$. Let $m = max\{k_0, 2k+2, 2n+2\}$.

Clearly, $\neg\Delta\rho_i \in \Gamma_m$, since either $\neg\Delta\rho_i \in \Gamma_{2k+2}$ or $\neg\Delta\rho_i \in \Gamma_{2n+2}$ and $\Gamma_s \subseteq \Gamma_m$ for $s \leq m$.

On the other hand, $\rho_i \in \Gamma_m$ too, so $\Gamma_m \vdash_L \bot$, which is a contradiction since $\Gamma_m \nvdash_L \alpha$ by construction. ⊠

The following is the remarkable corollary that can be proven following the same reasoning that in Theorem 4.11.

Corollary 4.17. *As a corollary, we have that any logic L like in the Theorem, i.e., an expansion of MTL_Δ such that*

- *L is implicative for $\to$,*
- *L is axiomatized by a countable amount of inference deduction rules, each one closed under $\vee$ and with a finite number of variables.*

is strongly complete with respect to the linearly ordered algebras from its algebraic counterpart.

The previous one is an interesting result that opens the door to a systematic and uniform study of some infinitary logics. Moreover, it is as strong as it can be, in the sense that we will later see that there exist axiomatic systems fulfilling all the premises of the theorem but the limited cardinality on the set of inference rules which are not semilinear (Lemma 5.8).

4.4 Expansions by operations from $[0,1]$

As a side effect of studying the possible axiomatizations of $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$ we realized the results obtained were applicable not only for the logic of this particular algebra. It is clear that we can use the D^∞ as the "main" infinitary rule of Hilbert-style axiomatizations not only for the logic of $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$, but also for the more general case of the logics arising from the standard algebra of a left-continuous t-norm expanded with Δ and *an arbitrary set of operations* defined in $[0,1]$ that respect some -but not so strict- regularity conditions and the subalgebra generated by all the operations from the rationals in $[0,1]$. This work complements in some sense the one presented by Cintula in [34], showing an alternative way of axiomatizing logics Pavelka-complete.

The general problem: representable operations

In [34], Cintula studies the extensions of the standard MTL algebras with rational constants by argument-wise monotonic operations (i.e. those which, fixed all variables except one result in a (unary) increasing or decreasing operation). Our approach allows us to partially generalize those results, working with a family of operations with different restrictions. We rely on the density rule to approach the values on the new operations added to the system, so it is reasonable to restrict our studies to operations for which this can be done: those whose images can be reached as limits of the values taken at the constants. These turn to be operations with a domain that can be decomposed in argument-wise monotonic and directionally (i.e., left or right) continuous regions that can be determined in the language of the logic. However, in our approach we lose the capacity to work with some operations that are considered in [34]: for instance, the ones that have jump-type discontinuity points for which, for some argument, the value of the function does not coincide with with the left nor with the right limit. This is natural, since it is not clear how to deal with functions whose limit points cannot be reached through the rationals using the density rule presented before.

In order to unify the notation, given a n-ary function $\star$ that is componentwise monotonic and (left or right) continuous on $U = U^1 \times \ldots \times U^n \subseteq [0,1]^n$, we let

$$\eta_i^{\star U} = \begin{cases} + & \text{if } \star \text{ is increasing}^{6} \text{ in } U^i \\ - & \text{otherwise (decreasing)} \end{cases}$$

$$\delta_i^{\star U} = \begin{cases} L & \text{if } \star \text{ is left-continuous in } U^i \\ R & \text{otherwise (right-continuous)} \end{cases}$$

and we introduce the following notation:

$$impl(s,\varphi,\psi)=\begin{cases}\varphi\to\psi & \text{if } s=+ \text{ or } s=L\\ \psi\to\varphi & \text{if } s=- \text{ or } s=R\end{cases}$$

A family of operations that can be axiomatized using the density rule is the following one.

Definition 4.18. We say a n-ary function $\star$ is ***logically representable*** if there exists $I\subseteq\omega$ and $\{U_i\}_{i\in I}$, called a ***simplified universe*** (and we refer to the U_i's as ***regions*** of this simplified universe) of $\star$ such that

1. $\bigcup_{i\in I}U_i=[0,1]^n$ and for each $i\in I$, $U_i=U_i^1\times\cdots\times U_i^n$ with U_i^j being a closed interval of $[0,1]$,[7]

2. For each $i\in I$, $\star$ is component-wise left or right continuous in U_i and component-wise monotonic in *the interior* of U_i;

3. For each $(x_1,...,x_n)\in[0,1]^n$, either it is a tuple of rational numbers or there exists U_j such that for each $1\leq i\leq n$, $x_i\in U_j^i$ and $\inf U_j^i<x_i$ if $\delta_i^{\star U_j}=L$ and $\sup U_j^i>x_i$ otherwise. [8]

We call these operations logically representable because of two logic-oriented characteristics. First, it is clear that splitting the universe on intervals we can write, in the syntax (using the rational constants), a set of formulas that represents logically the idea of a certain point belonging in that part. For instance, the truth of formulas like $(\overline{0.4}\to\varphi)\&(\varphi\to\overline{0.7})$ expresses that the value of φ belongs to the interval $[0.4,0.7]$. For simplicity, we use the symbol $\in$ in the logic to write these kind of formulas. In the previous example, the formula would be equivalently expressed as "$\varphi\in[0.4,0.7]$". Similarly, an expression "$(\varphi,\psi)\in U$", where $U=[a,b]\times[c,d]$ is used as a shorthand for "$\varphi\in[a,b]\wedge\psi\in[c,d]$". Second, the regularity of the function in the regions of its simplified universe is characterizable with rules, as we will see in the following section. This allows to propose an axiomatic system whose associated linearly ordered algebras interpret the new operations in a way that the monotonicity conditions is satisfied. We conjecture that logically representable operations are not the only ones that can be treated using the density rule approach. Determining the regions using intervals is an elegant and clean approach, but other kind of definitions can be expressed syntactically. For instance, it would also be possible to address a binary function $f(x,y)$ with two regions determined by $x\leq y$ and $y<x$ (since these can be expressed in the syntax by $x\to y$ and $\neg\Delta(x\to y)$. The full characterization of these operations is ongoing work and not complete, and for this

[7] For simplicity we assume the extreme points of the interval to be rational numbers, but this is not necessary. In other case, some deduction rules that will be defined later would have an infinite set of premises and some small modifications must be done in the axiomatic system, but the methodology is the same that in the studied case.

[8] This last condition implies that x_i does not coincide with the edge point that is not covered by the continuity direction.

reason we will stick along this work with the logically representable operations defined here, that are nevertheless enough to understand how can the density rule be used for other operations.

Figure 4.1 gives an intuitive idea of somelogically representable operations. On the other hand, functions that do not belong to this class are, for instance, those that have a discontinuity jump in a non-rational point.

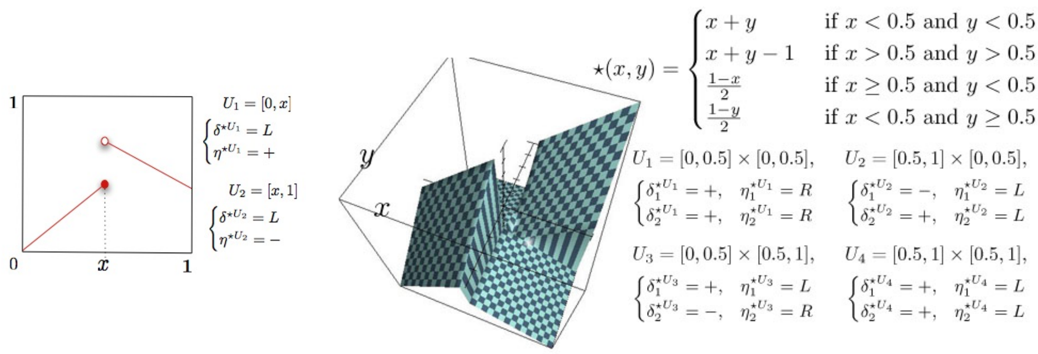


Figure 4.1: Examples of representable operations

Given a set OP of logically representable operations, we consider a new language $\mathfrak{L}(OP)$, given by [9]

$$\langle \&/2, \rightarrow /2, \wedge/2, \Delta/1, \{\overline{\star}/\Lambda(\star)\}_{\star \in OP}, \{\overline{c}/0\}_{c \in \mathcal{C}_*(OP)}\rangle$$

where $\mathcal{C}_*(OP)$ is the subalgebra generated by the rational numbers in $[0,1]$ using the operations $*, \Rightarrow_*, \delta$ and $\star$ for each $\star \in OP$. Accordingly we let $[\mathbf{0},\mathbf{1}]_*(\mathbf{OP})$ to be the algebra $\langle [0,1], *, \Rightarrow_*, \wedge, \delta, \{\overline{\star}\}_{\star \in OP}, \{c\}_{c \in \mathcal{C}_*(OP)}\rangle$.

We shall now study which inference rules are needed in order to axiomatize a set of representable operations OP. In the first place, since we are working over propositional expansions of MTL (in the sense that the new operations are functions over the standard $*$-algebra), it is natural to require that an axiomatic system for the logic induced by $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}(\mathbf{OP})$ to be an *implicative logic* in the sense of Rasiowa. To ensure this, we have to add to our axiomatic system, for each new connective of the logic $\overline{\star}$, the following congruence rule (from the definition of Rasiowa implicative logic):

$$(\vee\mathsf{CONG}^\star)\frac{r \vee \{p_1 \leftrightarrow q_1, ..., p_n \leftrightarrow q_n\}}{r \vee (\overline{\star}(p_1, ..., p_n) \rightarrow \overline{\star}(q_1, ..., q_n))}$$

Besides this, we need two new families of rules in order to control the behaviour of the operation on the "non-rational" elements of the algebra (i.e. elements that do not coincide with the interpretation of any rational truth-constant). One type of rules copes with the monotonicity of the functions and

[9]recall that $\Lambda(\star)$ stands for the arity of the $\star$ operation.

the other refers to the continuity. While the intuition behind them is quite natural, the formal rules finally defined could seem a bit unclear due to the necessity of addressing the different regions and components of each function. For this reason, along the following definitions, we propose a simple example to allow the reader to interpret more easily the regularity rules.

We let $\divideontimes$ to be a binary representable operation whose simplified universe has the following form:

$U_1 \cup U_2$ where $\{U_1 = [0,1] \times [0,b], U_2 = [0,1] \times [b,1]\}$ and with

$$\begin{cases} \delta_1^{\star U_1} = L, & \eta_1^{\star U_1} = + \\ \delta_2^{\star U_1} = L, & \eta_2^{\star U_1} = - \end{cases} \text{and} \begin{cases} \delta_1^{\star U_2} = L, & \eta_1^{\star U_2} = - \\ \delta_2^{\star U_2} = R, & \eta_2^{\star U_2} = - \end{cases}$$

The rules concerning the monotonicity are finitary rules, since their meaning is that of fixing, point wise, and order relation. Their formal version is a bit more complex than that, in order to keep under control the extreme points of the regions from the simplified universe, since there exists the possibility of one of the extreme points behaving non-monotonically (if the function is non-continuous in it).

Formally, the rules that characterize the monotonicity of a n-ary operation $\star$ are of the following form: for each region U from its simplified universe and each coordinate $1 \leq i \leq n$ we consider the following rule

$$(\vee \mathsf{M}_i^{\star \mathsf{U}}): \ \frac{\begin{array}{c} r \vee \{(p_1, ..., p_i, ..., p_n) \in U, s \in U^i, \\ impl(\eta_i^{\star U}, s, q), impl(\eta_i^{\star U}, q, p_i)\} \end{array}}{r \vee \chi(s) \vee (\overline{\star}(p_1, ..., q, ..., p_n) \to \overline{\star}(p_1, ..., p_i, ..., p_n))}$$

where $\chi(s) = s \leftrightarrow \overline{extr}$, and

$$extr = \begin{cases} \inf U^i & \text{if } \delta_i^{\star U} = L \\ \sup U^i & \text{if } \delta_i^{\star U} = R \end{cases}$$

Observe that the meaning of the formula $\chi(s)$ is just to check if s coincides with the edge of the region – opposite to that covered by the continuity of the operation in that component.

In terms of the $\divideontimes$ operation, this results in the consideration of the following four rules (one for each pair region-component):

$$\frac{r \vee \{(p_2 \to \overline{b}, s \to q, q \to p_1)\}}{r \vee \neg s \vee (\overline{\divideontimes}(q, p_2) \to \overline{\divideontimes}(p_1, p_2))} \qquad \frac{r \vee \{(p_2 \to \overline{b}, s \to \overline{b}, q \to s, p_2 \to q)\}}{r \vee \neg s \vee (\overline{\divideontimes}(p_1, q) \to \overline{\divideontimes}(p_1, p_2))}$$

$$\frac{r \vee \{(\overline{b} \to p_2, q \to s, p_1 \to q)\}}{r \vee \neg s \vee (\overline{\divideontimes}(q, p_2) \to \overline{\divideontimes}(p_1, p_2))} \qquad \frac{r \vee \{(\overline{b} \to p_2, \overline{b} \to s, q \to s, p_2 \to q)\}}{r \vee s \vee (\overline{\divideontimes}(p_1, q) \to \overline{\divideontimes}(p_1, p_2))}$$

On the other hand, we also need rules that determine the continuity of the function in the regions of the simplified universe. That can be done by translating some of the information about the operation to the axiomatic system, in terms of fixing the behaviour of limit points. For this reason, we need to resort to infinitary rules, given that limits are a naturally infinitary concept. In particular, the intuitive meaning of the two rules below capture the fact that, for a given point in a continuity fragment of a function, if the value of the function is smaller/greater than a certain value in all rationals from that (continuity) fragment, then so is the image of that point. Figure 4.2 shows the intuition behind the elements in the following rules. Formally, for each region U of the simplified universe of $\star$ and each component $1 \leq i \leq n$, we consider the following two rules: (with χ is as above):

- If $\star$ is ***left-continuous and increasing*** in U^i ($\delta_i^{\star U} = L, \eta_i^{\star U} = +$) or ***right-continuous and decreasing*** ($\delta_i^{\star U} = R, \eta_i^{\star U} = -$):

$$(\vee \mathsf{C}_\mathsf{i}^{\star \mathsf{U}}): \quad \frac{\begin{array}{c} r \vee \{(p_1, ..., p_n) \in U, q \to \overline{\star}(p_1, ..., p_n), \\ \{\chi(\overline{d}) \vee impl(\delta_i^{\star U}, p_i, \overline{d}) \vee \\ \overline{\star}(p_1, ..., \overline{d}, ..., p_n) \to q\}_{d \in U^i \cap \mathcal{C}_*}\} \end{array}}{r \vee (\overline{\star}(p_1, ..., p_n) \to q)}$$

- If $\star$ is ***left-continuous and decreasing*** in U^i ($\delta_i^{\star U} = L, \eta_i^{\star U} = -$) or ***right-continuous and increasing*** ($\delta_i^{\star U} = R, \eta_i^{\star U} = +$):

$$(\vee \mathsf{C}_\mathsf{i}^{\star \mathsf{U}}): \quad \frac{\begin{array}{c} r \vee \{(p_1, ..., p_n) \in U, \overline{\star}(p_1, ..., p_n) \to q, \\ \{\chi(\overline{d}) \vee impl(\delta_i^{\star U}, p_i, \overline{d}) \vee \\ q \to \overline{\star}(p_1, ..., \overline{d}, ..., p_n)\}_{d \in U^i \cap \mathcal{C}_*}\} \end{array}}{r \vee (\overline{c} \to \overline{\star}(p_1, ..., p_n))}$$

Continuing with the example of operation $\circledast$, this results in the consideration of the following four rules (one for each pair region-component):

$$\frac{r \vee \{p_2 \to \overline{b}, q \to \overline{\circledast}(p_1, p_2), \{\neg \overline{e} \vee p_1 \to \overline{e}) \ \vee \overline{\circledast}(\overline{e}, p_2) \to q\}_{e \in \mathcal{C}_*}\}}{r \vee (\overline{\circledast}(p_1, p_2) \to q)}$$

$$\frac{r \vee \{p_2 \to \overline{b}, \overline{\circledast}(p_1, p_2) \to q, \{\neg \overline{e} \vee p_2 \to \overline{e}) \ \vee q \to \overline{\circledast}(p_1, \overline{e})\}_{e \in [0,b]_\mathbb{Q}}\}}{r \vee (\overline{\circledast}(p_1, p_2) \to q)}$$

$$\frac{r \vee \{\overline{b} \to p_2, \overline{\circledast}(p_1, p_2) \to q, \{\neg \overline{e} \vee p_1 \to \overline{e}) \ \vee q \to \overline{\circledast}(\overline{e}, p_2)\}_{e \in \mathcal{C}_*}\}}{r \vee (\overline{\circledast}(p_1, p_2) \to q)}$$

$$\frac{r \vee \{\overline{b} \to p_2, q \to \overline{\circledast}(p_1, p_2), \{(\overline{e} \leftrightarrow \overline{b}) \vee p_2 \to \overline{e}) \ \vee \overline{\circledast}(p_1, \overline{e}) \to q\}_{e \in \mathcal{C}_*}\}}{r \vee (\overline{\circledast}(p_1, p_2) \to q)}$$

These rules enforce that the value of a function in a point can be approached through the values on rational constants near it (in the direction in which the function is continuous).

Strong standard completeness

In the previous section we have defined the necessary rules to provide a formal definition of an axiomatic system that is complete with respect to $[\mathbf{0}, \mathbf{1}]^{\mathbb{Q}}_{*}(\mathbf{OP})$, for a set OP of representable operations.

Definition 4.19. Let $*$ be a left continuos t-norm and let OP a set of logically representable operations. Then the axiomatic system $L^{\infty}_{*}(OP)$ is defined on the language $\mathfrak{L}(OP)$ adding to $MTL^{\mathbb{Q}}_{*}$ the following axioms and rules:

- book-kepping axioms (Book-$\star$), for each $\star \in OP$ and constants in $\mathcal{C}_{*}(OP)$,
- density rule (D^{∞})
- congruence rule $(\vee\mathsf{CONG}^{\star})$, for each $\star \in OP$
- monotonicity rules $(\vee\mathsf{M}_{\mathsf{i}}^{\star^{\mathsf{U}}})$, for each $\star \in OP$ and region U of its universe
- continuity rules $(\vee\mathsf{C}_{\mathsf{i}}^{\star^{\mathsf{U}}})$, for each $\star \in OP$ and region U of its universe

First, it is an exercise to check that all the regularity rules are sound. Indeed, the only case that could be somewhat not obvious is the last family of formulas, but observe that they hold in the standard algebra with the corresponding operations $[\mathbf{0}, \mathbf{1}]^{\mathbb{Q}}_{*}(\mathbf{OP})$: if $c < \star(x_1, ..., x_n)$, there is c_i, with $c_i \leq x_i$ if $\delta_i^{\star U} = +$ or with $x_i \leq c_i$ if $\delta_i^{\star U} = -$, such that $c < \star(x_1, ..., c_i, ..., x_n)$. Figure 4.2 shows some examples of this.

In order to prove completeness of $L^{\infty}_{*}(OP)$, we begin by proving they are strongly complete with respect to the linearly ordered algebras from the algebraic companion. For this, we just need to check that the rules added to MTL_{Δ} fulfil the requirements of Theorem 4.16.

Observe that we have added a rule for each new operation in order to get an implicative logic (thus, finite congruence rules), and since the definition of representable universe is limited to a countable amount of regions, so there is a countable amount of inference rules concerning the regularity conditions. Moreover, all the rules are closed under the $\vee$ operation (by definition) and satisfy the restriction over the variables. As a corollary of the previous theorem, we obtain the completeness with respect to linearly ordered algebras of the class, reasoning as in Theorem 4.11.

Theorem 4.20. *For any set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm$, the following are equivalent:*

1. $\Gamma \vdash_{L^{\infty}_{*}(OP)} \varphi$
2. $\Gamma \models_{\mathbf{C}} \varphi$ *for all $L^{\infty}_{*}(OP)$-chain $\mathbf{C}$.*

What remains is then to study the relationship of the linearly ordered $L^{\infty}_{*}(OP)$-algebras with respect to the one defined over the real unit interval, i.e., $[\mathbf{0}, \mathbf{1}]^{\mathbb{Q}}_{*}(\mathbf{OP})$.

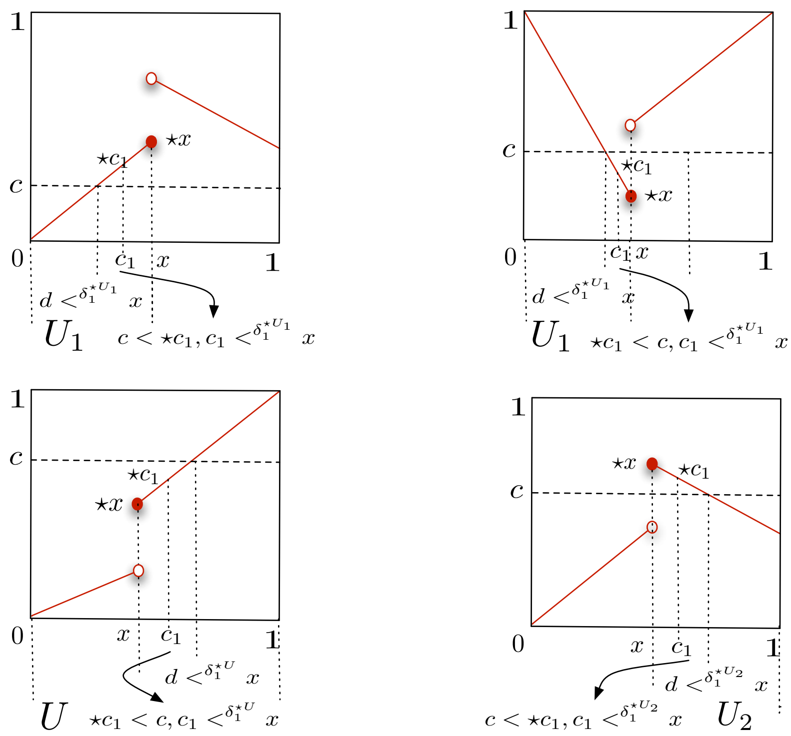


Figure 4.2: Examples for the rule $\mathsf{C}_i^{\star \mathsf{U}}$.

To show that, for an arbitrary set OP of logically representable operations, $L_*^\infty(OP)$ enjoys the strong standard completeness we can resort again to the same method as in the case of L_*^∞: it is possible again to embed any linearly ordered $L_*^\infty(OP)$ into $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}(\mathbf{OP})$.

Having the density rule, the natural approach is to consider again the mapping defined in 4.13 and study whether it is an embedding in the new algebraic setting (with the new operations from OP).

In order to show this, we first observe that the regularity conditions of the operations from OP (as defined in $[0,1]$) are properly translated to their correspondent operation symbols in the logic.

Lemma 4.21. *Let OP be a set of logically representable operations in $[0,1]$ and let $\mathbf{A}$ be a linearly ordered $L_*^\infty(OP)$-algebra. Let $\star \in OP$ be a n-ary operation with simplified universe $U = \bigcup_{i\in I} U_i \subseteq [0,1]^n$, and for some $k \in I$, let $x_1, ..., x_n \in U_k$ in the sense of point 3 from the definition of simplifiable universe and such that for some $1 \leq i \leq n$, $x_i \neq \overline{c}^{\mathbf{A}}$ for any $c \in \mathcal{C}_*(OP)$.* [10]

Then

[10]The fully rational points can have an irregular behaviour, in the sense that they are completely determined by the book-keeping axioms.

$$\overline{\star}^{\mathbf{A}}(x_1, ..., x_n) =$$
$$\Sigma_1\{...\Sigma_n\{\overline{\star}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, ..., \overline{c_n}^{\mathbf{A}}) : c_n \in C_n\}... : c_1 \in C_1\}$$

where

$$\Sigma_i = \begin{cases} \sup & \text{if } \eta_i^{\star U_k} = +, \delta_i^{\star U_k} = L \\ & \text{or } \eta_i^{\star U_k} = -, \delta_i^{\star U_k} = R \\ \inf & \text{otherwise} \end{cases}$$
$$C_i = \begin{cases} \{a \in U_k^i \cap \mathcal{C}_*(OP) : \overline{a}^{\mathbf{A}} \leq x_i\} & \text{if } \delta_i^{\star U} = L \\ \{a \in U_k^i \cap \mathcal{C}_*(OP) : \overline{a}^{\mathbf{A}} \geq x_i\} & \text{otherwise} \end{cases}$$

Proof. For the sake of readability, we write the proof assuming $\star$ is the $\circledast$ operation used as an example along this section, and studying one of its components (the first one). Other cases are proven analogously.

The statement of the lemma (for $k = 1$) can be written in this case as $\overline{\star}^{\mathbf{A}}(x1, x2) = \sup\{\inf\{\overline{\star}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, \overline{c_2}^{\mathbf{A}}) : c_2 \in (0, b]_{\mathbb{Q}}, \overline{c_2}^{\mathbf{A}} \leq x_2\} : c_1 \in (0, 1]_{\mathbb{Q}}, \overline{c_1}^{\mathbf{A}} \leq x_1\}$.

First, assume towards a contradiction, that there is $x \in A$ such that $\overline{\star}^{\mathbf{A}}(x1, x2) < x < \sup\{\inf\{\overline{\star}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, \overline{c_2}^{\mathbf{A}}) : c_2 \in (0, b]_{\mathbb{Q}}, \overline{c_2}^{\mathbf{A}} \leq x_2\} : c_1 \in (0, 1]_{\mathbb{Q}}, \overline{c_1}^{\mathbf{A}} \leq x_1\}$.
From $\overline{\star}^{\mathbf{A}}(x1, x2) < x$ we know that the rule $(\vee \mathsf{C}_2^{\star \mathsf{U}_1})$ cannot be applied and thus, its premises shall be false. This implies that it exists $d_2 \in [0, b]_{\mathbb{Q}}$ such that

$$d_2 > 0,\ \overline{d_2}^{\mathbf{A}} < x_2 \text{ and } x > \overline{\star}^{\mathbf{A}}(x_1, \overline{d_2}^{\mathbf{A}}) \text{ [Condition 1].}$$

On the other hand, $x < \sup\{\inf\{\overline{\star}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, \overline{c_2}^{\mathbf{A}}) : c_2 \in (0, b]_{\mathbb{Q}}, \overline{c_2}^{\mathbf{A}} \leq x_2\} : c_1 \in (0, 1]_{\mathbb{Q}}, \overline{c_1}^{\mathbf{A}} \leq x_1\}$ implies, by definition, that there is $d_1 \in (0, 1]_{\mathbb{Q}}$ such that $\overline{d_1}^{\mathring{\mathbf{A}}} \leq x_1$ and $x < \inf\{\overline{\star}^{\mathbf{A}}(\overline{d_1}^{\mathbf{A}}, \overline{c_2}^{\mathbf{A}}) : c_2 \in (0, b]_{\mathbb{Q}}, \overline{c_2}^{\mathbf{A}} \leq x_2\}$. By definition, we have that for all $c_2 \in (0, b]_{\mathbb{Q}}$ such that $\overline{c_2}^{\mathbf{A}} \leq x_2$, it holds that $x < \overline{\star}^{\mathbf{A}}(\overline{d_1}^{\mathbf{A}}, \overline{c_2}^{\mathbf{A}})$. Applying the monotonicity rule for this region on the first component (the function was increasing) and since $\overline{d_1}^{\mathbf{A}} \leq x_1$, we have that $x < \overline{\star}^{\mathbf{A}}(\overline{d_1}^{\mathbf{A}}, \overline{c_2}^{\mathbf{A}}) \leq \overline{\star}^{\mathbf{A}}(x_1, \overline{c_2}^{\mathbf{A}})$ which contradicts [Condition 1].

Analogously, assume that there is $x \in A$ such that $\sup\{\inf\{\overline{\star}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, \overline{c_2}^{\mathbf{A}}) : c_2 \in (0, b]_{\mathbb{Q}}, \overline{c_2}^{\mathbf{A}} \leq x_2\} : c_1 \in (0, 1]_{\mathbb{Q}}, \overline{c_1}^{\mathbf{A}} \leq x_1\} < x < \overline{\star}^{\mathbf{A}}(x1, x2)$

From $x < \overline{\star}^{\mathbf{A}}(x1, x2)$, we know that the rule $(\vee \mathsf{C}_1^{\star \mathsf{U}_1})$ cannot be applied and thus, it exists $d_1 \in \mathcal{C}_*$ such that

$$d_1 > 0,\ \overline{d_1}^{\mathbf{A}} < x_1 \text{ and } x < \overline{\star}^{\mathbf{A}}(\overline{d_1}^{\mathbf{A}}, x_2) \text{ [Condition 2].}$$

On the other hand, $\sup\{\inf\{\overline{\star}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, \overline{c_2}^{\mathbf{A}}) : c_2 \in (0, b]_{\mathbb{Q}}, \overline{c_2}^{\mathbf{A}} \leq x_2\} : c_1 \in (0, 1]_{\mathbb{Q}}, \overline{c_1}^{\mathbf{A}} \leq x_1\} < x$ implies that for all $c \in (0, 1]_{\mathbb{Q}}$ with $\overline{c}^{\mathbf{A}} \leq x_1$ it holds that $\inf\{\overline{\star}^{\mathbf{A}}(\overline{c}^{\mathbf{A}}, \overline{c_2}^{\mathbf{A}}) : c_2 \in (0, b]_{\mathbb{Q}}, \overline{c_2}^{\mathbf{A}} \leq x_2\} < x$. In particular, this also holds for d_1. This implies that it exists $d_2 \in (0, b]_{\mathbb{Q}}$ such that $\overline{d_2}^{\mathbf{A}} \leq x_2$ and $\overline{\star}^{\mathbf{A}}(\overline{d_1}^{\mathbf{A}}, \overline{d_2}^{\mathbf{A}}) < x$. Applying the monotonicity rule for this region on the second component (decreasing), we have that $\overline{\star}^{\mathbf{A}}(\overline{d_1}^{\mathbf{A}}, x_2) \leq \overline{\star}^{\mathbf{A}}(\overline{d_1}^{\mathbf{A}}, \overline{d_2}^{\mathbf{A}}) < x$, which contradicts [Condition 2]. ⊠

From this result we can easily prove that the map $\theta : A \to [0,1]$ defined by

$$\theta(a) = \inf\{c \in \mathcal{C}_* : \overline{c}^{\mathbf{A}} \geq a\} = \sup\{c \in \mathcal{C}_* : \overline{c}^{\mathbf{A}} \leq a\}$$

is an embedding from a linearly ordered $L_*^\infty(OP)$-algebra $\mathbf{A}$ into $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}(\mathbf{OP})$.

Lemma 4.22. *Let* $\mathbf{A}$ *be a* $L_*^\infty(OP)$ *chain. Then, the function* θ *defined above is an embedding from* $\mathbf{A}$ *into* $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}(\mathbf{OP})$.

Proof. First note that for any constant $\overline{d}$, $d = \min\{c \in \mathcal{C}_* : \overline{c}^{\mathbf{A}} \geq \overline{d}^{\mathbf{A}}\} = \max\{c \in \mathcal{C}_* : \overline{c}^{\mathbf{A}} \leq \overline{d}^{\mathbf{A}}\}$ and so $\theta(\overline{d}^{\mathbf{A}}) = d = \overline{d}^{[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}(\mathbf{OP})}$. On the other hand, it is immediate to see that it is strictly order preserving: if $a < b \in A$, there exists $c \in \mathcal{C}_*$ such that $a < \overline{c}^{\mathbf{A}} < b$ and thus, $\theta(a) < c < \theta(b)$. This shows that θ is one-to-one.

Regarding the homomorphic conditions for the operations, we exploit the density of the constants in $\mathbf{A}$ and in $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}(\mathbf{OP})$. Observe that the left-continuous t-norm and its residuum are, by definition, logically representable operations, so the proof can be done in general for any logically representable operation $\star$.[11]

As for the $\leq$ direction, let $c \in \mathcal{C}_*$ such that $c < \theta(\overline{\star}^{\mathbf{A}}(x_1, ..., x_n))$. By definition and given that θ preserves the order, $\overline{c}^{\mathbf{A}} \leq \overline{\star}^{\mathbf{A}}(x_1, ..., x_n)$. By the previous lemma, it follows that $\overline{c}^{\mathbf{A}} \leq \Sigma_1\{...\Sigma_n\{\overline{\star}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, ..., \overline{c_n}^{\mathbf{A}}) : c_n \in C_n\}...x_1 \in C_1\}$. Then, $\overline{c}^{\mathbf{A}} \leq \overline{\star}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, ..., \overline{c_n}^{\mathbf{A}})$ for some $c_i \in C_i$ if $\Sigma_i = \sup$ and for all $c_i \in C_i$ if $\Sigma_i = \inf$ (for each $1 \leq i \leq n$).

We can use now the book-keeping axioms to get that $c \leq \star(c_1, ..., c_n)$ for c_i as above. Now, we can use the properties of $\star$ in $[0,1]$ (monotonicity and left-right continuity), take limits and conclude that $c \leq \star(\theta x_1, ..., \theta x_n)$.

In order to prove the $\geq$ inequality, let $c \in \mathcal{C}_*$ be such that $\star(\theta x_1, ..., \theta x_n) < c$. Then, as before (since $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}(\mathbf{OP})$ is linearly ordered), from the previous lemma we get $\Sigma_1\{...\{\Sigma_n\{\star(c_1, ..., c_n) : c_n \in C_n\}...\} : x_1 \in C_1\} < c$. Then, $\star(c_1, ..., c_n) < c$ for the families of c_i as above.

From the book-keeping axioms we have that $\overline{\star}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, ..., \overline{c_n}^{\mathbf{A}}) < \overline{c}^{\mathbf{A}}$ for c_i as above. We can now clearly take suprema and infima to get $\Sigma_1\{...\Sigma_n\{\overline{\star}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, ..., \overline{c_n}^{\mathbf{A}}) : c_n \in C_n\}...x_1 \in C_1\} \leq \overline{c}^{\mathbf{A}}$. Again from the previous lemma, it follows that $\overline{\star}^{\mathbf{A}}x_1, ..., x_n \leq \overline{c}^{\mathbf{A}}$. Since θ is order preserving, we finally have $\theta(\overline{\star}^{\mathbf{A}}x_1, ..., x_n) \leq \theta(\overline{c}^{\mathbf{A}}) = c$. ⊠

Strong standard completeness of $L_*^\infty(OP)$ follows straightforwardly.

[11] Nevertheless, the case of the left-continuous t-norm operation has a more direct approach, that does not need any of the $\vee\mathsf{CONG}^\star$, $\vee\mathsf{M}_i^{\star\mathsf{U}}$ nor $\vee\mathsf{C}_i^{\star\mathsf{U}}$ rules and that relies on the MTL-axiomatization of a residuated operation.

Theorem 4.23 (Strong Standard Completeness of $L_*^\infty(OP)$)**.** *For any set of formulas $\Gamma \cup \{\varphi\}$ the following are equivalent:*

1. $\Gamma \vdash_{L_*^\infty(OP)} \varphi$
2. $\Gamma \models_{[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}(\mathbf{OP})} \varphi$.

Proof. One direction (from 1 to 2) is soundness, that is easy to prove. As for the other implication, suppose that $\Gamma \nvdash_{L_*^\infty(OP)} \varphi$. Then, by Theorem 4.11 there is a linearly ordered $L_*^\infty(OP)$-algebra $\mathbf{A}$ and a $\mathbf{A}$-evaluation h such that $h([\Gamma]) \subseteq \{1\}$ and $h(\varphi) < 1$. It is immediate that $h([Fm])$ is a countable subalgebra of $\mathbf{A}$ (thus linearly ordered) and so it can be embedded into the standard algebra $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}(\mathbf{OP})$ by the embedding θ from the previous lemma. Then, it is clear that $\theta \circ h$ is a $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}(\mathbf{OP})$-evaluation such that $\theta \circ h(\Gamma) \subseteq \{1\}$ and $\theta \circ h(\varphi) < 1$. This concludes the proof. $\boxtimes$

Chapter 5

Modally well-behaved axiomatizations

The density rule presented in the previous chapter is not very comfortable to use when we expand the L_*^∞ axiomatic system with modalities. This is mainly due to its disjunctive form, that does not commute with the $\Box$ operator in any way, so it is not clear if it is possible to prove it is closed under the $\Box$. This turns to be quite problematic when facing the completeness proof of the modal expansion.

For this reason, we have searched for axiomatizations equivalent to L_*^∞ that instead of the density rule use inference rules that are modally well behaved in the sense that they can be proven to be closed under the $\Box$ operator (see Theorem 6.9). It turns out that a family of rules presented in [34] is modally well behaved in the previous sense, but it is only possible to use them to get systems equivalent to L_*^∞ for a particular family of left-continuous t-norms. We characterize this class, and give some examples of t-norms belonging to it.

5.1 Conjunctive inference rules

The main characteristic obtained from the density rule was the density of the constants on the linearly ordered algebras of the class. We can approach the problem of finding alternative axiomatizations for L_*^∞ studying different characterizations of this property on these algebras, and seeing whether these characterizations can be used in order to axiomatize a logic that is semilinear.

From [34] we can select a family of infinitary inference rules that are, from the point of view of the modal expansion, well behaved. They are inspired on the rules presented in the equations 4.2 and 4.1 from the previous chapter, but correspond only to the implication operation over all the points of the diagonal.

Definition 5.1. We call ***conjunctive inference rules*** to the set of infinitary rules given by

$$\mathsf{R}^\infty_\mathsf{x} : \frac{\{(p \to \overline{c}) \wedge (\overline{d} \to q)\}_{d\in[0,x)_\mathbb{Q}, c\in(x,1]_\mathbb{Q}}}{p \to q}$$

for each $x \in [0,1]$.

We will see that, for what concerns MTL_Δ, these rules have the same effect that the whole family given in 4.2, 4.1, in the sense that an axiomatic system that satisfies the conjunctive inference rules and which is, moreover, semilinear (and includes the book-keeping axioms) is strongly standard canonical complete.

We can first check that these rules are weaker than the density rule.

Lemma 5.2. *For each $x \in [0,1]$, the rule $\mathsf{R}^\infty_\mathsf{x}$ can be derived in L^∞_*.*

Proof. Let $\Gamma \cup \{\varphi, \psi\} \subseteq Fm$ and $x \in [0,1]$ such that $\Gamma \vdash_{L^\infty_*} (\varphi \to \overline{c}) \wedge (\overline{d} \to \psi)$ for all $c \in (x,1] \cap \mathcal{C}_*, d \in [0,x) \cap \mathcal{C}_*$. It follows that $\Gamma \vdash_{L^\infty_*} \varphi \to \overline{c}$ for all $c \in (x,1] \cap \mathcal{C}_*$ and $\Gamma \vdash_{L^\infty_*} \overline{d} \to \psi$ for all $d \in [0,x) \cap \mathcal{C}_*$.

Consider first the case when $x \in \mathcal{C}_*$, i.e., $\overline{x}$ is an element of the language. Then, $\Gamma \vdash_{L^\infty_*} \varphi \to \overline{c}$ for all $c \in (x,1] \cap \mathcal{C}_*$ implies that $\Gamma \vdash_{L^\infty_*} (\varphi \to \overline{c}) \vee (\overline{c} \to \overline{x})$ for all $c \in \mathcal{C}_*$. Similarly, we also get that $\Gamma \vdash_{L^\infty_*} (\overline{x} \to \overline{c}) \vee (\overline{c} \to \psi)$ for all $c \in \mathcal{C}_*$. Applying the density rule, D^∞ on each case, we get $\Gamma \vdash_{L^\infty_*} \varphi \to \overline{x}$ and $\Gamma \vdash_{L^\infty_*} \overline{x} \to \psi$. By transitivity we conclude that $\Gamma \vdash_{L^\infty_*} \varphi \to \psi$.

On the other hand, suppose that $x \notin \mathcal{C}_*$, then clearly any $c \in \mathcal{C}_*$ is either smaller or greater than x. Then, for each $c \in \mathcal{C}_*$, $\Gamma \vdash_{L^\infty_*} \varphi \to \overline{c}$ or $\Gamma \vdash_{L^\infty_*} \overline{c} \to \psi$. Then, $\Gamma \vdash_{L^\infty_*} (\varphi \to \overline{c}) \vee (\overline{c} \to \psi)$ for each $c \in \mathcal{C}_*$. Applying the density rule now, we get that $\Gamma \vdash_{L^\infty_*} \varphi \to \psi$. ⊠

Our aim is to determine whether the conjunctive inference rules are equivalent to D^∞. We can first show that any chain satisfying all the conjunctive inference rules has the constants densely distributed.

Lemma 5.3. *Let us assume that $\mathbf{A}$ is a $MTL^\mathbb{Q}_\Delta$-chain. Then, the following conditions are equivalent:*

1. *$\mathbf{A}$ validates the generalized quasi-equation arising from D^∞, i.e.,*

$$\mathcal{Q} : \bigwedge_{c\in\mathcal{C}_*} [(x \to \overline{c}) \vee (\overline{c} \to y) \approx \overline{1}] \quad \Longrightarrow \quad [(x \to y) \approx \overline{1}]$$

2. *$\mathbf{A}$ validates all generalized quasi-equations arising from the conjunctive inference rules, i.e., for each $x \in [0,1]$,*

$$\mathcal{Q}_x : \bigwedge_{c\in(x,1]\cap\mathcal{C}_*} [(x \to \overline{c}) \approx \overline{1}] \wedge \bigwedge_{c\in[0,x)\cap\mathcal{C}_*} [(\overline{c} \to y) \approx \overline{1}] \quad \Longrightarrow \quad [(x \to y) \approx \overline{1}]$$

3. *For each $a < b \in A$ there is $c \in \mathcal{C}_*$ such that $a < \overline{c}^\mathbf{A} < b$.*

Proof. ($1 \Leftrightarrow 3$): while $3 \Rightarrow 1$ is trivial by definition, the other direction was proven in Lemma 4.12.

($1 \Leftrightarrow 2$): By 5.2 it is enough to show that ($2 \Rightarrow 1$). Let us assume that $\mathbf{A} \not\models \mathcal{Q}$, and let us prove that $\mathbf{A} \not\models \mathcal{Q}_x$ for some $x \in [0, 1]$.

Since $\mathbf{A} \not\models \mathcal{Q}$ there is $a, b \in A$ such that $a > b$ (recall that A was a chain) but for each $c \in \mathcal{C}_*$ either $a \leq \overline{c}^{\mathbf{A}}$ or $\overline{c}^{\mathbf{A}} \leq b$. This implies that $\inf\{c \in \mathcal{C}_* : a \leq \overline{c}^{\mathbf{A}}\} = \sup\{c \in \mathcal{C}_* : \overline{c}^{\mathbf{A}} \leq b\}$. Let x be this value, and observe that for each $c \in (x, 1] \cap \mathcal{C}_*$, (i.e., with $c > x = \inf\{c \in \mathcal{C}_* : a \leq \overline{c}^{\mathbf{A}}\}$) it holds that $a \leq \overline{c}^{\mathbf{A}}$, so $a \Rightarrow \overline{c}^{\mathbf{A}} \approx \overline{1}^{\mathbf{A}}$. Similarly, for each $d \in [0, x) \cap \mathcal{C}_*$ (i.e., with $d < x = sup\{c \in \mathcal{C}_* : \overline{c}^{\mathbf{A}} \leq b\}$) it holds that $\overline{d}^{\mathbf{A}} \Rightarrow b \approx \overline{1}^{\mathbf{A}}$. Therefore, $\mathbf{A} \not\models \mathcal{Q}_x$. ⊠

This makes of the conjunctive inference rules good candidates for getting an alternative axiomatization of L_*^∞. In fact, as an immediate corollary we can state a result similar to [34, Cor. 23]:

Corollary 5.4. *Let $*$ be a left-continuous t-norm, and AS an axiomatic system in $\mathfrak{L}$ extending MTL_Δ such that*

- *AS derives all book-keeping axioms,*
- *AS derives all the conjunctive inference rules,*
- *AS is semilinear,*

Then AS is strongly complete with respect to $[\mathbf{0}, \mathbf{1}]_*^{\mathbb{Q}}$.

The semilinearity condition is, as it might be expected, a problematic one. We cannot simply extend $MTL_*^{\mathbb{Q}}$ with the conjunctive inference rules, since the resulting system may not be semilinear. In Theorem 4.16 we proved semilinearity of some logics with up to a denumerable set of inference rules, so this could seem a promising starting point to approach the semilinearity of the axiomatic systems as above, but we will see that it is not enough.

On the one hand, it is not hard to see that the conjunctive inference rules are closed under the $\vee$ operation.

Lemma 5.5. *Let R be a set of conjunctive inference rules, and $MTL_*^{\mathbb{Q}}(\mathsf{R})$ be the extension of $MTL_*^{\mathbb{Q}}$ with the rules in R. Then, for all $x \in [0, 1]$ such that $\mathsf{R_x} \in \mathsf{R}$ the rule*

$$\mathsf{R}'_\mathsf{x} : \frac{\{r \vee ((p \to \overline{c}) \wedge (\overline{d} \to q))\}_{c \in (x,1] \cap \mathcal{C}_*, d \in [0,x) \cap \mathcal{C}_*}}{r \vee (p \to q)}$$

can be derived.

Proof. In MTL_Δ, from $r \vee ((p \to \overline{c}) \wedge (\overline{d} \to q))$ it can be deduced $(((p \wedge \neg\Delta r) \to \overline{c}) \wedge (\overline{d} \to (q \vee \Delta r))$. Applying the corresponding infinitary rule, this implies that $(p \wedge \neg\Delta q) \to (q \vee \Delta r)$, and from here it follows that $r \vee (p \to q)$. ⊠

However, this was not the only premise of Lemma 4.16. There we proved that using a countable set of rules like the above ones, the resulting system is semilinear, but that is not necessarily true for non-countable sets of rules, as we will see in the next section.

A natural idea is that of seeing whether a denumerable subset of conjunctive inference rules is enough to prove this density property. Sadly, in general, this does not hold, as we can show a case where all the conjunctive inference rules are valid except for one but the constants are no longer dense.

Lemma 5.6. *For every $x \in [0,1]$ it holds that there is a Gödel chain* $\mathbf{A}$ *such that:*

- $\mathbf{A}$ *does not model $\mathcal{Q}_x$,*
- $\mathbf{A}$ *models $\mathcal{Q}_s$ for all $s \in [0,1] \setminus \{x\}$,*
- $\mathbf{A}$ *does not model $\mathcal{Q}$.*

Proof. Let us distinguish three cases.

Case $x \in [0,1] \setminus [0,1]_{\mathbb{Q}}$: Consider the Gödel chain given by the universe $[0,1]_{\mathbb{Q}}$ expanded with two points $\{x_-, x_+\}$ which play the role of the irrational x, and $x_- < x_+$. In particular, $[0,x) < x_- < x_+ < (x,1]$, and so, $[0,x)_{\mathbb{Q}} < x_- < x_+ < (x,1]_{\mathbb{Q}}$.

Consider first $s \in [0,1] \setminus \{x\}$. Observe than by construction, $\sup\{d \in [0,s)_{\mathbb{Q}}\} = \inf\{c \in (s,1]_{\mathbb{Q}}\}$. Then, let a, b be elements of the algebra such that $a \leq c$ for all $c \in (s,1]_{\mathbb{Q}}$, and $d \leq b$ for all $d \in [0,s)_{\mathbb{Q}}$. By definition of infimum and supremum, it follows that $a \leq \inf\{c \in (s,1]_{\mathbb{Q}}\}$ and $\sup\{d \in [0,s)_{\mathbb{Q}}\} \leq b$. But by the previous observation this implies that $a \leq \inf\{c \in (s,1]_{\mathbb{Q}}\} = \sup\{d \in [0,s)_{\mathbb{Q}}\} \leq b$, so the generalised quasi-equation $\mathcal{Q}_s$ is valid.

On the other hand, we have that by construction $x_+ \leq c$ for all $c \in (x,1]_{\mathbb{Q}}$, and also $d \leq x_-$ for all $d \in [0,x)_{\mathbb{Q}}$. However, we defined the algebra so $x_- < x_+$, so the generalised quasi-equation $\mathcal{Q}_x$ does not hold. Similarly, for any $c \in \mathcal{C}_*$, either $x_+ \leq c$ or $c \leq x_-$ (since there is no constant in between them), so the generalised quasi-equation arising from the density rule, i.e., $\mathcal{Q}$ does not hold either.

Case $x \in \mathcal{C}_*, x < 1$: Consider the Gödel chain given by the universe $\mathcal{C}_*$ expanded with one point $\{x_+\}$ which separates the sets $[0,x]$ and $(x,1]$. It follows that $[0,x]_{\mathbb{Q}} < x_+ < (x,1]_{\mathbb{Q}}$.

We can prove as in the previous case that $\mathcal{Q}_s$ holds for any $s \neq x$. To see that the algebra does not model $\mathcal{Q}_x$ nor $\mathcal{Q}$ it is enough to consider the points x and x_+. For the first one observe that $x_+ \leq c$ for all $c \in (x,1]_{\mathbb{Q}}$ and $d \leq x$ for all $d \in [0,x)_{\mathbb{Q}}$, but $x < x_+$ by definition. Concerning $\mathcal{Q}$, it is clear that for any $c \in \mathcal{C}_*$, $x_+ \leq c$ or $c \leq x$, but again, the consequence of the quasi-equation does not hold.

Case $x = 1$: Consider the Gödel chain given by the universe $\mathcal{C}_*$ expanded with a new element 1_- which separates the sets $[0, 1)$ and $\{1\}$. As before, $[0, 1)_{\mathbb{Q}} < 1_- < 1$.

The proof is again analogous to the previous cases, considering now 1 and 1^-. ⊠

This last lemma implies that if there is derivation of the density rule using conjunctive inference rules such derivation must be really long: it cannot be countably long[1] because it has to take into account the conjunctive inference rules for all real numbers in $[0, 1]$. Nevertheless, we will see in the next section that there is no such derivation.

5.2 Algebraic models of the density rule

In what follows we show that, in general, the logic axiomatized using all the conjunctive inference rules is not semilinear, i.e., it does not coincide with the one axiomatized using the density rule. Before providing a counterexample let us notice a case where, also for arbitrary algebras, the system resulting from the addition of D^∞ is equivalent to the one obtained using all the conjunctive inference rules. Some conditions characterized here will guide us later in order to find a counterexample to the equivalence between D^∞ and $\{\mathsf{R}^\infty_\mathsf{x}\}_{x\in[0,1]}$.

Lemma 5.7. *Let* $\mathbf{A}$ *be a* $MTL^{\mathbb{Q}}_*$*-algebra (not necessarily a chain), and let us assume that* $\mathbf{A}$ *has a subdirect representation* $\mathbf{A} \subseteq \prod_{i\in I} \mathbf{A_i}$ *(with* $\mathbf{A_i}$ *subdirectly irreducible, and thus, chain) such that for every* $i \in I$*, the element* $\vec{e}_i$ *given by*

$$\vec{e}_i[j] = \begin{cases} 0 & \textit{if } j \neq i \\ 1 & \textit{otherwise} \end{cases}$$

belongs to A*. Then, the following statements are equivalent:*

1. $\mathbf{A}$ *validates* $\mathcal{Q}$
2. *For all* $x \in [0, 1]$*,* $\mathbf{A}$ *validates* $\mathcal{Q}_x$*.*

Proof. It is enough to prove that $(2 \Rightarrow 1)$. Assume that $\mathbf{A} \not\models \mathcal{Q}$. Then, there is some $i \in I$ such that $\mathbf{A_i} \not\models \mathcal{Q}$, because the class of algebras is a generalised quasi-variety and the generalized quasi-equations are preserved under direct products and subalgebras. By linearity and Lemma 5.5 if $\mathbf{A_i} \not\models \mathcal{Q}$ then $\mathbf{A_i} \not\models \mathcal{Q}_x$ for some $x \in [0, 1]$ (we have see that, over the linearly ordered algebras, these two sets of rules are interderivable). Then, since $\mathbf{A_i} \not\models \mathcal{Q}_x$ and $\vec{e}_i \in A$, then it is clear that there exists a substitution σ such that $\vec{e}_i \to \sigma(\gamma) = 1$ for each γ in the premises of R_x, while $\vec{e}_i \to \sigma(\delta)$ for δ being the consequence of the same rule. But this means that $\mathbf{A} \models (\neg\delta^{\mathbf{A}}\vec{e}_i \vee \sigma(\gamma)) \approx \overline{1}^{\mathbf{A}}$, but $\mathbf{A} \not\models (\neg\neg\delta^{\mathbf{A}}\vec{e}_i \vee \sigma(\delta)) \approx \overline{1}^{\mathbf{A}}$, having that $\mathcal{Q}_x$ does not hold in $\mathbf{A}$. ⊠

[1] But it is well-founded.

Now we are ready to provide the counterexample to the equivalence between D^∞ and $\{\mathsf{R}^\infty_x\}_{x\in[0,1]}$. As expected, such counterexample will not satisfy the assumptions in 5.7. Indeed, the construction we do is inspired by the construction of the unique, up to isomorphism, countable atomless Boolean algebra (see for instance [64, Chapter 16]).

Lemma 5.8. *Let $*$ be the Gödel t-norm. Then, there is a $MTL^{\mathbb{Q}}_*$-algebra $\mathbf{A}$ such that $\mathbf{A} \models \mathcal{Q}_x$ for all $x \in [0,1]_*$ while $\mathbf{A} \not\models \mathcal{Q}$.*

Proof. Let I be the interval $[0,1)_{\mathbb{Q}}$. For every $q \in I$ we consider the Gödel-chain $\mathbf{A_q}$ defined by:[2]

- the universe is $[0,1]_{\mathbb{Q}}$ enlarged with a new element $\widetilde{q}$.
- the universe is linearly-ordered with the expansion of the linear order among rational numbers such that $\widetilde{q}$ is strictly between the elements in $[0,q]_{\mathbb{Q}}$ and $(q,1]_{\mathbb{Q}}$. In other words, $\widetilde{q}$ is the sucessor (next element) of q. We emphasize that we do not consider $\widetilde{q}$ as a rational element.
- the operations of the Δ-Gödel chain $\mathbf{A_q}$ are the ones determined by the linear order in the previous item.
- for every $c \in [0,1]_{\mathbb{Q}}$, the interpretation of the constant $\bar{c}$ in $\mathbf{A_q}$ is the rational number c.

It is worth noticing that all chains $\mathbf{A_q}$ (with $q \in I$) satisfy that

- for every $x \in [0,1]_{\mathbb{R}} \setminus \{q\}$, there are no elements $a, b \in A_q$ which simultaneously satisfy 1) $a \leq c$ for every $c \in (x,1]_{\mathbb{Q}}$ and 2) $c \leq b$ for all $c \in [0,x)_{\mathbb{Q}}$ and 3) $b < a$. In other words, for every $x \in [0,1] \setminus \mathcal{C}_*$, it holds that $\mathbf{A_q} \models \mathcal{Q}_x$.
- $\mathbf{A_q} \not\models \mathcal{Q}_q$. Indeed, there is only one pair of elements $a, b \in A_q$ which simultaneosly satisfy 1) $a \leq c$ for every $c \in (x,1]_{\mathbb{Q}}$ and 2) $c \leq b$ for all $c \in [0,x)_{\mathbb{Q}}$ and 3) $b < a$; such a pair is the one given by $x := \widetilde{q}$ and $y := q$.

By 5.7 it is obvious that the direct product $\mathbf{B} := \prod_{q\in I} \mathbf{A_q}$ is not an algebra such that $\mathbf{B} \models \mathcal{Q}_x$ for all $x \in [0,1]$ and $\mathbf{B} \not\models \mathcal{Q}$.

Next we define $\mathbf{A}$ as the subalgebra of $\mathbf{B}$ whose universe is given by the elements $f \in B$ (seen as maps from I) such that there is a finite sequence $q_0 < q_1 < q_2 < \cdots < q_{n+1}$ of rational numbers with

- $q_0 := 0$ and $q_{n+1} := 1$ (and $n \in \mathbb{N}$),
- for $0 \leq i \leq n$, $f \restriction [q_i, q_{i+1})$ is either a constant function given by a rational number or the function given by $fq) = \widetilde{q}$ or the function given by $f(q) = q$.

[2] These chains were previously considered inside the proof of 5.6.

It is quite simple[3] to verify that such set A is closed under all operations, and so A is the support of a MTL_Δ-chain $\mathbf{A}$.

It is worth noticing here that such $\mathbf{A}$ has the subdirect product representation given by $\mathbf{A} \subseteq \prod_{q \in I} \mathbf{A_q}$ (i.e., all projections are surjective), and that for every $q \in I$ the element e_i considered in 5.7 does not belong to $\mathbf{A}$. Thus, the assumptions in 5.7 do not hold for this particular algebra $\mathbf{A}$.

Next we check the following claims.

- $\mathbf{A}$ is not a model of $\mathcal{Q}$. To show this let us consider the elements $s, t \in A$ defined by $s := (\widetilde{q})_{q \in I}$ and $t := (q)_{q \in I}$. It is obvious that $s \Rightarrow t = t \neq 1$. Moreover, for every $c \in \mathcal{C}_*$ it holds that $s \Rightarrow \overline{c}^{\mathbf{A}}$ and $\overline{c}^{\mathbf{A}} \Rightarrow t$ are the elements given by

$$(s \Rightarrow \overline{c}^{\mathbf{A}})(q) := \begin{cases} 1 & \text{if } q \in [0, c)_{\mathbb{Q}} \\ c & \text{if } q \in [c, 1)_{\mathbb{Q}} \end{cases} \quad \text{and}$$
$$(\overline{c}^{\mathbf{A}} \Rightarrow t)(q) := \begin{cases} q & \text{if } q \in [0, c)_{\mathbb{Q}} \\ 1 & \text{if } q \in [c, 1)_{\mathbb{Q}} \end{cases}$$

  Therefore, for every $c \in [0,1]_{\mathbb{Q}}$ it holds that $(s \Rightarrow \overline{c}^{\mathbf{A}}) \vee (\overline{c}^{\mathbf{A}} \Rightarrow t) = \overline{1}^{\mathbf{A}}$. Thus, $\mathbf{A} \not\models \mathcal{Q}$ under the interpretation sending φ to the element s and ψ to the element t.

- for every $x \in [0,1] \setminus [0,1]_{\mathbb{Q}}$, it holds that $\mathbf{A} \models \mathcal{Q}_x$. This is trivial because all algebras in $\{\mathbf{A_q} : q \in I\}$ validate such generalized quasiequation $\mathcal{Q}_x$.

- for every $r \in [0,1]_{\mathbb{Q}}$, it holds that $\mathbf{A} \models \mathcal{Q}_r$. We can equivalently (an more easily) prove that $\mathbf{A} \models \mathcal{Q}_r^1$ and $\mathbf{A} \models \mathcal{Q}_r^2$, where

$$\mathcal{Q}_r^1 \quad := \quad \bigwedge_{c \in (r,1]_{\mathbb{Q}}} [(x \to \overline{c}) \approx \overline{1}] \quad \Longrightarrow \quad [(x \to \overline{r}) \approx \overline{1}]$$
$$\mathcal{Q}_r^2 \quad := \quad \bigwedge_{c \in [0,r)_{\mathbb{Q}}} [(\overline{c} \to x) \approx \overline{1}] \quad \Longrightarrow \quad [(\overline{r} \to x) \approx \overline{1}]$$

  It is clear that from these two, if $\mathbf{A} \models ((x \to \overline{c}) \wedge (\overline{d} \to y)) \approx \overline{1}$ for all $c \in (r,1]_{\mathbb{Q}}$ and all $d \in [0,r)_{\mathbb{Q}}$, they both follow that $\mathbf{A} \models (x \to \overline{r}) \approx \overline{1}$ and $\mathbf{A} \models (\overline{r} \to y) \approx \overline{1}$, so $\mathbf{A} \models (x \to y) \approx \overline{1}$.

  We will prove that $\mathbf{A} \models \mathcal{Q}_r^1$ and $\mathbf{A} \models \mathcal{Q}_r^2$. Let us fix a rational number in $[0,1]$ r.

[3] For the reader interested to check the details we suggest to start considering the following three elements in $\mathbf{A}$:

$$t_1 := (\tfrac{1}{2})_{q \in [0,1)_{\mathbb{Q}}} \qquad \text{and} \qquad t_2 := (\widetilde{q})_{q \in [0,1)_{\mathbb{Q}}} \qquad \text{and} \qquad t_3 := (q)_{q \in [0,1)_{\mathbb{Q}}}$$

and checking that all possible combinations of these three elements under $\wedge, \vee, \to, \Delta$ are also elements in our universe A. Indeed, all difficulties to provide a general proof that A is closed under the operations are illustrated in the previous particular case.

Case $x \in A$ **such that** $x \leq r \uparrow$: We need to show that $\mathbf{A} \models \mathcal{Q}_r^1$, i.e., that $x \leq \bar{r}$. We will check this showing that for each one of the rational intervals $[q_i, q_{i+1})$ determined by the element $x \in A$, it holds that $x \upharpoonright [q_i, q_{i+1})$ is less or equal than $\bar{r} \upharpoonright [q_i, q_{i+1})$. The fact that $x \leq r \uparrow$ tells us that in each one of the intervals $[q_i, q_{i+1})$ one of the following conditions hold:

– $x \upharpoonright [q_i, q_{i+1})$ is a rational constant which is $\leq r$,
– $x \upharpoonright [q_i, q_{i+1})$ is a function given by $q \mapsto \widetilde{q}$, and moreover $q_{i+1} \leq r$
– $x \upharpoonright [q_i, q_{i+1})$ is a function given by $q \mapsto q$, and moreover $q_{i+1} \leq r$.

In all three cases, using that q_{i+1} is not an element of the interval $[q_i, q_{i+1})$, it follows that $x \upharpoonright [q_i, q_{i+1})$ is less or equal than $\bar{r} \upharpoonright [q_i, q_{i+1})$.

Case $x \in A$ **such that** $r \downarrow \leq x$: We need to show that $\mathbf{A} \models \mathcal{Q}_r^2$, that is to say, that $\bar{r} \leq x$. We will do this showing that for each one of the rational intervals $[q_i, q_{i+1})$ determined by the element $x \in A$, it holds that $\bar{r} \upharpoonright [q_i, q_{i+1})$ is less or equal than $x \upharpoonright [q_i, q_{i+1})$. The fact that $r \downarrow \leq x$ tells us that in each one of the intervals $[q_i, q_{i+1})$ one of the following conditions hold:

– $x \upharpoonright [q_i, q_{i+1})$ is a rational constant which is $\geq r$,
– $x \upharpoonright [q_i, q_{i+1})$ is a function given by $q \mapsto \widetilde{q}$, and moreover $q_i \geq r$
– $x \upharpoonright [q_i, q_{i+1})$ is a function given by $q \mapsto q$, and moreover $q_i \geq r$.

In all three cases it holds that $x \upharpoonright [q_i, q_{i+1})$ is greater or equal than $\bar{r} \upharpoonright [q_i, q_{i+1})$.

This finishes the proof that $\mathbf{A} \models \mathcal{Q}_r$ for the case that r is rational.

Therefore, we have just seen that $\mathbf{A} \not\models \mathcal{Q}$ while $\mathbf{A} \models \mathcal{Q}_x$ for all $x \in [0, 1]$. ⊠

With this we know that extending $MTL_*^{\mathbb{Q}}$ with the conjunctive inference rules does not provide in general an axiomatic system strongly complete with respect to the rational standard algebra of $*$.

5.3 T-norms accepting a conjunctive axiomatization

We can prove, nevertheless, some results over classes of left-continuous t-norms that still admit an axiomatization using conjunctive inference rules.

Definition 5.9. We say that a left-continuous t-norm $*$ ***accepts a conjunctive axiomatization*** if there exists an axiomatic system AS such that

- AS is strongly complete with respect to $[\mathbf{0}, \mathbf{1}]_*^{\mathbb{Q}}$,
- AS extends $MTL_*^{\mathbb{Q}}$,
- The only inference rules added to $MTL_*^{\mathbb{Q}}$ are conjunctive inference rules.

If a left-continuous t-norm $*$ accepts a conjunctive axiomatization, we denote it by $\overline{L_*^\infty}$.

It is worth remarking that the first and third points from the above definition are equivalents when all the conjunctive rules are considered (but then, as we have proven, the logic may not be semilinear).

Observe that, for a left-continuous t-norm accepting a conjunctive axiomatization, for any set of formulas $\Gamma \cup \{\varphi\}$ the following are equivalent:

- $\Gamma \vdash_{L_*^\infty} \varphi$,
- $\Gamma \vdash_{\overline{L_*^\infty}} \varphi$,
- $\Gamma \models_{[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}} \varphi$,

It is of course of great interest understand what kind of t-norms fall within the previous class. We can characterize some classes of t-norms that accept a conjunctive axiomatization.

First, we can prove that for an arbitrary left-continuous t-norm, the conjunctive inference rules corresponding to values x such that $\langle x, x\rangle$ is a discontinuity points of the diagonal of $\Rightarrow_*$ are enough to prove the whole set of conjunctive inferece rules on any linearly ordered algebra of the class. Recall that we write $\mathcal{Q}_x$ to denote the generalized quasi-equation associated to $\mathsf{R_x}$.

Lemma 5.10. *Let $*$ be a left-continuous t-norm, and $\mathbf{A}$ a linearly ordered $MTL_*^{\mathbb{Q}}$-algebra. Then if $\mathbf{A} \models \mathcal{Q}_x$ for all x such that $\langle x, x\rangle$ is a discontinuity point of $\Rightarrow_*$, then $\mathbf{A} \models \mathcal{Q}_y$ for all $y \in [0, 1]$.*

Proof. Observe first that $\varphi \to \psi, \chi \to \delta \vdash_{MTL} (\psi \to \chi) \to (\varphi \to \psi)$, so it is also true in $\overline{L_*^\infty}$ and thus in all the algebras of the class. Let $u \in (0, 1]$ and take $a, b \in A$ such that $a \Rightarrow \overline{c}^{\mathbf{A}}) = \overline{1}^{\mathbf{A}}$ for all $c > u$ and $(\overline{d}^{\mathbf{A}} \Rightarrow b) = \overline{1}^{\mathbf{A}}$ in $\mathbf{A}$ for all $d < u$. If u was a discontinuity point for $\Rightarrow_*$ the rule was satisfied by assumption. Otherwise, from the previous observation we have that $((\overline{c}^{\mathbf{A}} \Rightarrow \overline{d}^{\mathbf{A}}) \Rightarrow (a \Rightarrow b)) = \overline{1}^{\mathbf{A}}$. But then, since u was not a discontinuity point of $\Rightarrow_*$, and given that $(u \Rightarrow_* u) = 1$, then it holds that $\sup\{c \Rightarrow_* d : c > u > d\} = 1$. That is to say, for each $r < 1$, there are $c > u > d$ such that $r < c \Rightarrow_* d$. Using the book-keeping axioms, we get that $(\overline{r}^{\mathbf{A}} \Rightarrow (a \Rightarrow b)) = \overline{1}^{\mathbf{A}}$ for all $r < 1$. Then, using $\mathsf{R_1}$, we have that $(a \Rightarrow b) = \overline{1}^{\mathbf{A}}$.
If $u = 0$, and $\langle 0, 0\rangle$ is not a discontinuity point. Take $a \in A$ such that $a \Rightarrow \overline{c}^{\mathbf{A}} = \overline{1}^{\mathbf{A}}$ for all $c \in \mathcal{C}_* \setminus \{0\}$. From the previous observation, and using that $\overline{0} \to \overline{0}$ is a theorem, we have that $((\overline{c}^{\mathbf{A}} \Rightarrow \overline{0}^{\mathbf{A}}) \Rightarrow (a \Rightarrow \overline{0}^{\mathbf{A}})) = \overline{1}^{\mathbf{A}}$ for all $c \in \mathcal{C}_* \setminus \{0\}$. Since 0 is not a discontinuity point we know that $\sup\{c \Rightarrow_* 0 : c \in \mathcal{C}_* \setminus \{0\}\} = \overline{1}^{\mathbf{A}}$. Then, we have that $(\overline{r}^{\mathbf{A}} \Rightarrow (a \Rightarrow \overline{0}^{\mathbf{A}})) = \overline{1}^{\mathbf{A}}$ for all $r < 1$, and again by rule $\mathsf{R_1}$ we can conclude that $\neg a = \overline{1}^{\mathbf{A}}$. ⊠

The previous lemma gives as immediate corollary the following result.

Corollary 5.11. *Any left-continuous t-norm with a countable amount of discontinuity points on the diagonal of its residuum accept a conjunctive axiomatization.*

While the only two continuous t-norms that belong to the previous class are the Łukasiewicz and the Product t-norms, we do not know if there are left-continuous (non-continuous) t-norms that verify the previous statement.

A different class of left-continuous t-norms that accept a conjunctive axiomatization, that for what concerns the continuous t-norms is bigger than the previous one, is formed by all the ordinal sums of Product and Łukasiewicz components.

Lemma 5.12. *Any ordinal sum of Łukasiewicz an Product t-norms accepts a countable conjunctive axiomatization.*

Proof. Let $* = \bigoplus_{i\in I}\langle *_i, (b_i, t_i)\rangle$ be an ordinal sum of Łukasiewicz and Product components, and let $\mathcal{C}_*$ be the countable subalgebra generated by the rationals in $[0,1]$ and the idempotent elements of $*$, i.e., $\bigcup_{i\in I}\{b_i, t_i\}$ (the idempotents). We can consider the axiomatic system given by the extension of $MTL_*^{\mathbb{Q}}$ over the set of constants $\mathcal{C}_*$ by the following axiom and rules:

- $(\Delta(\overline{b} \to (p \to q)) \wedge \Delta(p \to \overline{b})) \to (p \to q)$ for each $b \in \bigcup_{i\in I}\{b_i, t_i\}$,
- For each $b \in \bigcup_{i\in I}\{b_i, t_i\}$, the rule $\mathsf{R_b}$, i.e.

$$\frac{\{(p \to \overline{c}) \wedge (\overline{d} \to q)\}_{c\in(b,1]\cap\mathcal{C}_*, d\in[0,b)\cap\mathcal{C}_*}}{p \to q}$$

We know the set I is countable by the definition of ordinal sum, so using Theorem 4.16 we get that the previous axiomatic systems are strongly complete with respect to the linearly ordered algebras of their corresponding algebraic companion. On the other hand, we can prove that any of these linearly ordered algebras validates all the conjunctive inference rules.

Let $\mathbf{A}$ be a linearly ordered $MTL_*^{\mathbb{Q}}$-algebra that satisfies the previous axiom and rules schemata, and let $a, b \in A$, $x \in [0,1]$ such that $((a \Rightarrow \overline{c}^{\mathbf{A}}) \wedge (\overline{d}^{\mathbf{A}} \Rightarrow b)) = \overline{1}^{\mathbf{A}}$ for all $c \in (x,1] \cap \mathcal{C}_*, d \in [0,x) \cap \mathcal{C}_*$. By construction, we know that x belongs to some component i. Moreover, we are only interested in the case when x belongs to the interior of this component (because otherwise, $x = b_i$ or $x = t_i$, and the corresponding infinitary rule holds trivially by definition of the axiomatic system. Let then $b_i < x < t_i$.

From $((a \Rightarrow \overline{c}^{\mathbf{A}}) \wedge (\overline{d}^{\mathbf{A}} \Rightarrow b)) = \overline{1}^{\mathbf{A}}$ it follows that $((\overline{c}^{\mathbf{A}} \Rightarrow \overline{d}^{\mathbf{A}}) \Rightarrow (a \Rightarrow b)) = \overline{1}^{\mathbf{A}}$. From the behaviour of the residuum in an ordinal sum of Łukasiewicz and Product components, we know that, on the standard algebra it holds that

$$\lim_{\{c\in(x,1]\cap\mathcal{C}_*, d\in[0,x)\cap\mathcal{C}_*\}} \{c \Rightarrow_* d\} = t_i.$$

Then, from the book-keeping axioms, we get that $(\overline{r}^{\mathbf{A}} \Rightarrow (a \Rightarrow b)) = \overline{1}^{\mathbf{A}}$ for all $r \in [0, t_i) \cap \mathcal{C}_*$. Applying the generalised quasi-equation arising from the inference rule $\mathsf{R}_{\mathsf{t_i}}$ we get that $(\overline{t_i}^{\mathbf{A}} \Rightarrow (a \Rightarrow b)) = \overline{1}^{\mathbf{A}}$.

On the other hand, since $x < t_i$ and $(a \Rightarrow \overline{c}^{\mathbf{A}}) = \overline{1}^{\mathbf{A}}$ for all $c > x$, in particular $(a \Rightarrow \overline{t_i}^{\mathbf{A}}) = \overline{1}^{\mathbf{A}}$. Then, applying MP and the axiom we originally added to the system, we get that $(a \Rightarrow b) = \overline{1}^{\mathbf{A}}$. ⊠

Part III

Modal expansions of some t-norm based logics

Chapter 6

Many-valued modal logics: $\mathbf{K}_*$ and $\mathbf{K}_*^{\mathbf{g}}$

In this chapter we will focus on the study of the local and global modal logics associated to certain many-valued Kripke frames. We consider Kripke models with a crips accesibility relation. Moreover, the evaluation of formulas in these Kripke models is done over algebras belonging to the generalized quasi-variety generated by $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$ for $*$ being a left continuous t-norm that accepts a conjunctive axiomatization (see Definition 5.9). The main reason for restricting the class of studied t-norms is that, while we know how to syntactically prove certain characteristics for the conjunctive inference rules of $\overline{L_*^\infty}$ (in particular, Theorem 6.9), we have not been able to do the same for the density rule. In this and the following chapter, $*$ **stands for a left-continuous t-norm accepting a conjunctive axiomatization**. Recall that we know cases like ordinal sums of Łukasiewicz and Product t-norms fall within this class (Lemma 5.12) We will remark thisfact in the more relevant results in order to avoid confusion.

In this dissertation, we understand a modal logic as a consequence operator. For this reason the definitions are not based on the set of theorems but on the deduction relation itself. It is possible to move from the notion of logic as sets of theorems to the notion of consequence operator if considering finitary deductions only (see for instance [78] or [18], where the logics are defined as sets of formulas). However, we think this approach gives a new intuition on what the derivation relation of these logics mean and it is also more uniform with respect to the works on abstract algebraic logic.

The main result we present here is the definition of axiomatic systems that are (respectively) strongly complete with respect to the intended semantics defined in Section 6.1, arising from the Kripke models with a crisp accesibility relation evaluated over L_*^∞-algebras. We begin by defining these modal logics in semantic terms, following the usual definitions presented at this respect in the previous literature (see Section 3.2 at Chapter 3) and in Section 6.2 we propose two equivalent axiomatizations for them. We prove this logic enjoys strong completeness with respect to the semantic characterizations of the modal logics. In Section 6.3 we present several results concerning the behaviour and characteristics of the many-valued modal logics presented and we conclude this chapter proposing a method to axiomatize the possibilistic logics over L_*^∞-algebras (for left-continuous t-norms accepting a conjunctive axiomatization).

6.1 The semantics: crisp local and global modal logics over $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$

First, as usual we let the ***modal language*** $\mathfrak{L}_{\Box,\Diamond}$ be the expansion of the non-modal language $\mathfrak{L}$ defined in the previous chapter with two unary operators, $\Box$ and $\Diamond$. From now on, unless specified otherwise $\mathbf{Fm}$ will denote the the algebra of formulas built from a countable set of variables Var and the truth constants generated from the rationals $\{\overline{c}\}_{c\in\mathcal{C}_*}$ using the non-modal and the modal connectives of $\mathfrak{L}_{\Box,\Diamond}$.

Within this dissertation, the semantics of the modal logic built over the L_*^∞-algebras and in particular over $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$, is given following most of the previous works on many-valued modal logics. We will deal with the problem where the accessibility relation of the model is crisp and the fuzziness is given though the evaluation of the formulas within the worlds. For this reason, the frames over which the logic is defined do in fact coincide with the ones from the classical case: pairs of non-empty sets of worlds W and crisp accessibility relations between them, $R \subseteq W \times W$. For simplicity, we will write Rvw for $\langle v, w\rangle \in R$. The definition of a (crisp) many-valued Kripke model is as follows.

Definition 6.1. Let $\mathbf{A} = \langle A, \odot, \Rightarrow, \wedge, \delta^{\mathbf{A}}, \{\overline{c}^{\mathbf{A}}\}_{c\in\mathcal{C}_*}\rangle$ be a L_*^∞-algebra.[1] An ***(crisp)*** $\mathbf{A}$***-Kripke model*** $\mathfrak{M}$ (and in general, a ***many-valued Kripke model***) is a triplet $\langle W, R, w\rangle$ where W is a non-empty set of worlds called ***the universe of the model***, $R \subseteq W\times W$ and it is called the ***accessibility relation*** and e is an evaluation of propositional variables in $\mathbf{A}$ for each world, i.e., $e\colon W \times Var \to \mathbf{A}$. e is extended to non-modal formulas by the corresponding operation in $\mathbf{A}$,

$$\begin{aligned}
e(w,\overline{c}) &:= \overline{c}^{\mathbf{A}}\\
e(w,\varphi\wedge\psi) &:= e(w,\varphi)\wedge e(w,\psi)\\
e(w,\varphi\,\&\,\psi) &:= e(w,\varphi)\odot e(w,\psi)\\
e(w,\varphi\to\psi) &:= e(w,\varphi)\Rightarrow e(w,\psi)\\
e(w,\Delta\varphi) &:= \delta^{\mathbf{A}}e(w,\varphi)
\end{aligned}$$

and to modal formulas by

$$\begin{aligned}
e(w,\Box\varphi) &:= \inf\{e(v,\varphi) : Rwv\}\\
e(w,\Diamond\varphi) &:= \sup\{e(v,\varphi) : Rwv\}
\end{aligned}$$

[1] The generalized quasi-variety generated by $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$.

whenever these values exist and left undefined otherwise.

A many-valued Kripke model $\mathfrak{M}$ is called ***safe*** whenever the evaluation of $\Box\varphi$ and $\Diamond\varphi$ is defined over any world for any $\varphi \in Fm$. We will say that a model $\mathfrak{M}$ is based on a frame $\mathfrak{F}$ whenever the underlying frame of $\mathfrak{M}$ coincides with $\mathfrak{F}$.

For simplicity, the class of safe $\mathbf{A}$-Kripke models for $\mathbf{A}$ ranging over all the algebras in $\mathbb{GQ}([\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}})$ is called the ***class of $*$-Kripke models*** (and its elements, $*$-Kripke models).[2]

Over the previously defined models, different notions of truth arise, as it happened in the classical case: Given $\mathfrak{M}$ a $*$-Kripke model and $w \in W$, for any formula φ we write $(\mathfrak{M}, w) \Vdash \varphi$ and say that φ is ***valid in*** $\mathfrak{M}$ ***at state*** w whenever $e(w,\varphi) = 1$. On the other hand, we write $\mathfrak{M} \Vdash \varphi$ and say that φ is ***valid in*** $\mathfrak{M}$ whenever $\mathfrak{M}, w \Vdash \varphi$ for all $w \in W$. This notion is naturally extended to classes of formulas Γ, by saying that $\mathfrak{M}, w \Vdash \Gamma$ whenever $\mathfrak{M}, w \Vdash \gamma$ for each $\gamma \in \Gamma$.

It is also possible to extend the notion of validity to frames, as in the classical case. Given an Kripke frame $\mathfrak{F}$ we say that a formula φ is valid in $\mathfrak{F}$ and write $\mathfrak{F} \Vdash \varphi$ whenever $\mathfrak{M} \Vdash \varphi$ for all safe models $\mathfrak{M}$ with $\mathfrak{F}$ as frame.

Given a class of $*$-Kripke models, a local and a global deduction over it can be defined in an analogous way to the definitions of the classical local and global modal logics.

Definition 6.2. Let C be a class of $*$-Kripke models and $\Gamma \cup \varphi$ be a set of formulas. Then we say that:

- φ is ***local consequence*** of Γ over C and write $\Gamma \Vdash_{\mathsf{C}} \varphi$ whenever for all model $\mathfrak{M}$ from C and for all state w in $\mathfrak{M}$, if $\mathfrak{M}, w \Vdash \Gamma$ then $\mathfrak{M}, w \Vdash \varphi$.

- φ is ***global consequence*** of Γ over C and write $\Gamma \Vdash^g_{\mathsf{C}} \varphi$ whenever for all model $\mathfrak{M}$ from C, if $\mathfrak{M} \Vdash \Gamma$ then $\mathfrak{M} \Vdash \varphi$.

Notice that the local deduction is strictly weaker than the global one, since $\Gamma \Vdash_{\mathsf{KF}_*} \varphi$ implies that $\Gamma \Vdash^g_{\mathsf{KF}_*} \varphi$, but not the contrary. It is also remarkable that the set of theorems (i.e., deductions from the empty set) under the local and the global deduction coincide.

We study both these logics. We begin by the local one because from the results obtained in its study, the completeness results concerning the global modal logic follow quite naturally. Our aim is to provide an axiomatization for these logics, which will also result in a characterization of their algebraic semantics and so will allow us to treat these logics with different tools.

6.2 Axiomatization and completeness

In the previous chapters, we have developed two equivalent axiomatic systems strongly standard complete with respect to the corresponding rational standard

[2]Note that this are all safe models, even though we do not stress it in the name of the class.

algebra of a left-continuous t-norm. This has been done for the left-continuous t-norms that accepted a conjunctive axiomatization, which are the ones we are studying along this and the next chapters. It is now natural to expand these axiomatic systems in order to axiomatize the modal logics defined from the $*$-Kripke structures. Our first objective is to get two equivalent axiomatic systems, expanding respectively L_*^∞ and $\overline{L_*^\infty}$. Then, we will focus on proving completeness of the logic determined by these systems with respect to the intended semantics, using for this the axiomatization based on conjunctive inference rules.

Working over crisp frames, a natural intuition is that of including the K axiom schemata in the system. Moreover, with a language expanded with rational constant symbols, some interesting formulas relating modal operators and constants are valid and highly expressive, so they are also good candidates to consider in the axiomatic system. We have also realized that some axiom relating the Δ operator and the modal operators is necessary and finally, one of the two well-known Fisher-Servi axioms [56], formulas coming from the intuitionistic modal logic case, is also of remarkable importance for our axiomatic system.

Definition 6.3. We let $\mathcal{M}$ to be the following set of axiom schemata and rules over the modal language:

(K) $\Box(p \to q) \to (\Box p \to \Box q)$

($\Box$1) $\Box(\overline{c} \to p) \leftrightarrow (\overline{c} \to \Box p)$ for all $c \in \mathcal{C}_*$

($\Box$2) $\Delta\Box p \to \Box\Delta p$

($\Diamond$1) $\Box(p \to \overline{c}) \leftrightarrow (\Diamond p \to \overline{c})$ for all $c \in \mathcal{C}_*$

(FS1) $(\Diamond p \to \Box q) \to \Box(p \to q)$

($\mathsf{N}_\Box$) For any formula φ, from $\emptyset \vdash \varphi$ infer $\emptyset \vdash \Box\varphi$,

$\mathcal{M}^g$ is the set resulting from $\mathcal{M}$ by just replacing the local necessitation rule ($\mathsf{N}_\Box$) by the more general rule:

($N_\Box^g$) From p derive $\Box p$

We can now add the previous axioms and rules to the logical systems L_*^∞ (Definition 4.3) and $\overline{L_*^\infty}$ (Definition 5.9).

Lemma 6.4. *For any set $\Gamma \cup \{\varphi\}$ of modal formulas, the following are equivalent:*

- Γ *derives φ in the axiomatic system obtained from adding $\mathcal{M}$ to L_*^∞,*
- Γ *derives φ in the axiomatic system obtained from adding $\mathcal{M}$ to $\overline{L_*^\infty}$*

Proof. It is a trivial observation: from a proof of φ from Γ in any of the previous axiomatic systems, we can easily get a proof in the other one. For each rule applied in the proof, we leave it as such it is common to both systems (i.e., it is not the density rule or a conjunctive rule). Otherwise, since L_*^∞ and $\overline{L_*^\infty}$ are both the logic of $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$, we know that any inference rule from one logic has a derivation in the other system. Then, we include at that step the corresponding derivation of the applied rule in the new proof. ⊠

Then we can consider the previous logic and indistinctly use one or the other axiomatic systems.

Definition 6.5. The previous logic is denoted by K_*.

We can proceed analogously using $\mathcal{M}^g$ (i.e., consider the logic given equivalently by the axomatic system obtained from L_*^∞ plus $\mathcal{M}^g$ or from $\overline{L_*^\infty}$ plus $\mathcal{M}^g$) in order to get the logic we will name K_*^g.

As it happens at the semantical level, $\vdash_{K_*}$ is stronger than $\vdash_{K_*{}^g}$. Also their respective sets of theorems coincide, and we will denote them by Th_{K_*}.

It is easy to prove that both $\vdash_{K_*}$ and $\vdash_{K_*{}^g}$ are sound with respect to $\Vdash_{\mathsf{KF}_*}$ and $\Vdash_{\mathsf{KF}_*}^g$ correspondingly.

Lemma 6.6. *Let $\Gamma \cup \{\varphi\}$ be a set of formulas. Then*

1. $\Gamma \vdash_{K_*} \varphi$ *implies that* $\Gamma \Vdash_{\mathsf{KF}_*} \varphi$,
2. $\Gamma \vdash_{K_*^g} \varphi$ *implies that* $\Gamma \Vdash_{\mathsf{KF}_*}^g \varphi$.

Proof. By definition, all non-modal axioms and rules (that is, those from L_*^∞) are valid at any world of any $*$-model (by definition of this kind of model), so they are valid in general in any $*$-frame. Then, they trivially hold in $\Vdash_{\mathsf{KF}_*}$ and due to the fact that this logic is weaker than the global one, also hold in $\Vdash_{\mathsf{KF}_*}^g$). Then, it remains to check the soundness of $\mathcal{M}$ and $\mathcal{M}^g$ with respect to their corresponding semantics.

We first show soundness of the axioms of these sets (which are common). Let $\mathfrak{M}$ be an arbitrary safe Kripke model from KF_* and $v \in W$. Observe that we do not require for the algebra of evaluation to be linearly ordered.

K : By definition, $e(v, \Box(\varphi \to \psi)) = \inf\{e(w, \varphi \to \psi) : \ Rvw\} \leq e(w, \varphi) \Rightarrow e(w, \psi)$ for all $w \in W$ such that Rvw. By axiom $\mathsf{MTL1}$, we know $\to$ (and so the $\Rightarrow$ operation) is decreasing in the first component and thus $e(w, \varphi) \Rightarrow e(w, \psi) \leq \inf\{e(u, \varphi) : \ Rvu\} \Rightarrow e(w, \psi)$. Applying the definition and the previous family of inequalities, we get that $e(v, \Box(\varphi \to \psi)) \leq e(v, \Box\varphi) \Rightarrow e(w, \psi)$ for all $w \in W$ such that Rvw. By residuation this is equivalent to $e(v, \Box(\varphi \to \psi)) \odot e(v, \Box\varphi) \leq e(w, \psi)$ for all $w \in W$ such that Rvw and taking the infimum over w we get $e(v, \Box(\varphi \to \psi)) \odot e(v, \Box\varphi) \leq \inf\{e(w, \psi) : Rvw\} = e(v, \Box\psi)$. Again by residuation we get that $e(v, \Box(\varphi \to \psi)) \leq e(v, \Box\varphi) \Rightarrow e(v, \Box\psi)$ that is, that the K formula is valid in any state of any model from KF_*.

$\Box$1 : By definition $e(v, \Box(\overline{c} \to \varphi)) = \inf\{e(w, \overline{c} \to \varphi) : \ Rvw\} = \inf\{\overline{c}^{\mathbf{A}} \Rightarrow e(w, \varphi) Rvw\}$. On the one side, $\inf\{\overline{c}^{\mathbf{A}} \Rightarrow e(w, \varphi) : Rvw\} \leq \overline{c}^{\mathbf{A}} \Rightarrow e(w, \varphi)$ for all $w \in W$ such that Rvw. We can apply residuation and so get $e(v, \Box(\overline{c} \to \varphi)) \odot \overline{c}^{\mathbf{A}} \leq e(w, \varphi)$ for all $w \in W$ with Rvw and then take the infimum at the right side, getting (after using residuation again) that $e(v, \Box(\overline{c} \to \varphi)) \leq \overline{c}^{\mathbf{A}} \Rightarrow e(v, \Box\varphi) = e(v, \overline{c} \to \Box\varphi)$. On the other side, it is immediate that $\overline{c}^{\mathbf{A}} \Rightarrow \inf\{e(w, \varphi) : \ Rvw\} \leq \overline{c}^{\mathbf{A}} \Rightarrow e(w, \varphi)$ for all w such that Rvw and thus, taking the infimum at the right side, $\overline{c}^{\mathbf{A}} \Rightarrow$

$\inf\{e(w,\varphi) : Rvw\} \leq \inf\{\overline{c}^{\mathbf{A}} \Rightarrow e(w,\varphi) : Rvw\}$. Then we have that $\inf\{\overline{c}^{\mathbf{A}} \Rightarrow e(w,\varphi) : Rvw\} = \overline{c}^{\mathbf{A}} \Rightarrow \inf\{e(w,\varphi) : Rvw\}$, which concludes the proof.

$\Box$2 : By definition $e(v, \Delta\Box\varphi) = \delta^{\mathbf{A}} \inf\{e(w,\varphi) : Rvw\}$. We know $\delta^{\mathbf{A}}$ is monotonic increasing (from axiom Δ5), so $\delta^{\mathbf{A}} \inf\{e(w,\varphi) : Rvw\} \leq \delta^{\mathbf{A}} e(w,\varphi)$ for each $w \in W$ with Rvw. Then, we can take the infimum on the right side and get that $e(v, \Delta\Box\varphi) \leq \inf\{\delta^{\mathbf{A}} e(w,\varphi) : Rvw\} = e(v, \Box\Delta\varphi)$.

FS : By definition $e(v, \Diamond\varphi \to \Box\psi) = \sup\{e(w,\varphi) : Rvw\} \Rightarrow \inf\{e(w,\psi) : Rvw\}$. By the monotonicity of $\Rightarrow$ in each component, $\sup\{e(w,\varphi) : Rvw\} \Rightarrow \inf\{e(w,\psi) : Rvw\} \leq e(w,\varphi) \Rightarrow e(u,\psi)$ for each w, u such that Rvw and Rvu. Then, in particular, this is less or equal than $e(w,\varphi) \Rightarrow e(w,\psi)$ for each Rvw. Then, taking the infimum this proves that $e(v, \Diamond\varphi \to \Box\psi) \leq \inf\{e(w, \varphi \to \psi) : Rvw\} = e(v, \Box\varphi)$.

$\Diamond$1 : By definition $e(v, \Box(\varphi \to \overline{c})) = \inf\{e(w, \varphi \to \overline{c}) : Rvw\} = \inf\{e(w,\varphi) \Rightarrow \overline{c}^{\mathbf{A}} : Rvw\}$. As it happened in the ($\Box$1) case, it is not difficult to check that this is equal to $\sup\{e(w,\varphi) : Rvw\} \Rightarrow \overline{c}^{\mathbf{A}}$, which is, by definition, $e(v, \Diamond\varphi \to \overline{c})$. On the one hand, $\inf\{e(w,\varphi) \Rightarrow \overline{c}^{\mathbf{A}} : Rvw\} \leq e(w,\varphi) \Rightarrow \overline{c}^{\mathbf{A}}$ for each $w \in W$ such that Rvw. By residuation (twice) this is equivalent to $e(w,\varphi) \leq \inf\{e(w,\varphi) \Rightarrow \overline{c}^{\mathbf{A}} : Rvw\} \Rightarrow c$ and taking the supremum on the left we have that $\sup\{e(w,\varphi) : Rvw\} \leq \inf\{e(w,\varphi) \Rightarrow \overline{c}^{\mathbf{A}} : Rvw\} \Rightarrow \overline{c}^{\mathbf{A}}$, which is equivalent (again by residuation) to $\inf\{e(w,\varphi) \Rightarrow \overline{c}^{\mathbf{A}} : Rvw\} \leq \sup\{e(w,\varphi) : Rvw\} \Rightarrow \overline{c}^{\mathbf{A}}$. On the other hand, by (decreasing) monotonicity of the first component of the $\Rightarrow$ operation, $\sup\{e(w,\varphi) : Rvw\} \Rightarrow \overline{c}^{\mathbf{A}} \leq e(w,\varphi) \Rightarrow \overline{c}^{\mathbf{A}}$ for each $w \in W$ with Rvw. Then, we can take the infimum at the right side and get that $\sup\{e(w,\varphi) : Rvw\} \Rightarrow \overline{c}^{\mathbf{A}} \leq \inf\{e(w,\varphi) \Rightarrow \overline{c}^{\mathbf{A}} : Rvw\}$.

$\mathsf{N}_\Box$: To check that the inference rule (applied over theorems) is sound with respect to the local deduction, let φ be a formula such that for any $*$-model $\mathfrak{M}$ and any state $w \in W$, it holds that $e(w,\varphi) = \overline{1}^{\mathbf{A}}$. Then, for any $v \in W$, $e(v, \Box\varphi) = \inf\{e(w,\varphi) : Rvw\} = \overline{1}^{\mathbf{A}}$ from the previous remark.

$\mathsf{N}_\Box^{\mathbf{g}}$: To check that the inference rule is sound with respect to the global deduction, let φ be a formula such that $\mathfrak{M} \Vdash \varphi$, that is, at any state $w \in W$, $e(w,\varphi) = \overline{1}^{\mathbf{A}}$. Then, for any $v \in W$, it holds that $e(v, \Box\varphi) = \inf\{e(w,\varphi) : Rvw\}$ which coincides with $\overline{1}^{\mathbf{A}}$ by the previous assumption. ⊠

For proving completeness, on the other hand, the path is not direct. For now, we will focus on the local modal logic. Observe that the only modal rule added for the local logic in both cases is ($\mathsf{N}_\Box$), which is applied only over theorems of the logic. That allows to stablish a relation between the deductions in the modal logic and the ones over the propositional layer over an extended set of variables. It is necessary to add a particular infinite set of formulas to the premises, that will stablish the behaviour of these new variables (namely, a set of formulas containing the theorems of the modal logic). At this point it is where we can observe why it was important for the non-modal logic to enjoy strong standard completeness.

Formally, we denote by $Var^\star$ the extended set of variables $Var \cup \{\varphi_\Box, \varphi_\Diamond\}_{\varphi \in Fm}$,[3] and by $Fm^\star$ the set of formulas in the language of $\mathfrak{L}$ (*without* modal operators) over the extended set of variables $Var^\star$.

Definition 6.7. We inductively define a translation between Fm and the set of non-modal formulas $Fm^\star$ defined above $\star\colon Fm \to Fm^\star$ by

$$\begin{aligned} \overline{c}^\star &:= \overline{c} & (\varphi \wedge \psi)^\star &:= \varphi^\star \wedge \psi^\star \\ x^\star &:= x \text{ for } x \in Var & (\Delta\varphi)^\star &:= \Delta\varphi^\star \\ (\varphi \mathbin{\&} \psi)^\star &:= \varphi^\star \mathbin{\&} \psi^\star & (\Box\varphi)^\star &:= \varphi_\Box \\ (\varphi \to \psi)^\star &:= \varphi^\star \to \psi^\star & (\Diamond\varphi)^\star &:= \varphi_\Diamond. \end{aligned}$$

It is easy to see that $\star$ is a bijective mapping reasoning inductively. It is clearly injective by the definition: for any two different propositional variables from Var or constant symbols their images under $\star$ are clearly different ($\star$ behaves as the identity), and from that follows inductively for the rest of the propositional formulas. For what respect formulas beginning by a modal operator, it is also immediate since the images are two different variables from $Fm^\star$. To check that the mapping is also surjective, any propositional variable from Var or constant symbol is the image under $\star$ of itself and any variable $\varphi_\Box$ or $\varphi_\Diamond$ is the image under $\star$ of $\Box\varphi$ or $\Diamond\varphi$ respectively. From these basis, it is clear that any formula belonging to $Fm^\star$ can be built as the image under $\star$ of a formula from Fm.

Now, the modal formulas from $\mathcal{M}$ that take the role of modal axioms of K_* cannot be understood as schematas any more: the substitution should be done at the level of the non-modal part of the name of the variable, which is not the standard definition. Nevertheless, the translation through $\star$ of each one of the instances of the original schemata provides a (infinite) set of formulas in $Fm^\star$ that contain all the information from the original ones.

For arbitrary instances of the formulas of $\mathcal{M}$ with φ, ψ and $c \in \mathcal{C}_*$, we get the following translations:

$$\begin{aligned} (\mathsf{K})^\star &: (\varphi \to \psi)_\Box \to (\varphi_\Box \to \psi_\Box) & (\Box 2)^\star &: \Delta\varphi_\Box \to (\Delta\varphi)_\Box \\ (\Box 1)^\star &: (\overline{c} \to \varphi)_\Box \leftrightarrow (\overline{c} \to \varphi_\Box) & (\Diamond 1)^\star &: (\varphi \to \overline{c})_\Box \leftrightarrow (\varphi_\Diamond \to \overline{c}) \\ & & (\mathsf{FS})^\star &: (\varphi_\Diamond \to \psi_\Box) \to (\varphi \to \psi)_\Box \end{aligned}$$

This translation can be used to obtain an equivalence between deductions in the local modal logic and deductions in the respective non-modal logic.

It is crucial here that we did not add any inference rule to the modal expansion that is not limited to theorems. Otherwise, this result cannot be obtained and so the completeness proof would need a different approach.

[3] Recall that Fm stands for the formulas with modal operators.

Lemma 6.8. *Let $\Gamma \cup \{\varphi\} \subseteq Fm$. Then*

$$\Gamma \vdash_{K_*} \varphi \iff \Gamma^\star \cup (Th_{K_*})^\star \vdash_{L_*^\infty} \varphi^\star.$$

Proof. We can prove it by induction on the proof using the axiomatic system of K_*^∞ based on L_*^∞ (in both directions), taking as base cases the theorems of the logics.

To prove 1., first suppose that $\Gamma \vdash_{K_*} \varphi$. The cases for the induction are the following:

- If φ is a theorem of K_*, then $\varphi^\star \in (Th_{K_*})^\star$ (observe this case includes the deductions done using rule $\mathsf{N}_\Box$).
- Suppose the last rule used in the deduction was MP, that is to say, $\Gamma \vdash_{K_*} \psi$ and $\Gamma \vdash_{K_*} \psi \to \varphi$ and the Induction Hypothesis can be used over these cases (their proofs are shorter than the initial one). Then, $\Gamma^\star \cup (Th_{K_*})^\star \vdash_{L_*^\infty} \psi^\star$ and $\Gamma^\star \cup (Th_{K_*})^\star \vdash_{L_*^\infty} (\psi \to \varphi)^\star$. By definition of the $\star$ operation this second statement coincides with $\Gamma^\star \cup (Th_{K_*})^\star \vdash_{L_*^\infty} \psi^\star \to \varphi^\star$ and thus by MP (in L_*^∞), $\Gamma^\star \cup (Th_{K_*})^\star \vdash_{L_*^\infty} \varphi^\star$.
- Suppose the last rule used in the deduction was G_Δ. Then $\Gamma \vdash_{K_*} \psi$ for $\varphi = \Delta\psi$ and by the Induction Hypothesis $\Gamma^\star \cup (Th_{K_*})^\star \vdash_{L_*^\infty} \psi^\star$. Using G_Δ and the definition of $\star$, we get $\Gamma^\star \cup (Th_{K_*})^\star \vdash_{L_*^\infty} (\Delta\psi)^\star$.
- Suppose the last rule used in the deduction was D^∞. Then $\varphi = \psi \to \chi$ for some formulas ψ, χ and $\Gamma \vdash_{K_*} (\psi \to \overline{c}) \vee (\overline{c} \to \chi)$ for all $c \in \mathcal{C}_*$. By the Induction Hypothesis, $\Gamma^\star \cup (Th_{K_*})^\star \vdash_{L_*^\infty} ((\psi \to \overline{c}) \vee (\overline{c} \to \chi))^\star$ for each $c \in \mathcal{C}_*$, and by the definition of $\star$, $\Gamma^\star \cup (Th_{K_*})^\star \vdash_{L_*^\infty} (\psi^\star \to \overline{c}) \vee (\overline{c} \to \chi^\star)$ for each $c \in \mathcal{C}_*$. Applying the rule D^∞ (in L_*^∞) from this it follows that $\Gamma^\star \cup (Th_{K_*})^\star \vdash_{L_*^\infty} \psi^\star \to \chi^\star$ and again by the definition of $\star$, that $\Gamma^\star \cup (Th_{K_*})^\star \vdash_{L_*^\infty} (\psi \to \chi)^\star$.

For the other direction, assume that $\Gamma^\star \cup (Th_{K_*})^\star \vdash_{L_*^\infty} \varphi^\star$. Again by induction we can prove that $\Gamma \vdash_{K_*} \varphi$. For commodity, the base case will be now reduced to instances of the axioms of the logic. We will be using without further explanations of the fact that for every $\psi \in Fm^\star$, there is $\chi \in Fm$ such that $\psi = \chi^\star$ (since $\star$ is a bijection).

- If $\varphi^\star$ is an instance of an axiom of L_*^∞ (by formulas over the extended set of formulas $Var^\star$), φ is an instance of the same axiom in K_*.
- Suppose the last rule used in the deduction was MP, that is to say, $\Gamma^\star \cup (Th_{K_*})^\star \vdash_{L_*^\infty} \psi^\star$ and $\Gamma^\star \cup (Th_{K_*})^\star \vdash_{L_*^\infty} \psi^\star \to \varphi^\star$. By the Induction Hypothesis, $\Gamma \vdash_{K_*} \psi$ and $\Gamma \vdash_{K_*} \psi \to \varphi$, which immediately deduces (by MP in K_*) $\Gamma \vdash_{K_*} \varphi$.
- Suppose the last rule used in the deduction was G_Δ. Then $\Gamma^\star \cup (Th_{K_*})^\star \vdash_{L_*^\infty} \psi^\star$ for $\varphi^\star = \Delta\psi^\star = (\Delta\psi)^\star$. Note that this implies that $\Delta\psi = \varphi$. By Induction Hypothesis, $\Gamma \vdash_{K_*} \psi$ and thus by G_Δ from K_*, $\Gamma \vdash_{K_*} \Delta\psi$, which concludes this step.

- Suppose the last rule used in the deduction was D^∞. Then $\Gamma^\star \cup (Th_{K_*})^\star \vdash_{L_*^\infty} ((\psi \to \overline{c}) \vee (\overline{c} \to \chi))^\star$ for all $c \in \mathcal{C}_*$, for $\varphi^\star = \psi^\star \to \chi^\star$ (again, this implies that $\varphi = \psi \to \chi$). By the Induction Hypothesis, $\Gamma \vdash_{K_*} (\psi \to \overline{c}) \vee (\overline{c} \to \chi)$ for all $c \in \mathcal{C}_*$. By D^∞ from K_*, we can conclude that $\Gamma \vdash_{K_*} \psi \to \chi$. ⊠

At this point, one may wonder the necessity of presenting two equivalent axiomatizations of K_*^∞. We have found this to be the clearest way to prove an important result towards the completeness of these logics with respect to the Kripke semantics. It is easy to see (using that the accessibility relation is crisp) that, at the semantical level, deductions are closed under the $\Box$ operator, i.e., for any set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \Vdash_{\mathsf{KF}_*} \varphi \text{ implies that } \Box\Gamma \Vdash_{\mathsf{KF}_*} \Box\varphi$$

for $\Box\Gamma := \{\Box\gamma : \gamma \in \Gamma\}$. Then, aiming towards an axiomatization of this semantically defined modal logic, it should be possible to check this characteristic within the proposed axiomatic system. In fact, we will later see that this result is of great importance. However, while proving this for the density rule D^∞ (from L_*^∞) is not clear at all, the case of the $\overline{\mathsf{R}_\mathsf{x}^\infty}$ rules is not hard to check, as we show below.

Theorem 6.9. *Let $\Gamma \cup \{\varphi\} \subseteq Fm$. Then $\Gamma \vdash_{K_*} \varphi$ implies that $\Box\Gamma \vdash_{K_*} \Box\varphi$.*

Proof. We do it by induction on the last rule of the proof of φ from Γ using the axiomatic system of $K_*^i nfty$ based on $\overline{L_*^\infty}$. The basic case consists on the axioms of the logic and the elements of Γ and the induction steps coincide with checking the hypothesis over each deduction rule (for that is checking the last deduction rule used in the proof).

- If φ is an axiom of K_*^∞ or if it belongs to Γ, clearly $\Box\varphi$ is respectively, either a theorem of K_*^∞ (by rule $\mathsf{N}_\Box$) or an element of $\Box\Gamma$ by definition and so $\Box\Gamma \vdash_{K_*^\infty} \Box\varphi$.

- Suppose the last rule used for proving φ was MP. That is to say, $\Gamma \vdash_{K_*^\infty} \psi$ and $\Gamma \vdash_{K_*^\infty} \psi \to \varphi$ with shorter proofs. By the Induction Hypothesis, $\Box\Gamma \vdash_{K_*^\infty} \Box\psi$ and $\Box\Gamma \vdash_{K_*^\infty} \Box(\psi \to \varphi)$. From this last deduction, applying axiom K we get that $\Box\Gamma \vdash_{K_*^\infty} \Box\psi \to \Box\varphi$ and thus, by MP, $\Box\Gamma \vdash_{K_*^\infty} \Box\varphi$.

- Suppose the last rule used for proving φ was G_Δ, i.e., $\varphi \equiv \Delta\psi$ and $\Gamma \vdash_{K_*^\infty} \psi$. By the Induction Hypothesis $\Box\Gamma \vdash_{K_*^\infty} \Box\psi$. Then, combining the rule G_Δ with Axiom ($\Box$2) we get $\Box\Gamma \vdash_{K_*^\infty} \Box\Delta\psi$.

- Suppose the last rule used for proving φ was $\overline{\mathsf{R}_\mathsf{x}^\infty}$ for some $x \in [0,1]$. That is to say, $\varphi = \psi \to \chi$ and $\Gamma \vdash_{K_*^\infty} (\psi \to \overline{c}) \wedge (\overline{d} \to \chi)$ for each $d \in [0,x)$ and each $c \in (x,1]$. By Induction Hypothesis this implies that $\Box\Gamma \vdash_{K_*^\infty} \Box((\psi \to \overline{c}) \wedge (\overline{d} \to \chi))$ for each $d \in [0,x)$ and each $c \in (x,1]$. Observe that $(\Box(\varphi \wedge \psi)) \to (\Box\varphi \wedge \Box\psi)$ is a theorem of K_*.[4] Using this theorem and Axioms $(\Diamond 1)$ and $(\Box 1)$, we have that $\Box\Gamma \vdash_{K_*^\infty} (\Diamond\psi \to \overline{c}) \wedge (\overline{d} \to \Box\chi)$ for each $d \in [0, x)$ and each $c \in (x, 1]$. Now, rule $\overline{\mathsf{R}_\chi^\infty}$ can be applied to get $\Box\Gamma \vdash_{K_*^\infty} \Diamond\psi \to \Box\chi$. Now, using axiom $\mathsf{FS1}$ it is immediate that $\Box\Gamma \vdash_{K_*^\infty} \Box(\psi \to \chi)$.

$\boxtimes$

Getting to the previous result was all the motivation that lead to the proposal and study of the alternative axiomatic systems for K_*^∞. From now on, for the necessary proofs, we will use the axiomatic system of K_*^∞ based on L_*^∞.

Completeness: the Canonical Model

To prove strong standard completeness of K_* with respect to the local deduction $\Vdash_{\mathsf{KF}_*}$ we resort to a usual method (see Chapter 3). We define a $[\mathbf{0}, \mathbf{1}]_*^{\mathbb{Q}}$-Kripke model with the property that for any set of modal formulas $\Gamma \cup \{\varphi\} \subseteq Fm$ such that $\Gamma \nvdash_{K_*} \varphi$, there is a world in the model which assigns the value 1 to all $\gamma \in \Gamma$ but a value strictly smaller than 1 to φ. The existence of truth constants interpreted densely is of great importance to prove the existence of such model.

Definition 6.10. The ***canonical model of*** K_* is the $[\mathbf{0}, \mathbf{1}]_*^{\mathbb{Q}}$-Kripke model $\mathfrak{M}_{\mathfrak{c}}^* = \langle W_{\mathfrak{c}}^*, R_{\mathfrak{c}}^*, e_{\mathfrak{c}}^* \rangle$ where

- $W_{\mathfrak{c}}^* := \{h \in Hom(\mathbf{Fm}^\star, [\mathbf{0}, \mathbf{1}]_*^{\mathbb{Q}}) : \; h([(Th_{K_*})^\star]) \subseteq \{1\}\}$;
- $R_{\mathfrak{c}}^* vw$ if for any $\psi \in Fm$ such that $v(\psi_\Box) = 1$ it holds that $w(\psi^\star) = 1$;
- $e_{\mathfrak{c}}^*(w, p) := w(p)$, for every $p \in Var$.

One may wonder about the above definition of $R_{\mathfrak{c}}^*$ as it is only depends on the $\Box$ modality and on formulas evaluated to 1. However as next lemmas show, this suffices to prove the truth lemma for this canonical model.

Lemma 6.11. *$R_{\mathfrak{c}}^* vw$ if and only if for any $\psi \in Fm$, the following inequalities hold:*

- *$v(\psi_\Box) \leq w(\psi^\star)$;*
- *$v(\psi_\Diamond) \geq w(\psi^\star)$.*

Proof. To prove the non-direct case, assume $v(\psi_\Box) > w(\psi^\star)$ for some $\psi \in Fm$. Then there is $c \in \mathcal{C}_*$ such that $w(\psi^\star) < c < v(\psi_\Box)$, and so $1 = c \Rightarrow_* v(\psi_\Box) = v(\overline{c} \to \psi_\Box)$. Since the instances of the form $(\Box 1)^\star$ belong to $Th_{K_*})^\star$, v evaluates them to 1 getting that $1 = v((\overline{c} \to \psi)_\Box)$. However, $w((\overline{c} \to \psi)^\star) = c \Rightarrow_* w(\psi^\star) < 1$, and so it does not hold that $R_{\mathfrak{c}}^* vw$.

[4] From theorem $(\varphi \wedge \psi) \to \varphi$, the necessity rule and axiom K, we have that $\Box(\varphi \wedge \psi) \to \Box\varphi$ and the same for ψ.

To check the second statement the reasoning is analogous. Assume that $v(\psi_\Diamond) < w(\psi^\star)$. Then, there is $c \in \mathcal{C}_*$ such that $v(\psi_\Diamond) < c < w(\psi^\star)$. Then $1 = v(\psi_\Diamond \to \overline{c})$, but by the instances of ($\Diamond$1) this implies that $1 = v((\psi \to \overline{c})_\Box)$. However, by assumption $c < w(\psi^\star)$, so $w(\psi^\star \to \overline{c}) < 1$. Then, by definition v is not related to w under $R^*_\mathfrak{c}$. ⊠

In the way of proving the *Truth Lemma* for the canonical model, we first show a crucial result. Here, many of the important results proved up to now are used, in particular Lemma 6.9 and the fact that L^∞_* and $\overline{L^\infty_*}$ are strongly standard complete with respect to $[\mathbf{0},\mathbf{1}]^\mathbb{Q}_*$.

Lemma 6.12. *Let $v \in W^*_\mathfrak{c}$ and $\varphi \in Fm$ such that $w(\varphi^\star) = 1$ for all $w \in W^*_\mathfrak{c}$ such that $R^*_\mathfrak{c}vw$. Then $v(\varphi_\Box) = 1$.*

Proof. Let $w \in W^*_\mathfrak{c}$. Then, by definition, $R^*_\mathfrak{c}vw$ if and only if $w([(Th_{K_*})^\star]) \subseteq \{1\}$ and $w(\psi^\star) = 1$ for all $\psi \in Fm$ such that $v(\psi_\Box) = 1$. In other words, for any $w \in Hom(\mathbf{Fm}^\star, [\mathbf{0},\mathbf{1}]^\mathbb{Q}_*)$, $R^*_\mathfrak{c}vw$ if and only if $w([(Th_{K_*} \cup T)^\star]) \subseteq \{1\}$, where $T = \{\psi \in Fm : \ v(\psi_\Box) = 1\}$.

Therefore, the hypothesis of the lemma amounts to assume $(Th_{K_*} \cup T)^\star \models_{[\mathbf{0},\mathbf{1}]^\mathbb{Q}_*} \varphi^\star$. By strong standard completeness of L^∞_*, it follows that $(Th_{K_*} \cup T)^\star \vdash_{L^\infty_*} \varphi^\star$ and then, by Lemma 6.8, $T \vdash_{K_*} \varphi$ as well.

Now Theorem 6.9 can be applied, obtaining $\Box T \vdash_{K_*} \Box\varphi$. By the same reasoning as before in the opposite sense (Lemma 6.8 and then applying strong standard completeness), it follows that $(Th_{K_*} \cup \Box T)^\star \models_{[\mathbf{0},\mathbf{1}]^\mathbb{Q}_*} \varphi_\Box$. But then, given that $v \in W^*_\mathfrak{c}$, it follows by definition that $v([(Th_{K_*})^\star]) \subseteq \{1\}$. On the other hand, for each $\psi \in T$ it holds that $v(\psi_\Box) = 1$ too, hence $v([(\Box T)^\star]) \subseteq \{1\}$. Therefore, since the whole set of premises is evaluated to 1, it also holds that $v(\varphi_\Box) = 1$, which concludes the proof. ⊠

Using the density of the rational numbers within the reals, it is not hard to get the Truth Lemma as a consequence of the previous result.

Lemma 6.13 (Truth Lemma)**.** *For any $\varphi \in Fm$ and any $v \in W^*_\mathfrak{c}$ it holds that*

$$e(v, \varphi) = v(\varphi^\star).$$

Proof. This can be proven by induction on the structure of the formulas, and the only cases that are worth to be detailed are the modal ones.

To show that $v(\varphi_\Box) = \inf\{w(\varphi^\star) : R^*_\mathfrak{c}vw\}$, first notice that by Lemma 6.11 $v(\varphi_\Box) \leq w(\varphi^\star)$ for any w such that $R^*_\mathfrak{c}vw$, so $v(\varphi_\Box) \leq \inf\{w(\varphi^\star) : R^*_\mathfrak{c}vw\}$. To prove that $v(\varphi_\Box) \geq \inf\{w(\varphi^\star) : R^*_\mathfrak{c}vw\}$ assume towards a contradiction that $v(\varphi_\Box) < \inf\{w(\varphi^\star) : \ R^*_\mathfrak{c}vw\}$. Then there is $c \in \mathcal{C}_*$ such that $v(\varphi_\Box) < c < w(\varphi^\star)$ for any w such that $R^*_\mathfrak{c}vw$. This implies that $w((\overline{c} \to \varphi)^\star) = 1$ for any w such that $R^*_\mathfrak{c}vw$, and by Lemma 6.12 we have that $v((\overline{c} \to \varphi)_\Box) = 1$ as well. However, since all instances of the form $(\Box 1)^\star$ belong to Th_{K_*}, they are evaluated to 1 under v, so $v((\overline{c} \to \varphi)_\Box) = v(\overline{c} \to \varphi_\Box)$, and hence $c \Rightarrow_* v(\varphi_\Box) = 1$ too, which contradicts the fact that $v(\varphi_\Box) < c$.

To show that $v(\varphi_\Diamond) = \sup\{w(\varphi^\star) : R_\mathfrak{c}^* vw\}$, the proof is analogous. First, by Lemma 6.11 $v(\varphi_\Diamond) \geq w(\varphi^\star)$ for any w such that $R_\mathfrak{c}^* vw$, so $v(\varphi_\Diamond) \geq \sup\{w(\varphi^\star) : R_\mathfrak{c}^* vw\}$. To prove that $v(\varphi_\Diamond) \leq \sup\{w(\varphi^\star) : R_\mathfrak{c}^* vw\}$ assume towards a contradiction that $v(\varphi_\Diamond) > \sup\{w(\varphi^\star) : R_\mathfrak{c}^* vw\}$ Then there is $c \in \mathcal{C}_*$ such that $v(\varphi_\Diamond) > c > w(\varphi^\star)$ for any w such that $R_\mathfrak{c}^* vw$. This implies that $v(\varphi_\Diamond \to \overline{c}) < 1$ and since the instances of Axiom ($\Diamond 1$) hold in any world, $v((\varphi \to \overline{c})_\Box) < 1$. On the other hand, we also have that $w(\varphi^\star \to \overline{c}) = 1$ for any w such that $R_\mathfrak{c}^* vw$. Using Lemma 6.12, it follows that $v((\varphi \to \overline{c})_\Box) = 1$, which is a contradiction. ⊠

Having proved the Truth Lemma, it is easy to check completeness of K_* with respect to $\Vdash_{\mathsf{KF}_*}$ and in particular, with respect to subsets of this class of models: Kripke models evaluated over linearly ordered L_*^∞-algebras, Kripke models evaluated over $[\mathbf{0},\mathbf{1}]_*^{\mathbf{Q}}$ and the particular canonical model of K_* defined before.

Theorem 6.14 (Strong completeness of K_*)**.** *Let $*$ be a left-continuous t-norm accepting a conjunctive axiomatization. Then for any set of modal formulas $\Gamma \cup \{\varphi\}$ the following are equivalent:*

1. $\Gamma \vdash_{K_*} \varphi$;
2. $\Gamma \Vdash_{\mathfrak{M}_\mathfrak{c}^*} \varphi$;
3. $\Gamma \Vdash_{\mathsf{M}[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}} \varphi$, *where* $\mathsf{M}[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$ *denotes the class of* $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$*-Kripke models;*[5]
4. $\Gamma \Vdash_{\mathsf{CK}_*} \varphi$, *where* CK_* *denotes the class of safe* $\mathbf{A}$*-Kripke models such that* $\mathbf{A} \in \mathsf{L}_*^\infty$ *is a chain;*
5. $\Gamma \Vdash_{\mathsf{KF}_*} \varphi$.

Proof. Observe that the down-to-up implication chain from point 5. to 2. trivially holds (since the models form a chain of inclusions), i.e., 5. $\Longrightarrow$ 4. $\Longrightarrow$ 3. $\Longrightarrow$ 2. Soundness was checked before (see 6.6) for the whole $\Vdash_{\mathsf{KF}_*}$ case, so 1. implies 5.. Therefore it remains to prove 2. implies 1. and the rest follows directly.

Assume $\Gamma \nvdash_{K_*} \varphi$. By Lemma 6.8, this happens if and only if $\Gamma^\star \cup (Th_{K_*})^\star \nvdash_{L_*^\infty} \varphi^\star$. By strong standard completeness of L_*^∞, then there is an homomorphism v from $Fm^\star$ into $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$, such that $v([(Th_{K_*})^\star]) \subseteq \{1\}$ and $v([\Gamma^\star]) \subseteq \{1\}$ but $v(\varphi^\star) < 1$. Hence, $v \in W_\mathfrak{c}^*$ and by the Truth Lemma 6.13, $e(v,[\Gamma]) = v([\Gamma^\star]) \subseteq \{1\}$ and $e(v,\varphi) = v(\varphi^\star) < 1$. Then $\mathfrak{M}_\mathfrak{c}^* \models_v \Gamma$ and $\mathfrak{M}_\mathfrak{c}^* \not\models_v \varphi$, so by definition $\Gamma \nVdash_{\mathfrak{M}_\mathfrak{c}^*} \varphi$. ⊠

The global modal logic

For the proof of completeness of the global modal logic K_*^g, the approach we will follow is very similar to the one done for K_* and in fact we will resort to results proved in the previous section. The main difference is that of considering a more

[5]Observe $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$ is a complete algebra and so all the models defined over it are safe.

specific definition for the canonical model. Instead of proving completeness with respect to just *one* standard Kripke model as we did before (with respect to $\mathfrak{M}^*_{\mathfrak{c}}$), the definition of the canonical model depends on the premises of the deduction.

We begin by noticing that Lemma 6.8 does not work for the K^g_* logic. Nevertheless, the following obvious version will be enough for our purposes.

Remark 6.15. Let $\Gamma \cup \{\varphi\} \subseteq Fm$. Then

$$\Gamma \vdash_{K^g_*} \varphi \iff Cn_{K^g_*}(\Gamma)^\star \vdash_{L^\infty_*} \varphi^\star$$

where for a logic L, $Cn_L(\Gamma) \coloneqq \{\psi \in Fm : \Gamma \vdash_L \psi\}$.

At this point, a new definition of canonical model that depends on the set Γ of premises of the deduction naturally arises as a generalization of $\mathfrak{M}^*_{\mathfrak{c}}$.

Definition 6.16. Let $\Gamma \subseteq Fm$. The Γ***-canonical model of*** K^g_* is the $[\mathbf{0},\mathbf{1}]^{\mathbb{Q}}_*$-Kripke model $\mathfrak{M}^*_{\mathfrak{c}}[\mathbf{\Gamma}] = \langle W^*_{\mathfrak{c}}[\Gamma], R^*_{\mathfrak{c}}[\Gamma], e^*_{\mathfrak{c}}[\Gamma]\rangle$ where

- $W^*_{\mathfrak{c}}[\Gamma] \coloneqq \{h \in Hom(\mathbf{Fm}^\star, [\mathbf{0},\mathbf{1}]^{\mathbb{Q}}_*) : \; h([(Cn_{K^g_*}(\Gamma))^\star]) \subseteq \{1\}\}$;
- $R^*_{\mathfrak{c}}[\Gamma]vw$ if for any $\psi \in Fm$ such that $v(\psi_\Box) = 1$ it holds that $w(\psi^\star) = 1$;
- $e^*_{\mathfrak{c}}[\Gamma](w,p) \coloneqq w(p)$, for every $p \in Var$.

Observe that, if $\Gamma = \emptyset$, $\mathfrak{M}^*_{\mathfrak{c}}[\Gamma]$ coincides with the canonical model of K_*, i.e., with $\mathfrak{M}^*_{\mathfrak{c}}$, and that clearly, $(Th_{K_*})^\star = (Cn_{K^g_*}(\emptyset))^\star$.

In order to check the Truth Lemma for the models defined above we will use the tools developed for the local case. First, following the same reasoning that in Lemma 6.11 the same conclusion for $R^*_{\mathfrak{c}}[\Gamma]$ can be reached. That is to say, the following hold:

$$v(\varphi_\Box) \leq w(\varphi^\star) \text{ for each } w \text{ such that } R^*_{\mathfrak{c}}[\Gamma]vw$$
$$v(\varphi_\Diamond) \geq w(\varphi^\star) \text{ for each } w \text{ such that } R^*_{\mathfrak{c}}[\Gamma]vw$$

On the other hand, we can also prove the result corresponding to Lemma 6.12 in this new context using *the same* methods used there (and with that we mean resorting to the local modal logic K_* in the proof).

Lemma 6.17. *Let $v \in W^*_{\mathfrak{c}}[\Gamma]$ and $\varphi \in Fm$ such that $w(\varphi^\star) = 1$ for all $w \in W^*_{\mathfrak{c}}[\Gamma]$ such that $R^*_{\mathfrak{c}}[\Gamma]vw$. Then $v(\varphi_\Box) = 1$.*

Proof. As reasoned in Lemma 6.12, $w(\varphi^\star) = 1$ for all $w \in W^*_{\mathfrak{c}}[\Gamma]$ such that $R^*_{\mathfrak{c}}[\Gamma]vw$ is equivalent to have that $T^\star \cup (Cn_{K^g_*}(\Gamma))^\star \models_{[\mathbf{0},\mathbf{1}]^{\mathbb{Q}}_*} \varphi^\star$, for $T = \{\psi \in Fm : \; v(\psi_\Box) = 1\}$. Since $(Th_{K_*})^\star \subseteq (Cn_{K^g_*}(\Gamma))^\star$, we can apply Theorem 6.8 (after using strong standard completeness) and get that $T \cup Cn_{K^g_*}(\Gamma) \vdash_{K_*} \varphi$. As in Lemma 6.12, it is now enough to apply Theorems 6.9 and 6.8 and finally, standard completeness again, to get $(\Box T)^\star \cup (\Box Cn_{K^g_*}(\Gamma))^\star \models_{[\mathbf{0},\mathbf{1}]^{\mathbb{Q}}_*} \varphi_\Box$. But now observe that $v([(\Box T)^\star]) \subseteq \{1\}$ by the definition of T. Moreover, since deductions in K^g_* are closed under $\Box$ (immediate by the $\mathsf{N}^{\mathsf{g}}_\Box$ rule), $(\Box Cn_{K^g_*}(\Gamma))^\star \subseteq (Cn_{K^g_*}(\Gamma))^\star$, so by the definition of the worlds in the Γ-canonical model, $(\Box Cn_{K_*^g}(\Gamma))^\star$ is evaluated into $\{1\}$ in any world of the model (and in particular, in v). Then we can conclude that $v(\varphi_\Box) = 1$. ⊠

From here on, the proof of the Truth Lemma for $\mathfrak{M}_\mathfrak{c}^*[\Gamma]$ is the same as the one from Lemma 6.13 and is easy to conclude the completeness results for K_*^g.

Theorem 6.18 (Strong completeness of K_*^g)**.** *Let $*$ be a left-continuous t-norm accepting a conjunctive axiomatization. Then for any set of modal formulas $\Gamma \cup \{\varphi\}$ the following are equivalent:*

1. $\Gamma \vdash_{K_*^g} \varphi$;
2. $\Gamma \Vdash^g_{\mathfrak{M}_\mathfrak{c}^*[\Gamma]} \varphi$;
3. $\Gamma \Vdash^g_{\mathsf{M}[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}} \varphi$, *where* $\mathsf{M}[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$ *denotes the class of* $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$*-Kripke models;*
4. $\Gamma \Vdash^g_{\mathsf{CK}_*} \varphi$, *where* CK_* *denotes the class of safe* $\mathbf{A}$*-Kripke models such that* $\mathbf{A} \in \mathsf{L}_*^\infty$ *is a chain;*
5. $\Gamma \Vdash^g_{\mathsf{KF}_*} \varphi$.

Proof. Observe that $\Gamma \nvdash_{K_*^g} \varphi$ if and only if (by the remark above) $Cn_{K_*^g}(\Gamma)^\star \nvdash_{L_*^\infty} \varphi^\star$, i.e., if there is an homomorphism v from $Fm^\star$ to $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$ such that $v([Cn_{K_*^g}(\Gamma)^\star] \subseteq \{1\}$ (and thus, $v \in W_\mathfrak{c}^*[\Gamma]$ by definition) such that $v(\varphi^\star) < 1$. Appart from this fact, the rest of the proof is analogous to the one followed in the one concerning the completeness of the local logic (Theorem 6.14) ⊠

6.3 Extensions and further properties

Along the study of the logics K_* and K_*^g several interesting results have been found, either concerning these logics or referring to extensions of them. We will present within this section the main observations about the behaviour and characteristics of the modal logics defined in the previous sections and later we will present some results concerning some usual axiomatic extensions of them.

Interdefinability of $\Box$ and $\Diamond$

It seems clear that the usual interdefinability of $\Box$ and $\Diamond$ in this more general context does not exist in general. We can easily build a $[\mathbf{0},\mathbf{1}]_{\mathbf{\Pi}}^{\mathbb{Q}}$-Kripke model where $\Box\varphi \neq \neg\Diamond\neg\varphi$. Simply let $W = \mathbb{N}$ and $R = \{\langle 0, n+1\rangle : n \in \mathbb{N}\}$ and let $e(n+1, p) = \frac{1}{n+1}$ (and for instance, $e(0,p) = 1$). Clearly, $e(0, \Diamond\neg p) = 0$, because for all $n > 0$, $e(n,p) > 0$. Thus, $e(0, \neg\Diamond\neg p) = 1$, but, on the other hand, $e(0, \Box p) = 0$.

There are some special cases where the usual interdefinability keeps holding. For instance, the class of Kripke models built over IMTLs, i.e., the left continuous t-norms whose negation is involutive (for instance, the Łukasiewicz t-norm).

Indeed, since for all $\overline{c}$ it holds that $\Box(\varphi \to \overline{c}) \leftrightarrow (\Diamond\varphi \to \overline{c})$ (Axiom ($\Diamond$1)), in the particular case of $\overline{c} = \overline{0}$ this leads to $\Box\neg\varphi \leftrightarrow \neg\Diamond\varphi$. Having an involutive negation, this can be equivalently written as $\neg\Box\neg\varphi \leftrightarrow \Diamond\varphi$, that gives a definition of $\Diamond$ in terms of formulas without the $\Diamond$ operation. Similarly, in these t-norms it holds that $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$, since $\neg\Diamond\neg\varphi \leftrightarrow \Box(\neg\neg\varphi)$ (by Axiom Diamond1 again) and here the involutive behaviour of $\neg$ can be used.

Out of the scope of the IMTL class, results concerning interdefinability are not so clear. It is possible, however, to get partial characterizations of $\Box$ in terms of formulas without the $\Box$ operation and similarly for $\Diamond$, using the truth constants. We will do this analysing Axiom ($\Diamond$1), which is the only axiom of K_* and K_*^g that relates the $\Box$ and $\Diamond$ operators.

For similarity with respect to the IMTL case, we will denote by $\neg_c$ to the definable unary operation given by $\neg_c\varphi := \varphi \to \overline{c}$. It is natural now that if $\neg_c$ behaves in an involutive way over $\Diamond\varphi$ we can define $\Diamond\varphi$ as $\neg_c\Box\neg_c\varphi$ and similarly, if $\Box\neg_c\neg_c\varphi$ equals $\Box\varphi$ (semantically, $\neg_c$ is invoutive over φ in all the successors), we have that $\Box\varphi$ can be written as $\neg_c\Box\neg_c\varphi$. It is not hard to see that $\varphi \to \neg_c\neg_c\varphi$ is a theorem of K_* for any left-continuous t-norm $*$.

However, not for any left-continuous t-norm and any value a on $[0,1]$ exists a rational constant c such that $\neg_c\neg_c a = a$. For instance, it is clear that for what concerns the Gödel t-norm,

$$\neg_c\neg_c a = \begin{cases} 1 \text{ if } c > a \\ c \text{ if } c \leq a \end{cases}$$

for an arbitrary rational c and so for any irrational value $i \in [0,1]$, there is no rational c for which $\neg_c\neg_c i = i$. We can give some conditions on the t-norm that will allow to study a class of left-continuous t-norms from this point of view.

The following is a notion that allows us to uniformly study certain involutive operations and interdefinability of operations using them.

Definition 6.19. We say that a left-continuous t-norm $*$ is ***quasi-involutive*** whenever for any $x \in [0,1]$ and any $c \in \mathcal{C}_*$, if $x > c$ and $x \Rightarrow_* c > c$ then $((x \Rightarrow_* c) \Rightarrow_* c) = x$.

Some examples of quasi-involutive left-continuous t-norms are the product t-norm, the Łukasiewicz t-norm, and arbitrary ordinal sums of these two components. Thus, we can talk about the left-continuous t-norms accepting a conjunctive axiomatization (which are the t-norms that have been studied in this chapter) and that are, moreover, quasi-involutive. In fact, concerning the continuous t-norms, these two classes are the same. Moreover, for a left-continuous t-norm $*$, the notion of quasi-involutive can be equivalently characterized with the elements of the algebra (instead of those from $\mathcal{C}_*$.

For quasi-involutive t-norms it is possible to characterize a partial interdefinability of the modal operators, based on the constants.

Lemma 6.20. *Let $*$ be a quasi-involutive left-continuous t-norm that moreover accepts a conjunctive axiomatization. Then the following formulas are theorems of K_*.*

1. $(\neg\Delta\Box\neg_c\varphi \wedge \neg\Delta\neg_c\Box\neg_c\varphi) \to (\Diamond\varphi \leftrightarrow \neg_c\Box\neg_c\varphi)$

2. $(\neg\Diamond\Delta\neg_c\varphi \wedge \Diamond\neg\Delta\neg_c\neg_c\varphi) \to (\Box\varphi \leftrightarrow \neg_c\Diamond\neg_c\varphi)$.

Proof. We will prove these statement resorting to the completeness of K_* with respect to the local modal logic over the crisp $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$-Kripke models. Let $\mathfrak{M}$ be a crisp $[0,1]_*$-Kripke model and v be a world of its universe.

Concerning the first implication, observe that the premise is a bi-valued term, i.e., $v(\neg\Delta\Box\neg_c\varphi\wedge\neg\Delta\neg_c\Box\neg_c\varphi) = 1$ if and only if $v(\Box\neg_c\varphi) < 1$ and $v(\neg_c\Box\neg_c\varphi) < 1$, and it equals 0 otherwise (making the implication trivially true). In the first case, from the definition of $\neg_c$ and Axiom ($\Diamond 1$) it follows that $v(\Diamond\varphi) > c$ and $v(\Diamond\varphi) \Rightarrow_* c > c$. Since $*$ is quasi-involutive this implies that $v(\neg_c\neg_c\Diamond\varphi) = \Diamond\varphi$.

On the other hand, again using Axiom ($\Diamond 1$) we get that $v(\neg_c\Box\neg_c\varphi) = v(\neg_c\neg_c\Diamond\varphi)$ and from the above reasoning this is equal to $v(\Diamond\varphi)$.

For the second implication, we can observe as before that the premise of the implication is a bi-valued formula: $v(\neg\Diamond\Delta\neg_c\varphi \wedge \Diamond\neg\Delta\neg_c\neg_c\varphi) = 1$ if and only if $v(\Diamond\Delta\neg_c\varphi) = 0$ and $v(\Diamond\neg\Delta\neg_c\neg_c\varphi) = 1$ and it equals 0 otherwise.[6] In the first case, $v(\Diamond\Delta\neg_c\varphi) = 0$ if and only if for all w such that Rvw, $w(\Delta\neg_c\varphi) = 0$ and so if and only if for all such w, $w(\varphi) > c$. On the other hand, $v(\Diamond\neg\Delta\neg_c\neg_c\varphi) = 1$ means by definition that $\sup\{w(\neg\Delta\neg_c\neg_c\varphi) : Rvw\} = 1$ and by the definition of Δ this implies that there exists w_0 with Rvw_0 such that $w_0(\neg\Delta\neg_c\neg_c\varphi) = 1$. This leads to have that $w_0(\varphi) \Rightarrow_* c > c$ and more in general, that for all u such that Rvu and such that $u(\varphi) \leq w_0(\varphi)$ it holds that $u(\varphi) \Rightarrow_* c > c$. Then, applying that $*$ is quasi-involutive we have that for all u such that Rvu and $u(\varphi) \leq w_0(\varphi)$, it holds that $u(\neg_c\neg_c\varphi) = u(\varphi)$.

We can split the successor worlds of v in terms of $w_0(\varphi)$, i.e., $v(\Box\neg_c\neg_c\varphi) = \inf\{w(\neg_c\neg_c\varphi) : Rvw \text{ and } w(\varphi) \leq w_0(\varphi)\} \wedge \inf\{w(\neg_c\neg_c\varphi) : Rvw \text{ and } w(\varphi) > w_0(\varphi)\}$. From the previous reasoning we have that $\inf\{w(\neg_c\neg_c\varphi) : Rvw \text{ and } w(\varphi) \leq w_0(\varphi)\} = \inf\{w(\varphi) : Rvw \text{ and } w(\varphi) \leq w_0(\varphi)\}$. On the other hand, since $\varphi \to \neg_c\neg_c\varphi$ is a theorem of the logic we know that $\inf\{w(\varphi) : Rvw \text{ and } w(\varphi) > w_0(\varphi)\} \leq \inf\{w(\neg_c\neg_c\varphi) : Rvw \text{ and } w(\varphi) > w_0(\varphi)\}$ and so clearly $\inf\{w(\varphi) : Rvw \text{ and } w(\varphi) \leq w_0(\varphi)\} \leq \inf\{w(\varphi) : Rvw \text{ and } w(\varphi) > w_0(\varphi)\} \leq \inf\{w(\neg_c\neg_c\varphi) : Rvw \text{ and } w(\varphi) > w_0(\varphi)\}$. Then we can write that $v(\Box\neg_c\neg_c\varphi) = \inf\{w(\varphi) : Rvw\}$, so $v(\Box\neg_c\neg_c\varphi) = v(\Box\varphi)$. Now it is immediate to see that, applying Axiom ($\Diamond 1$), $v(\Box\neg_c\neg_c\varphi) = v(\neg_c\Diamond\neg_c\varphi)$, concluding the proof. ⊠

On the Finite Model Property

Another remarkable difference of the modal expansions of logics arising from a left-continuous t-norm with respect to the most general classical modal logic K is that the finite model property does no longer hold. We remarked in Chapter

[6] Indeed, $v(\Diamond\Delta\neg_c\varphi) > 0$ implies that there is w such that Rvw where $w(\Delta\neg_c\varphi) > 0$, that is to say, $w(\Delta\neg_c\varphi) = 1$. But then, $v(\Diamond\Delta\neg_c\varphi) = 1$ too and thus $v(\neg\Diamond\Delta\neg_c\varphi) = 0$. Similarly if $v(\Diamond\neg\Delta\neg_c\neg_c\varphi) < 1$ then for all w such that Rvw it hods that $w(\neg\Delta\neg_c\neg_c\varphi) < 1$ and thus $w(\Delta\neg_c\neg_c\varphi) > 0$. Then for such that w as before, $w(\Delta\neg_c\neg_c\varphi) = 1$ and so $w(\neg\Delta\neg_c\neg_c\varphi) = 0$.

3 that this was first observed for the case of the modal expansions of Gödel logic (see [24]), while in other works dealing with modal expansions of finitely-valued logics, the finite model property holds.

In the modal logics we have defined, depending on the t-norm defining the semantics, there exist formulas that are valid in the all finite frames while they are not theorems of the logic. Moreover, the use of Δ allows even more linguistic flexibility and we can prove that no left-continuous t-norm gives rise to a modal logic with finite model property (that is, considering the modal logic with Δ and rational constants).

A simple example can be given, instead of using theorems (where the fact of considering non-linear L_*^∞-algebras could lead to confusion), using deductions that are valid over all the finite frames and not in general. Consider the following case

$$\Box p \to \overline{c} \vdash \Diamond\Delta(p \to \overline{c})$$
$$\overline{c} \to \Diamond p \vdash \Diamond\Delta(\overline{c} \to p)$$

for any $c \in [0,1)_{\mathbb{Q}}$. Indeed, for an arbitrary $\mathbf{A} \in \mathbb{GQ}([\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}})$, at any finite $\mathbf{A}$-Kripke model $\mathfrak{M}$ and any state $w \in W$, the premise of the first deduction holds if and only if $e(w, \Box p) = \inf\{e(v,p) : Rwv\} \leq \overline{c}^{\mathbf{A}}$. In particular, $e(w, \Box p) < 1$, so there is some $v \in W$ with Rvw. Since the model has a finite number of states it holds that $\inf\{e(v,p) : Rwv\} = \min\{e(v,p) : Rwv\}$ and so there must exist $v \in W$ such that Rwv and $e(v,p) \leq \overline{c}^{\mathbf{A}}$. This implies that $e(w, \Diamond\Delta(p \to \overline{c})) = \overline{1}^{\mathbf{A}}$. However, consider $\mathfrak{M}$ the $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$-Kripke model with universe $\mathbb{N}$, with $R = \bigcup_{n\geq 1}\{\langle 0, n\rangle\}$ and such that $e(n+1, p) = c + \frac{1}{n+1}$. It is clear that $\mathfrak{M}, 0 \Vdash \Box p \to \overline{c}$, but it also hods that for any $n \in \mathbb{N} \setminus \{0\}$ (that is, all successors of 0 in the model) $e(n,p) > c$ and thus $e(n, \Delta(p \to \overline{c}) = 0$ and so $e(0, \Diamond\Delta(p \to \overline{c})) = 0$. The second example can be understood with an analogous reasoning.

It might seem that adding the previous set of deductions to the axiomatic system as inference rules could lead to a logic with finite model property (even if it would not be the minimum modal logic). However, observe that the previous deductions impose the notion of witness but only for the modal formulas and states of the model in which these modal formulas are evaluated by a rational number. When this is not the case (for instance, let $i \in [0,1] \setminus \mathcal{C}_*$ and consider a case where $e(v, \Box p \to \overline{c}) = 1$ for all $c \in (i,1] \cap \mathcal{C}_*$ and $e(v, \overline{d} \to \Box p)$ for all $c \in [0,i) \cap \mathcal{C}_*$), the previous rules do not imply the existence of a witness.

We consider this problem to be of great interest, since it offers a natural way of proving decidability of the addressed logics (concerning finitary deductions). However, we have not been able to find a suitable way, if it exists, to prove decidability of satisfiability, validity or consequence for the logics studied along this chapter. The approach followed in [21] does not seem to be applicable in our context. Nevertheless, it seems one of the more interesting approaches that should be taken into consideration when trying to solve the decidability problem.

Some usual axiomatic extensions

As it happens in classical modal logic, it is interesting to know which axiom schemata correspond to conditions over the frames and vice-versa, which classes of frames have an axiomatic representation, as an extension of K_* (or K_*^g). In general, this problem has a deep complexity, and we have not studied it for an arbitrary class of frames or formulas. Hansoul and Teheux study this problem in [78], in the case of the modal expansions of Łukasiewicz logic. However, it is not clear whether the methods used there can be applied in the more general problem of Kripke models evaluated over arbitrary L_*^∞-algebras, since some important characteristics of Łukasiewicz logic and MV-algebras do not hold in this new context.

Nevertheless, we have studied how or the most usual classes of frames (in the sense of having particular restrictions over the accessibility relation) are linked to their corresponding axiom schematas, also in the more general our case of many-valued models.

The main axioms and properties we are referring to are those from Table 3.1, from Chapter 3. We have already seen that the K axiom is valid in all the (crisp) many-valued Kripke models. On the other hand, concerning the other properties in the table, for $p \in \{\mathcal{D}, \mathcal{T}, \mathcal{B}, \mathit{5}, \mathit{4}\}$ we denote by ${\mathsf{KF}_*}^P$ the class of $*$-Kripke structures that enjoy the property P. Then, for $\mathcal{A}x \subseteq \{\mathcal{D}, \mathcal{T}, \mathcal{B}, \mathit{5}, \mathit{4}\}$, we let ${\mathsf{KF}_*}^{\mathcal{A}x} = \bigcap_{P \in \mathcal{A}x} {\mathsf{KF}_*}^P$. Analogously, for $Ax \subseteq \{(\mathsf{D}), (\mathsf{T}), (\mathsf{B}), (\mathsf{5}), (\mathsf{4})\}$ we will denote by ${K_*}^{Ax}$ the logic obtained by extending K_* with the set of axioms in Ax. It is obvious there is a 1-1 relation between subsets of structural axioms Ax and subsets of properties $\mathcal{A}x$, relating each axiom with its correspondent property over the Kripke models. Moreover, it is not hard to prove that indeed this relation preserves the completeness of the extensions with respect to the classes of models that enjoy the corresponding properties.

Theorem 6.21. *Let $Ax \subseteq \{(\mathsf{D}), (\mathsf{T}), (\mathsf{B}), (\mathsf{5}), (\mathsf{4})\}$ and $\Gamma \cup \{\varphi\} \subseteq Fm$. Then $\Gamma \vdash_{{K_*}^{Ax}} \varphi$ if and only if $\Gamma \models_{{\mathsf{KF}_*}^{\mathcal{A}x}} \varphi$.*

Proof. Since the axioms from Ax are clearly valid with respect to their corresponding frame conditions (by definition), stated in $\mathcal{A}x$, it is enough to check the soundness using the canonical model. For that, it is enough to define a canonical model $\mathfrak{M}_c^{Ax}$ for ${K_*}^{Ax}$ by letting its universe be $W_{\mathfrak{c}}^{\mathcal{A}x} := \{w \in W_{\mathfrak{c}} : w([Th_{{K_*}^{Ax}}]) \subseteq \{1\}\}$. The completeness of ${K_*}^{Ax}$ with respect to this canonical model is direct, by the Truth Lemma 6.13 To see that the extension is complete with respect to the class of models that enjoy a certain set of properties, it is enough to check that for each axiom in A, the canonical model $\mathfrak{M}_c^{Ax}$ enjoys the corresponding property. We will detail two cases, the rest can be done similarly.

1. Assume $\mathsf{D} \in Ax$. Suppose towards a contradiction that there is $v \in W_{\mathfrak{c}}^{Ax}$, for which there is no $w \in W_{\mathfrak{c}}^{Ax}$ such that $R_{\mathfrak{c}}^{Ax}vw$. Then, by definition it holds that $v(\Box\overline{0}) = 1$ and $v(\Diamond\overline{0}) = 0$, but this contradicts axiom (D), so $\mathfrak{M}_c^{Ax}$ is serial.

2. Assume $\mathsf{T} \in Ax$. Suppose towards a contradiction that there is $v \in W_{\mathfrak{c}}^{Ax}$ such that it does not hold that $R_{\mathfrak{c}}^{Ax}vv$, i.e., by definition of the canonical relation, there is $\theta \in Fm$ such that $v(\Box\theta) = 1$ but $v(\theta) < 1$. This contradicts axiom (T), so $\mathfrak{M}_c^{Ax}$ is reflexive.

3. Assume $\mathsf{B} \in Ax$. Suppose towards a contradiction that there are $v, w \in W_{\mathfrak{c}}^{Ax}$ such that $R_{\mathfrak{c}}^{Ax}vw$ but it does not hold that $R_{\mathfrak{c}}^{Ax}vw$ (i.e., that the model is not symmetric). Then, by definition of the accesibility relation of the canonical model, there is a formula φ such that $e(w, \Box\varphi) = 1$ but $e(v, \varphi) < 1$. But then, $e(v, \Diamond\Box\varphi) \geq e(w, \Box\varphi) = 1$, but $e(v, \varphi) < 1$, contradicting axiom B.

4. Assume $\mathsf{5} \in Ax$ and suppose towards a contradiction that there are $v, w, u \in W_{\mathfrak{c}}^{Ax}$ such that $R_{\mathfrak{c}}^{Ax}vw$ and $R_{\mathfrak{c}}^{Ax}vu$ but where it does not hold that $R_{\mathfrak{c}}^{Ax}wu$ (that is to say, that the canonical model is not serial). By definition, this means that there is a formula φ such that $e(w, \Box\varphi) = 1$ and $e(u, \varphi) < 1$. Then, it follows that $e(v, \Diamond\Box\varphi) = 1$ and $e(v, \Box\varphi) < 1$. But this contradicts axiom $\mathsf{5}$.

5. Assume $\mathsf{4} \in Ax$ and suppose towards a contradiction that there are $v, w, u \in W_{\mathfrak{c}}^{Ax}$ such that $R_{\mathfrak{c}}^{Ax}vw$ and $R_{\mathfrak{c}}^{Ax}wu$ but where it does not hold that $R_{\mathfrak{c}}^{Ax}vu$ (that is to say, that the canonical model is not transitive). By definition, this means that there is a formula φ such that $e(v, \Box\varphi) = 1$ and $e(u, \varphi) < 1$. This implies that $e(v, \Box\Box\varphi) \leq e(w, \Box\varphi) \leq e(u, \varphi) < 1$. However, this contradicts axiom $\mathsf{4}$.

⊠

6.4 Possibilistic many-valued logic

It is not in the scope of this dissertation to study fuzzy modal logics defined over non-crisp Kripke frames, that is, Kripke frames $\langle W, R\rangle$ where the accessibility relation R is a fuzzy relation $R\colon W \times W \to \mathbf{A}$ valued in an arbitrary $L_*\infty$-algebra $\mathbf{A}$ (for $*$ accepting a conjunctive axiomatization). The gap between the crisp MTL modal logics that have been studied in this chapter and these non-crisp modal logics is quite big (we comment this on Section 12.3 on future work). However, there is a particular kind of non-crisp modal logic over MTL algebras that can be studied with the tools developed in this paper, namely logics with possibilistic semantics. This is so thanks to the special characteristics of the accesibility relation of these logics.

Possibilistic logic (see e.g. [44, 45]) is a well-known uncertainty logic for reasoning with graded beliefs on classical propositions by means of necessity and possiblity measures. It deals with weighted formulas (φ, r), where φ is a classical proposition and $r \in [0, 1]$ is a weight, interpreted as a lower bound for the necessity degree of φ. The semantics of these degrees is defined in terms of possibility distributions $\pi : \Omega \to [0, 1]$ on the set Ω of classical interpretations of a given propositional language. A possibility distribution π on Ω ranks interpretations according to its plausibility level: $\pi(w) = 0$ means that w is rejected, $\pi(w) = 1$ means that w is fully plausible, while $\pi(w) < \pi(w')$ means that w' is more plausible than w. A possibility distribution $\pi : \Omega \to [0, 1]$ induces a pair of dual possibility and necessity measures on propositions, defined respectively as:

$$\Pi(\varphi) \coloneqq \sup\{\pi(w) \mid w \in \Omega, w(\varphi) = 1\}$$
$$\mathsf{N}(\varphi) \coloneqq \inf\{1 - \pi(w) \mid w \in \Omega, w(\varphi) = 0\}.$$

They are dual in the sense that $\Pi(\varphi) = 1 - N(\neg\varphi)$ for every proposition φ. From a logical point of view, possibilistic logic can be seen as a sort of graded extension of the non-nested fragment of the well-known modal logic of belief *KD45*.

When we go beyond the classical framework of Boolean algebras of events to generalized algebras of many-valued events, one has to come up with appropriate extensions of the notion of necessity and possibility measures for many-valued events, as explored in [43]. A natural generalization is to consider Ω as the set of propositional interpretations of some many-valued calculi defined by a t-norm $\odot$ and its residuum $\Rightarrow$. Then, a possibility distribution $\pi : \Omega \to [0, 1]$ induces the following generalized possibility and necessity measures over many-valued propositions:

$$\Pi(\varphi) \coloneqq \sup\{\pi(w) \odot w(\varphi) \mid w \in \Omega\}$$
$$\mathsf{N}(\varphi) \coloneqq \inf\{\pi(w) \Rightarrow w(\varphi) \mid w \in \Omega\}.$$

Actually, these definitions agree with the ones commonly used in many-valued modal logics with Kripke semantics based on frames (W, R) with R being a $[0, 1]$-valued binary relations $R : W \times W \to [0, 1]$ (see for example [18]), in the particular case where the many-valued accessibility relations R are of the form $R(w, w') = \pi(w')$, for some possibility distributions $\pi : W \to [0, 1]$. In the frame of this paper, we generalize this possibilistic semantics by replacing the unit real interval $[0, 1]$ by an arbitrary MTL algebra.

The set of formulas over the language of the L_*^∞ logic expanded with two unary operators, N and Π, will be denoted by Fm_{Pos_*} and we will refer to them as ***possibilistic formulas***.

Definition 6.22. Let $\mathbf{A}$ be a L_*^∞-algebra. An **$\mathbf{A}$-*possibilistic model*** is a structure $\langle W, \pi, e\rangle$ such that

- W is a non-empty set of worlds;
- $\pi\colon W \to \mathbf{A}$ is a $\mathbf{A}$-valued possibility distribution;
- $e : W \times \mathit{Var} \to \mathbf{A}$ is an evaluation of variables in each world. It extends to all the formulas by interpreting the propositional connectives by the corresponding operations of $\mathbf{A}$ and the possibilistic operations by

$$e(v, \Pi\varphi) = \sup_{w \in W} (\pi(w) \odot e(w, \varphi)\} \qquad e(v, \mathsf{N}\varphi) = \inf_{w \in W} (\pi(w) \Rightarrow e(w, \varphi)\}.$$

Again, if these latter two values exist for any formula in any world, the model is called *safe*. The class of safe Possibilistic models over L_*^∞-algebras will be denoted by Pos_*. For $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathsf{Pos}_*}$, we will write $\Gamma \Vdash_{\mathsf{Pos}_*} \varphi$ whenever for any $\mathfrak{M} \in \mathsf{Pos}_*$ and any $w \in W$, $e(w, [\Gamma]) \subseteq \{1\}$ implies that $e(w, \varphi) = 1$.

For the sake of uniformity with the notation used in existing literature, from now on we will denote ${\mathsf{KF}_*}^{\{T,5\}}$ (i.e., the set of reflexive and euclidean safe $*$-models, using the notation introduced in the previous section) by $\mathsf{S5}_*$ and ${K_*}^{\{\mathsf{T},5\}}$ by $S5_*$, the logic which is strongly complete with respect to that class of models.

In [76], it is shown how a possibilistic modal logic over a finitely-valued Łukasiewicz logic can be embedded in a $S5$-like extension (with axioms (K), (T) and (5)) over a language extended with a new propositional variable ρ, playing the special role of the possibility distribution. We follow the same idea here for arbitrary L_*^∞-algebras.

Let $\star$ be the translation from Fm_{Pos_*} (built from a set Var of propositional variables) to the set of modal[7] formulas built from $Var \cup \{\rho\}$, with $\{\rho\}$ a new fresh variable not in Var, defined as:

$$\begin{aligned}
\overline{c}^\star &:= \overline{c}, \quad p^\star := p \text{ for } p \in Var & (\Delta\varphi)^\star &:= \Delta\varphi^\star\\
(\varphi\&\psi)^\star &:= \varphi^\star\&\psi^\star & (\mathsf{N}\varphi)^\star &:= \Box(\rho \to \varphi^\star)\\
(\varphi \to \psi)^\star &:= \varphi^\star \to \psi^\star & (\Pi\varphi)^\star &:= \Diamond(\rho\&\varphi^\star).
\end{aligned}$$

Theorem 6.23. *For any set of possibilistic formulas $\Gamma \cup \{\varphi\}$ it holds that*

$$\Gamma \Vdash_{\mathsf{Pos}_*} \varphi \iff \Gamma^\star \Vdash_{\mathsf{S5}_*} \varphi^\star.$$

Proof. First, given $\mathfrak{M} = \langle W, \pi, e\rangle \in \mathsf{Pos}_*$, a model $\mathfrak{M}' = \langle W', R', e'\rangle \in \mathsf{S5}_*$ can be defined by taking $W' = W$, $R' = W \times W$, $e'(v, p) = e(v, p)$ for all $p \in Var$ and $e'(v, \rho) = \pi(v)$. By the definition of the evaluation of the modal formulas in a possibilistic model, and the behaviour of the translation $\star$ over them, it is immediate to see, by induction on the length of the formula, that the evaluations coincide, that is, $e(v, \psi) = e'(v, \psi^\star)$ for any $\psi \in Fm_{\mathsf{Pos}_*}$. Take for instance the possibilistic formula $\mathsf{N}\chi$. By definition, $e(v, \mathsf{N}\chi) = \inf_{w\in W}(\pi(w) \Rightarrow_* e(w, \chi)\} = \inf_{w\in W}\{e'(w, \rho) \Rightarrow_* e(w, \chi)\}$. By the induction hypothesis, this is equal to $\inf_{w\in W}\{e'(w, \rho) \Rightarrow_* e'(w, \chi^\star)\}$ and thus, to $\inf_{w\in W}\{e'(w, \rho \to \chi^\star)\} = e'(v, \Box(\rho \to \chi^\star)) = e'(v, (\mathsf{N}\chi)^\star)$.

Conversely, given a model $\mathfrak{M} \in \mathsf{S5_ast}$, define $\mathfrak{M}' \in \mathsf{Pos}_\Pi$ by letting $W' = W$, $\pi(v) = e(v, \rho)$ and $e'(v, p) = e(v, p)$ for all $p \in Var$ and $v \in W$. Then it is also easy to check, again by induction, that $e(v, \psi^\star) = e'(v, \psi)$ for all $\psi \in Fm_{\mathsf{Pos}_*}$ and $v \in W$. Similarly as before, consider the case where $\psi = \mathsf{N}\chi$, that is, $\psi^\star = \Box(\rho \to \chi^\star)$. Then, by definition, $e(v, \Box(\rho \to \chi^\star) = \inf_{w\in W}\{e(w, \rho \to \chi^\star)\} = \inf_{w\in W}\{e(w, \rho) \Rightarrow_* e(w, \chi^\star)\} = \inf_{w\in W}\{\pi(w) \Rightarrow_* e(w, \chi^\star)\}$. By the induction hypothesis, this is equal to $\inf_{w\in W}\{\pi(w) \Rightarrow_* e'(w, \chi)\}$ which is equal, by definition, to $e'(v, \mathsf{N}\chi)$. ⊠

[7] With usual $\Box$ and $\Diamond$ operators.

This directly provides a completeness theorem for the possibilistic logic over L_*^∞-algebras, which implicitly proposes an axiomatization of this logic, over an extended language.

Corollary 6.24 (Possibilistic strong completeness)**.** *For any set of possibilistic formulas $\Gamma \cup \{\varphi\}$ ($\subseteq Fm_{\mathsf{Pos}_*}$)*

$$\Gamma \Vdash_{\mathsf{Pos}_*} \varphi \iff \Gamma^\star \vdash_{S5_*} \varphi^\star.$$

Chapter 7

Many-valued modal algebras

In this chapter we study the algebraic semantics of the modal systems K_* and K_*^g presented in the previous chapter. We consider the understanding of this algebraic semantics can, among other things, bring light to the axiomatization of more general many-valued logics, like for instance those corresponding to Kripke models with a fuzzy accesibility relation, or to previous logics without rational constants.

After defining an algebraic semantics in Section 7.1, in Section 7.2 we deepen on the relation between the two semantics obtained for the modal logics, namely between the Kripke semantics and the Algebraic semantics. To conclude we present in Section 7.3 we present an algebraic completeness result of the logic K_* with respect to the order preserving logics over the algebraic semantics of K_*.

7.1 Algebraic semantics of K_* and GK_*

For the algebraic semantics of K_* and K_*^g it is natural to consider the class of algebras obtained through the expansion of the L_*^∞-algebras [1] by modal operators that satisfy the equations induced by the axioms in M. The deduction rules will affect the definition of the deductive filters.

Definition 7.1. An algebra $\mathbf{A} = \langle A, \odot, \Rightarrow, \wedge, \delta^{\mathbf{A}}, \Box, \Diamond, \{\bar{c}^{\mathbf{A}}\}_{c \in \mathcal{C}_*}\rangle$ is a ***(modal)*** K_****-algebra*** with (rational) constants and Δ whenever

$$\overline{\mathbf{A}} = \langle A, \odot, \Rightarrow, \wedge, \delta^{\mathbf{A}}, \{\bar{c}^{\mathbf{A}}\}_{c \in \mathcal{C}_*}\rangle$$

is a L_*^∞ algebra and the following equations hold in $\mathbf{A}$:

($\mathsf{E_K}$) $\Box(x \to y) \to (\Box x \to \Box y) \approx \bar{1}$;

[1] Recall this class coincides with the generalized quasi-variety generated by $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$.

($\mathsf{E}_{\Box 1}$) $\Box(x \to \overline{c}) \approx \Diamond x \to \overline{c}$, for every $c \in \mathcal{C}_*$;

($\mathsf{E}_{\Diamond 1}$) $\Box(\overline{c} \to x) \approx \overline{c} \to \Box x$, for every $c \in \mathcal{C}_*$;

($\mathsf{E}_{\Box 2}$) $\Delta\Box x \to \Box\Delta x \approx \overline{1}$;

($\mathsf{E}_{\mathsf{FS}1}$) $(\Diamond x \to \Box y) \to \Box(x \to y) \approx \overline{1}$

($\mathsf{E}_{\mathsf{N}_\Box}$) $\Box\overline{1} \approx \overline{1}$;

We will denote by K_* the class of K_*-agebras, and for each $\mathbf{A} \in \mathsf{K}_*$, $\overline{\mathbf{A}}$ will denote its non-modal reduct.

A ***modal evaluation*** e over a modal K_*-algebra $\mathbf{A}$ is an homomorphism from the algebra of modal formulas (with Δ and rational constants) into $\mathbf{A}$. From now on, unless stated otherwise, $\mathbf{Fm}$ stands for the algebra of modal formulas.

From the general definition of filter we know that, for what concerns the propositional (non-modal) logic L_*^∞ we can characterize the the L_*^∞-filters over a L_*^∞-algebra $\mathbf{A}$ by the subsets $F \subseteq \mathbf{A}$ that

- $\overline{1}^{\mathbf{A}} \in F$,
- If $x \in F$ and $x \Rightarrow y \in F$ then $y \in F$,
- If $x \in F$ then $\delta^{\mathbf{A}} x \in F$,
- If $(x \Rightarrow \overline{c}^{\mathbf{A}}) \vee (\overline{c}^{\mathbf{A}} \Rightarrow y) \in F$ for all $c \in (0,1)_{\mathbb{Q}}$ then $x \Rightarrow y \in F$,

In the local modal logic K_*, it follows from the definition of logical filter that the K_*-filters over a modal K_*-algebra $\mathbf{A}$ coincide with the L_*^∞-filters on $\overline{\mathbf{A}}$ that are also closed under $N_\Box$ (i.e., $\Box\overline{1}^{\mathbf{A}} \in F$). On the other hand, for the global modal logic we get that the K_*^g-filters on a modal K_*-algebra $\mathbf{A}$ are the L_*^∞-filters on $\overline{\mathbf{A}}$ closed under $N_\Box^g$ (i.e., if $x \in F$ then $\Box x \in F$, that is to say, F is ***open***).

Then, we can characterize more clearly the filters of K_* and K_*^g over a modal K_*-algebra $\mathbf{A} \in \mathsf{K}_*$.

Lemma 7.2. *Given* $\mathbf{A} \in \mathsf{K}_*$ *and* $F \subseteq A$

- *F is a filter of K_* over $\mathbf{A}$ if and only if $F \in \mathcal{F}i_{L_*^\infty}(\overline{\mathbf{A}})$.*
- *F is a filter of K_*^g over $\mathbf{A}$ if and only if $F \in \mathcal{F}i_{L_*^\infty}(\overline{\mathbf{A}})$ and F is an open set.*

It is routine to prove that $\{\overline{1}^{\mathbf{A}}\}$ is a filter of both K_* and K_*^∞ logics over all $\mathbf{A} \in \mathsf{K}_*$.

Algebraic completeness of the global modal logic

It turns out that the classification of the global modal logic K_*^g in the Leibniz Hierachy is now quite immediate, just by checking the definition of algebraizable logic (Definition 1.7).

Lemma 7.3. *K^g_* is algebraizable with algebraic semantics K_*, with equivalence formulas $\mathbf{\Delta}(x,y) := \{x \leftrightarrow y\}$ and with defining equations $E(x) = \{x \approx 1\}$.*

It is an obvious remark that $Alg^* K^g_* = \mathsf{K}_*$.

From this it follows that the class of reduced models of L^g_* is given by logical matrices of the form $\langle \mathbf{A}, \{\overline{1}^{\mathbf{A}}\}\rangle$ for $\mathbf{A} \in {\mathsf{K}_*}^{\infty}$. We can then resort to the usual notation of semantic deduction in an algebra. For an arbitrary set of formulas $\Gamma \cup \{\varphi\}$ and $\mathbf{A}$ a modal K_*-algebra we write $\Gamma \models_{\mathbf{A}} \varphi$ whenever for any modal evaluation e over $\mathbf{A}$, if $e(\Gamma) \subseteq \{\overline{1}^{\mathbf{A}}\}$ then $e(\varphi) = \overline{1}^{\mathbf{A}}$.

For later uses, we also consider the following more particular definition. For an arbitrary set of formulas $\Gamma \cup \{\varphi\}$, $\mathbf{A}$ a modal K_*-algebra and e a modal evaluation over $\mathbf{A}$, we write $\Gamma \models_{\mathbf{A},e} \varphi$ whenever, if $e(\Gamma) \subseteq \{\overline{1}^{\mathbf{A}}\}$ then $e(\varphi) = \overline{1}^{\mathbf{A}}$.

Theorem 7.4 (Algebraic completeness of K^g_*)**.** *Let $*$ be a left-continuous t-norm accepting a conjunctive axiomatization. For any set of modal formulas $\Gamma \cup \{\varphi\}$*

$$\Gamma \vdash_{K^g_*} \varphi \textit{ iff } \Gamma \models_{\mathsf{K}_*} \varphi.$$

This fact, together with the Isomorphism Theorem 1.8, ensures that for every modal K_*-algebra $\mathbf{A}$, the mappings

$$\boldsymbol{\rho}^+ \colon \mathcal{F}i_{K^g_*}(\mathbf{A}) \longleftrightarrow Co_{\mathsf{K}_*}(\mathbf{A}) \colon \boldsymbol{\tau}^+,$$

defined respectively as

$$\boldsymbol{\rho}^+(F) = \Omega^{\mathbf{A}} F \qquad \text{and} \qquad \boldsymbol{\tau}^+(\theta) = 1/\theta$$

for every $F \in \mathcal{F}i_{K^g_*}(\mathbf{A})$ and $\theta \in Co_{\mathsf{K}_*}(\mathbf{A})$, are complete lattice isomorphisms with one being the inverse of the other. We will make use of this isomorphism in the study of the local modal logic (given that the algebraic companion will be proven to be also K_*).

Algebraic completeness of the local modal logic

Concerning the local modal logic K_*, we begin by proving a property weaker than being algebraizable.

Lemma 7.5. *K_* is equivalential with congruence formulas $\mathbf{\Delta}(x,y) \equiv \{\Box^n(x \leftrightarrow y)\}_{n\in\omega}$, where $\Box^n\varphi = \Box\Box^{n-1}\varphi$ for $n > 0$ and $\Box^0\varphi = \varphi$.*

Proof. It suffices to see that $\mathbf{\Delta}(x,y)$ meets the first condition of Lemma 1.9 for K_*. It is routine to prove this using the already proven completeness of K_* with respect to the local logic over the class of crisp Kripke models over the standard L^{∞}_*-algebra. ⊠

This characterization classifies K_* within the Leibniz hierarchy, and moreover it is important for checking that the class K_* of modal K_*-algebras is indeed the algebraic semantics of the logic K_*.

Lemma 7.6. $Alg^*K_* = \mathsf{K}_*$.

Proof. Let $\mathbf{A} \in \mathsf{K}_*$. Taking into account that $\{\overline{1}^{\mathbf{A}}\}$ is a filter of K_* on $\mathbf{A}$, by (2) of Lemma 1.9, $\langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{A}}\{\overline{1}^{\mathbf{A}}\}$ iff $\boldsymbol{\Delta}(a, b) \subseteq \{\overline{1}^{\mathbf{A}}\}$. In particular, this implies that $(a \Leftrightarrow b) = \overline{1}^{\mathbf{A}}$ and so $a = b$. Therefore, $\boldsymbol{\Omega}^{\mathbf{A}}\{\overline{1}^{\mathbf{A}}\} = \mathrm{Id}_{\mathbf{A}}$ and thus, $\langle \mathbf{A}, \{\overline{1}^{\mathbf{A}}\}\rangle \in Mod^*K_*$.

For the converse inclusion, we know that $\mathbf{A} \in Alg^*K_*$ if and only if there exists $F \in \mathcal{F}i_{K_*}(\mathbf{A})$ for which $\boldsymbol{\Omega}^{\mathbf{A}}F = \mathrm{Id}_{\mathbf{A}}$. To see that $\mathbf{A} \in \mathsf{K}_*$ if $\mathbf{A} \in Alg^*K_*$ we have to check that the equations and quasiequations that define K_* hold in $\mathbf{A}$ as well.

Each equation $\alpha_i \approx \beta_i$ corresponding to one of the identities that must hold in an algebra belonging to K_* comes from an associated axiom $\alpha_i \leftrightarrow \beta_i$ of K_* (in general, of the form $\alpha_i \leftrightarrow \overline{1}$). By the $(\mathsf{N}_\Box)$ rule, it follows that $\Box^n(\alpha_i \leftrightarrow \beta_i)$ is also a theorem of the logic for each $n \in \omega$ and so for any $h \in Hom(\mathbf{Fm}, \mathbf{A})$ it holds that $\boldsymbol{\Delta}(h(\alpha_i), h(\beta_i)) \subseteq F$, with $\boldsymbol{\Delta}$ being the set of congruence formulas of K_* defined in the previous lemma. By (2) of Lemma 1.9, it follows that $\langle h(\alpha_i), h(\beta_i)\rangle \in \Omega^{\mathbf{A}}F$ for all $h \in Hom(\mathbf{Fm}, \mathbf{A})$ and given that by assumption $\Omega^{\mathbf{A}}F = \mathrm{Id}_{\mathbf{A}}$, it must hold that $h(\alpha_i) = h(\beta_i)$ for all $h \in Hom(\mathbf{Fm}, \mathbf{A})$ (so $\mathbf{A} \models \alpha_i \approx \beta_i$).

For the generalized quasiequation $(\mathcal{Q})$ the proof is similar using that the filters are closed under the infinitary rule D^∞. First observe that D^∞ can be equally written as

$$\frac{\{((p \to \overline{c}) \vee (\overline{c} \to q)) \leftrightarrow \overline{1}\}_{c \in \mathcal{C}_*}}{(p \to q) \leftrightarrow \overline{1}}.$$

By Theorem 6.9 it follows that for any $n \in \omega$, the following deduction is valid in K_*

$$\frac{\{\Box^n(((p \to \overline{c}) \vee (\overline{c} \to q)) \leftrightarrow \overline{1})\}_{c \in \mathcal{C}_*}}{\Box^n((p \to q) \leftrightarrow \overline{1})}.$$

By definition, filters of K_* are closed under valid deductions, so for any $a, b \in A$ and $n \in \omega$,

$$\text{if } \{\Box^n(((a \Rightarrow \overline{c}^{\mathbf{A}}) \vee (\overline{c}^{\mathbf{A}} \Rightarrow b)) \Leftrightarrow \overline{1}^{\mathbf{A}})\}_{c \in \mathcal{C}_*} \subseteq F \text{ then } \Box^n((a \Rightarrow b) \Leftrightarrow \overline{1}^{\mathbf{A}}) \in F. \tag{7.1}$$

To see that $\mathcal{Q}$ holds in $\mathbf{A}$, let $a, b \in A$ be such that $(a \Rightarrow \overline{c}^{\mathbf{A}}) \vee (\overline{c}^{\mathbf{A}} \Rightarrow b) = \overline{1}^{\mathbf{A}}$ for all $c \in \mathcal{C}_*$.

Since $\boldsymbol{\Omega}^{\mathbf{A}}F = \mathrm{Id}_{\mathbf{A}}$, $\langle (a \Rightarrow \overline{c}^{\mathbf{A}}) \vee (\overline{c}^{\mathbf{A}} \Rightarrow b), \overline{1}^{\mathbf{A}}\rangle \in \boldsymbol{\Omega}^{\mathbf{A}}F$ for all $c \in \mathcal{C}_*$. Given that K_* is equivalential with the congruence formulas given in Lemma 7.5, this is happens if and only if $\{\Box^n(((a \Rightarrow \overline{c}^{\mathbf{A}}) \vee (\overline{c}^{\mathbf{A}} \Rightarrow b)) \Leftrightarrow \overline{1}^{\mathbf{A}})\}_{c \in \mathcal{C}_*} \subseteq F$. From (7.1) it follows that $\{\Box^n((a \Rightarrow b) \Leftrightarrow \overline{1}^{\mathbf{A}})\}_{n \in \omega} \subseteq F$, i.e., $\boldsymbol{\Delta}^{\mathbf{A}}(a \Rightarrow b, \overline{1}^{\mathbf{A}}) \subseteq F$. Following the same reasoning as before this is equivalent to $\langle a \Rightarrow b, \overline{1}^{\mathbf{A}}\rangle \in \boldsymbol{\Omega}^{\mathbf{A}}F$, and so to $(a \Rightarrow b) = \overline{1}^{\mathbf{A}}$ in $\mathbf{A}$, which concludes the proof. ⊠

It is not hard to prove that, as in the case for the local classical modal logic, K_* is not algebraizable. It can be proven as a consequence of the fact that the classes of algebras associated to K_* and K_*^g coincide. There is a well known result from abstract algebraic logic that states that two algebraizable logics with re same algebraic semantics are either incomparable or they coincide. But on the one hand we know that K_* and K_*^g are comparable: for any set of formulas $\Gamma \cup \{\varphi\}$ it holds that

$$\Gamma \vdash_{K_*} \varphi \implies \Gamma \vdash_{K_*^g} \varphi.$$

On the other hand, we know they do not coincide (for instance, $\varphi \vdash_{K_*^g} \Box\varphi$ but $\varphi \nvdash_{K_*} \Box\varphi$), so it follows that K_* is not algebraizable (since K_*^g was proven to be so in Lemma 7.3.

We can also provide a constructive prove of this fact, obtaining a result analogous to the one from the local classical modal logic. We show this construction as an alternative proof (with respect to the previous more immediate non-constructive method) for proving the following result.

Lemma 7.7. *K_* is not algebraizable.*

Proof. For the sake of a simpler notation, in this proof we let $\mathcal{C}nX = \{\theta \in Fm : X \vdash_{K_*} \theta\}$, and we will write $\mathbf{\Omega}$ to denote the Leibniz operator $\mathbf{\Omega}^{\mathbf{Fm}}$ over the formula algebra.

It is clear that for $x, y \in \mathcal{V}ar \subseteq \mathbf{Fm}$ such that $x \neq y$, we have $\{x, y\} \nvdash_{K_*} \Box(x \leftrightarrow y)$ and so in particular, $\mathbf{\Delta}(x, y) \nsubseteq \mathcal{C}n\{x, y\}$. From the definition of equivalential logic and Lemma 1.10 it follows that $\langle x, y\rangle \notin \mathbf{\Omega}\mathcal{C}n\{x, y\}$, which in particular proves that $\mathcal{C}n\{x, y\}/\mathbf{\Omega}\mathcal{C}n\{x, y\}$ has at least two different elements: $x/\mathbf{\Omega}\mathcal{C}n\{x, y\}$ and $y/\mathbf{\Omega}\mathcal{C}n\{x, y\}$.

For simplicity, let $\mathbf{A}$ denote $\mathbf{Fm}/\mathbf{\Omega}\mathcal{C}n\{x, y\}$. Since $\mathcal{C}n\{x, y\} \in \mathcal{F}i_{K_*}\mathbf{Fm}$ it follows that $\mathbf{\Omega}^{\mathbf{A}}(\mathcal{C}n\{x, y\}/\mathbf{\Omega}\mathcal{C}n\{x, y\}) = \mathrm{Id}_{\mathbf{A}}$ and so

$$\langle \mathbf{A}, \mathcal{C}n\{x, y\}/\mathbf{\Omega}\mathcal{C}n\{x, y\}\rangle \in Mod^* K_*,$$

which implies that $\mathbf{A} \in Alg^* K_* = \mathsf{K}_*$. We know that $\{\bar{1}^{\mathbf{A}}\}$ is a filter of K_* over $\mathbf{A}$, so $\bar{1}^{\mathbf{A}}/\mathbf{\Omega}\mathcal{C}n\{x, y\} \in \mathcal{F}i_{K_*}(\mathbf{A})$. Now, by Lemma 1.10, we have that

$$\mathbf{\Omega}^{\mathbf{A}}(\bar{1}^{\mathbf{A}}/\mathbf{\Omega}\mathcal{C}n\{x, y\}) \subseteq \mathbf{\Omega}^{\mathbf{A}}(\mathcal{C}n\{x, y\}/\mathbf{\Omega}\mathcal{C}n\{x, y\}) = \mathrm{Id}_{\mathbf{A}},$$

and so $\mathbf{\Omega}^{\mathbf{A}}(\bar{1}^{\mathbf{A}}/\mathbf{\Omega}\mathcal{C}n\{x, y\}) = \mathrm{Id}_{\mathbf{A}}$ as well. Then, both

$$\langle \mathbf{A}, \mathcal{C}n\{x, y\}/\mathbf{\Omega}\mathcal{C}n\{x, y\}\rangle \quad \text{and} \quad \langle \mathbf{A}, \bar{1}^{\mathbf{A}}/\mathbf{\Omega}\mathcal{C}n\{x, y\}\rangle$$

are reduced models of K_*. Since their algebraic classes coincide, but they are different models (given that $\mathcal{C}n\{x, y\}/\mathbf{\Omega}\mathcal{C}n\{x, y\}$ has at least two elements and $\bar{1}^{\mathbf{A}}/\mathbf{\Omega}\mathcal{C}n\{x, y\}$ is a singleton), we have that $\mathbf{\Omega}$ is not injective and thus, it is not an isomorphism as the Isomorphism Theorem requires. ⊠

Recall that K_* is strongly complete with respect to its reduced models. Inspired by [92], where the analogous problem for the classical modal logic is studied, we can give a more specific characterization of this class of models, and thus, obtain a more concrete completeness result for K_* as an immediate corollary.

Theorem 7.8. *$\langle \mathbf{A}, F\rangle$ is a reduced model of K_* if and only if $\mathbf{A} \in \mathsf{K}_*$, $F \in \mathcal{F}i_{L_*^\infty}\mathbf{A}$ and $\{\overline{1}^{\mathbf{A}}\}$ is the only open filter included in F.*

Proof. For the left to right direction, let $\langle \mathbf{A}, F\rangle \in Mod^* K_*$. From Lemma 7.6 it follows that $\mathbf{A} \in \mathsf{K}_*$ and it also holds that $F \in \mathcal{F}i_{L_*^\infty}(\overline{\mathbf{A}})$. By way of contradiction, suppose there is an open K_*-filter G on $\mathbf{A}$ such that $\{\overline{1}^{\mathbf{A}}\} \subsetneq G \subseteq F$. Since both $\{\overline{1}^{\mathbf{A}}\}$ and G are open filters, they are also deductive K_*^g-filters on $\mathbf{A}$. Given that K_*^g is an algebraizable logic, $\mathbf{\Omega}^{\mathbf{A}}$ is a lattice isomorphism between $\mathcal{F}i_{K_*^g}(\mathbf{A})$ and $Co_{\mathsf{K}_*}(\mathbf{A})$ and, in particular, $\mathbf{\Omega}^{\mathbf{A}}\colon \mathcal{F}i_{K_*}\mathbf{A} \to Co_{\mathsf{K}_*}\mathbf{A}$ is injective. Then, given that $\{\overline{1}^{\mathbf{A}}\} \neq G$, $\mathbf{\Omega}^{\mathbf{A}}\{\overline{1}^{\mathbf{A}}\} \neq \mathbf{\Omega}^{\mathbf{A}}G$. It is clear that $\mathbf{\Omega}^{\mathbf{A}}\{\overline{1}^{\mathbf{A}}\} = \mathrm{Id}_{\mathbf{A}}$ and so $\mathbf{\Omega}^{\mathbf{A}}G \neq \mathrm{Id}_{\mathbf{A}}$. K_* is equivalential so $\mathbf{\Omega}^{\mathbf{A}}$ is monotone over the K_*-filters on $\mathbf{A}$. Given that both F and G are K_*-filters on $\mathbf{A}$ it follows that $\mathrm{Id}_{\mathbf{A}} \neq \mathbf{\Omega}^{\mathbf{A}}G \subseteq \mathbf{\Omega}^{\mathbf{A}}F$, so $\mathbf{\Omega}^{\mathbf{A}}F \neq \mathrm{Id}_{\mathbf{A}}$ which contradicts the assumption that $\langle \mathbf{A}, F\rangle \in Mod^* K_*$.

As for the other direction, let $F \in \mathcal{F}i_{K_*}(\mathbf{A})$ and suppose that $\mathbf{\Omega}^{\mathbf{A}}F \neq \mathrm{Id}_{\mathbf{A}}$. We know that $\mathbf{\Omega}^{\mathbf{A}}\colon \mathcal{F}i_{K_*^g}\mathbf{A} \to Co_{\mathsf{K}_*}(\mathbf{A})$ is an isomorphism and also that $\mathbf{\Omega}^{\mathbf{A}}F \in Co_{\mathsf{K}_*}(\mathbf{A})$. Then, there is $G \in \mathcal{F}i_{K_*^g}(\mathbf{A})$ such that $\mathbf{\Omega}^{\mathbf{A}}G = \mathbf{\Omega}^{\mathbf{A}}F$, and so $\mathbf{\Omega}^{\mathbf{A}}G \neq \mathrm{Id}_{\mathbf{A}}$. For any $a \in G$, $\langle a, \overline{1}^{\mathbf{A}}\rangle \in \mathbf{\Omega}^{\mathbf{A}}G$ and so $\langle a, \overline{1}^{\mathbf{A}}\rangle \in \mathbf{\Omega}^{\mathbf{A}}F$ too. Since $\overline{1}^{\mathbf{A}} \in F$ and $\mathbf{\Omega}^{\mathbf{A}}F$ is congruent with F (by definition), $a \in F$ and so $G \subseteq F$, which concludes the proof. ⊠

We shall now introduce a suitable notation in order to refer to deductions in this setting. For an arbitrary set of formulas $\Gamma \cup \{\varphi\}$ and $\mathbf{A}$ a modal K_*-algebra we write $\Gamma \models^L_{\mathbf{A}} \varphi$ whenever for any modal evaluation e over $\mathbf{A}$ and any K_*-filter F on $\mathbf{A}$ such that $\langle \mathbf{A}, F\rangle$ is a reduced model of the logic, if $e(\Gamma) \subseteq F$ then $e(\varphi) \in F$.

As in the global modal logic, we also introduce a more specific notation for later uses. For an arbitrary set of formulas $\Gamma \cup \{\varphi\}$, $\mathbf{A}$ a modal K_*-algebra and e a modal evaluation over $\mathbf{A}$, we write $\Gamma \models^L_{\mathbf{A},e} \varphi$ whenever for any K_*-filter F on $\mathbf{A}$ such that $\langle \mathbf{A}, F\rangle$ is a reduced model of the logic, if $e(\Gamma) \subseteq F$ then $e(\varphi) \in F$.

We can then more neatly rewrite the previous result.

Theorem 7.9. *Let $*$ be a left-continuous t-norm accepting a conjunctive axiomatization. For any set of modal formulas $\Gamma \cup \{\varphi\}$*

$$\Gamma \vdash_{K_*} \varphi \text{ iff } \Gamma \models^L_{\mathsf{K}_*} \varphi.$$

At this point we can study the linearly ordered modal K_*-algebras; the existence and behaviour of the constants in the algebras from K_* leads to a very neat characterization of all the chains from K_*.

Theorem 7.10. *If a modal K_*-algebra $\mathbf{A} = \langle A, \odot, \Rightarrow, \wedge, \delta^{\mathbf{A}}, \Box, \Diamond, \{\overline{c}^{\mathbf{A}}\}_{c \in \mathcal{C}_*}\rangle \in \mathsf{K}_*$ is linearly ordered then one of the following conditions hold:*

- $\Box = \Diamond = Id_{\mathbf{A}}$ *(the identity function on $\mathbf{A}$);*

- $\square = \overline{1}^{\mathbf{A}}$ *and* $\lozenge = \overline{0}^{\mathbf{A}}$ *(the constant functions of value* $\overline{1}^{\mathbf{A}}$ *and* $\overline{0}^{\mathbf{A}}$ *respectively).*

Proof. First we can easily see that for any $a \in A$ both $a \leq \square a$ and $\lozenge a \leq a$ hold. Indeed, if $\square a < a$, from Lemma 4.12, it follows that there is $c \in \mathcal{C}_*$ such that $\square a < \overline{c}^{\mathbf{A}} < a$. In that case, since $\overline{c}^{\mathbf{A}} \Rightarrow a = \overline{1}^{\mathbf{A}}$, it follows that $\square(\overline{c}^{\mathbf{A}} \Rightarrow a) = \overline{1}^{\mathbf{A}}$. At the same time, $\overline{1}^{\mathbf{A}} > \overline{c}^{\mathbf{A}} \Rightarrow \square a = \square(\overline{c}^{\mathbf{A}} \Rightarrow a)$, which is a contradiction. To check that $\lozenge a \leq a$, suppose analogously that $a < \lozenge a$, so there is $c \in \mathcal{C}_*$ with $a < \overline{c}^{\mathbf{A}} < \lozenge a$. As before, this implies that $\square(a \Rightarrow \overline{c}^{\mathbf{A}}) = \overline{1}^{\mathbf{A}}$, so by equation $\mathsf{E}_{\lozenge 1}$ it hods that $\lozenge a \Rightarrow \overline{c}^{\mathbf{A}} = \overline{1}^{\mathbf{A}}$. On the other hand, we had that $\overline{c}^{\mathbf{A}} < \lozenge a$, which contradicts the previous statement.

Now, we will prove by cases that if either $\square$ or $\lozenge$ are not the identity function, then $\square = \overline{1}^{\mathbf{A}}$ and $\lozenge = \overline{0}^{\mathbf{A}}$. First, suppose that $\square$ is not the identity function. Since for all $a \in A$ it holds that $a \leq \square a$, then there must exist $b \in A$ such that $b < \square b$ ($\mathbf{A}$ is a chain by assumption). By Lemma 4.12 there exists $c \in \mathcal{C}_*$ such that $b < \overline{c}^{\mathbf{A}} < \square b$, from where it follows that $\overline{1}^{\mathbf{A}} = \overline{c}^{\mathbf{A}} \Rightarrow \square b = \square(\overline{c}^{\mathbf{A}} \Rightarrow b) = \Delta\square(\overline{c}^{\mathbf{A}} \Rightarrow b)$. Given that Δ and $\square$ commute in $\mathbf{A}$, it follows that $\square\Delta(\overline{c}^{\mathbf{A}} \Rightarrow b) = \overline{1}^{\mathbf{A}}$. Now note that $(\overline{c}^{\mathbf{A}} \Rightarrow b) < \overline{1}^{\mathbf{A}}$ by assumption and being $\mathbf{A}$ a chain, $\Delta(\overline{c}^{\mathbf{A}} \Rightarrow b) = 0$. Then $\overline{1}^{\mathbf{A}} = \square\Delta(\overline{c}^{\mathbf{A}} \Rightarrow b) = \square\overline{0}^{\mathbf{A}}$, and using that $\square$ is an increasing function[2] we can conclude that $\square = \overline{1}^{\mathbf{A}}$ (since $\overline{0}^{\mathbf{A}}$ is the minimum element of $\mathbf{A}$). Given that $\square\neg x = \neg\lozenge x$, it is immediate that $\lozenge = \overline{0}^{\mathbf{A}}$.

On the other hand, suppose that for some $b \in A$, $\lozenge b < b$ and so $\lozenge b < \overline{c}^{\mathbf{A}} < b$ for some $c \in \mathcal{C}_*$. It follows that $\overline{1}^{\mathbf{A}} = \lozenge b \Rightarrow \overline{c}^{\mathbf{A}} = \square(b \Rightarrow \overline{c}^{\mathbf{A}})$, and an analogous reasoning as before can be followed to see that $\square\overline{0}^{\mathbf{A}} = \overline{1}^{\mathbf{A}}$. From there, again as above, one can prove that $\square = \overline{1}^{\mathbf{A}}$ and $\lozenge = \overline{0}^{\mathbf{A}}$ and this concludes the proof. ⊠

With this it is clear that neither K_* nor K_*^g are complete with respect to linearly ordered modal K_*-algebras. Indeed, from the previous lemma, $\square\overline{0}^{\mathbf{A}} \vee (\varphi \leftrightarrow \square\varphi)$ is valid in all linearly ordered modal K_*-algebras, but clearly it is not a theorem of K_*.

7.2 Kripke semantics and algebraic semantics

Since we have developed two semantics for our modal logics, namely Kripke and algebraic semantics, it seems natural to study their relationship. It is possible to build a modal K_*-algebra from a $*$-Kripke model, maintaining the behaviour of the logical deductions. Similarly, from a modal K_*-algebra it will be possible to construct an $*$-Kripke model and in particular, a Kripke model over the standard L_*^∞-algebra, whose deductions coincide with those over the algebra.

Moreover, the study of this relation has allowed to prove an interesting completeness result of the local modal logics K_* with respect to the order-preserving logics built over the modal algebras.

[2]It follows from (E_K) that for any $a, b \in A$, if $a \leq b$ then $\square a \leq \square b$

From Kripke models to modal algebras: complex algebras

Complex algebras are a family of algebras that arise from Kripke frames through a natural construction and were originally defined in the context of classical modal logic (see for instance [26]). The construction method can be easily generalized to the case of many-valued logics, taking into account the fact that the algebra over which the Kripke model evaluates formulas takes an explicit role in the construction of the algebra associated to the model.

In our case, we will describe a way of translating the Kripke semantics into the algebraic one by associating to each $*$-Kripke model a K_*-algebra and an evaluation over it.

Remember that for two sets A, B, A^B denotes the set of maps from B to A. In some cases we will denote by $[b \mapsto f(b)]$ the element $f \in A^B$ that sends $b \in B$ into $f(b) \in A$.

Definition 7.11. Let $\mathbf{A} \in \mathsf{L}_*^\infty$ and $\mathfrak{M} = \langle W, R, e\rangle$ be a $\mathbf{A}$-Kripke model. The ***complex algebra associated to*** $\mathfrak{M}$ is the modal K_*-algebra

$$\mathsf{A}lg(\mathfrak{M}) = \langle A^W, \odot^W, \Rightarrow_*^W, \wedge^W, (\delta^{\mathbf{A}})^W, \Box, \Diamond, \{\overline{c}^{\mathsf{A}lg(\mathfrak{M})}\}_{c\in\mathcal{C}_*}\rangle$$

where for every $f, g \in A^W$ the non-modal operations $*^W, \Rightarrow_*^W, \wedge^W, (\delta^{\mathbf{A}})^W$ and $\{\overline{c}^{\mathsf{A}lg(\mathfrak{M})}\}_{c\in\mathcal{C}_*}$ are defined component-wise from the ones of $\mathbf{A}$, and the modal operations are given by

$$\Box f := [v \mapsto \inf\{f(w) : \ w \in W, Rvw\}] \quad \Diamond f := [v \mapsto \sup\{f(w) : \ w \in W, Rvw\}]$$

The associated evaluation over $\mathsf{A}lg(\mathfrak{M})$ is $e_{\mathfrak{M}} \colon Fm \to A^W$ with

$$e_{\mathfrak{M}}(\varphi) = [v \mapsto e(v, \varphi)].$$

It is routine to see that for any $*$-Kripke model $\mathfrak{M}$, its complex algebra $\mathsf{A}lg(\mathfrak{M})$ is a modal K_*-algebra. Also it is easy to check that the evaluation associated to $\mathfrak{M}$ is indeed a modal evaluation over $\mathsf{A}lg(\mathfrak{M})$ (i.e., $e_{\mathfrak{M}} \in Hom(\mathbf{Fm}, \mathsf{A}lg(\mathfrak{M}))$) and that coincides with the homomorphism built from the mapping that sends each propositional variable x to $[v \mapsto e(v, x)]$.

We can study how the validity of deductions over a particular Kripke model are translated to the associated complex algebra and the correspondent evaluation. It is interesting that the more complete information of the linearly ordered algebras at the propositional level (in the sense that the non modal-logic is complete with respect to its linearly ordered algebras) arises here too, and the relation between the local deductions within a Kripke model and within its complex algebra is stronger when the basic algebra is linearly ordered.

Lemma 7.12. *Let* $\mathbf{A}$ *be a* L_*^∞*-algebra and* $\mathfrak{M}$ *a* $\mathbf{A}$*-Kripke model. For an arbitrary set of modal formulas* $\Gamma \cup \{\varphi\}$ *the following hold:*

1. $\Gamma \Vdash_{\mathfrak{M}}^{g} \varphi$ *if and only if* $\Gamma \models_{\mathsf{A}lg(\mathfrak{M}), e_{\mathfrak{M}}} \varphi$*,*

2. *$\Gamma \not\Vdash_{\mathfrak{M}} \varphi$ implies that $\Gamma \not\models^{L}_{\mathsf{A}lg(\mathfrak{M}),e_{\mathfrak{M}}} \varphi$,*[3]

3. *If $\mathbf{A}$ is linearly ordered, then $\Gamma \Vdash_{\mathfrak{M}} \varphi$ if and only if $\Gamma \models^{L}_{\mathsf{A}lg(\mathfrak{M}),e_{\mathfrak{M}}} \varphi$.*

Proof. 1. $\Gamma \Vdash^{g}_{\mathfrak{M}} \varphi$ if there is $v \in W$ such that $e(v,[\Gamma]) \not\subseteq \{\overline{1}^{\mathbf{A}}\}$ or if for all $v \in W$, $e(v,\varphi) = \overline{1}^{\mathbf{A}}$. In the first case, by definition of the evaluation over the modal algebra $e_{\mathfrak{M}}[\Gamma] \not\subseteq \{\overline{1}^{\mathsf{A}lg(\mathfrak{M})}\}$. In the second case, again by definition, it is clear that $e_{\mathfrak{M}}(\varphi) = \overline{1}^{\mathsf{A}lg(\mathfrak{M})}$ (since all its components are equal to $\overline{1}^{\mathbf{A}}$.

2. $\Gamma \not\Vdash_{\mathfrak{M}} \varphi$ whenever there is some $v \in W$ such that $e(v,[\Gamma]) \subseteq \{\overline{1}^{\mathbf{A}}\}$ and $e(v,\varphi) \neq \overline{1}^{\mathbf{A}}$. Then, consider the K_*-filter F on $\mathsf{A}lg(\mathfrak{M})$ generated by $e_{\mathfrak{M}}[\Gamma]$. Since the operations that contribute to the construction of the generated filter are not the modal ones they behave component-wise and so for any $v \in W$ such that $e(v,[\Gamma]) \subseteq \{\overline{1}^{\mathbf{A}}\}$ it also holds that, for all $f \in F$, $f(v) = \overline{1}^{\mathbf{A}}$. Thus, it is immediate that $e_{\mathfrak{M}}(\varphi) \notin F$, since $e_{\mathfrak{M}}(\varphi)(v) = e(v,\varphi) < \overline{1}^{\mathbf{A}}$.

3. Observe that $\Gamma \Vdash_{\mathfrak{M}} \varphi$ whenever for all $v \in W$ such that $e(v,[\Gamma]) \subseteq \{\overline{1}^{\mathbf{A}}\}$ it holds that $e(v,\varphi) = \overline{1}^{\mathbf{A}}$. Consider now an arbitrary filter F of K_* on $\mathsf{A}lg(\mathfrak{M})$ such that $e_{\mathfrak{M}}[\Gamma] \subseteq F$. Since the filters are closed under the deductive rules, it must hold that $\Delta e_{\mathfrak{M}}[\Gamma] \subseteq F$. Assuming now that the algebra $\mathbf{A}$ is linearly ordered, $\Delta e_{\mathfrak{M}}[\Gamma]$ is a sequence of 0s and 1s from $\mathbf{A}$. Since for all v such that $e_{\mathfrak{M}}[\Gamma](v) \subseteq \{\overline{1}^{\mathbf{A}}\}$ it holds that $e_{\mathfrak{M}}(\varphi)(v) = \overline{1}^{\mathbf{A}}$ and the other components hold trivially (since then $(\Delta e_{\mathfrak{M}}[\Gamma])(v) = 0$), we have that $\Delta e_{\mathfrak{M}}[\Gamma] \leq e_{\mathfrak{M}}(\varphi)$ and so $e_{\mathfrak{M}}(\varphi) \in F$. ⊠

When the non-modal algebra is not linearly ordered, the equivalence stated at point 2. of the previous lemma needs not to hold. For example, we can take the algebra $[\mathbf{0},\mathbf{1}]^{\mathbb{Q}}_{*} \times [\mathbf{0},\mathbf{1}]^{\mathbb{Q}}_{*}$ (where $\langle 1,0\rangle$ and $\langle 0,1\rangle$ are incomparable elements) and the Kripke model $\mathfrak{M}$ evaluated over it given by the following Kripke model

$$\begin{array}{l} \bullet\ w : p = \langle 0,1\rangle, q = \langle 1,0\rangle \\ \updownarrow \\ \bullet\ v : p = \langle 0,1\rangle, q = \langle 1,0\rangle \end{array}$$

formally defined by

$$\langle \{v,w\}, \{\langle v,w\rangle, \langle w,v\rangle\}, \left\{ \begin{array}{l} \left\{ \begin{array}{l} e(v,p) = \langle 0,1\rangle \\ e(w,p) = \langle 0,1\rangle \end{array} \right. \\ \left\{ \begin{array}{l} e(v,q) = \langle 1,0\rangle \\ e(w,q) = \langle 1,0\rangle \end{array} \right. \end{array} \right. \rangle.$$

[3]Recall that this means that there is a reduced model of K_* of the form $\langle \mathsf{A}lg(\mathfrak{M}), F\rangle$ such that $e_{\mathfrak{M}}[\Gamma] \subseteq F$ and $e_{\mathfrak{M}}(\varphi) \notin F$.

It is clear that $p \Vdash_{\mathfrak{M}} q$ (since p is always evaluated to less than $\langle 1,1\rangle$). However, consider the filter F of K_* on $\mathsf{A}lg(\mathfrak{M})$ generated by $e_{\mathfrak{M}}(p)$. We can easily see that $e_{\mathfrak{M}}(p) \in F$ whereas $e_{\mathfrak{M}}(q) \notin F$, and so $p \not\models^{L}_{\mathsf{A}lg(\mathfrak{M}),e_{\mathfrak{M}}} q$.

From modal algebras to Kripke models: canonical models

It is also natural to show how to build an $*$-Kripke model from a K_*-algebra $\mathbf{A}$ and a modal evaluation over it, in an inverse way to the complex algebra construction.[4]

Remember that from Lemma 4.13 it follows that if a L_*^∞-filter F on a L_*^∞-algebra $\mathbf{A} \in \mathsf{L}_*^\infty$ is prime (for any $a, b \in A$, it contains either $a \Rightarrow b$ or $b \Rightarrow a$ and thus, $\mathbf{A}/F$ is linearly ordered), then there is a σ-embedding ρ from $\mathbf{A}/F$ into $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$. Then, a suitable choice for the set of worlds of a Kripke model associated to a K_*-algebra $\mathbf{A}$ are the set of homomorphisms from $\overline{\mathbf{A}}$ into $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$.

It is important to note that the relation between the algebraic semantics and the Kripke one, as happened before, must be done between structures that are equivalent from the point of view of the logic. That is to say, we can establish a relation between algebras and Kripke frames, or between a pair of an algebra and a modal evaluation on it and a Kripke model. We will work with this second option, since being less general allows to prove a more specific result for the transfer of the deductions between the two semantics.

Definition 7.13. Let $\mathbf{A}$ be a modal K_*-algebra and $e \in Hom(\mathbf{Fm}, \mathbf{A})$. The ***canonical*** $\langle \mathbf{A}, e\rangle$***-Kripke model*** is the model

$$\mathfrak{Mod}(\mathbf{A}, e) = \langle W_{\mathbf{A},e}, R_{\mathbf{A},e}, e_{\mathbf{A},e}\rangle$$

where:

- $W_{\mathbf{A},e} = Hom(\overline{\mathbf{A}}, [\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}})$;
- $R_{\mathbf{A},e}vw$ if and only if $w(a) = 1$ for all $a \in A$ such that $v(\Box a) = 1$;
- $e_{\mathbf{A},e}(v, x) = v(e(x))$.

We can prove that the evaluation $e_{\mathbf{A},e}$ of the above defined canonical Kripke model $\mathfrak{Mod}(\mathbf{A}, e)$ verifies the Truth Lemma, i.e., that $e_{\mathbf{A},e}(v, \varphi) = v(e(\varphi))$ for any $\varphi \in Fm$ and any $e \in Hom(\mathbf{Fm}, \mathbf{A})$.

In fact, we can prove a stronger result that only depends on the frame: since $e \in Hom(\mathbf{Fm}, \mathbf{A})$, the previous formulation of the Truth Lemma can be proven as just a corollary of seeing that for any $a \in e(Fm)$[5], both $v(\Box a) = \inf\{w(a) : R_{\mathbf{A}}vw\}$ and $v(\Diamond a) = \sup\{w(a) : R_{\mathbf{A}}vw\}$ are true.

[4] It is remarkable that this construction leads to a more abstract proof of completeness of the logic K_* with respect to its correspondent canonical models. It turns out that the canonical model of the Lindenbaum-Tarski formula algebra of the correspondent logic is the canonical model of the logic. We do not however repeat the completeness proof in this fashion since the original one is clearer.

[5] Observe $e(Fm)$ is a countable subalgebra of $\mathbf{A}$.

First, observe that a generalization of Lemma 6.11 easily follows from the fact that each $w \in W_{\mathbf{A},e}$ is an homomorphism from $\overline{\mathbf{A}}$ into $[\mathbf{0},\mathbf{1}]^{\mathbb{Q}}_{*}$. Indeed, for any $a, b \in e(Fm)$ and any $v, w \in W_{\mathbf{A},e}$, if $v(a) \not\leq w(b)$ then $w(b) < v(a)$ and thus, there is $c \in \mathcal{C}_*$ such that $w(b) < c < v(a)$, i.e., $v(\overline{c}^{\mathbf{A}} \Rightarrow a) = 1$ and $w(\overline{c}^{\mathbf{A}} \Rightarrow b) < 1$. Using the equations $(\mathsf{E}_{\Box 1})$ and $(\mathsf{E}_{\Diamond 1})$ we get

- $v(\Box a) \leq w(a)$ for all $v, w \in W_{\mathbf{A},e}$ with $R_{\mathbf{A},e}vw$;
- $v(\Diamond a) \geq w(a)$ for all $v, w \in W_{\mathbf{A},e}$ with $R_{\mathbf{A},e}vw$.

As for the converse inequalities, a more general version of Lemma 6.12 can be proven. We can formulate it for an arbitrary countable algebra (which, in our context, is the countable subalgebra of $\mathbf{A}$ given by $e(Fm)$). This restriction on the cardinality of the algebra comes from the use of Lemma 4.10, which is proven for countable algebras.

Lemma 7.14. *Let $\mathbf{A}$ be a countable K_*-algebra and $e \in Hom(\mathbf{Fm}, \mathbf{A})$. Let $v \in W_{\mathbf{A},e}$ and $a \in A$. If $w(a) = 1$ for all $w \in W_{\mathbf{A},e}$ such that $R_{\mathbf{A}}vw$ then $v(\Box a) = 1$.*

Proof. First, it is easy to see that for any $w \in W_{\mathbf{A},e}$ the following hold:

- $w^{-1}(1)$ is a prime L_*^∞-filter on $\overline{\mathbf{A}}$ (remember that $\overline{\mathbf{A}}$ is the non-modal reduct of $\mathbf{A}$) and
- $\Box^{-1}w^{-1}(1)$ is a L_*^∞-filter on $\overline{\mathbf{A}}$. Indeed, it is just necessary to check that $\{a \in A : \ w(\Box a) = 1\}$ contains all the axioms and is closed under the derivation rules, which is easy to do using that w is an homomorphism from $\overline{\mathbf{A}}$ into $[\mathbf{0},\mathbf{1}]^{\mathbb{Q}}_{*}$ and that the equations and generalized quasi-equations arising from the axiomatization of K_* hold in $\mathbf{A}$. $1 \in \Box^{-1}w^{-1}(1)$ is immediate by definition of the $\Box$ operator over the K_* algebras (by $\mathsf{E}_{\mathsf{N}_\Box}$). Moreover, this set is closed under derivations as a consequence of Theorem 6.9.

On the other hand, we can prove the following claim:

> *CLAIM: The set of prime L_*^∞-filters on $\overline{\mathbf{A}}$ that contain $\Box^{-1}v^{-1}(1)$ coincides with the set $\{w^{-1}(1) : \ w \in W_{\mathbf{A},e}$ and $R_{\mathbf{A},e}vw\}$.*
>
> The fact that for each $w \in W_{\mathbf{A},e}$ such that $R_{\mathbf{A},e}vw$ it holds that $w^{-1}(1)$ is a prime filter that contains $\Box^{-1}v^{-1}(1)$ is immediate by definition. To see that each prime filter that contains $\Box^{-1}v^{-1}(1)$ coincides with $w^{-1}(1)$ for some $w \in W_{\mathbf{A},e}$ that is related with v, take for each filter F as before the homomorphism $h_F \in Hom(\overline{\mathbf{A}}, [\mathbf{0},\mathbf{1}]^{\mathbb{Q}}_{*})$ defined as $h_F = \rho_F \circ \Pi_F$, where $\Pi_F \colon \overline{\mathbf{A}} \to \overline{\mathbf{A}}/F$ is the projection over the quotient algebra and ρ_F is the embedding from $\overline{\mathbf{A}}/F$ into $[\mathbf{0},\mathbf{1}]^{\mathbb{Q}}_{*}$ built in Lemma 4.13 (since F is prime, $\overline{\mathbf{A}}/F$ is a linearly ordered). It is clear that $F = h_F^{-1}(1)$. To check that $R_{\mathbf{A},e}vh_F$, observe that $v(\Box a) = 1$ means that $a \in \Box^{-1}w^{-1}(1)$ and so $a \in F$. Then, $\Pi_F(a) = [1]_F$ and $h_F(a) = 1$.

By the previous claim and Lemma 4.10 (recall that the algebra was countable), it follows that $\Box^{-1}v^{-1}(1) = \bigcap\{w^{-1}(1) : w \in W_{\mathbf{A},e} \text{ and } R_{\mathbf{A},e}vw\}$. The assumption of the lemma was that $w(a) = 1$ for all w such that $R_{\mathbf{A},e}vw$ and thus $a \in \Box^{-1}v^{-1}(1)$, which concludes the proof. ⊠

From here it is straightforward to prove the following lemma (using the same reasoning in the proof of Lemma 6.13) and from which it follows the previously mentioned more general Truth Lemma that refers to the evaluation $e_{\mathbf{A}}$ of the canonical Kripke model $\mathfrak{Mod}(\mathbf{A}, e)$ for a K_*-algebra $\mathbf{A}$ and a modal $\mathbf{A}$-evaluation e.

Lemma 7.15. *Let $\mathbf{A}$ be a countable K_*-algebra. For any $a \in A$ and $v \in W_{\mathbf{A},e}$ the following hold.*

- $v(\Box a) = \inf\{w(a) : \ R_{\mathbf{A}}vw\}$;
- $v(\Diamond a) = \sup\{w(a) : \ R_{\mathbf{A}}vw\}$.

Observe that, for any algebra $\mathbf{A}$ and evaluation e on it, we can consider the countable subalgebra of $\mathbf{A}$ given by $e(Fm)$. We can straightforwardly conclude the following result relying on this fact.

Corollary 7.16. *Let $\mathbf{A}$ be a K_*-algebra. For any $e \in Hom(\mathbf{Fm}, \mathbf{A})$, any formula φ and any $v \in W_{\mathbf{A},e}$, it holds that $e_{\mathbf{A},e}(v, \varphi) = v(e(\varphi))$.*

It is now possible to obtain a relation between deductions in a modal algebra and within its associated canonical model, analogous to Lemma 7.12. It is remarkable that in this case we can prove the equivalence on the deductions without having to assume any condition on the (modal) algebra.

Lemma 7.17. *Let $\mathbf{A} \in \mathsf{K}_*$ and e a modal evaluation over it, i.e., $e \in Hom(\mathbf{Fm}, \mathbf{A})$. Then the following hold:*

1. $\Gamma \models_{\mathbf{A},e} \varphi$ *if and only if* $\Gamma \Vdash^g_{\mathfrak{Mod}(\mathbf{A},e)} \varphi$;
2. $\Gamma \models^L_{\mathbf{A},e} \varphi$ *if and only if* $\Gamma \Vdash_{\mathfrak{Mod}(\mathbf{A},e)} \varphi$.

Proof. 1. Assume $\Gamma \models_{\mathbf{A},e} \varphi$. Then, either $e[\Gamma] \not\subseteq \{\overline{1}^{\mathbf{A}}\}$ or $e(\varphi) = \overline{1}^{\mathbf{A}}$. In the first case, there exists an homomorphism h from $\overline{\mathbf{A}}$ to $[\mathbf{0},\mathbf{1}]^{\mathbb{Q}}_*$ such that $h(e[\Gamma]) \not\subseteq \{1\}$ so $e_{\mathbf{A},e}(h, [\Gamma]) \not\subseteq \{1\}$ and thus $\Gamma \Vdash^g_{\mathfrak{Mod}(\mathbf{A},e)} \varphi$. In the other case, $e(\varphi) = \overline{1}^{\mathbf{A}}$ implies that for any $h \in Hom(\overline{\mathbf{A}}, [\mathbf{0},\mathbf{1}]^{\mathbb{Q}}_*)$, $h(e(\varphi)) = 1$ and thus again $\Gamma \Vdash^g_{\mathfrak{Mod}(\mathbf{A},e)} \varphi$.

For the other direction, assume on the contrary that $\Gamma \not\models_{\mathbf{A},e} \varphi$. By definition, that means that $e[\Gamma] \subseteq \{\overline{1}^{\mathbf{A}}\}$ and that $e(\varphi) < \overline{1}^{\mathbf{A}}$. Then, there is $h \in Hom(\overline{\mathbf{A}}, [\mathbf{0},\mathbf{1}]^{\mathbb{Q}}_*)$ such that $h(e(\varphi)) < 1$, while for all $v \in Hom(\overline{\mathbf{A}}, [\mathbf{0},\mathbf{1}]^{\mathbb{Q}}_*)$, $v(e[\Gamma]) \subseteq \{1\}$. This means, by definition, that $\Gamma \nVdash^g_{\mathfrak{Mod}(\mathbf{A},e)} \varphi$.

2. Assume that $\Gamma \not\Vdash_{\mathfrak{Mod}(\mathbf{A},e)} \varphi$. This is equivalent to say that there is $v \in Hom(\overline{\mathbf{A}}, [\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}})$ such that $e_{\mathbf{A},e}(h,[\Gamma]) \subseteq \{1\}$ while $e_{\mathbf{A},e}(h,\varphi) \neq 1$. Let F to be the K_*-filter on $\mathbf{A}$ given by $h^{-1}(1)$. Then, by definition of $e_{\mathbf{A},e}$ it follows that $e[\Gamma] \subseteq F$ while $e(\varphi) \notin F$ and so $\Gamma \not\models_{\mathbf{A},e} \varphi$.

   For the other direction, suppose that $\Gamma \not\models_{\mathbf{A},e} \varphi$, i.e., there exists a K_*-filter F on $\mathbf{A}$ such that $e[\Gamma] \subseteq F$ and $e(\varphi) \notin F$. By Lemma 4.10, for all prime filter P such that $F \subseteq P$, $e[\Gamma] \subseteq P$, while there is a prime filter of this family such that $e(\varphi) \notin P$. Consider then the homomorphism h from $\overline{\mathbf{A}}$ to $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$ defined by $\rho \circ \Pi_P$, where Π_P is the projection from $\overline{\mathbf{A}}$ to $\overline{\mathbf{A}}/P$ and ρ is the embedding defined in 4.13. It is clear that $e_{\mathbf{A},e}(h,[\Gamma]) = h(e[\Gamma]) = 1$, while $e_{\mathbf{A},e}(h,\varphi) = h(e(\varphi)) < 1$ and so $\Gamma \not\Vdash_{\mathfrak{Mod}(\mathbf{A},e)} \varphi$.

$\boxtimes$

Correspondence results

It is now possible to study how the compositions of the $\mathfrak{Mod}()$ and the $\mathsf{A}lg()$ functions behave. Our aim is to prove that they are up to some level inverse one to the other, in the sense that applying these constructions one after the other over a structure of the corresponding type produces a new structure N of the same type in which the original one O can be "logically embedded". With this we mean that for an arbitrary set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \models_N \varphi \text{ implies that } \Gamma \models_O \varphi.$$

For the sake of a lighter notation, in what follows, given a $*$-Kripke model $\mathfrak{M}$, we will write $\mathfrak{M}'$ to denote the model $\mathfrak{Mod}(\mathsf{A}lg(\mathfrak{M}), e_{\mathfrak{M}})$ (the canonical model associated to its complex algebra), and we will name it its canonical model. Analogously, for a given $\mathbf{A} \in \mathsf{K}_*$ and any modal $\mathbf{A}$-evaluation e, we will write $\mathbf{A}'$ instead of $\mathsf{A}lg(\mathfrak{Mod}(\mathbf{A},e))$ (the complex algebra associated to the canonical model of $\mathbf{A}$) and name it its complex algebra. Note that the modal evaluation e does not affect the resulting algebra but only the evaluation associated to it, since for any two evaluations e_1, e_2, $\mathsf{A}lg(\mathfrak{Mod}(\mathbf{A},e_1)) = \mathsf{A}lg(\mathfrak{Mod}(\mathbf{A},e_2))$.

Applying the previous transfer results (Lemmas 7.12 and 7.17) it is not difficult to see how a Kripke model is related to its associated canonical one and respectively, how a modal algebra is related with its complex one.

Lemma 7.18. *Let $\mathbf{A} \in \mathsf{L}_*^{\infty}$, $\mathfrak{M} = \langle W, R, e\rangle$ be a $\mathbf{A}$-Kripke model and let $\Gamma \cup \{\varphi\} \subseteq Fm$. Then*

1. *$\Gamma \Vdash^g_{\mathfrak{M}} \varphi$ if and only if $\Gamma \Vdash^g_{\mathfrak{M}'} \varphi$.*
2. *$\Gamma \not\Vdash_{\mathfrak{M}} \varphi$ implies that $\Gamma \not\Vdash_{\mathfrak{M}'} \varphi$.*
3. *If $\mathbf{A}$ is linearly ordered, $\Gamma \Vdash_{\mathfrak{M}} \varphi$ if and only if $\Gamma \Vdash_{\mathfrak{M}'} \varphi$.*

Proof. All cases follow as immediate corollaries of Lemmas 7.12 and 7.17. $\boxtimes$

Point 3. from the previous Lemma is not true in general, as we remarked before.

On the other hand, we can also study the relation between a modal algebra $\mathbf{A}$ and its associated complex algebra $\mathbf{A}'$. Again, as a simple corollary of Lemmas 7.12 and 7.17 (observe that the algebra of evaluation of all the canonical models is the algebra $[\mathbf{0},\mathbf{1}]^{\mathbb{Q}}_{*}$, always linearly ordered) we can get the following result.

Lemma 7.19. *Let* $\mathbf{A} \in \mathsf{K}_*$ *and* e *a modal evaluation on it, i.e.,* $e \in Hom(\mathbf{Fm},\mathbf{A})$. *Then* $\Gamma \models^{L}_{\mathbf{A},e} \varphi$ *if and only if* $\Gamma \models^{L}_{\mathbf{A}',e'} \varphi$.

Moreover, it is possible to go a bit further and obtain a result that just depends on the algebra (and is not limited to the meaning of the modal algebra as algebra of evaluation).

Theorem 7.20. *Let* $\mathbf{A}$ *be a* K_**-algebra. Then* $\mathbf{A}$ *can be embedded (in the usual algebraic sense) into* $\mathbf{A}'$.

Proof. For an arbitrary modal evaluation on $\mathbf{A}$, note that the universe of the algebra $\mathsf{A}lg(\mathfrak{Mod}(\mathbf{A},v))$ is the set $[0,1]^{Hom(\overline{\mathbf{A}},[\mathbf{0},\mathbf{1}]^{\mathbb{Q}}_{*})}$. Let θ be the function that maps each element $a \in A$ to the function from $Hom(\overline{\mathbf{A}},[\mathbf{0},\mathbf{1}]^{\mathbb{Q}}_{*})$ to $[0,1]$ that sends each h to $h(a)$, i.e., $\theta(a) = [h \mapsto h(a)]$ for each $h \in Hom(\overline{\mathbf{A}},[\mathbf{0},\mathbf{1}]^{\mathbb{Q}}_{*})$. From the Truth Lemma 7.16 it follows that θ is a (modal) homomorphism from $\mathbf{A}$ to $\mathsf{A}lg(\mathfrak{Mod}(\mathbf{A},v))$. It is an exercise to see that it is injective (using that if for all $h \in Hom(\overline{\mathbf{A}},[\mathbf{0},\mathbf{1}]^{\mathbb{Q}}_{*})$ it holds that $h(a) = h(b)$, then $a = b$). ⊠

7.3 Order preserving logics

The construction of the complex algebras provides us with several concrete modal K_*-algebras. Moreover, using the completeness result of K_* with respect to $\Vdash_{\mathsf{K}_*}$, and in particular, with respect to the canonical Kripke model $\mathfrak{M}_{\mathfrak{c}}$, we will obtain another interesting algebraic completeness result for this logic.

The intuition behind the algebraic completeness result that can be achieved through this approach is as follows. For $\Gamma \cup \{\varphi\} \subseteq Fm$ it holds that $\Gamma \vdash_{K_*} \varphi$ if and only if for all $w \in W_c$ for which $w[\Gamma] \subseteq \{1\}$ it holds $w(\varphi) = 1$ as well. This condition is easily expressible within the associated complex algebra of the canonical model, $\mathsf{A}lg(\mathfrak{M}_{\mathfrak{c}})$. It means that for all $w \in W_c$ for which $e_{\mathfrak{M}_{\mathfrak{c}}}[\Gamma](w) \subseteq \{1\}$ then $e_{\mathfrak{M}_{\mathfrak{c}}}(\varphi)(w) = 1$. Given that the Δ operator is defined on the complex algebras component-wisely, it is clear that for any $w \in W_c$ and for any formula ψ either $e_{\mathfrak{M}_{\mathfrak{c}}}(\Delta\psi)(w) = 1$ or $e_{\mathfrak{M}_{\mathfrak{c}}}(\Delta\psi)(w) = 0$. Then, the previous condition can be rewritten as $\inf\{e_{\mathfrak{M}_{\mathfrak{c}}}[\Delta\Gamma]\} \leq e_{\mathfrak{M}_{\mathfrak{c}}}(\varphi)$.[6]

We first make a natural observation concerning the elements of the algebra that are images of the $\delta^{\mathbf{A}}$ operation. Given a L^{∞}_{*}-algebra $\mathbf{A}$, we will say that $b \in A$ is a ***Boolean element*** whenever there exists an element $b' \in \mathbf{A}$ called

[6] $\leq$ is defined component-wise, and thus for each $w \in W_c$, either $\varsigma[\Delta\Gamma](w) = 1$, in which case by assumption $e_{\mathfrak{M}_{\mathfrak{c}}}(\varphi)(w) = 1$, or $\inf\{e_{\mathfrak{M}_{\mathfrak{c}}}[\Delta\Gamma]\}(w) = 0$, and so the inequality is trivially true.

complement of b such that $b \vee b' = \overline{1}^{\mathbf{A}}$ and $b \wedge b' = \overline{0}^{\mathbf{A}}$. We will denote the set of Boolean elements of $\mathbf{A}$ by $\mathfrak{B}_{\mathbf{A}}$.

Remark 7.21. Let $\mathbf{A} \in \mathsf{L}_*^\infty$. Then for any $a, b \in A$, $\delta^{\mathbf{A}} a \in \mathfrak{B}_{\mathbf{A}}$ and $\delta^{\mathbf{A}} a \odot x = \delta^{\mathbf{A}} a \wedge x$.

With the aim of generalizing the above idea of logic preserving the Boolean degrees of truth to the whole class of K_*-algebras, we will refer to the usual definition of logic preserving degrees of truth (cf. [16, Def. 2.1]).

Definition 7.22. Let $\Gamma \cup \{\varphi\} \subseteq Fm$, $\mathsf{C} \cup \{\mathbf{A}\} \subseteq \mathsf{K}_*$ and $h \in Hom(\mathbf{Fm}, \mathbf{A})$.[7]

- $\Gamma \models^{\leq}_{\mathbf{A},h} \varphi$ if for any $a \in A$, if $a \leq h(\gamma)$ for all $\gamma \in \Gamma$ then $a \leq h(\varphi)$.
- $\Gamma \models^{\leq}_{\mathbf{A}} \varphi$ if $\Gamma \models^{\leq}_{\mathbf{A},h} \varphi$ for any $h \in Hom(\mathbf{Fm}, \mathbf{A})$.
- $\Gamma \models^{\leq}_{\mathsf{C}} \varphi$ if $\Gamma \models^{\leq}_{\mathbf{B}} \varphi$ for each $\mathbf{B} \in \mathsf{R}$.

In particular, if $\mathbf{A}$ is a complete algebra, the first definition is equivalent to say that $\inf\{h(\Gamma)\} \leq h(\varphi)$.

Now, a second algebraic completeness result for K_* can be obtained just checking the soundness of K_* with respect to the $\models^{\leq}_{\mathsf{K}_*}$ logic defined above preserving the order only for the Boolean elements.

Theorem 7.23 (Strong algebraic completeness of K_*)**.** *For any set $\Gamma \cup \{\varphi\}$ of modal formulas,*

$$\Gamma \vdash_{K_*} \varphi \textit{ iff } \Delta\Gamma \models^{\leq}_{\mathsf{K}_*} \varphi$$

where $\Delta\Gamma$ stands for the set $\{\Delta\gamma : \ \gamma \in \Gamma\}$.[8]

Proof. Completeness is direct because $\mathsf{A}lg(\mathfrak{M}_\mathfrak{c}) \in \mathsf{K}_*$. If $\Gamma \not\vdash_{K_*} \varphi$, by completeness with respect to $\mathfrak{M}_\mathfrak{c}$, there exists a world $v \in W_c$ such that $v[\Gamma] \subseteq \{1\}$ and $v(\varphi) < 1$. Then we have that $e_{\mathfrak{M}_\mathfrak{c}}[\Gamma](v) = 1$ (and so that $e_{\mathfrak{M}_\mathfrak{c}}[\Delta\Gamma](v) = 1$) but $e_{\mathfrak{M}_\mathfrak{c}}(\varphi)(v) < 1$. But taking into account how the order is defined in $\mathsf{A}lg(\mathfrak{M}_\mathfrak{c})$, this means that $e_{\mathfrak{M}_\mathfrak{c}}[\Delta\Gamma] \not\leq e_{\mathfrak{M}_\mathfrak{c}}(\varphi)$.

To prove soundness, let $\mathbf{A} \in \mathsf{K}_*$ and $h \in Hom(\mathbf{Fm}, \mathbf{A})$. Since all axioms of K_* are evaluated to $\overline{1}^{\mathbf{1}}$ under any homomorphism, we just need to check the soundness of the deduction rules of K_* in $\models^{\leq}_{\mathsf{K}_*}$:

MP : $h(\Delta\psi) \wedge h(\Delta(\psi \to \varphi)) = h(\Delta\psi) \odot h(\Delta(\psi \to \varphi)) \leq h(\Delta\psi) \odot (h(\Delta\psi) \Rightarrow h(\Delta\varphi)) \leq h(\Delta\varphi) \leq h(\varphi)$, by the Remark 7.21 and definitions and axioms of MTL logic.

G_Δ : It is direct by definition.

[7] Recall that $\mathbf{Fm}$ is the algebra of modal formulas.
[8] Do not forget that, in general, modal K_*-algebras are not linearly ordered.

D^∞ : Let $a \in A$ such that $a \leq h(\Delta((\varphi \to \overline{c}) \vee (\overline{c} \to \psi)))$ for all $c \in \mathcal{C}_*$. Then, by definition of order and distributing Δ over $\vee$ and applying the homomorphism properties, we have that $\overline{1}^{\mathbf{A}} = \delta^{\mathbf{A}}(a \Rightarrow (\delta^{\mathbf{A}}(h(\varphi) \Rightarrow \overline{c}^{\mathbf{A}})) \vee \delta^{\mathbf{A}}(\overline{c}^{\mathbf{A}} \Rightarrow h(\psi))))$. We know that the theorems of L_*^∞ (and so finite deductions, applying the deduction theorem) are valid equations in all the K_*-algebras and so applying the points 3. and 9. from Remark 4.2, we have the following chain of equalities:[9]

$$\begin{aligned}\overline{1}^{\mathbf{A}} = \delta^{\mathbf{A}}(a \Rightarrow \delta^{\mathbf{A}}(h(\varphi) \Rightarrow \overline{c}^{\mathbf{A}})) \vee \delta^{\mathbf{A}}(a \Rightarrow \delta^{\mathbf{A}}(\overline{c}^{\mathbf{A}} \Rightarrow h(\psi))) &= \\ \delta^{\mathbf{A}}((h(\varphi) \wedge \neg\delta^{\mathbf{A}}\neg a) \Rightarrow \overline{c}^{\mathbf{A}}) \vee \delta^{\mathbf{A}}(\overline{c}^{\mathbf{A}} \Rightarrow (h(\psi) \vee \delta^{\mathbf{A}}\neg a)) &= \\ ((h(\varphi) \wedge \neg\delta^{\mathbf{A}}\neg a) \Rightarrow \overline{c}^{\mathbf{A}}) \vee (\overline{c}^{\mathbf{A}} \Rightarrow (h(\psi) \vee \delta^{\mathbf{A}}\neg a))&\end{aligned}$$

for all $c \in \mathcal{C}_*$.
From the generalized quasi-equation $\mathcal{Q}^\infty$ it follows that

$$\overline{1}^{\mathbf{A}} = (h(\varphi) \wedge \neg\delta^{\mathbf{A}}\neg a) \Rightarrow (h(\psi) \vee \delta^{\mathbf{A}}\neg a).$$

Using again theorems of L_*^∞ (in particular, points 3., 4. and 5. from Remark 4.2), and reasoning in a similar fashion as above we get that

$$\overline{1}^{\mathbf{A}} = \delta^{\mathbf{A}}h(\varphi \to \psi) \vee \delta^{\mathbf{A}}h(\neg\varphi) \vee \delta^{\mathbf{A}}\neg a \vee \delta^{\mathbf{A}}h(\psi)$$

Since $\psi \to (\varphi \to \psi)$ is a theorem of the logic and applying also 6. from Remark 4.2, we have that $\overline{1}^{\mathbf{A}} = \delta^{\mathbf{A}}h(\varphi \to \psi) \vee \delta^{\mathbf{A}}\neg a$. With point 10. from Remark 4.2, we can conclude that $\overline{1}^{\mathbf{A}} = (a \Rightarrow h(\varphi \Rightarrow \psi))$, which by definition means $a \leq h(\varphi \Rightarrow \psi)$.

$(N_\Box)$ This rule is only applicable over theorems, so assume that for any modal L_*^∞-algebra $\mathbf{A}$ and for any evaluation e, $e(\psi) = \overline{1}^{\mathbf{A}}$. Then, by definition of modal K_*-algebra, for any modal morphism h we have $h(\Box\psi) = \Box h(\psi) = \Box\overline{1}^{\mathbf{A}} = \overline{1}^{\mathbf{A}}$, so $\models_{\mathsf{K}_*} \Box\psi$, and thus $\models_{\mathsf{K}_*}^{\leq} \Box\psi$ as well. ⊠

[9] In several cases, we write $=$ instead of $\leq$ because the smaller element is $\overline{1}^{\mathbf{A}}$.

Part IV

Aplications

Chapter 8

Satisfiability Modulo Theories

Intuitively, Satisfiability Modulo Theories (SMT) can be seen as a generalization of the SAT problem by adding the ability to handle arithmetic and other theories. For what concerns this dissertation, SMT form an interesting layer over which build non-classical logics applications. We found studies concerning the adequacy of SMT solvers for solving Constraint Satisfaction Problems [6], and we considered these applications could be of great use to work with the Basic Logic and its axiomatic extensions simply by interpreting in them the constraints imposed by these logics.

This chapter is devoted to explain and provide a formal description of Satisfiability Modulo Theories (SMT) and detail some results concerning the implementation and design of solvers for SMT. It is a preliminaries chapter, with no original content and its localization out of the preliminaies chapter is due to the fact that the notions presented here are of great importance in showing the correctnesss of SMT, but they are not necessary for the comprehension of the dissertation (and, over all, they do not play any role in the previous parts of this work).

8.1 The satisfiability problem

Satisfiability, namely the problem of determining whether a formula (or in real world applications, a constraint) has a model in a certain logic (semantically speaking, an evaluation that sends the formula to the top element of the algebra) is one of the most fundamental problems in theoretical computer science.

Constraint satisfaction problems arise in many diverse areas including graph and game theory problems, planning, scheduling, software and hardware verification, extended static checking, optimization, test case generation or type inference. The most well-known constraint satisfaction problem is propositional satisfiability SAT: decide whether a classical logic formula can be made true by choosing true/false values for its variables.

Many of these constraint satisfaction problems can be encoded by Boolean formulas and solved using Boolean satisfiability (SAT) solvers. However, other problems require the additional expressiveness of equality, uninterpreted function symbols, arithmetic, arrays, datatype operations or quantifiers. For example, many applications of formal methods that rely on generating first order formulas over theories of the real numbers or integers (including fuzzy logics) are in need of more expressive logical languages and solvers.

Thus, a formalism extending SAT, called Satisfiability Modulo Theories (SMT), has also been developed to deal with these more general decision problems. An SMT instance can informally be understood as a first order Boolean formula in which some propositional variables are replaced by predicates with predefined interpretations from background theories. Namely, these predicates are binary-valued functions over non-binary variables.

It is known that the complexity of the satisfiability problem of Hájek's Basic Logic and its extensions is **NP-complete**. This is not an exception, as many of the Constraint Satisfaction Problems reach at least the **NP** complexity class. Due to this high computational complexity and to the fact that real problems are not interested in validity in general, but with respect to a fixed background theory, the idea is not to build a procedure that can solve arbitrary SMT problems, but to focus on specialized SMT solvers.

As done with efficient SAT solvers, when working with concrete problems, the procedures can be highly simplified and fastened by paying attention to implementation details. In recent years, there has been an enormous progress in the scale of problems that can be solved, thanks to innovations in core algorithms, data structures, heuristics and other methods, and for example, modern SAT procedures can check formulas with hundreds of thousands variables and millions of clauses.

In the case of SMT, similar progress has been observed in the procedures for the more commonly occurring theories that not only work with FOL but also in fragments of it (for instance, quantifier free formulas). For many of these, specialized methods actually yield decision procedures for the validity of ground formulas or some subset of them. This is for instance the case, thanks to classical results in mathematics, for the theory of real numbers or the theory of integer numbers with only addition and substraction. In the last two decades however, specialized decision procedures have also been discovered for a large and still growing, list of theories of other important data types such as certain theories of arrays and strings, variants of the theory of finite sets, the theory of several classes of lattices, the theories of finite, regular and infinite trees and the theory of lists, tuples, records, queues, hash tables and bit vectors of a fixed or arbitrary finite size.

8.2 Formal definitions

Based on pioneer works proposing the use of SMT solvers in formal methods in the 80s [102, 112, 19], on the last ten years we have lived an increasing interest on this field and research on the foundational and practical aspects of SMT has rapidly grown. Several SMT solvers have been developed in academia and industry with continually increasing scope and performance. We can cite here examples integrated into interactive theorem provers for high-order logic (such as HOL and Isabelle), extended static checkers (such as CAsCaDE, Boogie and ESC/Java 2), verification systems (such as ACL2, Frama-c, SAL and UCLID) or model checkers (such as BLAST) among others. In industry, as important centers with SMT-related projects we can name Microsoft Research, Cadence Berkeley Labs, Intel Strategic CAD Labs and NEC Labs.[1]

Most approaches for automated deduction tools rely on case-analysis for its core system. In the case of SMT, most of the solvers exploit SAT procedures for performing case-analysis efficiently.

In this section, basic techniques used in state-of-the-art SAT solvers and the more common approaches to the SMT problem are detailed.

SAT encodings

Most state-of-the-art SAT solvers (Glucose [8], Minisat [47], BerkMin [66]) today are based on Conflict-Driven Clause Learning algorithm (CDCL), originally grown from the Davis-Putnam-Logemann-Loveland (DPLL) procedure [38, 37].

The DPLL algorithm is a complete, backtracking-based search algorithm for deciding the satisfiability of propositional logic formulae in conjunctive normal form, i.e. for solving the CNF-SAT problem. The basic backtracking algorithm runs by choosing a literal, assigning a truth value to it, simplifying the formula and then recursively checking if the simplified formula is satisfiable; if this is the case, the original formula is satisfiable; otherwise, the same recursive check is done assuming the opposite truth value. This is known as the splitting rule, as it splits the problem into two simpler sub-problems. The simplification step essentially removes all clauses which become true under the assignment from the formula and all literals that become false from the remaining clauses.

The great improvements in the performance of DPLL-based SAT solvers achieved in the last years are due, on the one hand, to better implementation techniques and on the other, to several conceptual enhancements on the original DPLL procedure, aimed at reducing the amount of explored search space, such as backjumping (a form of non-chronological backtracking), conflict-driven lemma learning, and restarts. On the other hand, correctness has been proved for each of these methods, ensuring the coherence of their usage.

[1] For the interested reader, see for instance https://isabelle.in.tum.de/, http://research.microsoft.com/en-us/projects/boogie/, http://www.cs.utexas.edu/users/moore/acl2/, http://frama-c.com/, http://sal.csl.sri.com/, http://uclid.eecs.berkeley.edu/, http://forge.ispras.ru/projects/blast/.

These advances make it now possible to decide the satisfiability of very complex *SAT* problems.

Detailed description of existing procedures is out of the scope of this dissertation, but a uniform, declarative framework for describing DPLL-based solvers, *Abstract DPLL*, can be found in [103].

SMT-solvers approach

Following the more recent SMT literature, given a signature Σ, we define a theory T over Σ as just one or more (possibly infinitely many) Σ-models. Then, a ground Σ-formula φ is satisfiable in a Σ-theory or is T-satisfiable) if and only if there is an element of the set T that satisfies φ. Similarly, a set Γ of ground Σ-formulas T-entails a ground formula φ ($\Gamma \models_T \varphi$) if and only if every model of T that satisfies all formulas in Γ satisfies φ as well.

We say the satisfiability problem for theory T is decidable if there is a procedure Υ that checks whether any ground (and hence, quantifier free) formula is satisfiable or not. In this case, we say Υ is a decision procedure for T or a T-solver.

The proof of correctness of SMT methods for decidable background theories is a step that has to be checked for each solver. Satisfiability procedures must be proved sound and complete; while soundness is usually easy, completeness requires specific model construction arguments showing that, whenever the procedure finds a formula satisfiable, a satisfying theory interpretation for it does indeed exist. This means that each new procedure in principle requires a new completeness proof.

Working with a decidable theory T, there are two main approaches for determining the satisfiability of a formula with respect to T, each with its own pros and cons: *eager* and *lazy*.

Eager SMT approach

This approach consists on ad-hoc translations from an input formula and relevant data from the theory T into a set of propositional formulas that is satisfied in the same points that the original formula (see for instance [114] for this procedure), which is then checked by a SAT solver for satisfiability.

The good point of this approach is that it can always make use of the lasts existing SAT solvers, overcoming the problem of relatively big translation results.

However, the problem lies on the exponential cost of the translation operation for making the problem treatable by a SAT solver. For this, sophisticated adhoc translations are needed for each theory and experiments have shown the explosion on needed resources when escalating the problems (see [41]). Also it has been studied the difficulty of combining several theories. To address these issues, the latest research on SAT encodings focuses on general frameworks that allow incremental translations and calls to the *SAT* solver, and general mechanisms for combining encodings for different theories.

From a theoretical point of view, proving soundness and completeness is relatively simple because it reduces to proving that the translation is satisfiability invariant (but that proof needs to be done for each defined translation).

Lazy SMT approach

Instead of an ad-hoc translation for each theory into a SAT problem, a specialized T-solver for deciding the satisfiability of conjunctions of theory literals can be defined. Then, the objective is combining the strength of this solver with the existing SAT-solvers to produce an efficient SMT-solver.

A lot of research has been done in the matter of combining these two solvers. The most widely used approach in the last few years is usually referred to as the *lazy* approach [40, 10, 7].

The idea behind this approach is such that each atom occurring in a formula φ to be checked for satisfiability is initially considered simply as a propositional symbol, not taking into account the theory T. Then, the formula is processed by a SAT solver, which determines its propositional satisfiability. If φ was found unsatisfiable by the SAT solver, then it also is T-unsatisfiable. In other cases, the SAT solver returned a propositional model M of φ and then this assignment will be checked by a T-solver. If M is found T-consistent, it is a T-model of φ. Otherwise, the T-solver generates a ground clause rejecting that assignment. This formula is then added to φ by propositional conjunction and the SAT solver is started again. This process is repeated until a T-model is found or the SAT solver returns the formula is unsatisfiable.

For details on satisfiability in first-order theories and results on combining several theories for working under the *lazy approach* see for instance [39].

Most of the currently implemented solvers follow this approach, and include a high number of available theories like linear, difference and non-linear arithmetics, bit-vectors, arrays and free functions among others. By modularity and correctness of the SAT solvers, the correctness for each particular theory is the main point proving the correctness of the solver (up to exactness of the implementation).

A general definition for SMT has been studied in [103] providing the $DPLL(T)$ approach, a general modular architecture based on a general DPLL engine parametrized by a solver for a theory T of interest. Here details about the theories used in our particular case will be given.

- **Linear arithmetic**
  Linear arithmetic (LA) constraints have the form $c_0+\Sigma_{i=1}^{n}c_i\cdot x_i \leq 0$, where each c_i for $0 \leq i \leq n$ is a rational constant and the variables x_i range over $\mathbb{R}$. The LA-solver algorithm implemented by the SMT solver used for the experiments, z3 [42], is based on the method proposed by de Moura and Dutertre in [46].

- **Arrays**
  This theory was introduced by McCarthy in [94] and its functions are

reduced to *read (a, i, v)* and *write(a, i, v)*. Depending on the theory over which the arrays are used and the dimension specified to the array in its creation, a and i will differ on type. In our case, the implementation of the solver is built over arrays of dimension two of real values, so operations are defined as follows:

$$\begin{aligned} write &: \mathbb{R}^2 \times \mathbb{R} \times \{0,1\} \longrightarrow \mathbb{R}^2 \\ read &: \mathbb{R}^2 \times \{0,1\} \longrightarrow \mathbb{R} \end{aligned}$$

with

$$\begin{aligned} write([a,b],c,i) &:= \begin{cases} [c,b] & \text{if } i=0, \\ [a,c] & \text{if } i=1. \end{cases} \\ read([a,b],i) &:= \begin{cases} a & \text{if } i=0, \\ b & \text{if } i=1. \end{cases} \end{aligned}$$

8.3 Standardisation: language and solvers

It seems natural that for different SMT-solvers, given their specific treatment of problems and so their different working methods, different interfaces and input formats are given. Until 2002, no standardization existed and so comparing different SMT solvers was a difficult task.

To reduce this drawback, the SMT community launched in 2002 the SMT-LIB initiative [11] which is currently backed by the vast majority of research groups in SMT. SMT-LIB defines standard input/output formats and interfaces for SMT solvers and also provides an on-line repository of benchmarks for several theories.

The standardisation led to the creation of a an annual competition for SMT-solvers, SMT-COMP [113], where state of the art SMT solvers show their strengths in different kind of tests. We decided to use one of the winners of several competitions, Z3 [42], a solver dveloped by Microsoft Research, not only for its efficiency but also for the versatility it allows.

Chapter 9

The theory behind the solver

As we commented in the introductory part of this dissertation, the software application presented in this third part is oriented to open the practical use of many-valued logics to a public not exclusively belonging to the logic research community, but also to other fields. For this reason we considered it was important to allow the use of a large family of logics but without neglecting the speed when reasoning over these systems: the answering time is often as important as the exactness of the solution (in the sense that getting a solution near the objective in a fast way can be as interesting as getting the exact solution on a considerably longer time) or of the modelling of the problem. The solver presented in the following sections will not cope with the whole family of many-valued logics studied in Parts II and II of this dissertation for this motivation. It supports a smaller family of many-valued logics. We will call them "Nice BL Logics" (because they allow a nice representation) and comprehend, among others, a large family of continuous t-norm based logics.

The motivation behind the use of this logics is mainly practical. For an arbitrary left-continuous t-norm there is no general form to simplify the reasoning and thus, a reasoner for a logic based on it would just consist on coding the operation specified by the user (and also its residuum) into the SMT solver. However, a preprocessing can be done for these Nice BL logics, codifying the reasoning over them in such a way that a much faster behaviour (that the general approach commented above) can be obtained. On the other hand, we think that the potential users of this software will have, with this amount of logics, a large enough basis to start exploiting the versatility of the many-valued fuzzy logics.

Concerning the modal expansion of these logics, a similar reasoning has lead us to face the problem in what we think is the most practical way. On the one hand, while it seems natural to think of problems that could be modelled within a Kripke structure (for instance, those that resort to graphs, or those related with the field of temporal logics), for what concerns non-theoretical uses it does not seem clear which kind of problems would need of a structure with an infinite number of worlds. Moreover, the time for obtaining a model -if it exists- seems likely to be quite high, at least if a deep study on the complexity of the problem and on the optimization of the hardest parts is not done beforehand. Since the objectives of this dissertation do not aim of coping with these issues we think that a first practical and useful solution is consider the modal logics arising from models with a finite set of worlds.

This chapter is divided in two main parts. Within the first one, we begin by giving an overview of the most important definitions for understanding and working, from an automatic (and no symbolic) point of view, with BL and its main schematic extensions. For the whole theoretical system comprehension, we refer to Chapter 2, an in particular to sections 2.3 and 2.4. We define the class of Nice BL logics that will form the propositional layer of the reasoner and we will also provide several important results about these logics that will be later applied to the design and implementation of the reasoner.

The second part focuses on studying the implementability of modal expansions of the Nice BL logics previously defined. It contains the main definitions and details a also describes a simple and efficient way to reason over them.

9.1 The propositional level

The non-modal logics over which the solver is settled belong to a large class of *BL* logics.

We begin by presenting these logics, and then present some theoretical results that will allow to gain efficiency on the reasoner. Later, we comment on how the previous results affect the treatment of the constant symbols in the logic and detail how have we approached have the problems derived from them.

Nice *BL* logics

Recall that any continuous t-norm can be expressed as an ordinal sum with a denumerable cardinality of the Łukasiewicz, Gödel and Product t-norms (see Theorem 2.4). From this result, we know that a reasoner for the axiomatic extensions of the BL logic can be designed focusing only over intervals of $[0, 1]$ with the three previous t-norms. On the other hand, a computer application treating logics arising from one of these t-norms needs to take as an argument the t-norm itself. Since this is a function in $[0, 1]$, it does not seem clear how can this value can be specified if not as a list of Łukasiewicz Gödel and Product components (each one associated with an interval in $[0, 1]$ determining the universe of that component), or with a unique name for some particular cases. This naturally limits the possible t-norms, expressed as lists, to those that are a finite ordinal sums of the three basic t-norms.

However, the logics arising from ordinal sums of the $*_{\text{Ł}}$, $*_G$ and $*_\Pi$ t-norms with a finite number of components are a part, but not the totality of the logics that will be accepted by our software application.

First, thinking in the possible applications, we have also included an additional family of operations (not strictly speaking continuous t-norms) that, in the same way that above, have naturally an associated (semantical) logic. We are talking about ordinal sums (with a finite number of components!) whose components can be either the previously commented three basic ones, or range over a finite universe, with uniformly distributed points. While it is well known that there do not exist finite linearly ordered product algebras different from the boolean one (see for instance [31]), and thus there is no way to reason with the product operations over a finite universe, the cases of the Gödel and Łukasiewicz logics can have this behaviour. Then, it is possible to consider as components of an ordinal sum restrictions of the Łukasiewicz and Gödel ones to a finite universe over which the operations are closed. In order to simplify the notation, in this case we denote the operations in the representation of the ordinal sum by $*_{\text{Ł}n}$ and $*_{Gn}$, for n being the number of elements considered in the universe.

We extend the definition of ordinal sum in order to include these new operations as possible components in such a way that the sum of just one of them respectively coincides with the usual algebraic definition of the n-valued Łukasiewicz and Gödel logics over the real interval $[0,1]$:

- $\mathbf{Ł_n}$ is the subalgebra of $[\mathbf{0},\mathbf{1}]_{\mathbf{Ł}}$ with universe $\{0, \frac{1}{n-1}, ..., \frac{n-1}{n-1}\}$.
- $\mathbf{G_n}$ is the subalgebra of $[\mathbf{0},\mathbf{1}]_{\mathbf{G}}$ with universe $\{0, \frac{1}{n-1}, ..., \frac{n-1}{n-1}\}$.

Now, we will abuse notation regarding ordinal sums and generalize that name to a wider family. Given $\{*_i\}_{i\in I}$ a set of operations in $\{*_{\text{Ł}}, *_G, *_\Pi\} \cup \{*_{\text{Ł}n}, *_{Gn}\}_{n\in\mathbb{Z},n>1}$, and $\{(b_i,t_i)\}_{i\in I}$ a family of pairwise disjoint open intervals of $[0,1]$, we call ordinal sum of $\{\langle *_i, (b_i,t_i)\rangle\}_{i\in I}$ and will be denoted by

$$* = \bigoplus_{i\in I} \langle *_i, (b_i,t_i)\rangle$$

to the operation defined as in 2.3 but with restricted universe $U \subseteq [0,1]$ given by

$$U := \bigcup_{i\in I} \begin{cases} [b_i,t_i] & \text{if } *_i \in \{*_{\text{Ł}}, *_G, *_\Pi\} \\ b_i + (t_i - b_i)\cdot\{0, \frac{1}{n-1}, ..., \frac{n-1}{n-1}\} & \text{if } *_i \text{ is either } *_{\text{Ł}n} \text{ or } *_{Gn} \end{cases}$$

where for $X \cup \{y\} \subseteq [0,1]$, we write $y\cdot X$ to denote the set $\{y\cdot x\}_{x\in X}$.

This new notion allows to specify the well known n-valued Łukasiewicz and Gödel logics (which seem likely to be useful from an applied point of view), and also combinations of these with infinitely valued components.

On the other hand, thanks to several theoretical results concerning BL, we will be also able to deal with this logic too. As we remarked in Chapter 2 (see Theorem 2.5) *BL* logic coincides with the logic of the t-norm obtained as ordinal sum of infinite Łukasiewicz components. Only with this, it could seem that *BL* does not fall in the cases detailed above, but when dealing with a particular deduction (with a finite amount of variables), whether is valid or not on BL coincides with the answer to the same question on an ordinal sum of finite components. Indeed, from Theorem 2.5, follows following corollary, more naturally applicable to our case.

Corollary 9.1. *(cf. [3]) Given a formula φ and a finite set of formulas Γ,*

$$\Gamma \models_{\mathrm{BL}} \varphi \Longleftrightarrow \Gamma \models_{(n+1)[\mathbf{0},\mathbf{1}]_{\text{Ł}}} \varphi$$

where n is the number of different variables in $\Gamma \cup \{\varphi\}$ and by $(n+1)[\mathbf{0},\mathbf{1}]_{\text{Ł}}$ we denote the BL*-algebra* $\underset{i\in\{0,\dots,n\}}{\boxplus}[\mathbf{0},\mathbf{1}]_{\text{Ł}}$.

With this corollary, the validity of a formula φ in the logic BL is reduced to validity in the logic defined from the algebra of $(n+1)$ copies of $[\mathbf{0},\mathbf{1}]_{\text{Ł}}$, where n is the number of variables in φ.

After all the previous details, we can finally provide a description of the logics that are allowed into the reasoner.

Definition 9.2. We say that a logic L is a **Nice** BL **Logic** when one of the following cases holds:

1. L is equivalent to BL. That is to say, for any set of formulas $\Gamma \cup \{\varphi\}$

$$\Gamma \vdash_L \varphi \text{ if and only if } \Gamma \vdash_{\mathrm{BL}} \varphi$$

2. L coincides with the logic associated to the ordinal sum $*$ of $\{\langle *_i, (b_i, t_i)\rangle\}_{i\in I}$ for $|I| < \omega$, $\{*_i\}_{i\in I}$ a set of operations in $\{*_{\text{Ł}}, *_G, *_\Pi\} \cup \{*_{\text{Ł}n}, *_{Gn}\}_{n\in\mathbb{Z},n>1}$ and $\{(b_i, t_i)\}_{i\in I}$ a family of pairwise disjoint open intervals of $[0,1]$. That is to say, for $*$ ordinal sum as above, and for any set of formulas $\Gamma \cup \{\varphi\}$

$$\Gamma \vdash_L \varphi \text{ if and only if } \Gamma \vdash_{[\mathbf{0},\mathbf{1}]_*} \varphi$$

Efficiency issues: the product components case

Studying previous works towards the development of a solver for fuzzy logics, we found out that the reasoners that had implemented the product logic case (see [6]) showed much worse results, in terms of reasoning time, than the other cases (Łukasiewicz and Gödel).

After some research, we realized this is a problem intrinsic to the operations of the associated algebra: while reasoning with linear operations (sums, subtractions and minimum/maximum operations) is fast in general and also in the particular case of the SMT-solvers, finding solutions for equation systems with multiplication and division operations is a much harder problem (since it makes the solver face non-linear equations) .

Nevertheless, Cignoli and Torrens presented in [31] important studies concerning the product logic. In particular, they proved the following result, which states the equivalence of the theorems over the standard product algebra and those over a fragment of the so-called Presburger arithmetic, which entirely omits multiplication

Theorem 9.3. *Let $\mathbb{Z}_\bullet^-$ be the product algebra given by $\langle\{x \in \mathbb{Z} : x \leq 0\} \cup \{-\infty\}, +, -', 0, -\infty\rangle$ with*

$$x + y \quad := \quad \begin{cases} x + y & \text{if } x, y \neq -\infty \\ -\infty & \text{otherwise} \end{cases}$$

$$x -' y \quad := \quad \begin{cases} 0 & \text{if } x \leq y \\ -\infty & \text{if } x > y \text{ and } y = -\infty \\ y - x & \text{otherwise} \end{cases}$$

For any formula φ it holds that

$$\emptyset \models_{[\mathbf{0},\mathbf{1}]_\mathbf{\Pi}} \varphi \text{ if and only if } \emptyset \models_{\mathbb{Z}_\bullet^-} \varphi.$$

Gaining inspiration in the ideas behind the proof of the previous result, we can resort to an alternative and more efficient codification of the product logic and also, as shown below, of the product components of any Nice BL logic. Motivated by the possibility of the addition of rational constants to the language, we rather considered to use the extension over the negative real numbers, $\mathbb{R}_\bullet^-$, instead of $\mathbb{Z}_\bullet^-$ itself. Formally,

$$\mathbb{R}_\bullet^- := \langle\{x \in \mathbb{R} : x \leq 0\} \cup \{-\infty\}, +, -', 0, -\infty\rangle$$

where the operations are defined as in $\mathbb{Z}_\bullet^-$, but over the extended universe of the negative real numbers.

This is also a product algebra and its operations keep being linear as in the $\mathbb{Z}_\bullet^-$ case, which is what we need in order to gain practical efficiency.

It is natural to see that the standard product algebra is in fact isomorphic to $\mathbb{R}_\bullet^-$ and so the logics arising from them coincide.

Lemma 9.4. *For any pair $\langle a, b\rangle \in (0,1) \times \{z \in \mathbb{R} : z < 0\}$, the function $\sigma_{\langle a,b\rangle} : [0,1] \to \{x \in \mathbb{R} : x \leq 0\} \cup \{-\infty\}$ defined by*

$$\sigma_{\langle a,b\rangle}(x) = \begin{cases} -\infty & \text{if } x = 0 \\ b \cdot log_a x & \text{otherwise} \end{cases}$$

is an isomorphism between $[\mathbf{0},\mathbf{1}]_\mathbf{\Pi}$ and $\mathbb{R}_\bullet^-$ (sending $\cdot$ to $+$ and $\to$ to $-$).

Proof. It is first clear that σ is order preserving: being $a \in (0,1)$, the function log_a is monotonically decreasing (in $(0,1]$) and being b a negative number, $b \cdot log_a$ is monotonically increasing. With the same basic calculus we know that it is also a bijection and extending it by mapping 0 to $-\infty$ results in a bijective mapping between $[0,1]$ and $\{x \in \mathbb{R} : x \leq 0\} \cup \{-\infty\}$.

In order to prove it is an homomorphism, it is first clear that the top and bottom elements are properly mapped between the two algebras. For what concerns the operations, the results follow naturally using basic properties of the logarithm function.

Let $x, y \in [0,1]$. If $x = 0$, then clearly $\sigma_{\langle a,b\rangle}(x \cdot y) = \sigma_{\langle a,b\rangle}(0) = -\infty = \sigma_{\langle a,b\rangle}(0) + \sigma_{\langle a,b\rangle}(x)$. On the other hand, if both $x, y > 0$, then $\sigma_{\langle a,b\rangle}(x \cdot y) = b \cdot log_a(x \cdot y)$. By the properties of the logarithm, this is equal to $b \cdot (log_a x + log_b y)$ and so to $\sigma_{\langle a,b\rangle}(x) + \sigma_{\langle a,b\rangle}(y)$.

For what respects the $\rightarrow$ operation, consider $x, y \in [0,1]$. If $x \leq y$, then $\sigma_{\langle a,b\rangle}(x \rightarrow_\Pi y) = \sigma_{\langle a,b\rangle}(1) = 0$. On the other hand, since $\sigma_{\langle a,b\rangle}$ is order preserving, we know that $\sigma_{\langle a,b\rangle}(x) \leq \sigma_{\langle a,b\rangle}(y)$ and so $\sigma_{\langle a,b\rangle}(x) - \sigma_{\langle a,b\rangle}(y) = 1$ too. If $x > y$, then by definition we know that $\sigma_{\langle a,b\rangle}(x \rightarrow_\Pi y) = \sigma_{\langle a,b\rangle}(y/x)$. If $y = 0$ then clearly $\sigma_{\langle a,b\rangle}(y/x) = \sigma_{\langle a,b\rangle}(0) = -\infty = \sigma_{\langle a,b\rangle}(y) - \sigma_{\langle a,b\rangle}(x)$. Otherwise, $\sigma_{\langle a,b\rangle}(y/x) = b \cdot log_a(y/x)$. Again, by the properties of the logarithm function, this is equal to $b \cdot (log_a y - log_a x)$ which is the definition of $\sigma_{\langle a,b\rangle}(x) -' \sigma_{\langle a,b\rangle}(y)$. ⊠

With the previous result in mind, it becomes clear how to translate each product component from an ordinal sum to a component formed by $\langle \mathbb{R}^-_\bullet, i\rangle$, where i denotes the index of the original product component. In fact and due to the behaviour of the product algebras, it is enough to translate only the interior of the product components (to the interior of copies of $\mathbb{R}^-_\bullet$), and thus we avoid the problem of having the same element addressed in two ways.[1]

Let $* = \bigoplus_{i\in I}\langle *_i, (b_i, t_i)\rangle$ for I a finite indexes set, $\{*_i\}_{i\in I}$ a set of operations in $\{*_{\text{L}}, *_G, *_\Pi\} \cup \{*_{\text{L}n}, *_{Gn}\}_{n\in\mathbb{Z}, n>1}$, and $\{(b_i, t_i)\}_{i\in I}$ a family of pairwise disjoint open intervals of $[0,1]$. Our objective is to substitute the product fragments (i.e., (b_i, t_i) such that $*_i = *_\Pi$) by copies of the $\mathbb{R}^-_\bullet$ properly positioned. First of all, the universe of the new conjunction operation shall be no longer $[0,1]$, but is given by

$$S = \big([0,1] \setminus \bigcup_{i\in I: *_i = *_\Pi} \{(b_i, t_i)\}\big) \cup \bigcup_{i\in I: *_i = *_\Pi} \{\langle x, i\rangle : x \in \mathbb{R}, x < 0\}$$

The order relation in S is the natural one, understanding that the $\langle x, i\rangle$ elements (for a fixed i) are placed (strictly) in between b_i and t_i Formally the definition is a follows.

Definition 9.5. Let S be the universe defined above and $x, y \in U$. Then x is smaller or equal to y in S ($x \leq_S y$) whenever one of the following cases holds:

- For $x, y \in [0,1]$, then $x \leq y$ with the usual order of the reals
- For $x = \langle z, i\rangle$ and $y \in [0,1]$, then $y \geq t_i$
- For $x \in [0,1]$ and $y = \langle z, i\rangle$, then $x \leq b_i$
- For $x = \langle z_1, i_1\rangle$ and $y = \langle z_2, i_2\rangle$ with $i_1 \neq i_2$, then $t_{i_1} \leq b_{i_2}$
- For $x = \langle z_1, i\rangle$ and $y = \langle z_2, i\rangle$, then $z_1 \leq z_2$ with the usual order of the reals.

[1] Observe that, naturally, the elements b_i of component of these type could either be referred to by $b_i \in [0,1]$, as belonging to the component below, or by $\langle -\infty, i\rangle$ and the same happens for t_i, which could lead to problems in the computation of the problem.

From here, we can define a new conjunction operation $*'$, over the universe S as follows:

$$x *' y := \begin{cases} x * y & \text{if } x, y \in [0,1] \\ min\{x,y\} & \text{if } \begin{cases} x \in [0,1] \text{ and } y = \langle z, i\rangle & \text{or} \\ x = \langle z, i\rangle \text{ and } y \in [0,1] & \text{or} \\ x = \langle z_1, i_1\rangle, y = \langle z_2, i_2\rangle \text{ and } i_1 \neq i_2 \end{cases} \\ \langle z_1 + z_2, i\rangle & \text{if } x = \langle z_1, i\rangle \text{ and } y = \langle z_2, i\rangle \end{cases}$$

Clearly, the corresponding residuated operation is given by:

$$x \to_{*'} y := \begin{cases} 1 & \text{if } x \leq_S y \\ x \to y & \text{if } x >_S y \text{ and } \ x, y \in [0,1] \\ y & \text{if } x >_S y \text{ and } \begin{cases} x \in [0,1] \text{ and } y = \langle z, i\rangle & \text{or} \\ x = \langle z, i\rangle \text{ and } y \in [0,1] & \text{or} \\ x = \langle z_1, i_1\rangle, y = \langle z_2, i_2\rangle \text{ and } i_1 \neq i_2 \end{cases} \\ \langle z_2 - z_1, i\rangle & \text{if } x >_S y \text{ and } \ x = \langle z_1, i\rangle \text{ and } y = \langle z_2, i\rangle \end{cases}$$

We will denote $\mathbf{S}_{*'}$ the BL-algebra defined from these two operations over S, i.e.,

$$\mathbf{S}_{*'} := \langle S, *', \to', 0, 1\rangle$$

Clearly, in this algebra, the interpretation of & is $*'$ and that of $\to$, $\to_{*'}$.

Theorem 9.6. *Let $*$ be a Nice BL Logic conjunction operation. Then $[\mathbf{0},\mathbf{1}]_*$ and $\mathbf{S}_{*'}$ are isomorphic BL-algebras.*

Proof. It is easy to see that we can define an embedding from $[\mathbf{0},\mathbf{1}]_*$ into $\mathbf{S}_{*'}$ that moreover is surjective, by adjusting the proof of Lemma 9.4. It is just necessary to take into account a normalization of the values of the product components before applying the σ mapping defined before (and the identity function as the mapping from the elements outside the products components).

For each $i \in I$ with $*_i = *_\Pi$, define the normalization function $\mathbf{n_i}\colon (b_i, t_i) \to (0,1)$ by $\mathbf{n_i}(x) = \frac{x-b_i}{t_i-b_i}$. Note that for $x, y \in (b_i, t_i)$, it holds that $\mathbf{n_i}(x * y) = \mathbf{n_i}(x) \cdot \mathbf{n_i}(y)$.

Given that in the case of Lemma 9.4 each pair of elements from $(0,1) \times \{z \in \mathbb{R} : z < 0\}$ determines a different isomorphism, it is natural that now each set with one of this pairs for each product component determines different isomorphisms between the two algebras.

Let then P be a set of pairs of values from the product components at each side, i.e., for each $P = \{\langle x_i, \langle -v_{x_i}, i\rangle\rangle : i \in I \text{ with } *_i = *_\Pi, x_i \in (b_i, t_i), v_{x_i} \in \mathbb{R}, v_{x_i} > 0\}$.

We define the function $\sigma_P\colon [0,1] \to S$ by

$$\sigma_P(x) := \begin{cases} \langle -v_{x_i} \cdot log_{\mathbf{n_i}(x_i)} \mathbf{n_i}(x), i\rangle & \text{if } x \in (b_i, t_i) \text{ for some } i \in I \text{ with } *_i = *_\Pi \\ x & \text{otherwise} \end{cases}$$

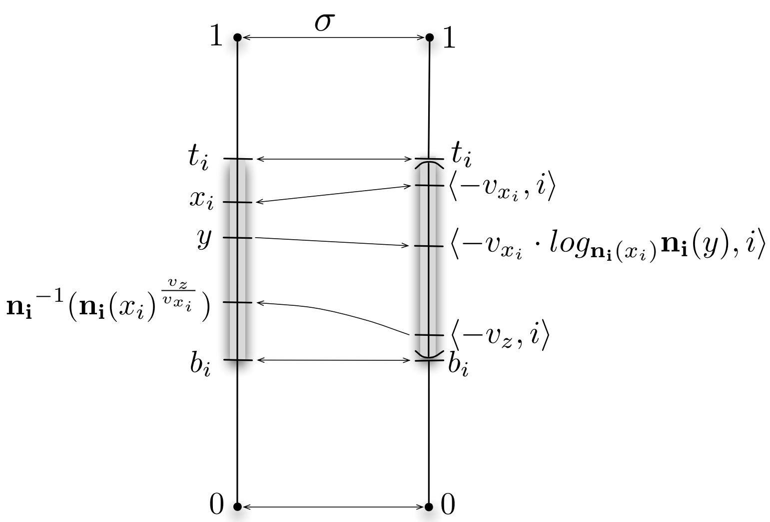


Figure 9.1: Diagram of σ isomorphism over a product component

Following the same reasoning that in the proof of Lemma 9.4, and taking in consideration that outside the product components we have the identity, it is easy to check that σ_P is injective and surjective because the identity function and the logarithm are so. As for proving that σ_P is an homomorphism, the methods coincide with those of Lemma 9.4, taking into consideration the possible types of pairs of elements depending on their position in $[0, 1]$ for what concerns the components.

Observe that the inverse of σ_P is the identity function for all the elements outside a product component and for the rest, it is

$$\sigma_P^{-1}(\langle -v, i\rangle \coloneqq \mathbf{n_i^{-1}}(\mathbf{n_i}(x_i^{\frac{v}{v_{x_i}}})).$$

Figure 9.1 represents the σ isomorphism in both directions, where (b_i, t_i) is a product component.

⊠

From the previous theorem, it is immediate that the logics arising from $[\mathbf{0}, \mathbf{1}]_*$ and those from $\mathbf{S}_{*'}$ coincide, which allows us to use the later algebra in order to compute the solutions.

Rational constants and $\mathbf{\Delta}$ operator

Two different ways to expand at propositional level the previously presented logics are very interesting to point out: truth constants and the Monteiro-Baaz Δ projection operator. Both are natural extensions of the classical fuzzy language and have been widely studied from a theoretical point of view. On the other hand, the applicability, or we could even say necessity, of this enhanced expressibility level from the point of view of applications is clear: it allows to address a particular variable of the system and fix its value to a previously known one (that is, use a constant), or being able to reason differently if a variable is equal to 1 or not (which is reachable using the Δ operator).

For what this work is concerned, it is clear how to work with the Δ operation, since it has a very determined semantic definition (over linearly ordered algebras, which is our case). As explained in Chapter 2, in any BL-chain,

$$\Delta(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

In the case of Nice BL-logics, the two associated algebras we are concerned with (namely $[\mathbf{0},\mathbf{1}]_*$ and $\mathbf{S}_{*'}$) are linearly ordered and so the definition of the Δ operation coincides with the one above in both cases. Moreover, it is clear that the σ and σ' mappings used in the proof of Theorem 9.6 keep behaving as restricted embeddings when the Δ operation is also considered.

The case of working with truth constants and in particular, with rational constants that fall within a product component, was already remarked at the end of Theorem 9.6. Indeed, using constants was the main reason behind the use of the negative cone of the real numbers instead of that of the naturals, which, on the other hand, led to a more constructive proof in order to show the logical equivalence of these two algebras. The key fact in order to understand the treatment of constants in these kind of components is that the function $f(x) = x^k$ is an endomorphism of the standard product algebra $[\mathbf{0},\mathbf{1}]_{\mathbf{\Pi}}$ for any $k \in \mathbb{R}$ with $k > 0$. This means that the deductions over $[\mathbf{0},\mathbf{1}]_{\mathbf{\Pi}}$ extended with rational constants interpreted by its name coincide with those over $[\mathbf{0},\mathbf{1}]_{\mathbf{\Pi}}$ with the constants "moved" arbitrarily (but consistently among them). That is to say, the interpretation of one arbitrary constant $\overline{c}$ can be set to any value (different from the top and bottom elements), and the other ones will take their values depending on it (and on the relation of their names).

As usual, we denote by $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$ the standard algebra of $*$ expanded with rational constant symbols interpreted by its name.

On the other hand, for our purposes, we need not to address a unique rational expansion of $\mathbf{S}_*$ (that is to say, fix a unique interpretation of the rational constants). It is only needed that, while the constants that fall out of any product component maintain their interpretations as rationals from $[0,1]$, the other ones will have a different particular behaviour. Let $C_i = [0,1]_{\mathbb{Q}} \cap (b_i, t_i)$ for $i \in I$.

Definition 9.7. We say that an algebra $\mathbf{S}_{*'}^{\mathbb{Q}}$ is a **rational expansion** of $\mathbf{S}_{*'}$ when $\mathbf{S}_{*'}^{\mathbb{Q}}$ has the form $\langle S, *', \rightarrow_{*'}, 0, 1, \{\overline{c}^{\mathbf{S}_{*'}^{\mathbb{Q}}}\}_{c \in [0,1]_{\mathbb{Q}}}\rangle$ and the following holds:

- For $i \in I$ such that $*_i = *_\Pi$, there is $c_0^i \in C_i$ with $\overline{c_0^i}^{\mathbf{S}_{*'}^{\mathbb{Q}}} = \langle -v^i, i\rangle$ with $v^i \in \mathbb{R}$, $v^i > 0$ and $\overline{d}^{\mathbf{S}_{*'}^{\mathbb{Q}}} = \langle -v^i \cdot log_{\mathbf{n}(c_0^i)}\mathbf{n}(d), i\rangle$ for each $d \in C_i$ (where

$\mathbf{n}$ stands for the normalization function defined in the proof of Theorem 9.6).

- For each d in $[0,1]_{\mathbb{Q}}$ such that $d \notin C_i$ for any $i \in I$ such that $*_i = *_\Pi$, $\overline{d}^{\mathbf{S}^{\mathbb{Q}}_{*'}} = d$.

Now, the addition of constants slightly changes the formulation of Theorem 9.6 and its proof.

Lemma 9.8. $[\mathbf{0},\mathbf{1}]^{\mathbb{Q}}_*$ *is isomorphic to any rational expansion of* $\mathbf{S}_{*'}$*.*

Proof. The proof is analogue to that of Theorem 9.6 in almost all aspects. The only different point is that of not having an embedding for each pair of values of the product components, but now the set of pairs is limited to pairs of constants from each side. For simplicity on the calculus, we will take these constants to be the c^i_0 outlined at the definition of free rational expansion of $\mathbf{S}_{*'}$. Let $\mathbf{S}^{\mathbb{Q}}_{*'}$ be a free rational expansion of $\mathbf{S}_{*'}$ and $P = \{\langle c^i_0, \overline{c^i_0}^{\mathbf{S}^{\mathbb{Q}}_{*'}}\rangle : i \in I \text{ with } *_i = *_\Pi, c^i_0 \in (b_i, t_i) \cap [0,1]_{\mathbb{Q}}\}$ where $\overline{c^i_0}^{\mathbf{S}^{\mathbb{Q}}_{*'}} = \langle -v^i, i\rangle$ for some $v_i \in \mathbb{R}, v_i > 0$.

The rest of the proof coincides with the one of Theorem 9.6 and the only point that needs to be checked is that σ_P is an homomorphism for the constants too, which is immediate.

First, it is clear that σ_P behaves as the identity for all the constants whose name is a rational value out of all the product components, and so they are sent to their interpretation in $\mathbf{S}^{\mathbb{Q}}_{*'}$.

Consider a constant $\overline{c}$ such that $c \in (b_i, t_i)$ where $*_i = *_\Pi$. By definition of σ we know that $\sigma(\overline{c}) = \sigma(c) = \langle -v^i \cdot log_{\mathbf{n_i}(c^i_0)}\mathbf{n_i}(c), i\rangle$. This coincides with the definition of $\overline{c}^{\mathbf{S}^{\mathbb{Q}}_{*'}}$, which concludes the proof. ⊠

As before, this implies that the logic arising from $[\mathbf{0},\mathbf{1}]^{\mathbf{Q}}_*$ coincides with that of any free rational expansion of $\mathbf{S}_{*'}$. It is natural to see that using the inverse of σ, we can translate a particular evaluation over some $\mathbf{S}^{\mathbb{Q}}_{*'}$ to $[\mathbf{0},\mathbf{1}]^{\mathbb{Q}}_*$.

9.2 Some modal expansions

Within this section, we will detail the approach followed towards the implementation of some modal expansion of the Nice BL logics defined in the previous section. Our objective is to design the methods to treat finite structures with a crisp accessibility relation and evaluate over Nice BL logics.

As explained in Chapter 6, the L_* logics do not enjoy in general the finite model property. It is then unclear whether the process to decide if a formula is a theorem (or satisfiable) in L_* is decidable. It was not in the scope of this dissertation to study this problem, but our intuition is that, given the strong relation of these logics with first order fuzzy logics, it is semi-decidable and thus, only in some cases the solver would produce an answer.

Since our main objective was producing an application useful from a practical point of view, we also wondered whether a real-world problem would truly require modal structures with an infinite number of worlds. Thus we decided to restrict the solver to reason over finite Kripke structures, even though these do not fully coincide with those of K_*. Starting from a Nice BL logic L, we will refer to this modal expansion K_L^ω and its definition is given semantically as expected.

Definition 9.9. Let L be a Nice BL-logic. For a finite set of formulas $\Gamma \cup \{\varphi\}$, $\Gamma \vdash_{K_L^\omega} \varphi$ if for any L-valued Kripke model $\mathfrak{M}$ with crisp accessibility relation and a finite set of worlds it hods that $\Gamma \Vdash_{\mathfrak{M}} \varphi$.

To treat the problem of generating a finite model (either to prove that a certain formula is not a theorem, or to prove that a set of equations are satisfiable), we exploit the notion of witness of a modal formula in a world. This is, the existence of a particular world among the successors of the of the original one where the effective value of the modal formula on the original world is taken by the non-modal version of the formula. This clearly exists, since working over a finite model, the definitions of the $\Box$ and $\Diamond$ operations now become *min* and *max* of a set of values.

Formally and in order to get a clear design of the application and low computing times, for a given formula (or equivalently, a finite set of formulas) we generate a Kripke frame with some attached information, which will generate all possible models in terms of the evaluation of the formula in a certain world. This allows to quickly generate a counter-model for a formula that is not a theorem (and also for a set $\Gamma \cup \{\varphi\}$ such that $\Gamma \not\models_{K_L^\omega} varphi$.

The construction of this frame is similar to that done by Hájek in [73] in the context of fuzzy description logics. For our purposes, we will define it here in a purely modal way. It is based on the decomposition of the formulas up to modal level, which are the elements that end up determining the structure and complexity of this general frame.

Definition 9.10. Let φ be a formula. The set of **propositional subformulas** of φ, $\mathrm{PS}(\varphi)$ is inductively defined by:

$$\begin{array}{rcl}
\mathrm{PS}(p) & := & \{x\} \text{ for } p \text{ being a propositional variable or a constant symbol} \\
\mathrm{PS}(\mathtt{M}\psi) & := & \{\mathtt{M}\psi\} \text{ for } \mathtt{M} \in \{\Box, \Diamond\} \\
\mathrm{PS}(\psi \,\&\, \chi) & := & \mathrm{PS}(\psi) \cup \mathrm{PS}(\chi) \\
\mathrm{PS}(\psi \to \chi) & := & \mathrm{PS}(\psi) \cup \mathrm{PS}(\chi) \\
\mathrm{PS}(\Delta\psi) & := & \mathrm{PS}(\psi)
\end{array}$$

Using the previously defined set and taking into account that any formula has a finite number of subformulas (in the usual sense of the word), it is possible to generate recursively the structure commented above. It will consist on a tree where all the worlds except for the root one are pairs of the form $\langle n, \mathtt{M}\psi\rangle$ with $n \in \mathbb{N}$, $\mathtt{M} \in \{\Box, \Diamond\}$ and $\varphi \in Fm$. The $\mathtt{M}\psi$ modal formula associated to each world indicates that the value of $\mathtt{M}\psi$ at the father of the world (it is only one, since it is a tree) coincides with the value of ψ in the current world.

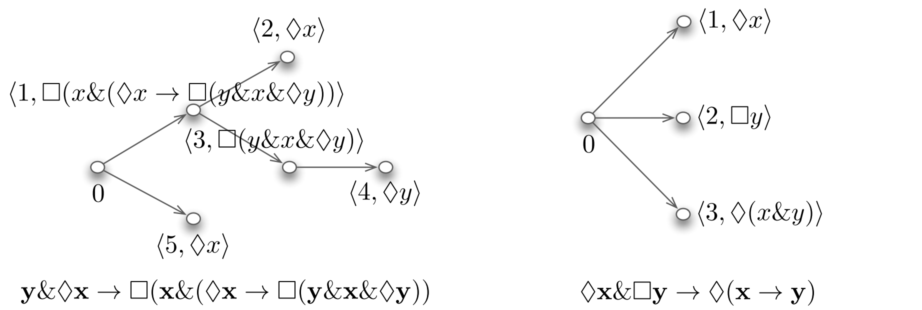


Figure 9.2: Examples of the Skeleton tree

The following algorithm shows how this structure can be constructed recursively, starting from a structure with only a root note $S = \{\langle 0, \emptyset\rangle\}$, the index of that (root) world labelled by 0 and from an empty set of accessibility relations $R = \emptyset$. We will write mod(χ) to the denote the function that returns true if χ is of the form $\mathtt{M}\psi$ and false otherwise. Also, for a set of worlds as the above one, we will denote by $last(S)$ the greatest index in the worlds from S.

Listing 9.1: Skeleton(Formula S R f)

```
NewS = S
NewR = R
MPS(Formula) = {g in PS(Formula) such that mod(g)}
if MPS(Formula) is empty, return S, R

otherwise do:
  for each g in MPS(Formula) do:
    newIndex = last(NewS)+1
    NewS = NewS \cup \langle newIndex, g \rangle
    NewR = NewR \cup \langle f, newIndex \rangle
    NewS, NewR$ = Skeleton(g NewS NewR newIndex)
```

Figure 9.2 gives some examples of this construction.

This algorithm generates a finite tree for a given formula φ, with maximum depth given by the maximum number of nested modalities in φ ($MMD(\varphi)$). Following the algorithm, it is easy to check that that the size of the skeleton tree (i.e., the number of worlds of the frame) is given by the amount of modal operators appearing in the formula. Indeed, a new world is only added for each formula beginning by $\Box$ or $\Diamond$.

We can see that this skeleton is enough in order to "witness" the value of a formula in a particular world from an arbitrary model.

Lemma 9.11. *Let $\mathfrak{M} = \langle W, R, e\rangle$ be a finite $[\mathbf{0},\mathbf{1}]_*$-kripke model, $w \in W$ and φ a formula. Then, there is a model $\mathfrak{Sk} = \langle W_{sk}, R_{sk} e'\rangle$ where $\langle W_{sk}, R_{sk}\rangle$ is a restriction of the Skeleton(φ) frame and*

$$e(w, \varphi) = e'(0, \varphi)$$

(where 0 *is the root of the skeleton tree).*

Proof. Since W is finite, we know the model is witnessed. This implies that the set $WS(\mathfrak{M}, w, \mathtt{M}\psi) = \{v \in W \text{ such that } Rwv \text{ and } e(w, \mathtt{M}\psi) = e(v, \psi)\}$ is empty if and only if it does not exist any $v \in W$ with Rwv. Then, define the function

$$wit(\mathfrak{M}, w, \mathtt{M}\psi) = \begin{cases} \emptyset & \text{if } WS(\mathfrak{M}, w, \mathtt{M}\psi) = \emptyset \\ \text{any } v \in WS(\mathfrak{M}, w, \mathtt{M}\psi) & \text{otherwise} \end{cases}$$

For practical reasons, we will define $wit(\mathfrak{M}, \emptyset, \mathtt{M}\psi) = \emptyset$. [2]

It is now easy to formalize the strong relation between the worlds in W_{sk} and those chosen by the *wit* function. Let σ be the mapping from W_{sk} into $W \cup \{\emptyset\}$ defined by:

$$\begin{aligned} \sigma(0) &= w \\ \sigma(\langle v, \mathtt{M}\psi\rangle) &= wit(\mathfrak{M}, \sigma(father(\langle v, \mathtt{M}\psi\rangle)), \mathtt{M}\psi) \end{aligned}$$

Now, consider the restriction of $Skeleton(\psi)$ to the universe $W' = \{w \in W_{sk} : \sigma(w) \neq \emptyset\}$, and let e' be the evaluation over this frame given by

$$e'(w, x) = e(\sigma(w), x)$$

It is clear that $\sigma(w) \in W$ for all $w \in W'$. Moreover, observe that $wit(\mathfrak{M}, \sigma(v), \mathtt{M}\psi) = \emptyset$ if and only if $\sigma(v)$ has no successors in $\mathfrak{M}$. Moreover, $wit(\mathfrak{M}, \sigma(v), \mathtt{M}\psi) = \emptyset$ implies that $\sigma(w) = \emptyset$ for any $w \in W_{sk}$ such that $R_{sk}vw$. From here it follows that for any $v, w \in W'$ such that $R'_{sk}vw$, $R\sigma(v)\sigma(w)$ (in $\mathfrak{M}$). In particular, it is interesting to remark that for any $v \in W'$, $wit(\mathfrak{M}, \sigma(father(\langle v, \mathtt{M}\psi\rangle)), \mathtt{M}\psi) \neq \emptyset$.

Now, for any $\langle v, \mathtt{M}\psi\rangle \in W'$ we can prove the following chain of equalities

$$\begin{aligned} e'(\langle v, \mathtt{M}\psi\rangle, \psi) = e(\sigma(\langle v, \mathtt{M}\psi\rangle), \psi) &= e(wit(\mathfrak{M}, \sigma(father(\langle v, \mathtt{M}\psi\rangle)), \mathtt{M}\psi), \psi) = \\ e(\sigma(father(\langle v, \mathtt{M}\psi\rangle)), \mathtt{M}\psi) &= e'(father(\langle v, \mathtt{M}\psi\rangle), \mathtt{M}\psi) \end{aligned}$$

In particular, this implies that $e'(0, \varphi) = e(w, \varphi)$, concluding the proof. ⊠

Having a structure defined in the above way, we can see that the computational complexity of the problem of determining whether a formula is true in all the finite $[\mathbf{0}, \mathbf{1}]^{\mathbb{Q}}_{*}$-Kripke models is of the same order than in the propositional case.

[2] If $\emptyset \in W$, consider in the whole proof the model $\mathfrak{M}$ with the world $\emptyset$ renamed to $\emptyset'$.

Chapter 10

mNiBLoS: modal Nice BL Logics Solver

To the best of our knowledge, little attention has been paid to the development of a generic solver for systems of mathematical fuzzy logic. Some theoretical works about computations (mainly for L logic) have been presented and there is also an important number of studies on complexity and proof theory on fuzzy logics too, but a working application that copes with the most common fuzzy logics has still not been implemented. We consider this is a problem that limits the use of fuzzy logics in real applications, mainly in the field of Artificial Intelligence, where several theoretical projects that resort to these logics can been found.

In [6], a new approach for implementing a theorem prover for L, Gödel and Product fuzzy logics using Satisfiability Modulo Theories was proposed. The idea of using Satisfiability Modulo Theories solvers for this problem was new and the results were interesting for L and Gödel because the results were optimistically efficient. Also, the modularity this approach enjoys allows to cope with several fuzzy logics. However, in the case of Product logic, the results were far from being satisfactory. In [121] a solver for the whole family of logics based on a continuous t-norm was presented and the particular case of the product logic was strongly enhanced using theoretical equivalences for reasoning over an equivalent but less hard -computationally talking- algebra that the original one. For what respects modal expansions of these logics, almost no work seems to have been developed. Some notes towards the implementation of a solver (that works with the problem of positive satisfiability) for product description logic (a fragment of product modal logic) over the standard product algebra can be found in [5]. However, the work presented there uses the possibility of reducing a problem of positive satisfiability to the satisfiability of a quantifier free boolean formula, which is not proved to work in more general cases. Moreover, the resulting formula has non-linear real arithmetic properties, making it very challenging from an efficiency point of view.

In the current chapter, we provide details on the theoretical basis necessary for the extension of Ansótegui et al.'s work. We present an application that is able to solve problems involving reasoning over a wide family of BL-chains, namely the Nice BL logics we defined in the previous chapter, and also supports modal operators whose semantic behaviour will be determined by finite crisp Kripke structures (evaluated over Nice BL chains). We present some implementation details and the obtained results of a new solver which is a generalization of the one shown in [121], integrating more efficient ways of reasoning for the product logic in the cases where this can be done (i.e., whenever there is a product component on the definition of the t-norm).

10.1 What does mNiBLoS do?

The software application presented here, mNiBLoS, performs, over a certain modal logic, either the task of checking whether a given formula is a theorem (valid for all possible evaluations over all possible finite frames), whether a pair formed by a sequence of premises and a formula hold in the logical consequence relation in the same sense that above, or finally, whether a given set of equations is locally satisfiable (i.e., in a world of a structure) and provides a model for this if the answer is affirmative. Explicitly, as a concise presentation of the solver, its use can be summarized in terms of the inputs and outputs through the following lines:

- First, a nice BL logic L is chosen. The exact format is the one specified in the piece of code 10.1
- Then, a particular task **op** to execute must be specified by the user. This determines the remaining arguments of the program.
  - With **op** =`th`, the specified task is theoremhood checking. A formula will be asked and after the execution, the program will show a message confirming whether the formula is or not a theorem in the logic L.
  - With **op** =`d`, the specified task is logical consequence checking (`c`). A formula and a set of formulas (premises) shall be given. After the execution, the program will show a message confirming whether the formula is or not a logical consequence of the premises set in the logic L.
  - With **op** =`s`, the specified task is satisfiability checking. A set of equations is asked to the user and the result of the execution will be a message confirming whether the set of equations is locally satisfiable or not in the logic L and if it is so it will show a model (the relation matrix and the assignment of each variable in each world of the models) that satisfies it in world #0.

The notation in which the formulas and equations are asked for is the Polish prefix notation (which is much simpler to treat internally), using the variables, constants, operations and relation symbols formally described in the piece of code 10.2.

10.2 Design of the application

In this work we generalize the idea presented in [6], exploiting results presented on Chapter 9 to expand the solver tasks and to enhance its performance. The metodology behind the design of the application is based in the possibility of defining the strong conjunction as the ordinal sum of the basic cases (Łukasiewicz, Gödel and product ones) specified over certain intervals, as detailed in 2.3. Moreover, while the BL logic was not considered in [6], it is not hard to include in our reasoner, making use of Theorem 9.1: we can reduce reasoning over BL, when dealing with a particular set of formulas, to work over the logic defined with the t-norm given by an ordinal sum of $(n+1)$ Łukasiewicz components, where n is the number of different variables in the set of formulas involved.

To implement the ordinal sum as defined above we initially considered the classical definition of the three basic t-norms, Łukasiewicz, Gödel and Product. Realizing the slow behaviour of the product logics, due to its non-linear nature, we developed some theoretical results that allow us to obtain a more efficient encoding (see Theorem 9.6).

As a negative effect of doing this change, there exist some potential problems with the treatment of the constant symbols from the product components. This is due to the fact that it is problematic from a computational point of view, to use the logarithmic function, which is a keystone in the construction of the isomorphism from Theorem 9.6. Computer applications are not able to instantiate all the real numbers for a simple reason it is not possible to provide a finite codification for them. Since it is not known whether if for two arbitrary a, b, $log_a b$ is rational or not, this takes us to a problematic point. While the theory tells how to move $[\mathbf{0},\mathbf{1}]_{\mathbf{\Pi}}$ to $\mathbb{R}^-_\bullet$ and vice-versa (that is, the isomorphism from Theorem 9.8 and its inverse), if there exist several constant symbols specified by the user in the same product component, the codification of the book-keeping axioms in $\mathbb{R}^-_\bullet$ (that is to say, point one from Definition 9.7) is not possible in general. This happens whenever that logarithm is not a rational -or it is a rational with more than 16 decimals, the maximum sensibility of Python, our programming language-. We though of several options to overcome this obstacle, all passing through giving to the user a minimal inexactness or imposing some limitations in the set of constants allowed. We comment the possible solutions here, with their respective pros and cons, and justify the chosen one.

- Imposing to the user to provide a "seed" within each product component with constants and then expressing the constants as any rational number, which will be the power to which the seed is elevated to in order to get the real value. **Pros:** No inexactness can happen (the logarithm in base "seed" of each constant is a rational number, easy to deal with). **Cons:** It imposes a limitation on the language allowed and, overall, it forces the users to do a very complex and quite unnatural codification of the constants.

- Looking for the previously commented seed. **Pros:** Again, no inexactness can happen. **Cons:** This process is not even decidable (this seed can not exist), and for some tested cases with solution, takes a really long time even with only a pair of values.

- Forcing a slight modification (of the order of 10^{-5}) the constants given by the user, in order to get the previous value. **Pros:** There is no inexactness in the reasoning *over the new constants*. **Cons:** We modify the user data and to a larger extent to that of the following point.

- **The chosen option.** Again, forcing a slight modification of the values given by the user (except of one, c_0), but in order to get that the logarithm in base c_0 of these new values are a rational numbers. The modification in this case depends on the sensibility of the programming language: the value given by the user and the one used in the reasoner will be indiscernible in the used language. In our case, using python, we have a sensibility of 16 decimal numbers.**Pros:** As before, There is no inexactness in the reasoning *with the new constants*. **Cons:** As before, we modify the user data, but in a very slightly way.

With our codification, there exists the possibility of getting wrong results, for cases when `z3` values a variable in a wrong interval, that is, somewhere between the real value and the approximated one. It is an open work to fix this problem: since the sensibility allowed to the user constants is, again, that of python, it is possible to make use of a discrete universe (instead of that of $\mathbb{R}^-_\bullet$) with step equal to the minimum distance expressible in python (that is, 10^{-16}). This would fix the uncontrolled results and we think is the highest accuracy that can be achieved with a computer application. However, some tests have resulted in highly inefficient systems and so they are not a satisfactory solution.

Finally, from a purely technical point of view, the theory from SMT we have used in mNiBLoS includes also the theory of the arrays, which allow a cleaner design of the application. First, arrays serve the purpose of storing the values of the product components (that is, the natural number associated to the index of the product component and the negative real value of the variable). Moreover, when dealing with the modal expansions, the use of arrays simplifies the practical approach and we can define the modal variables as lists (of length determined by the number of worlds created with the Algorithm 9.1), each item of the list being the variable in the world determined by that index. Even though, being this list of finite length, this is not the only way to deal with the problem, we though it was the clearest one and also, one of the fastest, since SMT is optimized to deal with the theories it has implemented (while, on the other hand, we are no experts on theory of arrays and data accessibility).

10.3 Use of mNiBLoS

By now, the interface of our solver is limited to command lines. This allows a simpler codification and a more direct access to the options of the solver. Moreover, intensive and automated executions can be easily coded. The development of a graphical interface to make the application more accessible and user-friendly is left for future work.

Pre execution

The program is implemented in python and calls the SMT-solver `z3` developed by Microsoft Research, and it is meant to be used from the terminal or command line.

Before its use, a configuration file should be filled by the user. Since the SMT solver is internally used, in file
`configuration.py` the line

```
Z3_LOCATION = "/.../z3"
```

must be modified to meet the user's `z3` folder. It shall have the relative (to the reasoner main folder) or the absolute path in the user's computer to the `z3` solver general folder, obtained after downloading and decompressing the `z3` solver ([130]).

Inputs

From the main solver, the application is called as

```
>python mniblos.py
```

The previous command starts the interactive application, that is, the arguments are asked to the user in order and help messages can be generated (with the form of the arguments required). It is important to remark that we have no implemented error control for what concerns the inputs from the used. This means that if the inputs are not properly given, i.e., following the specifications we remark, the results will be uncontrolled (and, in most cases, the program will crash).

First, mNiBLoS asks for the logic **L** to be used. The following help message can be reached writing `h`, which contains all the information concerning the form of **L**:

Listing 10.1: help on logic formatting message

```
The logics accepted by mNiBLoS have two main possible formats:
    * bl  (to indicate Hajek's Basic Logic)
      * (b1, t1)v1+(b2,t2)v2+...+(bn, tn)vn (to indicate the ordinal
           sum of the logics vi in the intervals (tk, t(k+1)) where:
         - bi,ti are rational values in [0,1] with bi < ti and ti
             <= b(i+1)
         - vi are of the following form:
               l (for denoting a Lukasiewicz component)
             ln with n natural number >= 2 (for denoting an n-
                 valued Lukasiewicz component)
             g (for denoting a Godel component)
             gn with n natural number >= 2 (for denoting an n-
                 valued Godel component)
             p (for denoting a Product component)
```

Note that if the logic `bl` is selected, it will be internally computed as $n+1$ (equally distributes in $[0,1]$) copies of the Łukasiewicz t-norm, for n being the number of variables appearing in the formulas considered (following Corollary 9.1). Thus, the use of rational constants in this case must be careful: the user must be aware of the kind of logic used internally and to which extent two constants belong to the same component or not.

After specifying the logic, the user is asked for the task that will be done over **L**. That is, he must choose between `th`, `d` or `s` to select the operation, corresponding each one of these to, respectively, theoremhood proving of a formula, checking if a formula can be deduced from a set of premises or looking for a model that locally satisfies a set of equations.

Depending on the option selected by the user, the rest of the program will be slightly different, starting from asking for different data. In any case, the rest of the data corresponds to formulas or equations of the logic. The following message with the specification of the notation of the formulas and equations accepted by mniblos can be obtained writing `h`.

Listing 10.2: help on formulas formatting message

```
The format of the formulas accepted  by mNiBLoS is as follows:
    1) The variables must be named by xi, where i are natural
        numbers
  2) The constants can be either rational numbers in (0,1)
  3) The formulas are built inductively from variables and constant
      symbols as follows:
    *The notation must be Polish (prefix) and in parenthesis, i.e.,
        (operation arg1... argn) for an n-ary operation and argi
        valid formulas
    * The possible operations with their corresponding arity and
        usual meaning is the following:
      - wcon/2 (weak conjunction = minimum)
      - wdis/2 (weak disjunction = maximum)
      - con/2 (conjuntion = t-norm)
      - impl/2 (implication = residuum)
      - neg/1 (negation = (impl x 0))
      - delta/1 (Monteiro-Baaz Delta)
      - box/1 (modal box operation)
      - diamond/1 (modal diamond operation)
The equations (for the satisfiability operation) must be of the form
     (R form1 form2), where
          * form1 and form2 are formulas in the previous format
        * R is a relation operation among the following ones:
            - eq (equal)
            - leq (less or equal)
            - l (strictly less)
            - ge (greater or equal)
            - g (strictly greater)
```

If the operation selected by the user is `th`, only one formula is asked for. In case the operation is `d`, mniblos asks for a non-empty set of formulas which will conform the premises set and for a final formula that will play the role of the consequence of the deduction. Finally, if the operation selected is `s`, the user needs to provide mniblos with a non-empty set of equations.

In order to easily allow the execution of the solver from a bash script (or other program), it is also possible to include all the previously commented parameters as arguments of the program. That is, it is also accepted the following format for executing mniblos (when the formatting of each argument is as detailed above).

non-interactive mniblos

```
>python mniblos.py L op dataOperation
```

where `L` is the Nice BL logic, `op` is the operation to perform and dataOperation is, depending on `op`:

- If `op` is `th`, `dataOperation` must be a formula
- If `op` is `th`, `dataOperation` is of the form `[prem1,..., premn, cons]`, where `premi` are formulas (for each $1 \leq i \leq n$) which will denote the premises set and `cons` is a formula referring to the consequence.
- If `op` is `th`, `dataOperation` is of the form `[eq1,..., eqn]`, where `eqi` are equations.

The generated SMT-code

Running the program with the desired options, an auxiliary output file named `smtcode` will be generated in the `z3codes` folder. This file will contain the necessary SMT-code to call the `z3` solver over it and get the desired answer.

The generated file will be different depending on all the attributes given. We will sketch here the most remarkable parts.

The first part of the file will be the same for all tasks and logics. In the case of working with the option to generate a model, an extra line

```
(set-option :produce-models true)
```

will be added in the head of the file to specify that we are willing to get the models in `z3`.

The first section of code begins by the definition of the *propositional variable's type* with its corresponding access methods (value and component of each variable) and the definition of the top and bottom propositional constants. Observe that we fix the component of the non-product components to -1 (i.e., `(  1.0))`, in order to simplify further codifications. Below, the order relations are specified between pairs of these variables (implementing the order definition from 9.5 and so depending deeply on the product components appearing in the logic). Finally, we conclude this "propositional" part by defining the operations over propositional variables. The common ones, minimum and maximum, that directly refer to the order relations previously defined and then, depending on the logic, the more specific definitions of the conjunction, the implication and the negation, that resort to the theoretical definition of the algebra $\mathbf{S}_{*'}$ described in 9.

The following code illustrates a simple example of this code, for the particular logic $\mathbf{L} =$ `(0.1, 0.4)l+(0.4,0.7)p+(0.9,1)l4.` [1]. [2]

[1]For a more easy comprehension of the following code, observe that the `(ite` sentence stands for `if... then... else..` and that `z3` is a programming laguage that relies on the Polish prefix (or polish) notation

[2]The indentation is here optimized for the comprehension of the code. Since the `z3` is not intentation-sensible, this code is to all effects equal to the one generated by mniblos.

```
;propositional variable type definitions
(define-sort pvar () (Array Int Real))
(define-fun component ((x pvar)) Real
  (select x 1))
(define-fun value ((x pvar)) Real
    (select x 2))
(define-fun setComponent ((x pvar) (comp Real)) pvar
    (store x 1 comp))
(define-fun setValue ((x pvar) (val Real)) pvar
    (store x 2 val))

;truth const
(declare-const pT pvar)
(assert (and (= (component pT) (~ 1.0)) (= (value pT) 1.0)))
(declare-const pF pvar)
(assert (and (= (component pF) (~ 1.0)) (= (value pF) 0.0)))

;;RELATIONS
;eq(x,y) T if x = y, F else
(define-fun eq ((x pvar) (y pvar)) Bool
  (and (= (component x) (component y)) (= (value x) (value y))))

;g (x,y) true if x > y, false otherwise
(define-fun g ((x pvar) (y pvar)) Bool
    (ite (= (component x) (component y))
        ;same component
        (< (value y) (value x))
        ;values in different components
        ;non-prod, prod
        (ite (and (= (component x) (~ 1.0)) (> (component y) (~ 1.0)
            ))
            (ite (= (component y) 1.0)
                (<= 0.7(value x))
                ;this case in never reached
              false)
            ;prod, non-prod
            (ite (and (> (component x) (~ 1.0)) (= (component y) (~
                1.0)))
                (ite (= (component x) 1.0)
                  (<=  (value y) 0.4)
                    ;this case in never reached
                false)
              ;prod-prod
              (< (component y) (component x))))))

;geq (x,y) true if x >= y, false otherwise
(define-fun geq ((x pvar) (y pvar)) Bool
  (or (eq x y) (g x y)))

;l (x,y) true if x < y, false otherwise
(define-fun l ((x pvar) (y pvar)) Bool
  (not (geq x y)))

;leq (x, y) true if x <= y, false otherwise
(define-fun leq ((x pvar) (y pvar)) Bool
  (not (g x y)))
```

```
;;OPERATIONS
;pmin(x,y)
(define-fun pmin ((x pvar) (y pvar)) pvar
  (ite (leq x y) x y))

;pmax(x,y)
(define-fun pmax ((x pvar) (y pvar)) pvar
  (ite (geq x y) x y))

;weak conjunction = pmin (x, y)
(define-fun pwcon ((x pvar) (y pvar)) pvar
  (pmin x y))

;weak disjunction pmax (x, y)
(define-fun pwdis ((x pvar) (y pvar)) pvar
  (pmax x y))

;delta pdelta = 1 iff x = 1, 0 oth.
(define-fun pdelta ((x pvar)) pvar
  (ite (= (value x) 1.0) pT pF))

;pcon(x, y)
(define-fun pcon ((x pvar) (y pvar)) pvar
  (ite (and (<= 0.1 (value x)) (<= 0.1 (value y))
       (<= (value x) 0.4) (<= (value y) 0.4))
        ;they are in the same luk-component
        (setValue x
            (ite (<= 0.1 (- (+ (value x) (value y)) 0.4))
                (- (+ (value x) (value y)) 0.4)
                0.1))
        (ite (and (= (component x) 1.0) (= (component y) 1.0))
            ;they are in the same prod-component
            (setValue x (+ (value x) (value y)))
            (ite (and (<= 0.9 (value x)) (<= 0.9 (value y))
                 (<= (value x) 1.0) (<= (value y) 1.0))
                ;they are in the same luk-component
                 (setValue x
                 (ite (<= 0.9 (- (+ (value x) (value y)) 1.0))
                  (- (+ (value x) (value y)) 1.0) 0.9))
                  ;different components
                    (pmin x y)))))

;pimpl(x y) -residuum-
(define-fun pimpl ((x pvar) (y pvar)) pvar
  (ite (leq x y)
    pT
    (ite (and (<= 0.1 (value x)) (<= 0.1 (value y))
         (<= (value x) 0.4) (<= (value y) 0.4))
          ;they are in the same luk-component
            (setValue x (+ 0.4 (- (value y) (value x))))
        (ite (and (= (component x) 1.0) (= (component y) 1.0))
            ;they are in the same prod-component
            (setValue x (- (value y) (value x)))
            (ite (and (<= 0.9 (value x)) (<= 0.9 (value y))
                 (<= (value x) 1.0) (<= (value y) 1.0))
                ;they are in the same luk-component
```

```
                  (setValue x (+ 1.0 (- (value y) (value x))))
                  ;different components
                  y)))))

;negation (pneg x = x -> 0)
(define-fun pneg ((x pvar)) pvar
  (pimpl x pF))
```

The next section of the generated SMT-code comprehends the definitions and methods associated with the modal expansion of the logic given by the user. The modal extent of the codifications is given by the structure generated by the modal skeleton (see Algorithm 9.1) generated by all the formulas in the data corresponding to the operation. That is to say, if the operation is `th`, the skeleton associated to the formula; if the operation is `d`, the union by the root of the skeletons of all the formulas from the premises set and the consequence; and if the operation is `s`, similarly, the union by the root of all the formulas involved in the set of equations. Among other things, this implies that even if there are no modal operations in the formulas, the solver will work with a modal structure, which will only have one world (the root).

At the beginning of the section we have defined the variable's type, and the access functions to get the value of the variable on a certain world of the structure. Similarly to the propositional level, we define the top and bottom constant elements, as variables that are evaluated to 1 or 0 in all the worlds of the structure. Next, the modal version of the previously defined propositional operations is given, by just calling to the respective propositional version component-wise for all the effective values of the variable. That is, the number of worlds, which we know after running the skeleton algorithm.

It is noticeable that `z3` does not allow the use of loops, or even the definition of recursive functions and so the length of the variables must be explicitely given in each operation that concerns it. In the previously commented operations, while the definition would be nice and clean using recursion, we have had to resort to the case-by-case definition that generates a different code depending on such length.

Later on, we have defined the real modal operations. For that, first we have defined the accessibility square matrix `R` of boolean elements, which is of length equal to the number of worlds of the skeleton structure in each side. Then, the values on coordinates $\langle i, j\rangle$ for which $\langle i, j\rangle$ was not in the accesibility relation of the skeleton structure are set to false. This way, any assignation of `R` is a restriction of the skeleton structure and so we will be in the premises of Lemma 10.3. Later on, some methods related with the definition of the `box` and `diamond` operations are defined (those that get the minimum/maximum of the list of values corresponding to the successors of a certain world, those setting respectively to 1/0 the values associated to non-related worlds) and from these, the final definitions of the modal operators are given.

While the part of code associated with the propositional level commented above depended on the logic, this part depends on the skeleton structure generated. The following is the codification associated with the formula `(con (box (con x2 x1)) (box (con x1 (diamond x2))))`, whose skeleton is built over the frame

$$\langle\{0, 1, 2, 3\}, \{\langle 0, 1\rangle, \langle 0, 2\rangle, \langle 2, 3\rangle\}\rangle$$

Listing 10.3: Modal level STM-code

```
;modal types and functions
(define-sort var () (Array Int pvar))
```

```
(define-fun valueInWorld ((x var) (world Int)) pvar
  (select x world))

;truth constants
(declare-const T var)
(assert (= (valueInWorld T 0) pT))
(assert (= (valueInWorld T 1) pT))
(assert (= (valueInWorld T 2) pT))
(assert (= (valueInWorld T 3) pT))
(declare-const F var)
(assert (= (valueInWorld F 0) pF))
(assert (= (valueInWorld F 1) pF))
(assert (= (valueInWorld F 2) pF))
(assert (= (valueInWorld F 3) pF))

;auxiliar var to generate the results
(declare-const results var)

;weak conjunction
(define-fun wcon ((x var) (y var)) var
  (store (store (store (store results
            3 (pwcon (select x 3) (select y 3)))
          2 (pwcon (select x 2) (select y 2)))
       1  (pwcon (select x 1) (select y 1)))
    0  (pwcon (select x 0) (select y 0))))

;weak disjunction
(define-fun wdis ((x var) (y var)) var
  (store (store (store (store results
            3 (pwdis (select x 3) (select y 3)))
          2  (pwdis (select x 2) (select y 2)))
       1  (pwdis (select x 1) (select y 1)))
  0  (pwdis (select x 0) (select y 0))))

;delta
(define-fun delta ((x var)) var
  (store (store (store (store results
              3 (pdelta (select x 3)))
             2  (pdelta (select x 2)))
        1  (pdelta (select x 1)))
  0  (pdelta (select x 0))))

;con(x, y)
(define-fun con ((x var) (y var)) var
  (store (store (store (store results
            3 (pcon (select x 3) (select y 3)))
         2  (pcon (select x 2) (select y 2)))
      1  (pcon (select x 1) (select y 1)))
    0  (pcon (select x 0) (select y 0))))

;impl(x y) -residuum-
(define-fun impl ((x var) (y var)) var
  (store (store (store (store results
            3 (pimpl (select x 3) (select y 3)))
         2  (pimpl (select x 2) (select y 2)))
      1  (pimpl (select x 1) (select y 1)))
    0  (pimpl (select x 0) (select y 0))))
```

```
;negation (neg x = x -> 0)
(define-fun neg ((x var)) var
  (store (store (store (store results
             3 (pneg (select x 3)))
          2  (pneg (select x 2)))
       1  (pneg (select x 1)))
    0  (pneg (select x 0))))


;MODAL OPERATIONS

;accessibility relation matrix
;the values will meet conditions from [skeleton]
(declare-fun R () (Array Int (Array Int Bool)))
;some relations are always false from the skeleton construction:
(assert (not (select (select R 0) 0)))
(assert (not (select (select R 0) 3)))
(assert (not (select (select R 1) 0)))
(assert (not (select (select R 1) 1)))
(assert (not (select (select R 1) 2)))
(assert (not (select (select R 1) 3)))
(assert (not (select (select R 2) 0)))
(assert (not (select (select R 2) 1)))
(assert (not (select (select R 2) 2)))
(assert (not (select (select R 3) 0)))
(assert (not (select (select R 3) 1)))
(assert (not (select (select R 3) 2)))
(assert (not (select (select R 3) 3)))

;; auxiliary functions

;box related functions

;given a var and an index i, returns a list of reals of length
    nworlds (i.e., a var) with
;1s in the possitions p where R[p][i] = false and the original value
     of the variable otherwise
(define-fun tfunc ((x var) (orig Int)) var
  (store (store (store (store results
             3 (ite (not (select (select R orig) 3))
                pT
                (select x 3)))
          2 (ite (not (select (select R orig) 2))
                pT
                (select x 2)))
        1 (ite (not (select (select R orig) 1))
                pT
                (select x 1)))
     0 (ite (not (select (select R orig) 0))
            pT
            (select x 0))))

;returns the minimum of a list of reals of lentgh nworlds
(define-fun minList ((x var)) pvar
  (pmin (pmin (pmin (select x 0) (select x 1)) (select x 2)) (select
       x 3)))
```

```
;minSuccessors
;takes a variable and an index and returns the corresponging pVar
    with the value of the minimum
;over the successors of the world given by the index
(define-fun minSuccessors ((x var) (i Int)) pvar
  (minList (tfunc x i)))

;box
(define-fun box ((x var)) var
  (store  (store  (store  (store  results
             3 (minSuccessors x 3))
           2 (minSuccessors x 2))
       1 (minSuccessors x 1))
    0 (minSuccessors x 0)))

;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;;
;diamond functions

;given a var and an index i, returns a list of reals of length
    nworlds (i.e., a var) with
;0s in the positions p where R[p][i] = false and the original value
    of the variable otherwise
(define-fun bfunc ((x var) (orig Int)) var
  (store (store (store (store  results
            3 (ite (not (select (select R orig) 3))
                pF
                (select x 3)))
          2 (ite (not (select (select R orig) 2))
              pF
              (select x 2)))
       1 (ite (not (select (select R orig) 1))
                pF
                (select x 1)))
    0 (ite (not (select (select R orig) 0))
       pF
       (select x 0))))

;returns the maximum of a list of reals of lentgh nworlds
(define-fun maxList ((x var)) pvar
  (pmax (pmax (pmax (select x 0) (select x 1)) (select x 2)) (select
       x 3)))

;maxSuccessors
;takes a variable and an index and returns the corresponging pVar
    with the value of the maximum
;over the successors of the world given by the index
(define-fun maxSuccessors ((x var) (i Int)) pvar
  (maxList (bfunc x i)))

;diamond
(define-fun diamond ((x var)) var
  (store  (store  (store  (store  results
             3 (maxSuccessors x 3))
           2 (maxSuccessors x 2))
       1 (maxSuccessors x 1))
   0 (maxSuccessors x 0)))
```

Within a third part of the generated SMT code we have first included the specific instantiations of the variables appearing in the formulas/equations and the definition of the constant symbols. For the first ones, the values allowed for each variable in each world are determined by the logic (in the sense of definition $\mathbf{S}_*$ from Chapter 9, i.e., propositional variables with component equal to `(   1.9)` and value, a real number in [0,1] for the Łukasiewcicz, Godel and non-determined components; the same with value in the corresponding finite set of (bi, ti) for the finitely valued components; and finally, for the product components in (bi, ti), propositional variables with component equal to the index of the component in the ordinal sum and value a negative real number.

For what respects the constants, if the rational specified is not within a product component, the associated variable is evaluated in every world of the structure to that value. On the other hand, for the constants within a product component, the method followed is that of leaving free one of the constants (that is, evaluating it in world 0 to an arbitrary negative number and copying this value to the other worlds) and define the other constants in terms of this one using the relation among the rational values with the logaritmic function. Since `z3` does not support the logarithm nor the exponential function, we had been forced to compute this values (i.e., $log_{c_0}d$ for c_0 being the free constant and d any other constant in the component) in the python codification and just giving to `z3` an approximation of 16 decimals to the real number. We have remarked in the previous section the possible irregularities inherited from this treatment.

Finally, in this part of code, we include also some lines oriented purely towards the efficiency of the modal computations. As it was seen in the construction of the skeleton model (Algorithm 9.1) and also in the proof of Lemma 10.3, the main point behind each world of this frame is that of becoming the witness from a certain modal formula of the father node. In our implementation, we have stored these pairings (of modal formula and father node) when building the skeleton and thus it is possible to fix some of the values of the formulas.

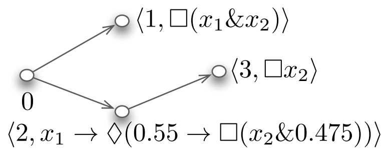
In this part, both the logic and the formulas involved in the reasoning are relevant: the values allowed to each variable and constant are determined by the logic, while the skeleton construction is intrinsic to the shape of the formulas.

The following piece of code illustrates the SMT-code generated for the logic `(0.1, 0.4)l+(0.4,0.7)p+(0.9,1)l4` and the formula `(con (impl 0.3 (box (con x2 x1))) (impl x1 (diamond (impl 0.55 (con (box x2) 0.475)))))`, whose skeleton is shown in Figure 10.1. We pick here this rather complicated formula in order to show the codification of all the possible different cases. On the one hand, it can be seen definition of the three kind on possible universes (i.e., infinite valued Łukasiewiczor Gödel components, product components and finitely valued components). It is also shown the treatment of the constant symbols, both within and outside product components. And finally, the nested modalities give an idea of what we mean by fixing the values of the modal formulas following the skeleton definition.

In order to not include unnecessary repetitive code, we present here, for what respects the variables, only the code corresponding to variable `x1` and world `0`. The complete codification includes the repetition of that piece of code, changing the `0` world to all the other possible indexes (in this case, `1,2` and `3`) and the same for what respects variable `x2`.

Listing 10.4: Declarations SMT-code

```
;;DECLARATIONS
```



$$(\mathbf{0.3} \to \Box(\mathbf{x_1} \& \mathbf{x_2}))\&(\mathbf{x_1} \to \Diamond(\mathbf{0.55} \to (\Box\mathbf{x_2} \& \mathbf{0.475})))$$

Figure 10.1: skeleton tree

```
;;;assingments of the variables
(declare-fun x1 () var)
(assert (or
        ;real values
        (and (= (component (valueInWorld x1 0)) (~ 1.0))
             (or (and (<= 0.0 (value (valueInWorld x1 0))) (<= (
                 value (valueInWorld x1 0)) 0.4))
                 (and (<= 0.7 (value (valueInWorld x1 0))) (<= (
                     value (valueInWorld x1 0)) 1.0))))
        ;non real values, i.e., those from the product components
        (or (and (= (component (valueInWorld x1 0)) 1.0)
                 (< (value (valueInWorld x1 0)) 0.0)))))
;limitations for the finitely valued components
(assert (ite (and (<= 0.9 (value (valueInWorld x1 0))) (>= 1.0 (
    value (valueInWorld x1 0))))
            (or  (= (value (valueInWorld x1 0)) (+  0.9 (* 0.1 (/
                0.0 3.0))))
                 (= (value (valueInWorld x1 0)) (+  0.9 (* 0.1 (/
                     1.0 3.0))))
                 (= (value (valueInWorld x1 0)) (+  0.9 (* 0.1 (/
                     2.0 3.0))))
                 (= (value (valueInWorld x1 0)) (+  0.9 (* 0.1 (/
                     3.0 3.0)))))
             true))

/******* repetition of the previous code moving 0 through {1,2,3}
and also considering the case with x2 instead of x1 *********/

;declaration of the constants

;constant outside any product component
;WARNING: no control of constants in the finitely-valued components
    is done!
(declare-const c1 var )
(assert (and (= (value (valueInWorld c1 0)) 0.3)
             (= (component (valueInWorld c1 0)) (~ 1.0))))
(assert (= (valueInWorld c1 1) (valueInWorld c1 0)))
(assert (= (valueInWorld c1 2) (valueInWorld c1 0)))
(assert (= (valueInWorld c1 3) (valueInWorld c1 0)))

;constant in product component 1
(declare-fun c2() var )
(assert (= (component (valueInWorld c2 0)) 1.0))
```

```
;it is the reference constant of the component
(assert (< (value (valueInWorld c2 0)) 0.0))

(assert (= (valueInWorld c2 1) (valueInWorld c2 0)))
(assert (= (valueInWorld c2 2) (valueInWorld c2 0)))
(assert (= (valueInWorld c2 3) (valueInWorld c2 0)))

;constant in product component 1
(declare-fun c0() var )
(assert (= (component (valueInWorld c0 0)) 1.0))
;constant relativized to the reference one
(assert (= (value (valueInWorld c0 0)) (* (value (valueInWorld c2 0)
    ) 2.0)))

(assert (= (valueInWorld c0 1) (valueInWorld c0 0)))
(assert (= (valueInWorld c0 2) (valueInWorld c0 0)))
(assert (= (valueInWorld c0 3) (valueInWorld c0 0)))


;the values of parts of the functions are fixed following the
    skeleton definition

(assert (or (not (select (select R 0) 1))
            (= (select (box (con x2 x1)) 0)
                 (select (con x2 x1) 1))))
(assert (or (not (select (select R 0) 2))
            (= (select (diamond (impl c2 (con (box x2) c0))) 0)
               (select (impl c2 (con (box x2) c0)) 2))))
(assert (or (not (select (select R 2) 3))
            (= (select (box x2) 2)
               (select x2 3))))
```

The last part of code is the one that formalizes, finally, the problem to treat. We will present here an example of each case, to show how each one of them is faced.

The following code is generated in order to ask whether `(con (impl 0.3 (box (con x2 x1))) (impl x1 (diamond (impl 0.55 (con (box x2) 0.475)))))` is a theorem.

Listing 10.5: Theoremhood example

```
(assert (< (value (valueInWorld (con (impl c1 (box (con x2 x1))) (
    impl x1 (diamond (impl c2 (con (box x2) c0))))) 0)) 1.0))

(check-sat)
```

If `z3` concludes the previous code is unsatisfiable, that means (applying Lemma) that `(con (impl 0.3 (box (con x2 x1))) (impl x1 (diamond (impl 0.55 (con (box x2) 0.475)))))` is a theorem. Otherwise, there is a model and a world that evaluates the formula to less than 1 and thus, is is not a theorem.

For what respects checking if a formula is logical deduction of a certain set of premises, the approach is quite similar. The following code is generated to check whether `(con (box x1) x2)` is consequence of `(impl (box x1) (con (impl 0.3 (diamond x1)) x2))`

Listing 10.6: Deduction example

```
(assert (= (value (valueInWorld (impl (box x1) (con (impl c0 (
    diamond x1)) x2)) 0)) 1.0))
(assert (< (value (valueInWorld (con (box x1) x2) 0)) 1.0))

(check-sat)
```

As before, if there is a model satisfying the previous assignments it means that the deduction consequence does not hold over the given formulas and vice-versa.

Finally, the operation of looking for a model that locally satisfies a set of equations is similar to the previous cases, but considering the value in world `0` for the two formulas of each equation. Moreover, in this case, an extra piece of code is attached in order to get the model, if it exists. The following code illustrates this case for the equation `(eq (con (box x1) x2) (impl (box x1) (con (impl 0.3 (diamond x1)) x2)))`

Listing 10.7: Local satisfiability example

```
(assert (eq (valueInWorld (con (box x1) x2) 0)
      (valueInWorld (impl (box x1) (con (impl c0 (diamond x1)) x2))
          0)))

(check-sat)

;SHOWS
(get-value((select (select R 0) 0)))
(get-value((select (select R 0) 1)))
(get-value((select (select R 0) 2)))
(get-value((select (select R 0) 3)))
(get-value((select (select R 1) 0)))
(get-value((select (select R 1) 1)))
(get-value((select (select R 1) 2)))
(get-value((select (select R 1) 3)))
(get-value((select (select R 2) 0)))
(get-value((select (select R 2) 1)))
(get-value((select (select R 2) 2)))
(get-value((select (select R 2) 3)))
(get-value((select (select R 3) 0)))
(get-value((select (select R 3) 1)))
(get-value((select (select R 3) 2)))
(get-value((select (select R 3) 3)))
(get-value((component (select x2 0))))
(get-value((value (select x2 0))))
(get-value((component (select x1 0))))
(get-value((value (select x1 0))))
(get-value((component (select x2 1))))
(get-value((value (select x2 1))))
(get-value((component (select x1 1))))
(get-value((value (select x1 1))))
(get-value((component (select x2 2))))
(get-value((value (select x2 2))))
(get-value((component (select x1 2))))
(get-value((value (select x1 2))))
(get-value((component (select x2 3))))
(get-value((value (select x2 3))))
```

```
(get-value((component (select x1 3))))
(get-value((value (select x1 3))))
```

Output

Depending on the task we are performing, the output of the program will be different.

- If the chosen operation was that of proving theoremhood (`th`), the output message will be either the confirmation or the negation that the specified formula is a theorem in the given logic. Namely,

  ```
  THE SPECIFIED FORMULA (IS/ IS NOT) A THEOREM
  ```

- Similarly, if the task to be done was proving if a certain formula was derivable from a set of premises, the output message will be either the confirmation or the negation that the specified formula is a logical consequence of the premises in the given logic. Namely,

  ```
  THE FORMULA SPECIFIED (IS/ IS NOT) DERIVABLE FROM THE GIVEN SET
       OF PREMISES
  ```

- If the selected operation was that of checking satisfiability and generating a model of a set of equations, the output message will either confirm or deny that the equations set is locally satisfiable (i.e., that there exists a kripke model evaluated on the given logic and a world within this model that satisfies the set of equations). If the previous answer is affirmative, said model will be shown, taking into account that the differentiated world is `0`. The format of the model will be that of first showing the accessibility matrix (an square matrix of side equal to the number of worlds of the model, with the position $[i][j]$ being either 1 or 0 respectively if world i is related to j or not. Then, for each variable, a list of values is given, indicating each one of these the value of the variable in the world indexed by the position in the list.

  To explain it with an example, the output of running the satisfiability operation with logic `(0.1, 0.4)l+(0.4,0.7)p+(0.9,1)l4` and equation `(eq (con (box x1) x2) (impl (box x1) (con (impl 0.3 (diamond x1)) x2)))` is the following one:

  Output example

  ```
  A POSSIBLE MODEL FOR THE SET OF EQUATIONS IS THE FOLLOWING ONE:
  The matrix of accessibility relations of the Kripke Structure
      is given by:
     | 0 | 1 | 2 | 3 |
  ---+---+---+---+---+
   0 | 0 | 0 | 0 | 1 |
   1 | 0 | 0 | 0 | 0 |
   2 | 0 | 0 | 0 | 0 |
   3 | 0 | 0 | 0 | 0 |
  ---+---+---+---+---+
  The tuple of values of each variable on each world
  (ordered from the value on world 0 on) is the following:
  ```

```
x2: <0.55, 0.55, (/ 7.0 10.0), (/ 7.0 10.0)>
x1: <0.55, 0.55, 1.0, (/ 7.0 10.0)>
```

The model represented by the previous codification can be simply reduced (seeing that, in fact, only worlds 0 and 3 are relevant for the evaluation) to

$$\mathfrak{M} = \langle \{0,3\}, \{\langle 0,3\rangle\}, e\rangle$$

with $e(0, x1) = 0.55 = e(0, x2)$ and $e(3, x1) = 0.7 = e(3, x2)$.

On the other hand, for what respects intensive and automatic testing procedures, we have only proposed an output of the program accounting on the boolean answer of the process, that is, whether, respect to the specified task, the data associated to the operation meet the requirements. This is because the model generated when the operation is satisfiability checking is not useful from the point of view of an intensive testing oriented to check not the correctness of the software (already checked) but its efficiency.

Formally, mNiBLoS will return 1 whenever:

- The operation selected is theoremhood proving and the formula specified **is a theorem** of the given logic.
- The operation selected is deduction checking and the consequence formula specified **is derivable** from the proposed set of premises in the given logic.
- The operation selected is satisfiability and the set of equations **is locally satisfiable** in the given logic.

On the other hand, the program will return 0 in the other cases, namely:

- The operation selected is theoremhood proving and the formula given **is not a theorem** of the given logic.
- The operation selected is deduction checking and the consequence formula specified **is not derivable** from the proposed set of premises in the given logic.
- The operation selected is satisfiability and the set of equations **is not locally satisfiable** in the given logic.

Examples

For an easier comprehension on the usage of the mNiBLoS, we include here a collection of examples of use, with their respective output messages. We express them by specifying the logic, the operation and the data required by this last one.

- $\models_{\mathrm{BL}} (x_1 * x_2) \to (x_2 * x_1)$

```
> logic: bl
> operation: th
> formula : (impl (con x1 x2) (con x2 x1))
```

will provide the output

```
THE SPECIFIED FORMULA IS A THEOREM
```

- $\not\models_{\mathrm{BL}} \neg\neg x_1 \to x_1$

```
> logic: bl
> operation: th
> formula : (impl (neg (neg x1)) x1)
```

will provide the output

```
THE SPECIFIED FORMULA IS NOT A THEOREM
```

- $\models_{[\mathbf{0},\mathbf{1}]_{\mathbf{Ł}}} \neg\neg x_1 \to x_1$

```
> logic: (0,1)l
> operation: th
> formula : (impl (neg (neg x1)) x1)
```

will provide the output

```
THE SPECIFIED FORMULA IS A THEOREM
```

- $\models_{(0,0.5)_{\mathrm{Ł}}} (\neg\Delta(0.5 \to x_1)) \vee \neg\neg x_1$

```
> logic: (0,0.5)l
> operation: th
> formula : (wdis (neg (delta (impl 0.5 x1))) (neg (neg x1)))
```

will provide the output

```
THE SPECIFIED FORMULA IS A THEOREM
```

- $\neg\Delta(x_1 \to 0.5) \models_{(0.5,1)_{\mathrm{Ł}2}} x_1$

```
> logic: (0.5,1)l2
> operation: d
> premises : (neg (delta (impl (x1 0.5))))
> consequence : x1
```

will provide the output

```
THE FORMULA SPECIFIED IS DERIVABLE FROM THE GIVEN SET OF
    PREMISES
```

- $\models_{\mathbf{K}^{\omega}_{\mathrm{BL}}} (\Diamond x_1 \to \Box x_2) \to \Box(x_1 \to x_2)$

```
> logic: bl
> operation: th
> formula : (impl (impl (diamond x1) (box x2)) (box (impl x1 x2
    )))
```

will provide the output

```
THE SPECIFIED FORMULA IS A THEOREM
```

- $\neg\Delta\neg\Box x_1, \Diamond x_1 \to 0.5 \models_{\mathbf{K}^{\omega}_{(\mathbf{0},\mathbf{0.6})_{\mathbf{\Pi}}}} \Box\neg\neg x_1$

```
> logic: (0,0.6)p
> operation: d
> premises: (neg (delta (neg (box x1)))), (impl (diamond x1)
    0.5)
> consequence: (box (neg (neg x1)))
```

will provide the output

```
THE FORMULA SPECIFIED IS DERIVABLE FROM THE GIVEN SET OF
    PREMISES
```

- $\neg\Delta\neg\Box x_1, \Diamond x_1 \to 0.5 \not\models_{\mathbf{K}^{\omega}_{(\mathbf{0},\mathbf{0.6})_{\mathbf{L}}}} \Box\neg\neg x_1$

```
> logic: (0,0.6)l
> operation: d
> premises: (neg (delta (neg (box x1)))), (impl (diamond x1)
    0.5)
> consequence: (box (neg (neg x1)))
```

will provide the output

```
THE FORMULA SPECIFIED IS NOT DERIVABLE FROM THE GIVEN SET OF
    PREMISES
```

- exists? $\mathfrak{M}$ $(0,0.6)_{\mathrm{L}}$-Kripe stucture such that $\mathfrak{M} \models_0 \{\Box x_1 > 0$ and $\Diamond x_1 \leq 0.5$ and $\Box\neg\neg x_1 < 1\}$

```
> logic: (0,0.6)l
> operation: s
> equations: (l F (box x1)), (leq (diamond x1) 0.5), (l (box (
    neg (neg x1))) T)
```

will provide the output

```
A POSSIBLE MODEL FOR THE SET OF EQUATIONS IS THE FOLLOWING ONE:
The matrix of accessibility relations of the Kripke Structure
    is given by:
   | 0 | 1 | 2 | 3 |
---+---+---+---+---+
 0 | 0 | 1 | 1 | 1 |
 1 | 0 | 0 | 0 | 0 |
 2 | 0 | 0 | 0 | 0 |
 3 | 0 | 0 | 0 | 0 |
---+---+---+---+---+
The tuple of values of each variable on each world (ordered
    from the value on world 0 on) is the following:
x1: <0.0, (/ 7051.0 33010.0), (/ 26857.0 66020.0), (/ 7051.0
    33010.0)>
```

Chapter 11

mNiBLoS Statistics

We devote this chapter to show the results obtained after running intensive testing of the mNiBLoS application detailed in the previous chapters. We begin showing some results of intensive tests over families of *BL* theorems that have been already used as benchmarks in some previous works in the literature, and thus we can compare our results with the existing ones. Since we can see some objections concerning the generality of the previous benchmark (because the number of variables is constant), we later present some other studies. First, we propose a family of *BL* whose number of variables increases with the parameter. Next, we present some results obtained over a randomly generated set of formulas, seeing TO DO. Finally and inspired by the classical logic case, where the formulas expressed in clausal form are the most widely used in terms of satisfiability problems and testing, we devote the last section of this chapter to study this approach from the many-valued point of view.

11.1 Tests of BL theorems

Since all possible theorems on BL so are on any of its extensions, experiments over two different families of BL-theorems were conducted, see (11.1) and (11.2) below. First, for comparison reasons with [6], the following generalizations (based on powers of the & connective) of the first seven Hájek's axioms of BL [71] were considered:

$$\begin{array}{ll}
(\mathrm{A1}) & (p^n \to q^n) \to ((q^n \to r^n) \to (p^n \to r^n)) \\
(\mathrm{A2}) & (p^n \& q^n) \to p^n \\
(\mathrm{A3}) & (p^n \& q^n) \to (q^n \& p^n) \\
(\mathrm{A4}) & (p^n \& (p^n \to q^n)) \to (q^n \& (q^n \to p^n)) \\
(\mathrm{A5}a) & (p^n \to (q^n \to r^n)) \to ((p^n \& q^n) \to r^n) \\
(\mathrm{A5}b) & ((p^n \& q^n) \to r^n) \to (p^n \to (q^n \to r^n)) \\
(\mathrm{A6}) & ((p^n \to q^n) \to r^n) \to (((q^n \to p^n) \to r^n) \to r^n)
\end{array} \tag{11.1}$$

where p, q and r are propositional variables and $n \in \mathbb{N} \setminus \{0\}$. It is worth noticing that the length of these formulas grows linearly with the parameter n.

In [6] the authors refer to [110] to justify why these formulas can be considered a good test bench for (at least) Łukasiewicz logic. In our opinion, these formulas have the problem to be a good evaluator set of using only three variables. We consider this is a serious drawback because the known results on BL complexity state that Łukasiewicz-SAT is an NP-complete problem when the number of variables in the input is not fixed. However, we consider that proving that tautologicity for formulas with three variables can be solved in polynomial time could be done.

With this in mind, to overcome the drawback of the bounded number of variables, we present a new family of BL-theorems to be used as a bench test.

For every $n \in \mathbb{N} \setminus \{0\}$,

$$\bigwedge_{i=1}^{n} (\&_{j=1}^{n}\, p_{ij}) \;\rightarrow\; \bigvee_{j=1}^{n} (\&_{i=1}^{n}\, p_{ij}) \tag{11.2}$$

is a BL-theorem which uses n^2 variables; the length of these formulas grows quadratically with n. As an example, we note that for $n = 2$ we get the BL-theorem $\big((p_{11}\&p_{12}) \wedge (p_{21}\&p_{22})\big) \;\rightarrow\; \big((p_{11}\&p_{21}) \vee (p_{12}\&p_{22})\big)$. These formulas can be considered significantly harder than the ones previously proposed in [110]; and indeed, the experimental results support this claim. It is important to notice that the natural way to compare this new formula with parameter n with the previous set is to consider the formulas in [110] with the integer part of $\sqrt{n}$ as parameter.

Experiments were run on a machine with a i5-650 3.20GHz processor and 8GB of RAM. Evaluating the validity in Łukasiewiczand Gödel logics of the generalizations of the *BL* axioms (11.1), ranging n from 0 to 500 with increments of 10, throws better results than the ones obtained in [6], but since the new solver is, on these logics, an extension of their work, this can be assumed to be due to the use of different machines. For Product Logic, very good timings were obtained. They are still worse than the ones for Łukasiewicz and Gödel logics in several cases, but the difference with the previous approach is clear: complex formulas are solved in a comparatively short time, whereas in [6] they could not even be processed. In Figure 11.1 one can see and compare solving times (given in seconds) for some of the axioms of the test bench for the cases of *BL*, Łukasiewicz, Gödel and Product logics. It is also interesting to observe how irregularly the computation time for Product Logic varies depending on the axiom and the parameter. This probably happens due to the way the z3-solver internally works with the integer arithmetic theory.

The experiments done with the other family of BL-theorems (11.2) (see Figure 11.2 for the results) suggests that here the evaluation time is growing non-polynomially on the parameter n. In the graphs we give here, only those answers (for parameters $n \leq 70$) obtained in at most 3 hours of execution are shown (e.g. for the *BL* case answers could be reached within this time only for the problems with $n \leq 4$). The high differences in time when evaluating the theorems were expectable: Łukasiewicz and Gödel are simpler than *BL* when proving the theoremhood because of the method used for *BL* (considering $n^2 + 1$ copies of Łukasiewicz, where n is the parameter of the formula). On the other hand, the computation times for Product logic modelled over $\mathcal{Z}^-$ are also smaller than for *BL*.

11.2 Randomly generated formulas: number of variables vs. length of the formula

With the aim of understanding the behaviour of mNiBLoS when faced with a more irregular set of formulas, we designed and implemented a test that produces random formulas of varying length and number of variables. Using these, we have executed several kind of tests over the solver, in order to determine particular behaviours or patterns.

The following tests show, for logics expanding Łukasiewicz, Gödel, Product and BL logics, the times of response of mNiBLoS trying to determine whether a formula with certain length and number of variables is a theorem of the logic or not.

In figure 11.2, we show the times considering the previous four logics alone, that is, without accepting constants in the language and not using modalities.

Some characteristics of these graphs fit into our expectations, but we can only guess concerning some points of the experiment. Seeing the two first graphs (of Product and Gödel logics), it is clear that alternative semantics of the product components pays a central role in the system: mNiBLoS treats Product logic almost as fast as Gödel logic. We consider this to be a very interesting result, since in previous works product logic was the slowest one and its faster computation can open the door a more demanding applications. Moreover the increasing in the reasoning time of BL are naturally expected from the fact that the t-norm used internally in that case is much more complex than just one component of the basic ones (it depends on the number of variables).

On the other hand, we find intriguing the irregular behaviour of, over all, Łukasiewicz. We think this can be due to the internal methods used by z3, since the external part is a general code (in the sense that there are no high differences in between executions) that does not have the capacity of changing that times. Moreover, it is also interesting to remark that, except in the BL case (where there is an explanation), the increasing of the number of constants in the formulas do not uniformly increase the output times. This is also a behaviour we do not know how to explain and we think it would be very interesting some further study on this matter and find the reason for it.

Concerning the addition of constants to the language, the results are clearly fastened, see Figure 11.3.

Using constants simplifies in some sense the calculus because, in a sense, they behave like variables that do not need to be tested by the solver. For this, the ratio length vs. number of variables of the formula gets much higher: nor formulas include arbitrary constants, so they can be larger without adding new variables.

On the other hand, concerning the modal tests, the first result that we got was that the time of pre-processing of the modal formulas (that is, the creation of the skeleton structure) does not add any valuable time to the total reasoning time. The graphs in Figure 11.4 show some examples when adding modal operators to the formula's language.

The processing times of formulas with modalities are much higher. This is reasonable, since the calculus are in some sense multiplied by the modal complexity of the formula (the skeleton it generates). However, it is remarkable that the solver works fine up to formulas of length up to 80 elements and up to 25-30 variables, which is a good result for a first attempt on automated reasoning for many-valued modal logics.

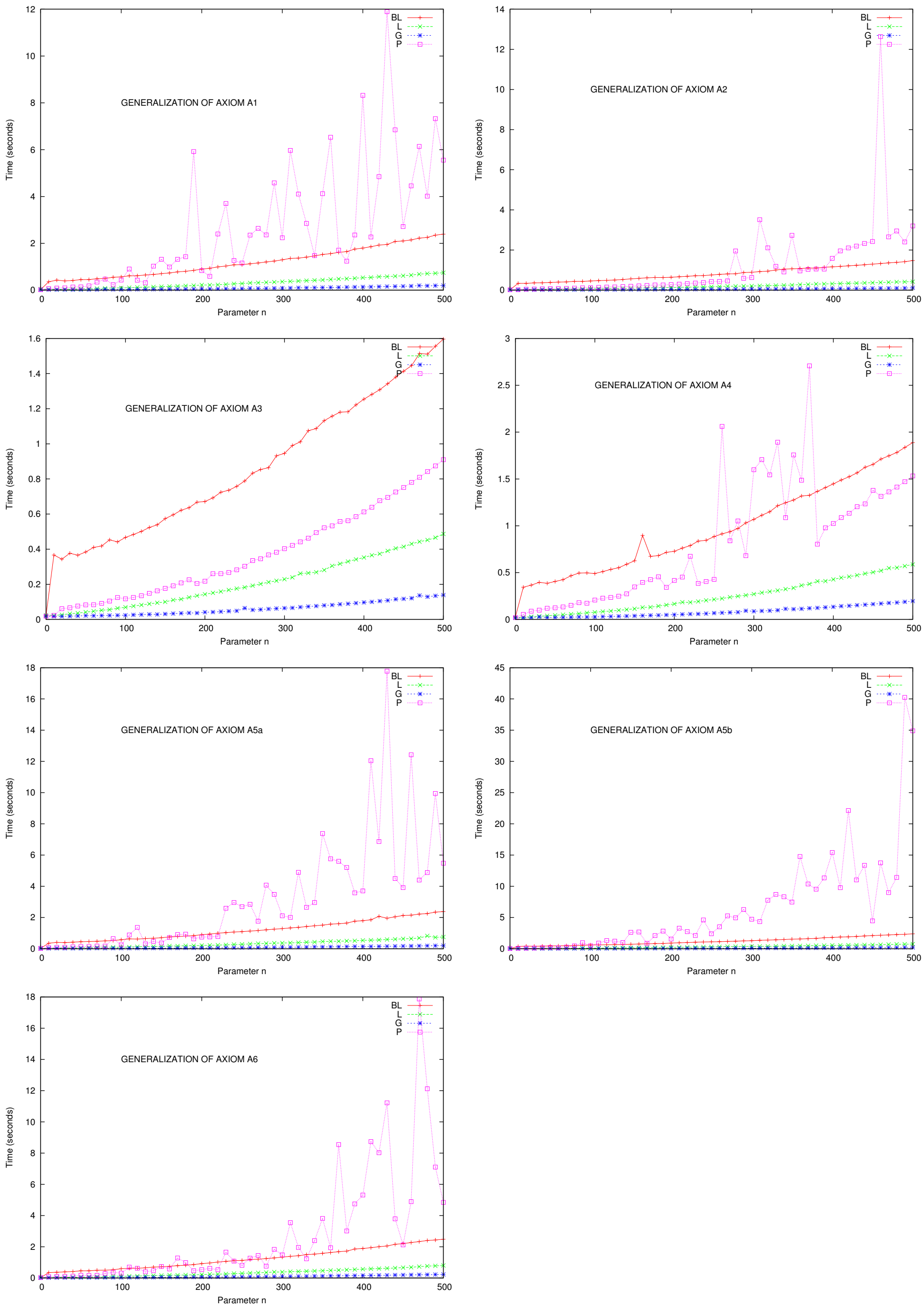


Figure 11.1: Generalizations of BL-axioms given in (11.1).

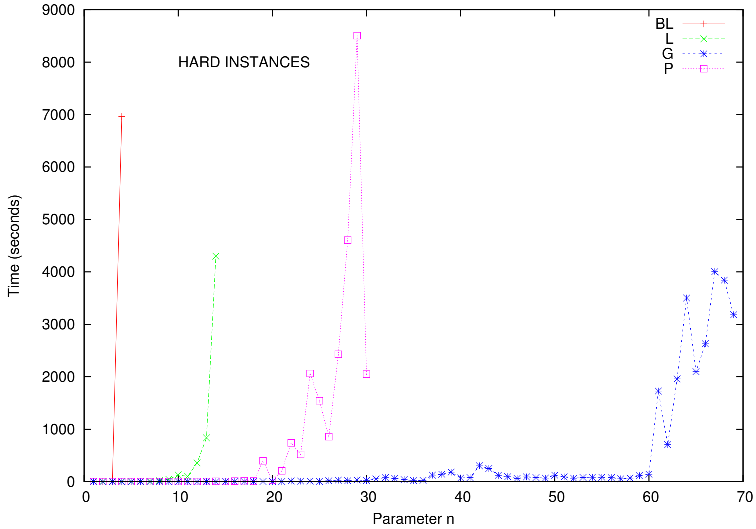
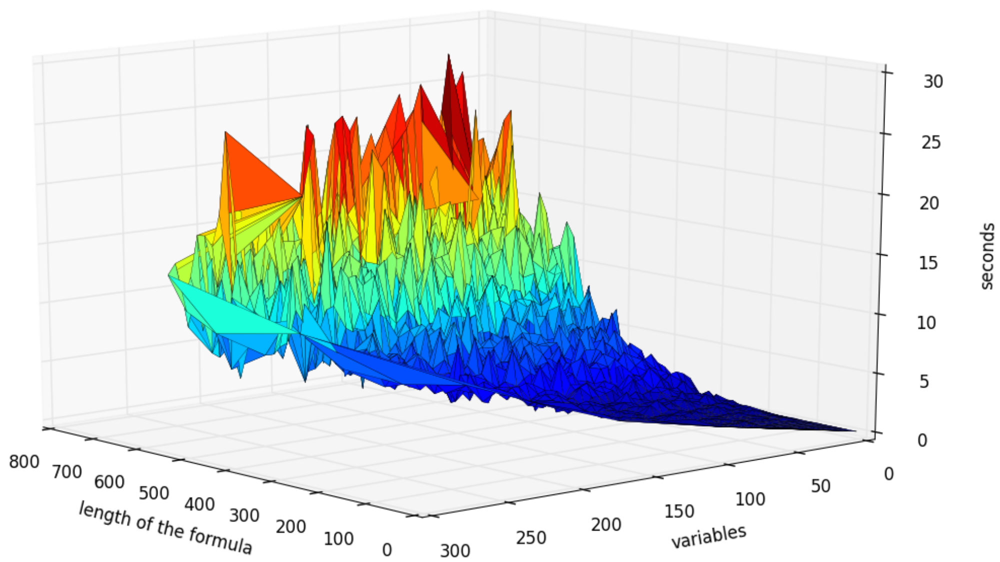
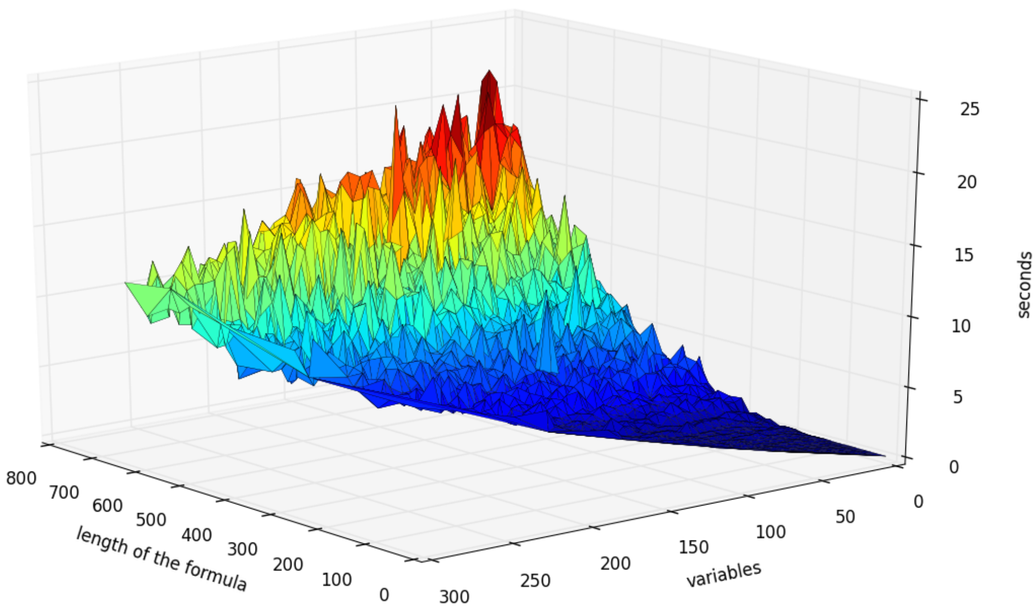


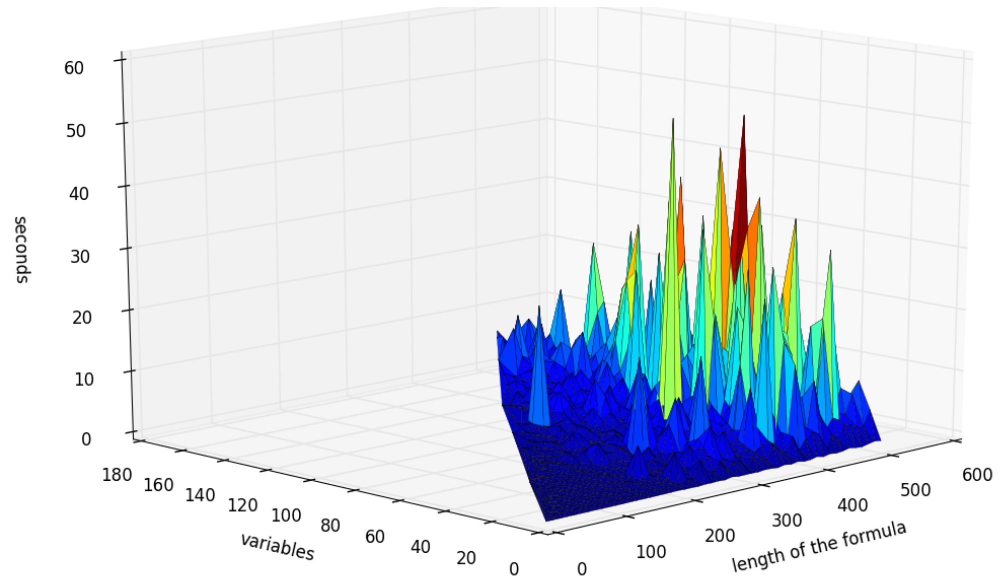
Figure 11.2: Our proposed BL-theorems given in (11.2).



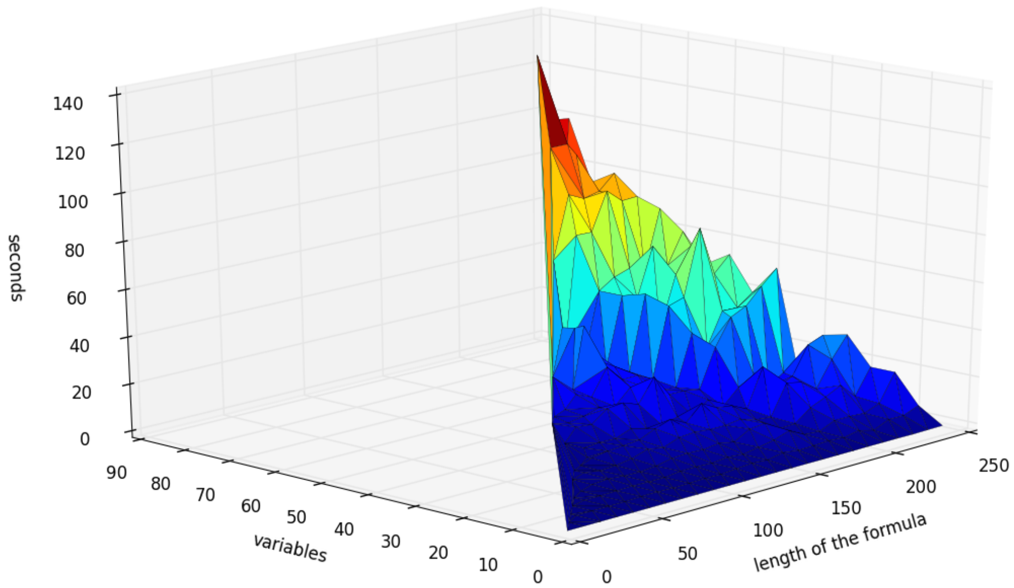
Product logic



Gödel logic

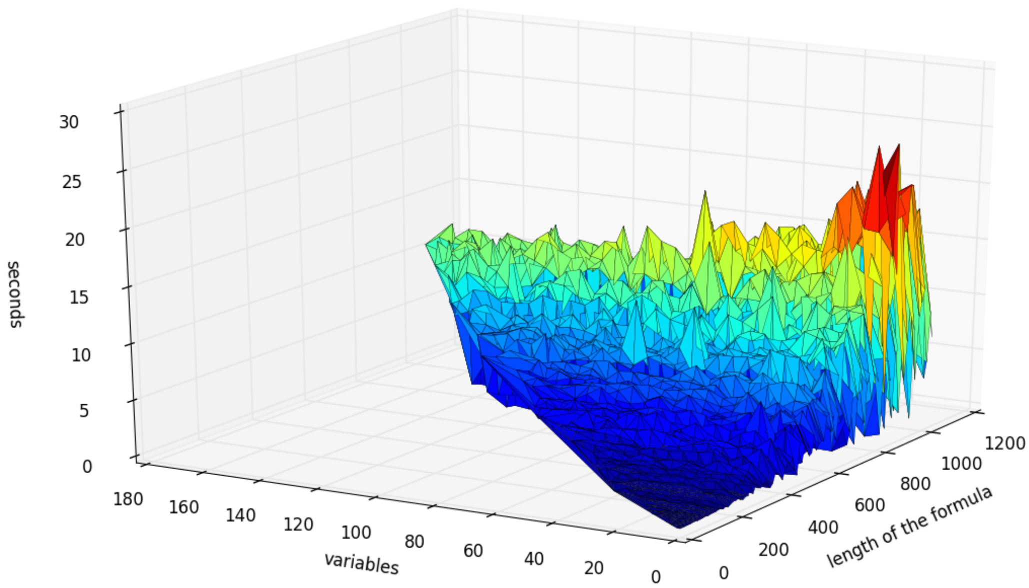


Łukasiewicz logic

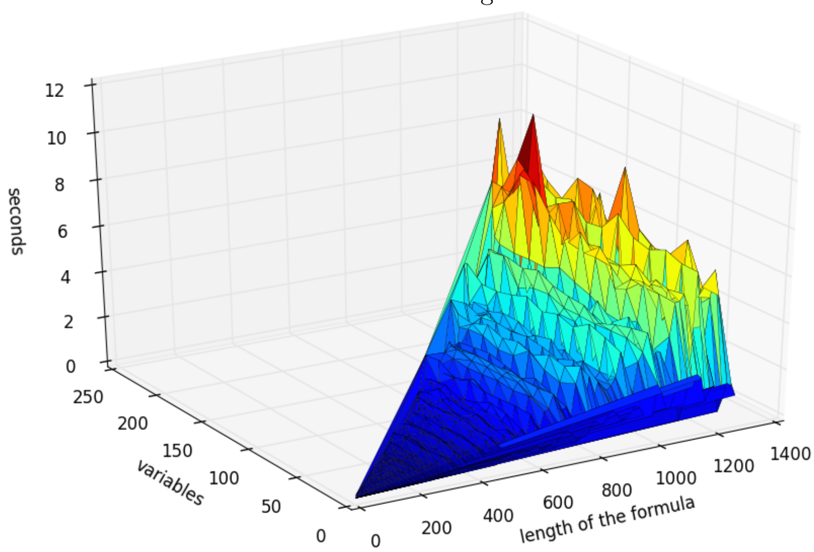


BL logic

Figure 11.2: Times of response, no constants and no modalities in the language.

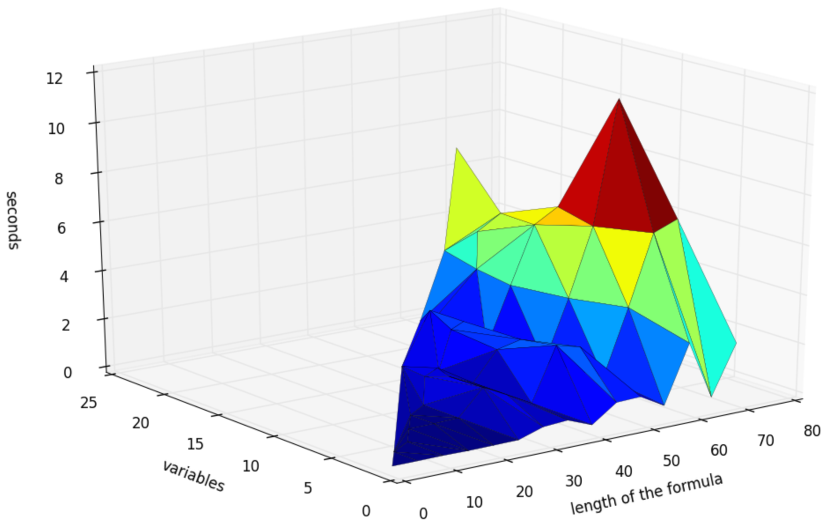


Łukasiewicz logic

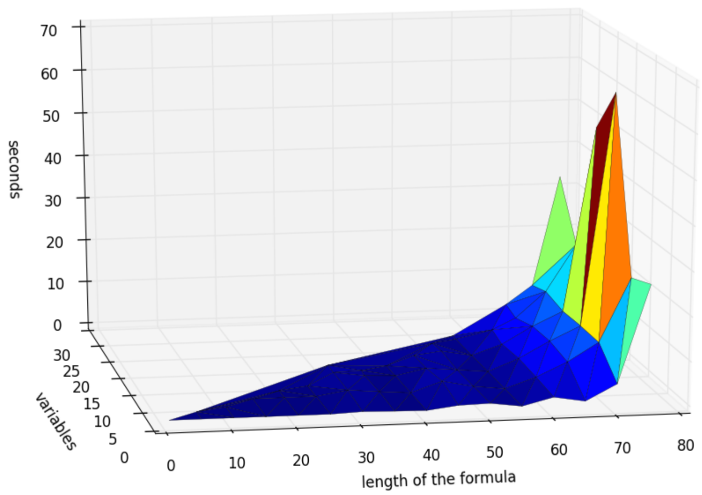


Gödel logic

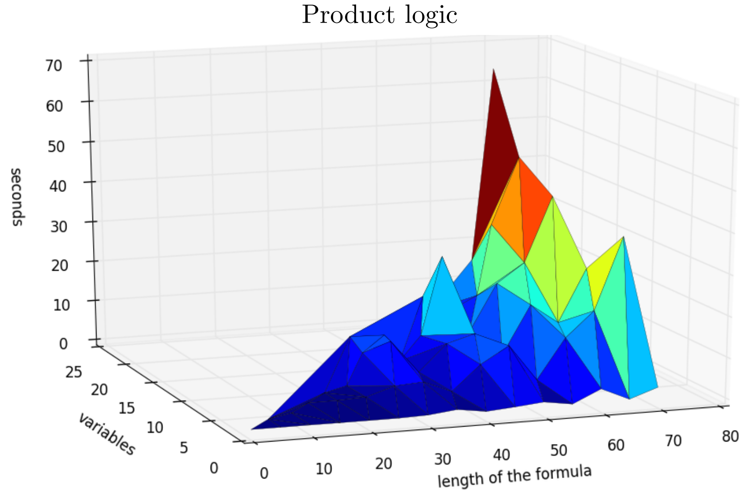
Figure 11.3: Times of response, constants and no modalities in the language.



Łukasiewicz logic



Product logic



BL logic

Figure 11.4: Times of response,modalities but no constants in the language.

Chapter 12

Conclusions and open problems

12.1 Main contributions

In this dissertation, we have focused on the study of the modal logics arising from Kripke semantics with a crisp accessibility relation and evaluation ranging over *MTL*-algebras. We are concerned with this problem both from a theoretical and from an applied point of view.

From a theoretical point of view, the study of these modal logics has revealed some characteristics that are desirable to hold at the propositional level in order to axiomatize the modal expansions. Namely, that the logic is strongly standard complete and, moreover, that it has a set of truth constants that are interpreted densely in the standard algebra. The characterization and study of logics enjoying these properties have enlarged the current work with results not exactly linked to the modal logics but rather motivated by them. For this reason, regarding theoretical developments, the results presented in this dissertation are of two types. First, those that study the strong standard completeness of *MTL* logics with truth constants and second, those that are concerned with the modal expansion of the previous logics. More precisely

- We have proposed, for each left-continuous t-norm $*$, an axiomatic system L_*^∞ that is strongly complete (i.e., for deductions with infinite premises) with respect to the standard canonical algebra of $*$ expanded with the Δ operator. It is based on an infinitary rule called *density rule* that determines the density of the constants on the linearly ordered algebras. We also present a general result concerning semilinearity of some infinitary logics. In particular, we show that the *prime theory extension property* holds for a large family of logics extending MTL_Δ with up to a denumerable set of infinitary inference rules. Up to now, proofs of this property when infinitary inference rules were considered in the axiomatization of the logic were done ad-hoc, so we think this more general solution can be quite useful to the community. Moreover, we have studied how this infinitary rule allows to axiomatize also any expansion of the previous standard algebra by additional operations defined in $[0, 1]$ that only need to satisfy some regularity conditions. These results have closed several problems, like the (strongly complete) axiomatization of the Gödel logic with truth constants and Δ and the same problem concerning arbitrary ordinal sums. These works were presented in [125] and [126].

- Motivated by the bad behaviour of the previous systems when expanded with modalities (in the sense that it is not clear how to prove the density rule is closed under the $\Box$ operator), we consider the left-continuous t-norms whose associated infinitary logic can be axiomatized using, instead of the infinitary rule from before, a larger set of infinitary inference rules that have a better-behaved schemata (from the modal expansion point of view). These are the t-norms that *accept a conjunctive axiomatization*. We prove not all left-continuous t-norms have this kind of representation, but we also observe that for instance all the ordinal sums formed with Łukasiewicz and Product t-norms belong to this class. The fragment of this work referring to product logic was presented in [122].
- We propose axiomatic systems for the local and global modal logics arising from the class of Kripke models with crisp accessibility relation and worlds evaluated over algebras associated to a t-norm that accepts a conjunctive axiomatization. We prove these systems are strongly complete with respect to their corresponding intended semantics and study some of their characteristics. With these results we solve some open problems in the literature; in particular, the product modal logic had not been axiomatized before and nor had any ordinal sums of continuous t-norms. The fragment of this work referring to product logic was presented in [124].
- We study the algebraic semantics of the previously defined families of modal logics. We characterize their algebraic companion, classify them within the Leibniz hierarchy of abstract algebraic logic and present algebraic completeness results. We also study some relations between the Kripke and the algebraic semantics of these modal logics and show how the canonical model and complex algebra construction is generalized to the MTL context. These studies contribute to the state of the art on many-valued modal algebras and also leave several results to the community, enlarging the knowledge about many-valued Kripke frames, modal algebras and their relationship. The previous results (including the previous point) regarding the product logic were presented in [123] and have also been published in [127].

On the other hand, we have tried to bring closer the many-valued modal logics studied before to an applied scenery. For this reason, we have developed a solver, mNiBLoS, to reason automatically over a large family of continuous t-norm based logics, allowing also modal operations that are computed up to finite structures. There were very few implementations of solvers for many-valued (and particularly, for infinite-valued) logics and the existing ones solved the Łukasiewicz, Gödel and Product logics only (this latter one, with important efficiency problems). Our work generalizes some previous approaches and implements a solver for most continuous t-norm based fuzzy logics and further considers modal operators up to finite structures. Moreover, it improves the state of the art on the Product logic and related cases.

In the design of mNiBLoS we have paid great attention to two main points: efficiency and versatility. Concerning the first point, we have proven that for our concerns, it is equivalent to reason (after the corresponding translations) with a linear arithmetic (over the negative real numbers with + and - operations) than with the non-linear algebra that naturally arises from the product components of the t-norm. This has strongly shortened the reasoning times of mNiBLoS in all the logics involving the product one.

Moreover, we have developed an algorithm that given a formula builds up the smaller (finite) modal structure in which it could fail, which is also an efficiency enhancement concerning the modal reasoning. On the other hand, we have tried to make mNiBLoS as expressive as possible and for that allow the user freedom to choose the logic and the operation that he wishes to execute. mNiBLoS enables the specification of a large family of BL-logics (slightly limited only because of technical reasons coming from the finitary character of a computer) and implements three different operations that can be tested over these logics: theoremhood proving, deduction proving (of a formula from a set of premises) and model generation. A partial version (without including modalities) of this work was presented in [121].

12.2 Publications and Comunications

The development of this doctoral dissertation has lead to the publication of some results in conferences and journals. We include their references here.

- Amanda Vidal, Francesc Esteva and Lluís Godo, *On modal extensions of product fuzzy logic*,**Journal of Logic and Computation**, (To appear, 2015)
- Miquel Bofill, Felip Manyà, Amanda Vidal and Mateu Villaret, *Finding Hard Instances of Satisfiability in Lukasiewicz Logics.* In Proceedings, **12th International Conference on Modelling Decisions for Artificial Intelligence (MDAI)**, Skövde, (Sweden), [28-30]/10/2015. pp. in Press.
- Amanda Vidal, Francesc Esteva and Lluís Godo, *On strong standard completeness of $MTL_*^{\mathbb{Q}}$ expansions.* In **Topology, Algebra and Categories in Logic (TACL)**, Ischia (Italy), [21-26]/06/2015. pp. 275-276
- Amanda Vidal, Francesc Esteva and Lluís Godo, *On Strongly Standard Complete Fuzzy Logics: $MTL_*^{\mathbb{Q}}$ and its expansions.* In **9th Conference of the European Society for Fuzzy Logic and Technology (IFSA-EUSFLAT)**, Gijón (Spain), [30/06-3/07]/2015, pp. 828-835.
- Miquel Bofill, Felip Manyà, Amanda Vidal and Mateu Villaret, *Finding Hard Instances of Satisfiability in Lukasiewicz Logics.* In Proceedings, **45th International Symposium on Multiple-Valued Logics (ISMVL)**, Waterloo, Canada, [18-20]/05/2015. IEEE CS Press, pp. 30-35
- Amanda Vidal, Francesc Esteva and Lluís Godo, *Axiomatising a fuzzy modal logic over the standard product algebra.* In **Logic, Algebra and Truth Degrees (LATD)**, Vienna (Austria), [16-19]/07/2014. Abstract Booklet: Logic Colloquium and Logic, Algebra and Truth Degrees, M. Baaz, A. Ciabattoni, S. Hetzl (eds.), pp. 275-279.
- Amanda Vidal, Francesc Esteva and Lluís Godo, *A product modal logic.* In **35th Linz Seminar on Fuzzy Set Theory**, Linz (Austria), [18-22]/02/2014. T. Flaminio et al. (eds.), pp. 127-130.
- Amanda Vidal, Francesc Esteva and Lluís Godo, *About standard completeness of Product logic.* In Proceedings, **XVII Congreso Español sobre Tecnologías y Lógica Fuzzy (ESTYLF)**, Zaragoza (Spain), [5-7]/02/2014. F. Bobillo et al. (eds.), pp. 423-428.
- Amanda Vidal and Félix Bou, *Image-finite first-order structures.* In **ManyVal**, Prage (Czech Republic) [4-6]/09/2013. Abstracts Volume, T. Kroupa (eds.), pp. 52-53.

- Amanda Vidal, Félix Bou and Lluís Godo, *An SMT-based solver for continuous t-norm based logics*. In Proceedings, **Scalable Uncertainty Management (SUM)**, Marburg (Germany), [17-19]/09/2012. Volume 7520 of **Lecture Notes in Computer Science**, E. Hüllermeier, S.Link, T.Fober and B.Seeger (eds.), pp. 633–640. Springer Berlin Heidelberg.

12.3 Future work

During the development of this research several interesting problems have appeared. Some of them are still open and we think worth of future works. We present here some of the most remarkable ones:

- We have resorted to the use of rational constants (and Δ operation) in order to be able to successfully axiomatize the modal logics based on left-continuous t-norms. Are these truly necessary (in order to get a finite axiomatic system)? Previous works have remarked the importance that these constants seem to have ([18]), but we are not certain there are no other approaches that could avoid the truth constants. In particular, it is still open to axiomatize the modal expansion of Product logic.
- We have limited the modal expansions to a family of left-continuous t-norms with a certain "good behaviour" talking from the modal expansion point of view. It would be interesting to know if this process can be done, more in general, for all left-continuous t-normss, or be able to determine (if that is the case) that there is not a recursively enumerable axiomatic system for some of these other modal logics. It would also be interesting to know more examples of left-continuous t-norms that belong to this class.
- We have studied the modal expansion considering Kripke models with a crisp accessibility relation. The problem of axiomatizing the modal logics when the accessibility is a mapping from pairs of worlds into the many valued algebra is a problem not addressed in this dissertation. The solution of this problem does not seem to follow from the methods developed in our work, but its resolution would be very interesting being it the more general modal fuzzy logic. It does not seem like a simple task, since it seems clear that different techniques -and even, possibly, a different approach- to the ones used in the current work must be followed. For the interested reader, we comment here some points that stress out this differences. First, as we commented in the Chapter 3, the K axiom does no longer hold and nor does the Lemma 6.8. Moreover, the definition of the canonical model shall be different to the one given in this chapter, in order to capture the many-valued character of the accessibility relation. This fact, together with the previous point, make it seem unlikely to be able to prove an analogous result to Lemma 6.12 and so reach the Truth Lemma prove with some tools similar to the current ones.
- It is known that, for some modal logics, duality results between the Kripke models associated to them and other categorical structures exist. We have already done some steps towards these kind of characterizations in Chapter 7, but it would be interesting to further develop them. Moreover, this could open the door to treat the canonicity problem more in general that what we do in Subsection 6.3 from Chapter 6, presenting a very promising set of results.

- Decidability and, if it proceeds, complexity studies concerning 1-validity, 1-satisfiability and other problems over the modal fuzzy logics studied along this dissertation have been not developed. It is clear that this problem is interesting when limiting the logic to the finitary one, but nevertheless these are problems very important to treat. It could be the case that, as in the classical case, modal fuzzy logics are a decidable fragment of firs-order fuzzy logics and also that the algorithmic studies of these logics can have (or not) a viable procedure (opening the door to implement a solver not only for finite modal structures).
- mNiBLoS is an application that has still many features unimplemented. The generalization of the modal part of the solver in order to cope with non-crisp accessibility relations seems a natural and not difficult step (if we still consider only finite models). Studies on how to implement the full modal logic (over possibly infinite Kripke structures) would be of great interest, but it seems to present more difficulties, since it is not known whether these logics are decidable.

Bibliography

[1] R. Adillon and V. Verdú. On a contraction-less intuitionistic propositional logic with conjunction and fusion. **Studia Logica, Special Issue on Abstract Algebraic Logic**, 65(1):11–30, 2000. 33

[2] P. Agliano and F. Montagna. Varieties of BL-algebras. I. General properties. **Journal of Pure and Applied Algebra**, 181(2-3):105–129, 2003. 36

[3] S. Aguzzoli and B. Gerla. On countermodels in Basic Logic. **Neural Network World**, 12(5):407–421, 2002. 154

[4] C. Alsina, E. Trillas, and L. Valverde. On some logical connectives for fuzzy sets theory. **Journal of Mathematical Analysis and Applications**, 93(1):15–26, 1983. 4

[5] T. Alsinet, D. Barroso, R. Béjar, F. Bou, M. Cerami, and F. Esteva. On the implementation of a fuzzy dl solver over infinite-valued product logic with smt solvers. In W. Liu, V.S. Subrahmanian, and J. Wijsen, editors, **Scalable Uncertainty Management**, volume 8078 of **Lecture Notes in Computer Science**, pages 325–330. Springer Berlin Heidelberg, 2013. 165

[6] C. Ansótegui, M. Bofill, F. Manyà, and M. Villaret. Building automated theorem provers for infinitely valued logics with satisfiability modulo theory solvers. In **Proceedings of the IEEE 42nd International Symposium on Multiple-Valued Logic (ISMVL 2012)**. IEEE Computer Society, 2012. 145, 154, 165, 167, 187, 188

[7] A. Armando, C. Castellini, E. Giunchiglia, and M. Maratea. A SAT-based decision procedure for the boolean combination of difference constraints. In **Proceedings of the 7th international conference on Theory and Applications of Satisfiability Testing**, SAT'04, pages 16–29. Springer-Verlag, 2005. 149

[8] G. Audemard and L. Simon. Predicting learnt clauses quality in modern SAT solvers. In **Proceedings of the 21st international jont conference on Artifical intelligence**, IJCAI'09, pages 399–404. Morgan Kaufmann Publishers Inc., 2009. 147

[9] Matthias Baaz. Infinite-valued Gödel logics with 0-1-projections and relativizations. 6:23–33, 1996. 36

[10] T. Ball, B. Cook, S. Lahiri, and L. Zhang. Zapato: Automatic theorem proving for predicate abstraction refinement. In **16th International Conference on Computer Aided Verification (CAV 2004)**, volume 3114 of **Lecture Notes in Computer Science**, pages 457–461. Springer, 2004. 149

[11] C. Barrett, A. Stump, and C. Tinelli. The SMT-LIB standard: Version 2.0. Technical report, Department of Computer Science, The University of Iowa, 2010. Available at `www.SMT-LIB.org`. 150

[12] P. Blackburn, M. de Rijke, and Y. Venema. **Modal logic**. Number 53 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, Cambridge, 2001. 41, 44

[13] W. J. Blok and B. Jónsson. Equivalence of Consequence Operations. **Studia Logica**, 83(1–3):91–110, 2006. 20

[14] W. J. Blok and D. Pigozzi. Protoalgebraic logics. **Studia Logica**, 45:337–369, 1986. 21

[15] W. J. Blok and D. Pigozzi. **Algebraizable logics**, volume 396 of **Mem. Amer. Math. Soc.** A.M.S., Providence, January 1989. 21, 25

[16] F. Bou, F. Esteva, J. M. Font, A. Gil, L. Godo, A. Torrens, and V. Verdú. Logics preserving degrees of truth from varieties of residuated lattices. **Journal of Logic and Computation**, 19(6):1031–1069, 2009. 141

[17] Exploring a syntactic notion of modal many-valued logics. **Mathware Soft Computing**15(2):175–188, 2008 52

[18] F. Bou, F. Esteva, L. Godo, and R. Rodríguez. On the minimum many-valued modal logic over a finite residuated lattice. **Journal of Logic and Computation**, 21(5):739–790, 2011. 6, 7, 8, 48, 52, 53, 55, 56, 57, 105, 124, 200

[19] R. S. Boyer and J. Strother Moore. **A computational logic**. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1979. ACM Monograph Series. 147

[20] S. Burris and H.P. Sankappanavar. **A course in Universal Algebra**. The Millennium, 2012 Update edition, 2012. 15

[21] X. Caicedo, G. Metcalfe, R. Rodríguez, and J. Rogger. A finite model property for gödel modal logics. In L. Libkin, U. Kohlenbach, and R. de Queiroz, editors, **Logic, Language, Information, and Computation**, volume 8071 of **Lecture Notes in Computer Science**. Springer Berlin Heidelberg, 2013. 6, 7, 58, 59, 121

[22] X. Caicedo and R. O. Rodríguez. A Gödel modal logic, 2009. Available at `http://arxiv.org/abs/0903.2767`.

[23] X. Caicedo and R. O. Rodriguez. Bi-modal Gödel logic over $[0, 1]$-valued kripke frames. **Journal of Logic and Computation**, 25(1):37–55, 2015. 6, 7, 48, 58

[24] X. Caicedo and R. Oscar Rodríguez. Standard Gödel modal logics. **Studia Logica**, 94(2):189–214, 2010. 6, 7, 8, 56, 58, 59, 121

[25] M. Cerami, F. Esteva, and F. Bou. Decidability of a description logic over infinite-valued product logic. In F. Lin, U. Sattler, and M. Truszczynski, editors, **Principles of Knowledge Representation and Reasoning: Proceedings of the Twelfth International Conference, KR 2010, Toronto, Ontario, Canada, May 9-13, 2010**, pages 203–213. AAAI Press, 2010. 59

[26] A. Chagrov and M. Zakharyaschev. **Modal Logic**, volume 35 of **Oxford Logic Guides**. Oxford University Press, 1997. 41, 44, 134

[27] B. F. Chellas. **Modal Logic: an introduction**. Cambridge University Press, Cambridge, Cambridge, 1980. 41, 44

[28] A. Ciabattoni, F. Esteva, and L. Godo. T-norm based logics with n-contraction. **Neural Network World**, 12(5):441–452, 2002. 34

[29] A. Ciabattoni, G. Metcalfe, and F. Montagna. Algebraic and proof-theoretic characterizations of truth stressers for MTL and its extensions. **Fuzzy Sets and Systems**, 161(3):369–389, 2010. 6, 48

[30] R. Cignoli, F. Esteva, L. Godo, and A. Torrens. Basic fuzzy logic is the logic of continuous t-norms and their residua. **Soft Computing**, 4(2):106–112, 2000. 5, 30

[31] R. Cignoli and A. Torrens. An algebraic analysis of product logic. **Multiple-valued logic**, 5:45–65, 2000. 153, 154

[32] P. Cintula. **From fuzzy logic to fuzzy mathematics**. Ph. D. thesis, Czech Technical University in Prague, 2004. 39, 65

[33] P. Cintula. Short note: on the redundancy of axiom (A3) in BL and MTL. **Soft Computing**, 9(12):942–942, 2005. 29

[34] P. Cintula. A note on axiomatizations of pavelka-style complete fuzzy logics. **Fuzzy Sets and Systems**, (To appear, 201x). 39, 65, 66, 68, 80, 91, 93

[35] P. Cintula, P. Hájek, and C. Noguera, editors. **Handbook of Mathematical Fuzzy Logic, 2 volumes**, volume 37 and 38 of **Studies in Logic. Mathematical Logic and Foundation**. College Publications, 2011. 27

[36] J. Czelakowski. **Protoalgebraic logics**, volume 10 of **Trends in Logic—Studia Logica Library**. Kluwer Academic Publishers, Dordrecht, 2001. 21

[37] M. Davis, G. Logemann, and D. Loveland. A machine program for theorem-proving. **Commun. ACM**, 5:394–397, 1962. 147

[38] M. Davis and H. Putnam. A computing procedure for quantification theory. **Journal of the Association for Computing Machinery**, 7:201–215, 1960. 147

[39] L. de Moura, B. Dutertre, and N. Shankar. A tutorial on satisfiability modulo theories. In W. Damm and H. Hermanns, editors, **CAV**, volume 4590 of **Lecture Notes in Computer Science**, pages 20–36. Springer, 2007. 149

[40] L. de Moura and H. Ruess. Lemmas on demand for satisfiability solvers. In **Proceedings of the 5th international conference on Theory and Applications of Satisfiability Testing**, SAT'02. Springer-Verlag, 2002. 149

[41] L. de Moura and H. Ruess. An experimental evaluation of ground decision procedures. In R. Alur and D. Peled, editors, **Proceedings of the 16th International Conference on Computer Aided Verification, CAV'04 (Boston, Massachusetts)**, volume 3114 of **Lecture Notes in Computer Science**, pages 162–174. Springer, 2004. 148

[42] L. Mendonça de Moura and N. Bjørner. Z3: An efficient SMT solver. In C. R. Ramakrishnan and J. Rehof, editors, **Tools and Algorithms for the Construction and Analysis of Systems, 14th International Conference, TACAS 2008, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2008, Budapest, Hungary, March 29-April 6, 2008. Proceedings**, pages 337–340, 2008. 10, 149, 150

[43] P. Dellunde, L. Godo, and E. Marchioni. Extending possibilistic logic over Gödel logic. **International Journal of Approximate Reasoning**, 52(1):63–75, 2011. 124

[44] D. Dubois, J. Lang, and H. Prade. Possibilistic logic. In **Handbook of logic in artificial intelligence and logic programming, Vol. 3**, Oxford Sci. Publ., pages 439–513. Oxford Univ. Press, New York, 1994. 123

[45] D. Dubois and H. Prade. Possibilistic logic: a retrospective and prospective view. **Fuzzy Sets and Systems**, 144(1):3–23, 2004. 123

[46] B. Dutertre and L. de Moura. A fast linear-arithmetic solver for DPLL(T). In **CAV**, pages 81–94. Springer, 2006. 149

[47] N. Eén and N. Sörensson. An extensible SAT-solver. In Enrico Giunchiglia and Armando Tacchella, editors, **SAT**, volume 2919 of **Lecture Notes in Computer Science**, pages 502–518. Springer, 2003. 147

[48] F. Esteva, J. Gispert, L. Godo, and C. Noguera. Adding truth-constants to logics of continuous t-norms: Axiomatization and completeness results. **Fuzzy Sets and Systems**, 158(6):597–618, March 2007. 39, 64, 66

[49] F. Esteva and L. Godo. Monoidal t-norm based logic: towards a logic for left-continuous t-norms. **Fuzzy Sets and Systems**, 124:271–288, 2001. 5, 28, 34

[50] F. Esteva, L. Godo and À. García-Cerdaña35
. On the hierarchy of t-norm based residuated fuzzy logics. **On the hierarchy of t-norm based residuated fuzzy logics**, Studies in Fuzziness and Soft Computing, Ed. M. Fitting and E. Orłowska. 114 251–272, 2003

[51] F. Esteva, L. Godo, P. Hájek, and M. Navara. Residuated fuzzy logics with an involutive negation. **Archive for Mathematical Logic**, 39(2):103–124, 2000. 34, 39

[52] F. Esteva, L. Godo, and M. Montagna. Equational Characterization of the Subvarieties of BL Generated by t-norm Algebras. **Studia Logica**, 76(2):161–200, 2004. 30, 64, 66

[53] F. Esteva, L. Godo, and C. Noguera. Expanding the propositional logic of a t-norm with truth-constants: completeness results for rational semantics. **Soft Computing**, 14:273–284, 2010. 39, 40

[54] F. Esteva, L. Godo, and C. Noguera. On expansions of WNM t-norm based logics with truth-constants. **Fuzzy Sets and Systems**, 161(3):347–368, 2010. 39

[55] C. G. Fermüller and H. Langsteiner. Tableaux for finite-valued logics with arbitrary distribution modalities. In **Automated reasoning with analytic tableaux and related methods (Oisterwijk, 1998)**, volume 1397 of **Lecture Notes in Computer Science**, pages 156–171. Springer, Berlin, 1998. 7

[56] G. Fischer-Servi. On modal logics with an intuitionistic base. **Studia Logica**, 36:141–149, 1977. 108

[57] M. Fitting. Many-valued modal logics. **Fundamenta Informaticae**, 15:235–254, 1992. 6, 7, 48

[58] M. Fitting. Many-valued modal logics, II. **Fundamenta Informaticae**, 17:55–73, 1992. 6, 7, 48

[59] J. M. Font and R. Jansana. **A general algebraic semantics for sentential logics**, volume 7 of **Lecture Notes in Logic**. Springer-Verlag, 1996. 135 pp. Presently distributed by the Association for Symbolic Logic. 21

[60] J. M. Font, R. Jansana, and D. Pigozzi. A survey of Abstract Algebraic Logic. Mathematics Preprint Series 329, IMUB (University of Barcelona), April 2003. 21

[61] D. M. Gabbay and J. Woods, editors. **Logic and the modalities in the Twentieth Century**, volume 7 of **Handbook of the History of Logic**. North-Holland (Elsevier), 2006. 47

[62] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. **Residuated Lattices: an algebraic glimpse at substructural logics**, volume 151 of **Studies in Logic and the Foundations of Mathematics**. Elsevier, Amsterdam, 2007. 33

[63] M. Gehrke and Y. Venema. Algebraic Tools for Modal Logic. Manuscript. ESSLLI'01 (Helsinki), 2001. 46

[64] S. Givant and P. Halmos. **Introduction to Boolean Algebras**. Undergraduate Texts in Mathematics. Springer, 2009. 96

[65] K. Gödel. Zur intuitionistischen Aussagenkalkül. 1932. English traslation, with an introduction note by A. S. Troelstra, in **{Kurt Gödel, Collected Works, Volume 1 4, 32**
, Solomon Feferman et. al. (eds.), Oxford University Press, 1986, 222–225.

[66] E. Goldberg and Y. Novikov. Berkmin: A fast and robust sat-solver. Discrete Appl. Math., **155(12):1549–1561, 2007. 147**

[67] R. I. Goldblatt. Metamathematics of modal logic, I. Reports on Mathematical Logic, **6:41–78, 1976. 47**

[68] R. I. Goldblatt. Metamathematics of modal logic, II. Reports on Mathematical Logic, **7:21–52, 1976. 47**

[69] R. I. Goldblatt and S. K. Thomason. Axiomatic classes in propositional modal logic. In J. Crossley, editor, Algebra and logic, **volume 450 of** Lecture Notes in Mathematics, **pages 163–173. Springer, 1974. 47**

[70] P. Hájek. Basic fuzzy logic and BL-algebras. Soft Computing, **2:124–128, 1998. 5**

[71] P. Hájek. Metamathematics of fuzzy logic, **volume 4 of** Trends in Logic—Studia Logica Library. **Kluwer Academic Publishers, Dordrecht, 1998. 5, 29, 30, 31, 34, 35, 36, 38, 39, 40, 67, 187**

[72] P. Hájek. Observations on the monoidal t-norm logic. Fuzzy Sets and Systems, **132(1):107–112, 2002. Possibility theory and fuzzy logic. 34**

[73] P. Hájek. Making fuzzy description logic more general. Fuzzy Sets and Systems, **154(1):1–15, 2005. 161**

[74] P. Hájek. On fuzzy modal logics $S5(\mathcal{C})$. Fuzzy Sets and Systems, **161(18):2389–2396, 2010. 59**

[75] P. Hájek, L. Godo, and F. Esteva. A complete many-valued logic with product-conjunction. Archive for Mathematical Logic, **35:191–208, 1996. 5, 33, 34**

[76] P. Hájek, D. Harmancová, F. Esteva, P. Garcia, and L. Godo. On modal logics for qualitative possibility in a fuzzy setting. In R. López de Mántaras and D. Poole, editors, Proceedings of the 10th Annual Conference on Uncertainty in Artificial Intelligence (UAI-94), **pages 278–285, San Francisco, CA, 1994. Morgan Kaufmann. 125**

[77] G. Hansoul and B. Teheux. Completeness results for many-valued Łukasiewicz modal systems and relational semantics, 2006. Available at http://arxiv.org/abs/math/0612542. 6

[78] G. Hansoul and B. Teheux. Extending łukasiewicz logics with a modality: Algebraic approach to relational semantics. *Studia Logica*, 101(3):505–545, 2013. 6, 7, 54, 55, 105, 122

[79] D. Harel. Dynamic logic. In D. M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic, Volume II: Extensions of Classical Logic*, volume 165 of *Synthese Library*, chapter II.10, pages 497–604. D. Reidel Publishing Co., Dordrecht, 1984. 41

[80] D. Harel, D. Kozen, and J. Tiuryn. *Dynamic logic*. Foundations of Computing Series. MIT Press, Cambridge, MA, 2000. 41

[81] A. Heyting. Die formalen regeln der intuitionistischen logik. *Sitzungsberichte der Preussischen Akademie der Wissenschaften*, pages 42–56, 1930. 4

[82] U. Höhle. Commutative, residuated l-monoids. In U. Höhle and E. P. Klement, editors, *Non-classical logics and their applications to fuzzy subsets (Linz, 1992)*, volume 32 of *Theory Decis. Lib. Ser. B Math. Statist. Methods*, pages 53–106. Kluwer Acad. Publ., Dordrecht, 1995. 33

[83] R. Jansana. Abstract modal logics. *Studia Logica*, 55(2):273–299, 1995. 46

[84] S. Jenei and F. Montagna. A proof of standard completeness for Esteva and Godo's logic MTL. *Studia Logica*, 70:183–192, 2002. 5, 28

[85] C. D. Koutras. A catalog of weak many-valued modal axioms and their corresponding frame classes. *Journal of Applied Non-Classical Logics*, 13(1):47–72, 2003. 7

[86] C. D. Koutras and C. Nomikos. The computational complexity of satisfiability in many-valued modal logic. In *Proceedings of the 3rd Panhellenic Logic Symposium*, Anogia, Greece, July 2001. 7

[87] C. D. Koutras, C. Nomikos, and P. Peppas. Canonicity and completeness results for many-valued modal logics. *Journal of Applied Non-Classical Logics*, 12(1):7–41, 2002. 7

[88] C. I. Lewis. *A survey of symbolic logic*. University of California Press, Berkeley, 1918. 2

[89] J. Łukasiewicz. Philosophische Bemerkungen zu mehrwertigen systemen der Aussagenlogik. *Comptes Rendus des Séances de la Société des ciences et des Lettres de Varsovie, Cl. III*, 23:51–77, 1930. English translation in [91], 153–178. 32

[90] J. Łukasiewicz. On three-valued logic (1920). In S. McCall, editor, *Polish Logic*, pages 16–18. Oxford University Press, 1967. 3

[91] J. Łukasiewicz. *Selected Works, edited by L. Borkowksi*. Studies in Logic and the Foundations of Mathematics. North-Holland, Amsterdam, 1970. 208

[92] J. Malinowski. Modal equivalential logics. *Journal of Non-Classical Logic*, 3:13–35, 1986. 46, 131

[93] Y. Maruyama. Algebraic study of lattice-valued logic an lattice-valued modal logic. In R. Ramanujam and S. Sarukkai, editors, Logic and Its Applications. Third Indian Conference, ICLA 2009 Chennai, India, January 7-11, 2009 Proceedings, volume 5378 of Lecture Notes in Computer Science, pages 170–184. Springer-Verlag, 2009. Lecture Notes in Artificial Intelligence. 46

[94] J. McCarthy. Towards a mathematical science of computation. In In IFIP Congress, pages 21–28. North Holland, 1962. 149

[95] G. Metcalfe and N. Olivetti. Towards a proof theory of Gödel modal logics. Logical Methods in Computer Science, 7(2):27, 2011. 58, 59

[96] J. Ch. Meyer and W. van der Hoek. Epistemic logic for AI and computer science, volume 41 of Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, Cambridge, 1995. 41

[97] A. M. Mironov. Fuzzy modal logics. Journal of Mathematical Sciences, 128(6):3641–3483, 2005. 6, 48

[98] F. Montagna. Generating the variety of BL-algebras. Soft Computing, 9(12):869–874, 2005. 36

[99] F. Montagna. Notes on strong completeness in Łukasiewicz, product and BL logics and in their first-order extensions. In Algebraic and Proof-theoretic Aspects of Non-classical Logics, pages 247–274, 2006. 66, 71

[100] F. Montagna, C. Noguera, and R. Horčík. On weakly cancellative fuzzy logics. Journal of Logic and Computation, 16(4):423–450, 2006. 34

[101] P. S. Mostert and A. L. Shields. On the structure of semigroups on a compact manifold with boundary. Annals of Mathematics. Second Series, 65:117–143, 1957. 5, 35

[102] G. Nelson and D. Oppen. Fast decision procedures based on congruence closure. J. ACM, 27(2):356–364, 1980. 147

[103] R. Nieuwenhuis, A. Oliveras, and C. Tinelli. Solving SAT and SAT modulo theories: From an abstract Davis–Putnam–Logemann–Loveland procedure to DPLL(T). Journal of the Association for Computing Machinery, 53:937–977, 2006. 148, 149

[104] P. Ostermann. Many-valued modal propositional calculi. Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, 34(4):343–354, 1988. 55

[105] J. Pavelka. On fuzzy logic. I. Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, 25(1):45–52, 1979. Many-valued rules of inference. 38

[106] J. Pavelka. On fuzzy logic. II. Enriched residuated lattices and semantics of propositional calculi. Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, 25(2):119–134, 1979. 38

[107] J. Pavelka. On fuzzy logic. III. Semantical completeness of some many-valued propositional calculi. Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, 25(5):447–464, 1979. 38

[108] H. Rasiowa. An algebraic approach to non-classical logics, volume 78 of Studies in Logic and the Foundations of Mathematics. North-Holland, Amsterdam, 1974. 24

[109] N. Rescher and A. Urquhart. *Temporal logic*. Springer-Verlag, 1971. 41

[110] R. Rothenberg. A class of theorems in Łukasiewicz logic for benchmarking automated theorem provers. In N. Olivetti and C. Schwind, editors, *TABLEAUX '07, Automated Reasoning with Analytic Tableaux and Related Methods, Position Papers*, number LSIS.RR.2007.002, pages 101–111, 2007. 187, 188

[111] P. Savický, R. Cignoli, F. Esteva, L. Godo, and C. Noguera. On product logic with truth-constants. *Journal of Logic and Computation*, 16(2):205–225, 2006. 39, 40

[112] R. Shostak. Deciding linear inequalities by computing loop residues. *J. ACM*, 28(4):769–779, October 1981. 147

[113] SMT-COMP. http://smtcomp.sourceforge.net/. 150

[114] O. Strichman, S. Seshia, and R. Bryant. Deciding separation formulas with SAT. In *Proceedings of the 14th International Conference on Computer Aided Verification*, CAV '02, pages 209–222, London, UK, UK, 2002. Springer-Verlag. 148

[115] G. Takeuti and S. Titani. Intuitionistic fuzzy logic and intuitionistic fuzzy set theory. *The Journal of Symbolic Logic*, 49(3):851–866, 1984. 68

[116] B. Teheux. A duality for the algebras of a Łukasiewicz $n+1$-valued modal system. *Studia Logica*, 87(1):13–36, 2008. 7, 54

[117] B. Teheux. *Algebraic approach to modal extensions of Łukasiewicz logics*. Ph. D. Dissertation, Université de Liège, 2009. Electronically available at: `http://orbi.ulg.ac.be/bitstream/2268/10887/1/These.pdf`. 6, 7, 54, 55

[118] J. van Benthem, J. van Eijck, and B. Kooi. Logics of communication and change. *Information and Computation*, 204(11):1620–1662, 2006. 41

[119] J. F. A. K. van Benthem, S. van Otterloo, and O. Roy. Preference logic, conditionals and solution concepts in games. Research Report PP-2005-28, Institute for Logic, Language and Computation, University of Amsterdam, Amsterdam, 2005. 41

[120] Y. Venema. Temporal logic. In L. Goble, editor, *The Blackwell guide to philosophical logic*, Blackwell Philos. Guides, pages 203–223. Blackwell, Oxford, 2001. 41

[121] A. Vidal, F. Bou, and L. Godo. An smt-based solver for continuous t-norm based logics. In E. Hüllermeier, S. Link, T. Fober, and B. Seeger, editors, *Scalable Uncertainty Management*, volume 7520 of *Lecture Notes in Computer Science*, pages 633–640. Springer Berlin Heidelberg, 2012. 165, 166, 199

[122] A. Vidal, F. Esteva, and L. Godo. About standard completeness of product logic. In F. Bobillo et al., editor, *XVII Congreso Español sobre Tecnologías y Lógica Fuzzy (ESTYLF)*, pages 423–428, 2014. 198

[123] A. Vidal, F. Esteva, and L. Godo. Axiomatising a fuzzy modal logic over the standard product algebra. In M. Baaz, A. Ciabattoni, and S. Hetzl, editors, *Logic, Algebra and Truth Degrees 2014. Booklet of Abstracts*, pages 275–279, 2014. 198

[124] **A. Vidal, F. Esteva, and L. Godo. A product modal logic. In T. Flaminio et al., editor,** 35th Linz Seminar on Fuzzy Set Theory, **pages 127–130, 2014.** 198

[125] **A. Vidal, F. Esteva, and L. Godo. On strong standard completeness of mtl*(q) expansions. In** Topology, Algebra and Categories in Logic, **pages 275–276, 2015.** 198

[126] **A. Vidal, F. Esteva, and L. Godo. On strongly standard complete fuzzy logics: Mtl*(q) and its expansions. In** Proceedings of the 9th Conference of the European Society for Fuzzy Logic and Technology (IFSA-EUSFLAT), **pages 828–835. Atlantis Press, 2015.** 198

[127] **A. Vidal, F. Esteva, and L. Godo. On modal extensions of product fuzzy logic.** Journal of Logic and Computation, **(In press, 2015).** 198

[128] **L. A. Zadeh. Fuzzy sets.** Information and Control, **8:338–353, 1965.** 3

[129] **L. A. Zadeh. Preface in Fuzzy Logic technology and Applications (R. J. Marks-II Ed.), IEEE Technical Activities Board, 1994.**

[130] **Z3. http://research.microsoft.com/en-us/um/redmond/projects/z3/index.html.** 169