A betting metaphor for belief functions on MV-algebras and fuzzy epistemic states

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Consider the following game. Two players, Bookmaker (**B**) and Gambler (**G**) agree in betting on a finite set of events described by functions $e_1, \ldots, e_n$ from a set of possible worlds $X = \{w_1, \ldots, w_k\}$ into $[0, 1]$ whose realizations are unknown now and which, in the future, will be evaluated in the possible worlds of X.

The events will be evaluated by the following stipulation: the two players will share a *common epistemic state* about the whole class of possible worlds which is represented by a map $\pi : X \to [0, 1]$ such that, for each $w_i \in X$, $\pi(w_i)$ represents the *feasibility degree* of w_i for both **B** and **G**. Therefore, given any epistemic state π and an event e_i, the aggregated value of e_i from π is computed by the following formula:

$$N_\pi(e_i) = \min\{\pi(w_j) \Rightarrow w_j(e_i) : j = 1, \ldots, k\}.$$

The game can hence be described by the following steps:

Stage 1 **G** fixes finitely many events $e_1, \ldots, e_n \in [0, 1]^X$ and publishes her book $\beta : e_i \mapsto \beta_i$ (for $i = 1, \ldots, n$).

Stage 2 **B** chooses stakes $\sigma_1, \ldots, \sigma_n$ each for each event in the book β and pays $\sum_{i=1}^n \sigma_i \cdot \beta_i$ to **G**.

Now, assume that an *epistemic state* $\pi : X \to [0, 1]$ is reached by both players as according to the rule previously described. Then the game proceeds in the following way:

Stage 3 Both players **B** and **G**, evaluate N_π of each event e_i of β in π. In other words they calculate $N_\pi(e_i)$ for each $i = 1, \ldots, n$.

Stage 4 **B** pays to **G** the amount $\sum_{i=1}^n \sigma_i \cdot N^\pi(e_i)$.

Definition 1. *According with the previous game, a book $\beta : e_i \mapsto \beta_i$ is called* B-coherent *iff there is no possible choice of stakes $\sigma_1, \ldots, \sigma_n$ ensuring* **G** *a sure win in every epistemic state π.*

In [8, 5] a generalization of belief function theory in the frame of MV-algebras has been proposed. The main idea of this approach is to define a belief function **b** over an MV-algebra of fuzzy sets $M = [0, 1]^X$ (where X is a finite set of cardinality k that represents the set of possible worlds we will take in consideration) as a state [12] over a separable

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MV-subalgebra $\mathcal{R}$ of $[0,1]^M$, that strictly contains the free MV-algebra over k generators $Free(k)$ (see [4]). More precisely, we call a mapping $\mathbf{b} : M \to [0,1]$ a *generalized belief function* if there is an state $\mathbf{s} : \mathcal{R} \to [0,1]$ such that, for every $f \in M$,

$$\mathbf{b}(f) = \mathbf{s}(\rho_f),$$

where $\rho_f : M \to [0,1]$ is defined as

$$\rho_f(g) = \inf_{x \in X} g(x) \Rightarrow f(x),$$

with $\Rightarrow$ being Łukasiewicz implication function in the standard MV-algebra $[0,1]_{MV}$.

The following result shows that B-coherence is a characterization of belief functions on MV-algebras in the same way as de Finetti's coherence [2] is a characterization for probability measures on Boolean algebras. It is worth recalling that a generalization of de Finetti's coherence criterion to the case of MV-algebras has been proved by Mundici [12] and Kühr and Mundici [9]. Moreover the following theorem generalizes classical results by Jaffray [6] and Paris [13].

Theorem 2. *Let X be a finite set of possible worlds, let $e_1, \ldots, e_n \in [0,1]^X$ be events and let $\beta : e_i \mapsto \beta_i$ be a book. Then the following are equivalent:*

- *β is B-coherent;*
- *There exists a belief function* $\mathbf{b}$ *on $[0,1]^X$ extending β.*

It is worth noticing that, if we restrict our attention to those particular possibility distributions like $\pi_w : X \to [0,1]$ such that $\pi_w(w') = 0$ if $w \neq w'$ and $\pi_w(w) = 1$, then $N^{\pi_w}(\cdot) = w(\cdot)$ and hence the resulting betting game coincides with the usual betting game for states. On the other hand, in the general case, a natural notion of *indeterminacy* of an event e in an epistemic state defined by a possibility distribution π, is given by the value $I^\pi(e) = \Pi^\pi(e) - N^\pi(e)$, where $\Pi^\pi(e) = 1 - N^\pi(\neg e)$.

In this setting, following [3, 10], we can consider a variant of the above discussed betting game in which, for every event e_i, the Bookmaker is obliged to give back to the Gambler a proportional amount of the balance regarding e_i according to $I_\pi(e_i)$. In particular, when $I_\pi(e_i) = 1$ (i.e. when there is total indeterminacy about e_i) the Bookmaker is obliged to call off the bet on e_i. The resulting game is hence a conditional game in which the realization of each event e_i is conditioned by its determinacy whose total balance is given by the expression

$$\sum_{i=1}^{n} (1 - I^\pi(e_i)) \cdot (\sigma_i \cdot (\alpha_i - N^\pi(e_i))).$$

The measure which characterizes the coherence of this variant of the game we have discussed can be regarded as a conditional probability on modal formulas. In particular a book $\beta : e_i \mapsto \beta_i$ is coherent iff there exists a conditional state $\mathbf{s}(\cdot \mid \cdot)$ in the sense of [7] on a suitably defined MV-algebra such that, for every $i = 1, \ldots, n$, $\beta_i = \mathbf{s}(\Box e_i \mid \Diamond e_i \to \Box e_i)$, where $\Box$ is the many-valued modal operator defined on Łukasiewicz logic as in [1] and as usual $\Diamond e = \neg\Box\neg e$.

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