On some substructural aspects of t-norm based logics*

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Abstract

We show in this paper that the (commutative integral bounded) semilatticed pseudocomplemented monoids $\mathbb{PM}^{s\ell}$ and the involutive algebras of this class, which are structures that appear in the semantical studies of t-norm based logics, are structures which are strongly linked with Gentzen systems without contraction and also with their corresponding external deductive systems.

Keywords: Substructural logics, t-norm based logics, algebraic logic, external deductive system, algebraization of Gentzen systems, residuated lattices, commutative integral bounded semilatticed pseudocomplemented monoids, Grišin algebras.

1 Introduction

Monoidal logic ML was introduced by Höhle [13] in the context of residuated structures and by Ono and Komori [21], under the name H_{BCK}, in the context of logics without contraction (see also [1, 10, 11]). It is usually given, up to equivalence, in the language $\langle \vee, \wedge, *, \rightarrow, \neg, 0, 1 \rangle$. When the axiom $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$ is added to ML we obtain the monoidal t-norm based logic MTL introduced by Esteva and Godo in [7]. Jenei and Montagna proved in [14] that this logic is the logic determined by left-continuous t-norms. It was shown in [1] that ML, denoted there by $IPC^*\backslash c$, is the external deductive system associated with the Gentzen system determined by the sequent calculus $\mathbf{FL_{ew}}$ [18, 20].

This paper contains a summary of the results obtained in [5] on the fragment of ML without implication and the fragment of ML without implication and without conjunction, and their relation with the axiomatic extension of ML obtained by adding the axiom $\neg\neg\varphi \rightarrow \varphi$. The main theorems of the paper are the ones that deal with algebraization of Gentzen systems, i.e., Theorems 20-22.

Let us describe the content of the paper. In Section 2 we remind the reader of the basic known facts about algebraization of Gentzen systems and the definitions of the logical systems from the literature that are used in this paper. In Section 3 we introduce the logical systems that we are interested in. In the next section, the algebras associated to these logical systems are introduced and we analyze several of their results. In Section 5 we discuss the algebraization theorems on the logical systems studied and several consequences of these theorems.

2 Basic concepts

2.1 Algebraizable Gentzen systems

First of all we recall some notions concerning Gentzen systems and their algebraization (see

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[22, 23, 8] for more information).

Let $Fm_{\mathcal{L}}$ be the set of $\mathcal{L}$-formulas of a propositional language $\mathcal{L}$, and $\alpha, \beta \subseteq \omega$. A *$\mathcal{L}$-sequent of type* $\langle \alpha, \beta \rangle$ is a pair $\langle \Gamma, \Delta \rangle$ of finite sequences of $\mathcal{L}$-formulas such that $length(\Gamma) \in \alpha$ and $length(\Delta) \in \beta$. We will write $\Gamma \Rightarrow \Delta$ instead of $\langle \Gamma, \Delta \rangle$. The set of $\mathcal{L}$-sequents of type $\langle \alpha, \beta \rangle$ will be denoted by $Seq_{\mathcal{L}}^{\langle \alpha, \beta \rangle}$. A *Gentzen system of type* $\langle \alpha, \beta \rangle$ is a pair $\mathcal{G} = \langle \mathcal{L}, \vdash_{\mathcal{G}} \rangle$, where $\vdash_{\mathcal{G}}$ is a finitary consequence relation on the set $Fm_{\mathcal{L}}$ that is invariant under substitutions.

An $(\mathcal{L}, \langle \alpha, \beta \rangle)$-*inference rule* is a set $r \subseteq \mathcal{P}_{fin}(Seq_{\mathcal{L}}^{\langle \alpha, \beta \rangle}) \times Seq_{\mathcal{L}}^{\langle \alpha, \beta \rangle}$ that is obtained as the closure under substitutions of a pair $\langle T, \Gamma \Rightarrow \Delta \rangle \in \mathcal{P}_{fin}(Seq_{\mathcal{L}}^{\langle \alpha, \beta \rangle}) \times Seq_{\mathcal{L}}^{\langle \alpha, \beta \rangle}$. We will use the pair $\langle T, \Gamma \Rightarrow \Delta \rangle$ to refer to the rule that it generates. We will often write this pair as

$$\frac{T}{\Gamma \Rightarrow \Delta}.$$

Axioms (also *initial sequents*) are a special kind of rule, the ones of the form $\langle \emptyset, \Gamma \Rightarrow \Delta \rangle$.

A rule $r = \langle T, \Gamma \Rightarrow \Delta \rangle$ is *derivable* in a Gentzen system $\mathcal{G} = \langle \mathcal{L}, \vdash_{\mathcal{G}} \rangle$ if $T \vdash_{\mathcal{G}} \Gamma \Rightarrow \Delta$. And a sequent $\Gamma \Rightarrow \Delta$ is called *derivable* in a Gentzen system if the rule $\langle \emptyset, \Gamma \Rightarrow \Delta \rangle$ is derivable in it. Most of the literature on Gentzen systems focusses only on the derivable sequents. The main difference between our approach and the standard one in the literature is that we aim to analyze all the consequence relation. A weaker notion than the derivability of a rule is its admissibility. A rule r is *admissible* in a Gentzen system if for every $\langle T, \Gamma \Rightarrow \Delta \rangle \in r$ and every substitution s, the derivability of all sequents in $\{s(\Gamma' \Rightarrow \Delta') : \Gamma' \Rightarrow \Delta' \in r\}$ imply the derivability of $s(\Gamma \Rightarrow \Delta)$.

An $(\mathcal{L}, \langle \alpha, \beta \rangle)$-*sequent calculus* is a set of $(\mathcal{L}, \langle \alpha, \beta \rangle)$-rules. Every $(\mathcal{L}, \langle \alpha, \beta \rangle)$-sequent calculus LX determines a Gentzen system $\mathcal{G}_{LX} = \langle \mathcal{L}, \vdash_{LX} \rangle$ of type $\langle \alpha, \beta \rangle$ in the following way: given $T \cup \{\Gamma \Rightarrow \Delta\} \subseteq Seq_{\mathcal{L}}^{\langle \alpha, \beta \rangle}$, $\Gamma \Rightarrow \Delta$ *follows from* T in Gentzen system $\mathcal{G}_{LX}$ (and we write $T \vdash_{LX} \Gamma \Rightarrow \Delta$) iff there is a finite sequence of sequents $\Gamma_0 \Rightarrow \Delta_0, \ldots, \Gamma_{n-1} \Rightarrow \Delta_{n-1}$ (called a *proof of* $\Gamma \Rightarrow \Delta$ *from* T) such that $\Gamma_{n-1} \Rightarrow \Delta_{n-1} = \Gamma \Rightarrow \Delta$, and for every $i < n$ some of the following conditions hold:

1. $\Gamma_i \Rightarrow \Delta_i \in T$,
2. $\Gamma_i \Rightarrow \Delta_i$ is obtained from the set $\{\Gamma_j \Rightarrow \Delta_j : j < i\}$ by means of a rule r of the calculus LX, i.e., $\langle T', \Gamma_i \Rightarrow \Delta_i \rangle \in r \in LX$ for some $T' \subseteq \{\Gamma_j \Rightarrow \Delta_j : j < i\}$.

In this case we will say that $\mathcal{G}_{LX}$ is the *Gentzen system determined by the sequent calculus LX*. Remember that we use the rules of the calculus to obtain sequents from sets of sequents (and not only from the empty set).

A Gentzen system $\mathcal{G} = \langle \mathcal{L}, \vdash_{\mathcal{G}} \rangle$ of type $\langle \alpha, \beta \rangle$ is *algebraizable* iff there is a quasivariety K, a map τ from $Seq_{\mathcal{L}}^{\langle \alpha, \beta \rangle}$ into subsets of $\mathcal{L}$-equations and a map ρ from $\mathcal{L}$-equations into subsets of $Seq_{\mathcal{L}}^{\langle \alpha, \beta \rangle}$ such that:

1. $T \vdash_{\mathcal{G}} \Gamma \Rightarrow \Delta$ iff $\{\tau(\Gamma' \Rightarrow \Delta') : \Gamma' \Rightarrow \Delta' \in T\} \models_{\mathsf{K}} \tau(\Gamma \Rightarrow \Delta)$,
2. $\varphi \approx \psi \dashv\models_{\mathsf{K}} \tau(\rho(\varphi \approx \psi))$,
3. $\Theta \models_{\mathsf{K}} \varphi \approx \psi$ iff $\rho(\Theta) \vdash_{\mathcal{G}} \rho(\varphi \approx \psi)$,
4. $\Gamma \Rightarrow \varphi \dashv\vdash_{\mathcal{G}} \rho(\tau(\Gamma \Rightarrow \varphi))$,

where $\models_{\mathsf{K}}$ is the equational logic associated with the quasivariety K. The quasivariety K is uniquely determined by the Gentzen system, and it is called the *equivalent algebraic semantics* for the Gentzen system. The definition that one can find in [23] is slightly different. There, some additional constraints are imposed on τ and ρ, which essentially say that these translations are given schematically. But as far as the results of this paper are concerned we can forget these constraints. One of the interesting consequences of the general theory on algebraization of Gentzen systems is that if K is the equivalent algebraic semantics for a Gentzen system, then there is a characterization of the K-congruences of the $\mathcal{L}$-algebras (see [22, Theorem 2.23] for the details).

Remark. Throughout the paper we will use the notion of deductive system and the algebraization of deductive systems, as in [4]. A deductive system is essentially the same as a Gentzen system of type $\langle \{0\}, \{1\} \rangle$, and its algebraization corresponds to the algebraization considered as a Gentzen system.

Let us recall the notion of the external deductive system associated with a Gentzen system [2]. The *external deductive system* associated with a Gentzen system $\mathcal{G}$ of type $\langle \alpha, \beta \rangle$, with $0 \in \alpha$, $1 \in \beta$, is the deductive system $\langle \mathcal{F}m_{\mathcal{L}}, \vdash \rangle$ defined in the following way: given $\Sigma \cup \{\varphi\} \subseteq \mathcal{F}m_{\mathcal{L}}$, $\Sigma \vdash \varphi$ iff there is a finite subset $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Sigma$ such that $\emptyset \Rightarrow \varphi_1, \ldots, \emptyset \Rightarrow \varphi_n \vdash_{\mathcal{G}} \emptyset \Rightarrow \varphi$. If the Gentzen system is determined by a sequent calculus, we also call it the *external deductive system associated with this sequent calculus.*

2.2 The calculi $\mathbf{FL_{ew}}$ and $\mathbf{CFL_{ew}}$

We now recall two Gentzen calculi that are well known in the literature. We will use them in the rest of the paper.

Definition 1. (Cf. [18]) Let $\mathcal{L}$ be the propositional language $\langle \vee, \wedge, *, \to, \neg, 0, 1 \rangle$ of type $\langle 2, 2, 2, 2, 1, 0, 0 \rangle$. $\mathbf{CFL_{ew}}$ is the calculus of $\mathcal{L}$-sequents of type (ω, ω) defined by the following axioms and rules[1]:

Axioms:

$$\varphi \Rightarrow \varphi \quad (\text{Ax.1}) \qquad 0 \Rightarrow \emptyset \quad (\text{Ax.2}) \qquad \emptyset \Rightarrow 1 \quad (\text{Ax.3})$$

Structural rules:

$$\frac{\Gamma \Rightarrow \varphi, \Theta \qquad \Sigma, \varphi, \Pi \Rightarrow \Delta}{\Sigma, \Gamma, \Pi \Rightarrow \Delta, \Theta} \quad (\text{Cut})$$

$$\frac{\Gamma, \varphi, \psi, \Sigma \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Sigma \Rightarrow \Delta} \quad (e \Rightarrow) \qquad \frac{\Gamma \Rightarrow \Lambda, \varphi, \psi, \Theta}{\Gamma \Rightarrow \Lambda, \psi, \varphi, \Theta} \quad (\Rightarrow e)$$

$$\frac{\Gamma, \Sigma \Rightarrow \Delta}{\Gamma, \varphi, \Sigma \Rightarrow \Delta} \quad (w \Rightarrow) \qquad \frac{\Gamma \Rightarrow \Lambda, \Theta}{\Gamma \Rightarrow \Lambda, \varphi, \Theta} \quad (\Rightarrow w)$$

Rules of introduction of connectives:

$$\frac{\Gamma, \varphi, \Sigma \Rightarrow \Delta \qquad \Gamma, \psi, \Sigma \Rightarrow \Delta}{\Gamma, \varphi \vee \psi, \Sigma \Rightarrow \Delta} \quad (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Lambda, \varphi, \Theta}{\Gamma \Rightarrow \Lambda, \varphi \vee \psi, \Theta} \quad (\Rightarrow \vee_1) \qquad \frac{\Gamma \Rightarrow \Lambda, \psi, \Theta}{\Gamma \Rightarrow \Lambda, \varphi \vee \psi, \Theta} \quad (\Rightarrow \vee_2)$$

$$\frac{\Gamma, \varphi, \Sigma \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi, \Sigma \Rightarrow \Delta} \quad (\wedge_1 \Rightarrow) \qquad \frac{\Gamma, \psi, \Sigma \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi, \Sigma \Rightarrow \Delta} \quad (\wedge_2 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Lambda, \varphi, \Theta \qquad \Gamma \Rightarrow \Lambda, \psi, \Theta}{\Gamma \Rightarrow \Lambda, \varphi \wedge \psi, \Theta} \quad (\Rightarrow \wedge)$$

$$\frac{\Gamma, \varphi, \psi, \Sigma \Rightarrow \Delta}{\Gamma, \varphi * \psi, \Sigma \Rightarrow \Delta} \quad (* \Rightarrow) \qquad \frac{\Gamma \Rightarrow \varphi, \Lambda \quad \Sigma \Rightarrow \psi, \Theta}{\Gamma, \Sigma \Rightarrow \varphi * \psi, \Lambda, \Theta} \quad (\Rightarrow *)$$

$$\frac{\Gamma \Rightarrow \Lambda, \varphi \qquad \Sigma, \psi, \Pi \Rightarrow \Delta}{\Sigma, \Gamma, \varphi \to \psi, \Pi \Rightarrow \Lambda, \Delta} \quad (\to\Rightarrow)$$

$$\frac{\varphi, \Gamma \Rightarrow \psi, \Theta}{\Gamma \Rightarrow \varphi \to \psi, \Theta} \quad (\Rightarrow\to)$$

$$\frac{\Gamma \Rightarrow \varphi, \Theta}{\neg\varphi, \Gamma \Rightarrow \Theta} \quad (\neg \Rightarrow) \qquad \frac{\Gamma, \varphi \Rightarrow \Theta}{\Gamma \Rightarrow \neg\varphi, \Theta} \quad (\Rightarrow \neg)$$

And $\mathbf{FL_{ew}}$ is the calculus of $\mathcal{L}$-sequents of type $(\omega, \{0, 1\})$ obtained by the same axioms and rules (now, the right-part of the sequents has at most one formula).

Note that the propositional fragment of the calculus L^0K in [12] and also the calculus LK^0 in [17] are the same, up to definitional equivalence, as $\mathbf{CFL_{ew}}$.

Theorem 2. *([18, Theorem 6]) The calculi* $\mathbf{FL_{ew}}$ *and* $\mathbf{CFL_{ew}}$ *both satisfy the cut elimination theorem.*

2.3 The deductive systems $IPC^*\backslash c$ and $CPC^*\backslash c$

Here we recall the deductive systems from the literature that we will need. These systems are intuitionistic propositional logic without contraction $IPC^*\backslash c$ and classical propositional logic without contraction $CPC^*\backslash c$.

Definition 3. (Cf. [1]) $IPC^*\backslash c$ is the deductive system in the language $\mathcal{L} = \langle \vee, \wedge, *, \to, \neg, 0, 1 \rangle$ of type $\langle 2, 2, 2, 2, 1, 0, 0 \rangle$, defined by the rule Modus Ponens and the following axioms:

(A1) $\varphi \to 1$
(A2) $(\varphi \to \psi) \to ((\gamma \to \varphi) \to (\gamma \to \psi))$
(A3) $(\varphi \to (\psi \to \gamma)) \to (\psi \to (\varphi \to \gamma))$
(A4) $\varphi \to (\psi \to \varphi)$
(A5) $(\varphi \wedge \psi) \to \varphi$
(A6) $(\varphi \wedge \psi) \to \psi$
(A7) $\varphi \to (\psi \to (\varphi \wedge \psi))$
(A8) $((\gamma \to \varphi) \wedge (\gamma \to \psi)) \to (\gamma \to (\varphi \wedge \psi))$
(A9) $\psi \to (\varphi \vee \psi)$
(A10) $\varphi \to (\varphi \vee \psi)$
(A11) $(\varphi \to \gamma) \to ((\psi \to \gamma) \to ((\varphi \vee \psi) \to \gamma))$
(A12) $\varphi \to (\psi \to (\varphi * \psi))$
(A13) $(\varphi \to (\psi \to \gamma)) \to ((\varphi * \psi) \to \gamma)$
(A14) $0 \to \varphi$
(A15) $\neg\varphi \to (\varphi \to \psi)$
(A16) $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$
(A17) $\varphi \to \neg\neg\varphi$.

And $CPC^*\backslash c$ is the deductive system, in the same language, obtained by adding the axiom

(A18) $\neg\neg\varphi \to \varphi$.

[1] Strictly speaking each of these rules is a family of rules, and not only a rule.

It can be shown that $IPC^*\backslash c$ is, up to definitional equivalence, the same deductive system as monoidal logic [13] and H_{BCK} [21]. And $CPC^*\backslash c$ is, up to definitional equivalence, the propositional fragment of the so called Grisǐn Logic (see [12] and the references therein) and also of the system aMALL, the *affine* Multiplicative Additive Linear Logic [9].

3 What we study

In this section we introduce the logical systems that we study throughout the paper. Let us begin with those that are introduced through a Gentzen calculus.

Definition 4. The calculus obtained by deleting from $\mathbf{FL_{ew}}$ the rules of introduction of the additive conjunction and the implication will be denoted by $\mathbf{FL_{ew}}[\vee, *, \neg]$[2], and the calculus obtained by deleting from $\mathbf{FL_{ew}}$ the rules for the implication will be denoted by $\mathbf{FL_{ew}}[\vee, \wedge, *, \neg]$. The Gentzen systems associated to the calculus $\mathbf{FL_{ew}}[\vee, *, \neg]$ and $\mathbf{FL_{ew}}[\vee, \wedge, *, \neg]$ will be denoted by $\mathcal{G}_{\mathbf{FL_{ew}}}[\vee, *, \neg]$ and $\mathcal{G}_{\mathbf{FL_{ew}}}[\vee, \wedge, *, \neg]$, respectively. The corresponding external deductive systems will be denoted by $\mathcal{S}_e[\vee, *, \neg]$ and $\mathcal{S}_e[\vee, \wedge, *, \neg]$.

Definition 5. In the same way, but replacing $\mathbf{FL_{ew}}$ with $\mathbf{CFL_{ew}}$, we can introduce $\mathbf{CFL_{ew}}[\vee, *, \neg]$, $\mathbf{CFL_{ew}}[\vee, \wedge, *, \neg]$, $\mathcal{G}_{\mathbf{CFL_{ew}}}[\vee, *, \neg]$ and $\mathcal{G}_{\mathbf{CFL_{ew}}}[\vee, \wedge, *, \neg]$.

It can easily be seen that $\mathcal{G}_{\mathbf{CFL_{ew}}}[\vee, *, \neg]$ and $\mathcal{G}_{\mathbf{CFL_{ew}}}[\vee, \wedge, *, \neg]$ are, up to definitional equivalence, the same Gentzen system. For this reason, in the rest of the paper we will refer only to $\mathcal{G}_{\mathbf{CFL_{ew}}}[\vee, *, \neg]$.

Obviously, as $\mathbf{FL_{ew}}$ and $\mathbf{CFL_{ew}}$ satisfy the cut elimination, $\mathbf{FL_{ew}}[\vee, *, \neg]$, $\mathbf{FL_{ew}}[\vee, \wedge, *, \neg]$, $\mathbf{CFL_{ew}}[\vee, *, \neg]$ and $\mathbf{CFL_{ew}}[\vee, \wedge, *, \neg]$ also satisfy it.

Now it is time to introduce the other deductive systems that we will study.

Definition 6. The $\langle \vee, *, \neg, 0, 1\rangle$-fragment of $IPC^*\backslash c$ is the deductive system $\langle\langle \vee, *, \neg, 0, 1\rangle, \vdash\rangle$ given by:

$$\Sigma \vdash \varphi \quad \text{iff} \quad \Sigma \vdash_{IPC^*\backslash c} \varphi.$$

In the same way we can introduce the $\langle \vee, \wedge, *, \neg, 0, 1\rangle$-fragment of $IPC^*\backslash c$, the $\langle \vee, *, \neg, 0, 1\rangle$-fragment of $CPC^*\backslash c$, and the $\langle \vee, \wedge, *, \neg, 0, 1\rangle$-fragment of $CPC^*\backslash c$.

Notice that we do not give Hilbert-style axiomatizations of the previous fragments of $IPC^*\backslash c$. In fact this is an open problem. The difficulties come from Theorem 27. Although we do not give Hilbert-style axiomatizations of the previous fragments of $CPC^*\backslash c$, it is quite simple to find them. The reason is that $CPC^*\backslash c$ and its $\langle \vee, *, \neg, 0, 1\rangle$-fragment are definitionally equivalent.

4 The associated algebras

Throughout the section, we introduce the algebraic structures needed to study the logical systems so far presented. First of all, we remember the definition of residuated lattice, the algebraic counterpart of intuitionistic propositional logic without contraction.

Definition 7. (Cf. [16]) An algebra $\mathbf{A} = \langle A, \vee, \wedge, *, \rightarrow, \neg, 0, 1\rangle$ of type $\langle 2, 2, 2, 2, 1, 0, 0\rangle$ is a *residuated lattice* if it satisfies:

1. $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded lattice with associated order $\leq$,
2. $\langle A, *, 1\rangle$ is a commutative monoid,
3. $x * y \leq z$ iff $y \leq x \rightarrow z$ (the law of residuation),
4. $x * y \leq 0$ iff $y \leq \neg x$ (the law of *pseudo-complementation* associated with $*$).

The class of residuated lattices is denoted by $\mathbb{RL}$.

Sometimes in the literature these structures have been called commutative integral bounded residuated lattice, e.g., [19, 20, 15]. The negation $\neg$ is called the pseudocomplement associated with $*$. It can be seen as a generalization of the concept of pseudocomplement defined traditionally in the context of the bounded distributive lattices [3, page 152], and it is characterized by the fact that $\neg x = x \rightarrow 0$.

[2] It would be better to talk about $\mathbf{FL_{ew}}[\vee, *, \neg, 0, 1]$ because this is the language where this calculus is given, but for the sake of simplicity we will not do so.

Now it is time to introduce the two new families of algebras that we need in order to analyze our logical systems.

Definition 8. An algebra $\mathbf{A} = \langle A, \vee, *, \neg, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ is a *(commutative integral bounded) semilatticed pseudocomplemented monoid* if it satisfies:

1. $\langle A, \vee, 0, 1\rangle$ is a bounded semilattice with associated order $\leq$,
2. $\langle A, *, 1\rangle$ is a commutative monoid,
3. $x * (y \vee z) = (x * y) \vee (x * z)$,
4. $x * y \leq 0$ iff $y \leq \neg x$ (the law of pseudocomplementation associated with $*$).

The class of semilatticed pseudocomplemented monoids is denoted by $\mathbb{PM}^{s\ell}$.

Definition 9. An algebra $\mathbf{A} = \langle A, \vee, \wedge, *, \neg, 0, 1\rangle$ of type $\langle 2, 2, 2, 1, 0, 0\rangle$ such that the reduct is a semilatticed pseudocomplemented monoid and $\langle A, \vee, \wedge\rangle$ is a lattice is called a *(commutative integral bounded) latticed pseudocomplemented monoid*. The class of these algebras is denoted by $\mathbb{PM}^{\ell}$.

The authors are indebted to Roberto Cignoli for a personal communication concerning the following result.

Theorem 10. *The classes $\mathbb{PM}^{s\ell}$ and $\mathbb{PM}^{\ell}$ are varieties. They are axiomatized by the equations involved in their definitions plus the following equations (which replace the law of pseudocomplementation):*

1. $\neg 1 \approx 0$,
2. $\neg 0 \approx 1$,
3. $(x * \neg(y * x)) \vee \neg y \approx \neg y$.

It is easy to see that the reducts of residuated lattices are $\mathbb{PM}^{s\ell}$-algebras and $\mathbb{PM}^{\ell}$-algebras. In fact, we have the following improvements.

Theorem 11. *Every $\mathbb{PM}^{s\ell}$-algebra is embeddable into a complete residuated lattice. Thus, the class $\mathbb{PM}^{s\ell}$ is the closure under isomorphism and subalgebras of the class of algebras $\{\mathbf{A} : \mathbf{A} = \langle A, \vee, *, \neg, 0, 1\rangle$ and there are operations $\wedge, \to$ such that $\langle A, \vee, \wedge, *, \to, \neg, 0, 1\rangle \in \mathbb{RL}\}$.*

Theorem 12. *Every $\mathbb{PM}^{\ell}$-algebra is embeddable into a complete residuated lattice. Therefore, the class $\mathbb{PM}^{\ell}$ is the closure under isomorphism and subalgebras of the class of algebras $\{\mathbf{A} : \mathbf{A} = \langle A, \vee, \wedge, *, \neg, 0, 1\rangle$ and there is an operation $\to$ such that $\langle A, \vee, \wedge, *, \to, \neg, 0, 1\rangle \in \mathbb{RL}\}$.*

Finally, the last class of algebras that we consider is the subclass in which the pseudocomplement is involutive.

Definition 13. We denote by $\mathbb{IPM}^{s\ell}$, $\mathbb{IPM}^{\ell}$ and $\mathbb{IRL}$ the subvarieties of $\mathbb{PM}^{s\ell}$, $\mathbb{PM}^{\ell}$ and $\mathbb{RL}$, respectively, obtained by adding the equation $\neg\neg x \approx x$ (the involutive law) to the equations defining these classes.

Theorem 14. *Let $\mathbf{A} \in \mathbb{PM}^{s\ell}$. The following conditions are equivalent:*

1. $\mathbf{A} \vDash x * z \leq y \Leftrightarrow z \leq \neg(x * \neg y)$,
2. $\mathbf{A} \vDash \neg\neg x \approx x$,
3. *For every* $a, b \in A$, $\neg(\neg a \vee \neg b) = Inf\{a, b\}$,
4. $\mathbf{A} \vDash x \leq y \Leftrightarrow \neg(\neg x \vee \neg y) \approx x$.

Observe that the first condition says that $\neg(x * \neg y)$ is the residuum with respect to the operation $*$. And the third condition tells us that in fact these semilattices are lattices. Therefore, we can see the following statement.

Theorem 15. *Every $\mathbb{IPM}^{s\ell}$-algebra is the reduct of a $\mathbb{IRL}$-algebra.*

Corollary 16. *The varieties $\mathbb{IPM}^{s\ell}$, $\mathbb{IPM}^{\ell}$ and $\mathbb{IRL}$ are definitionally equivalent.*

These involutive algebras are nothing essentially new; as we will now see they are very close to latticed Grišin algebras.

Definition 17. (Cf. [12, latticed L^0-algebra]) A *latticed Grišin algebra* is an algebra $\mathbf{A} = \langle A, \vee, \wedge, +, *, \neg, 0, 1\rangle$ of type $\langle 2, 2, 2, 2, 1, 0, 0\rangle$ satisfying the following conditions:

1. $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded lattice,
2. $\langle A, +, 0\rangle$ and $\langle A, *, 1\rangle$ are commutative monoids,
3. $\mathbf{A} \vDash x \leq y \Rightarrow x * z \leq y * z$ and $\mathbf{A} \vDash x \leq y \Rightarrow x + z \leq y + z$,
4. $\mathbf{A} \vDash \neg x + x \approx 1$ and $\mathbf{A} \vDash \neg x * x \approx 0$.

The class of latticed Grišin algebras is a well-known variety [12]. We have the following result.

Theorem 18. *Let $\mathbf{A} \in \mathbb{IPM}^{s\ell}$. If we define on A two new binary operations by means of the following equations:*

$$x + y \approx \neg(\neg x * \neg y), \tag{1}$$

$$x \wedge y \approx \neg(\neg x \vee \neg y), \tag{2}$$

then the algebra $\langle A, \vee, \wedge, +, *, \neg, 0, 1\rangle$ is a latticed Grišin algebra. Conversely, every latticed Grišin algebra satisfies the equations defining the variety $\mathbb{IPM}^{s\ell}$ and the equations (1) and (2). Hence, the variety $\mathbb{IPM}^{s\ell}$ and the variety of latticed Grišin algebras are definitionally equivalent.

For the classical case we also have an embedding into complete lattices.

Theorem 19. *Every $\mathbb{IPM}^{s\ell}$-algebra is embeddable into a complete $\mathbb{IRL}$-algebra. Therefore, every $\mathbb{IRL}$-algebra is embeddable into a complete $\mathbb{IRL}$-algebra.*

5 Connections between the logical systems and the new varieties

First of all we enumerate the results concerning algebraization of our Gentzen systems. The first two can be seen as an improvement of [19, Corollary 9] while the third one is an improvement of [17, Theorem 9].

Theorem 20. *$\mathcal{G}_{\mathbf{FL_{ew}}}[\vee, *, \neg]$ is algebraizable with equivalent algebraic semantics the variety $\mathbb{PM}^{s\ell}$, with translations τ from sequents to equations and ρ from equations to sequents defined as follows:*

$$\tau(\gamma_0, ..., \gamma_{m-1} \Rightarrow \delta) = \\ = \begin{cases} \{(\gamma_0 * ... * \gamma_{m-1}) \vee \delta \approx \delta\}, & \text{if } m \geq 1 \\ \{1 \approx \delta\}, & \text{if } m = 0 \end{cases}$$

$$\tau(\gamma_0, ..., \gamma_{m-1} \Rightarrow \emptyset) = \\ = \begin{cases} \{\gamma_0 * ... * \gamma_{m-1} \approx 0\}, & \text{if } m \geq 1 \\ \{1 \approx 0\}, & \text{if } m = 0 \end{cases}$$

$$\rho(\varphi \approx \psi) = \{\varphi \Rightarrow \psi,\ \psi \Rightarrow \varphi\}.$$

Theorem 21. *$\mathcal{G}_{\mathbf{FL_{ew}}}[\vee, \wedge, *, \neg]$ is algebraizable with equivalent algebraic semantics the variety $\mathbb{PM}^{\ell}$, with translations τ from sequents to equations and ρ from equations to sequents defined in Theorem 20.*

Theorem 22. *$\mathcal{G}_{\mathbf{CFL_{ew}}}[\vee, *, \neg]$ is algebraizable with equivalent algebraic semantics the variety $\mathbb{IPM}^{s\ell}$, with translations τ from sequents to equations and ρ from equations to sequents defined as follows:*

$$\tau(\gamma_0, ..., \gamma_{m-1} \Rightarrow \delta_0, ..., \delta_{n-1}) = \\ = \begin{cases} \{(\gamma_0 * ... * \gamma_{m-1}) \to \\ \quad \to (\delta_0 + ... + \delta_{n-1}) \approx 1\}, & \text{if } m \neq 0,\ n \neq 0 \\ \{1 \approx \delta_0 + ... + \delta_{n-1}\}, & \text{if } m = 0,\ n \neq 0 \\ \{\gamma_0 * ... * \gamma_{m-1} \approx 0\}, & \text{if } m \neq 0,\ n = 0 \\ \{1 \approx 0\}, & \text{if } m = 0,\ n = 0 \end{cases}$$

$$\rho(\varphi \approx \psi) = \{\varphi \Rightarrow \psi,\ \psi \Rightarrow \varphi\},$$

where $\varphi \to \psi := \neg(\varphi * \neg\psi)$ and $\varphi + \psi := \neg(\neg\varphi * \neg\psi)$.

As a consequence of the algebraization we obtain that the three Gentzen systems are contraction-free.

Theorem 23. *The contraction rule is not derivable either in $\mathcal{G}_{\mathbf{FL_{ew}}}[\vee, *, \neg]$ or in $\mathcal{G}_{\mathbf{FL_{ew}}}[\vee, \wedge, *, \neg]$ or in $\mathcal{G}_{\mathbf{CFL_{ew}}}[\vee, *, \neg]$. Nor it is admissible.*

In the rest of the paper we explain several results on the deductive systems introduced.

By using the fact that $IPC^*\backslash c$ is the external deductive system associated with $\mathbf{FL_{ew}}$ (see [1, 5]), the previous results about subreducts and the theorems on algebraization, we obtain that the external deductive system associated with $\mathbf{FL_{ew}}[\vee, *, \neg]$ ($\mathbf{FL_{ew}}[\vee, \wedge, *, \neg]$) is the $\langle \vee, *, \neg, 0, 1\rangle$-fragment (the $\langle \vee, \wedge, *, \neg, 0, 1\rangle$-fragment) of the deductive system $IPC^*\backslash c$.

Theorem 24. *Let $\Sigma \cup \{\varphi\} \subseteq Fm_{\langle \vee, *, \neg, 0, 1\rangle}$. Then,*

$$\Sigma \vdash_{IPC^*\backslash c} \varphi \quad \textit{iff} \quad \Sigma \vdash_{\mathcal{S}_e[\vee, *, \neg]} \varphi.$$

That is, $\mathcal{S}_e[\vee, *, \neg]$ is the $\langle \vee, *, \neg, 0, 1\rangle$-fragment of $IPC^*\backslash c$.

Theorem 25. *Let $\Sigma \cup \{\varphi\} \subseteq Fm_{\langle \vee, \wedge, *, \neg, 0, 1\rangle}$.*

Then,

$$\Sigma \vdash_{IPC^*\backslash c} \varphi \quad \textit{iff} \quad \Sigma \vdash_{\mathcal{S}_e[\vee,\wedge,*,\neg]} \varphi.$$

That is, $\mathcal{S}_e[\vee,\wedge,*,\neg]$ *is the* $\langle\vee,\wedge,*,\neg,0,1\rangle$*-fragment of* $IPC^*\backslash c$.

For the classical case we have the following.

Theorem 26. *The* $\langle\vee,*,\neg,0,1\rangle$*-fragment of* $CPC^*\backslash c$ *is the external deductive system associated with* $\mathcal{G}_{\mathbf{CFL_{ew}}}[\vee,*,\neg]$.

Now we will explain the position of our deductive systems in the Abstract Algebraic Logic hierarchy.

Theorem 27. *Neither the deductive system* $\mathcal{S}_e[\vee,*,\neg]$ *nor* $\mathcal{S}_e[\vee,\wedge,*,\neg]$ *are protoalgebraic.*

Recall that a deductive system $\mathcal{S}$ is protoalgebraic if the Leibniz operator Ω is monotonic on the set of $\mathcal{S}$-theories (see for example [6]). This condition is equivalent to the fact that there is a set of formulas $P(p,q)$ (in two variables at most) such that:

$$\begin{array}{ll} \emptyset \vdash_{\mathcal{S}} P(p,p), & \text{(Reflexivity)} \\ \{p\} \cup P(p,q) \vdash_{\mathcal{S}} q, & \text{(Modus Ponens)} \end{array}$$

As a consequence there is no defined binary connective $\to$ such that:

$$\begin{array}{ll} \emptyset \vdash_{\mathcal{S}} p \to p, & \text{(Identity)} \\ p, p \to q \vdash_{\mathcal{S}} q, & \text{(Modus Ponens).} \end{array}$$

On the other hand, the behavior in the classical case is much better.

Theorem 28. *The* $\langle\vee,*,\neg,0,1\rangle$*-fragment of* $CPC^*\backslash c$ *is algebraizable with equivalence formulas* $\{\neg(p*\neg q),\neg(q*\neg p)\}$ *and defining equation* $p \approx 1$*, and its equivalent algebraic semantics is the variety* $\mathbb{IPM}^{s\ell}$.

Although in the intuitionistic case we do not obtain algebraizable deductive systems, it can be seen that the algebraization results for our Gentzen systems give the following completeness statements.

Theorem 29. *The variety* $\mathbb{PM}^{s\ell}$ *is an algebraic semantics for the deductive system* $\mathcal{S}_e[\vee,*,\neg]$ *with defining equation* $p \approx 1$.

Remember that this last statement means that if $\Sigma \cup \{\varphi\} \subseteq Fm_{\langle\vee,*,\neg,0,1\rangle}$, then

$$\Sigma \vdash_{\mathcal{S}_e[\vee,*,\neg]} \varphi \quad \text{iff} \quad \{\sigma \approx 1 : \sigma \in \Sigma\} \models_{\mathbb{PM}^{s\ell}} \varphi \approx 1.$$

Theorem 30. *The variety* $\mathbb{PM}^{\ell}$ *is an algebraic semantics for the deductive system* $\mathcal{S}_e[\vee,\wedge,*,\neg]$ *with defining equation* $p \approx 1$.

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