Distinguished algebraic semantics for t-norm based fuzzy logics: methods and algebraic equivalencies

Petr Cintula[a,1], Francesc Esteva[b,2], Joan Gispert[c,2,3], Lluís Godo[b,2], Franco Montagna[d], Carles Noguera[d,2,4]

[a] *Institute of Computer Science, Academy of Sciences of the Czech Republic Pod vodárenskou věží 2, Prague, 182 07 Czech Republic*
[b] *Artificial Intelligence Research Institute (IIIA - CSIC) Campus de la UAB, 08193 Bellaterra, Catalonia, Spain*
[c] *Department of Probability, Logic and Statistics, University of Barcelona Gran Via de les Corts Catalanes 585, 08007 Barcelona, Catalonia, Spain*
[d] *Department of Mathematics and Computer Science, University of Siena Pian dei Mantellini 44, 53100 Siena, Italy*

Abstract

This paper is a contribution to the algebraic study of t-norm based fuzzy logics. In the general framework of propositional core and Δ-core fuzzy logics we consider three properties of completeness with respect to any semantics of linearly ordered algebras. Useful algebraic characterizations of these completeness properties are obtained and their relations are studied. Moreover, we concentrate on five kinds of distinguished semantics for these logics – namely the class of algebras defined over the real unit interval, the rational unit interval, the hyperreals (all ultrapowers of the real unit interval), the strict hyperreals (only ultrapowers giving a proper extension of the real unit interval) and finite chains, respectively – and we survey the known completeness methods and results for prominent logics. We also obtain new interesting relations between the real, rational and (strict) hyperreal semantics, and good characterizations for the completeness with respect to the semantics of finite chains. Finally, all completeness properties and distinguished semantics are also considered for the first-order versions of the logics where a number of new results are proved.

Key words: Algebraic Logic, Embedding properties, Fuzzy logics, Left-continuous t-norms, Mathematical Fuzzy Logic, MTL-algebras, Non-classical logics, Residuated lattices, Standard completeness, Substructural logics, Weakly implicative fuzzy logics

---

1. Introduction

In his seminal book [36], Hájek considered the problem of finding a common (well motivated) base for the most important fuzzy logics, namely Łukasiewicz, Gödel and product logics. There, he introduced a logic, named BL, and he proposed it for the role of basic fuzzy logic. Hájek's proposal was greatly supported by the proof that BL is the logic of all continuous t-norms[5] and

Email addresses: `cintula@cs.cas.cz` (Petr Cintula), `esteva@iiia.csic.es` (Francesc Esteva), `jgispertb@ub.edu` (Joan Gispert), `godo@iiia.csic.es` (Lluís Godo), `montagna@unisi.it` (Franco Montagna), `cnoguera@iiia.csic.es` (Carles Noguera)

URL: `http://www.cs.cas.cz/~cintula/` (Petr Cintula), `http://www.carlesnoguera.cat` (Carles Noguera)

[1]The work was supported by project 1M0545 of the Ministry of Education, Youth, and Sports of the Czech Republic.

[2]The work was supported by Spanish project MULOG2 (TIN2007-68005-C04).

[3]The work was supported by grant MTM2004-03101 of the Spanish Ministry of Education and Science, including FEDER funds of the European Union, an 2005SGR 00083 of D.U.R.S.I of Generalitat de Catalunya.

[4]The work was supported by grant 2006-BP-A-10043 of the Departament d'Educació i Universitats of the Generalitat de Catalunya.

[5]A t-norm is a non-decreasing commutative monoidal operation on $[0,1]$ with 1 as unit.

of their residua (see [35, 14]). But in [24] the authors observed that the minimal condition for a t-norm to have a residuum, and therefore to determine a logic, is left-continuity (continuity is not necessary). There, they proposed a weaker logic, called MTL (monoidal t-norm based logic), and conjectured that MTL is the logic of left-continuous t-norms and their residua. This conjecture was proved in [47]. Thus, it makes sense to propose it (instead of BL) as the real 'basic fuzzy logic' (this claim is also supported by an interesting methodological paper [6]). Another feature of MTL, which adds interest to it, is constituted by its relationship with substructural logics. Indeed, MTL is a logic without contraction (see [64]) and it can be characterized as FL_{ew} (i.e. Full Lambek calculus plus exchange and weakening, see [62, 32]) plus prelinearity. As the most important substructural logics, MTL can be formulated in a hypersequent calculus which enjoys cut-elimination (see [5]) by adding to the calculus for FL_{ew} Avron's *communication rule* [3], a rule which yields completeness with respect to linearly ordered (commutative, integral, bounded) residuated lattices. Therefore, MTL is a common base for (almost) all systems of fuzzy logic with the structural rules of exchange and weakening. Further generalizations are of course possible, mainly by removing these structural rules.

While BL and its extensions have been studied in a rather systematic way (see e.g. [14, 60, 20, 36, 35, 41]), the knowledge of MTL is still partial, in spite of a number of interesting publications, e.g. [22, 43, 42, 47, 56, 58, 55, 59]. The reason is that the structure of MTL-algebras is by far more complex than the structure of BL-algebras (which is more-or-less fully described in [36, 14, 15, 52, 20]). In any case, for some important expansions L of MTL, a *standard completeness* result has been shown, stating that for any set $T \cup \{\varphi\}$ of formulae, one has $T \vdash_{\mathrm{L}} \varphi$ iff for every L-algebra on $[0, 1]$ and for every evaluation v, if $v(\psi) = 1$ for all $\psi \in T$, then $v(\varphi) = 1$. This is a very important result, because for both the many-valued logics tradition (see e.g. [21, 50]) as well as for fuzzy logicians[6] (see e.g. [36, 34, 51]) the intended semantics is often real-valued.

Some logics, however, like BL, product logic and Łukasiewicz logic, satisfy the above completeness property only for finite T (to obtain it also for infinite T, one would need to use semantics based on L-algebras over either the rationals in $[0, 1]$ or a non-standard real interval). The situation for first-order fuzzy logics is even worse: for some logics like BL, Łukasiewicz or product logic, we do not even have standard completeness for $T = \emptyset$.

Another form of completeness, which is useful in order to obtain decidability, is completeness (for finite T) with respect to the class of finite L-algebras. This property and its variant consisting of the finite embeddability property (FEP), was used to show that many logics, like MTL, BL, Łukasiewicz logic and Gödel logic have a decidable consequence relation.

The purpose of this paper is to frame these completeness results, so far proved ad hoc in the literature, into a more general context where they will be consequences of new general theorems. Our generalization will proceed in two directions: taking an arbitrary class of L-chains instead of the class of standard L-chains or the class of finite chains, and considering a more general class of logics, that is, core and Δ-core fuzzy logics. These logics extend MTL, but may contain additional operators, provided that they are compatible with provable equivalence.

Then, after giving a number of general results concerning various types of completeness with respect to general classes of chains, we will state some consequences concerning some more familiar semantics based on the real unit interval, the rational unit interval, the hyperreals (all ultrapowers of the real unit interval), the strict hyperreals (only ultrapowers giving a proper extension of the real unit interval) and on finite chains respectively.

The paper is organized as follows. First, in Section 2 we introduce the necessary definitions, notation and preliminar results that we use throughout the paper, i.e. the notions of core and Δ-core fuzzy logics are defined together with their most prominent examples and further their algebraization and main properties are presented. In Section 3 three notions of semantical completeness ($\mathbb{K}$-completeness, finite strong $\mathbb{K}$-completeness and strong $\mathbb{K}$-completeness) are introduced and

[6]The central interest on real-valued semantics in Fuzzy Logic stems from the foundational work of Zadeh [71] on Fuzzy Sets and its extensions to other possible operations on $[0, 1]$, such as t-norms, to interpret operations between them (see e.g. [2]).

studied from a general point of view by letting the semantics $\mathbb{K}$ be any class of algebras. However, as all (Δ-)core fuzzy logics are complete with respect to the class of their linearly ordered algebras, we usually restrict our attention to classes of chain. With this restriction, we obtain good characterization results for some of these completeness properties; namely, we prove that a logic L has the strong $\mathbb{K}$-completeness iff every countable L-chain is embeddable into some chain of $\mathbb{K}$, and L has the finite strong $\mathbb{K}$-completeness iff every L-chain is partially embeddable into the class $\mathbb{K}$. These results show that some conditions that where used in the literature to prove standard completeness results were in fact not only sufficient but also necessary conditions. We also establish some interesting relations between the completeness properties. In Section 4 we introduce the five kinds of distinguished semantics for fuzzy logics. We survey the methods that have been used to prove completeness of the logics with respect some of these semantics and collect the known results for prominent fuzzy logics. In addition, by means of some algebraic and model-theoretic reasoning we show a strong link between the rational and the (strict) hyperreal semantics. On the contrary, we obtain a significant difference between standard and rational completeness properties, which is explained in algebraic terms. Finally, we show how completeness properties with respect to the semantics of finite chains are related to well-known algebraic properties: Finite Embeddability Property, Strong Finite Model Property and Finite Model Property.

The rest of the paper deals with the first-order versions of the logics. After recalling how they are uniformly produced from the corresponding propositional logics and the fundamentals of their model theory, we also consider the three completeness properties for these logics. We prove that a first-order fuzzy logic is strongly complete with respect to a class of models over the chains in $\mathbb{K}$ iff every countable model of the logic is elementarily equivalent to a model whose corresponding chain is in $\mathbb{K}$. Nevertheless, all known proofs of strong standard completeness for first-order fuzzy logics do not use this model-theoretic property, but an algebraic one: the fact that every countable chain of the variety is embeddable into a standard chain by a σ-embedding (an embedding which preserves all existing suprema and infima). We show that this algebraic condition is in general not equivalent to the strong $\mathbb{K}$-completeness but to another model-theoretic property: every countable model of the logic is elementarily equivalent to a model with an isomorphic domain whose corresponding chain is in $\mathbb{K}$. By making use of the theory of formal grammars and languages we present a constructive method that disproves completeness of a first-order logic starting from a counterexample to the completeness of the corresponding propositional logic. We complete the study by considering the five distinguished semantics also at the first-order level. Here, we collect again known results for prominent fuzzy logics and prove some interesting relations between different completeness properties. The paper ends with some conclusions and open problems.

2. Preliminaries

2.1. The logic MTL *and its prominent axiomatic extensions*

The term *t-norm based logic* usually refers to residuated systems of fuzzy logic with t-norm based semantics, i.e. where the conjunction connective is interpreted by a (left-continuous) t-norm and the implication operator by its residuum. In this framework, the weakest logic is the Monoidal T-norm based Logic (MTL). It is defined by Esteva and Godo in [24] by means of a Hilbert-style calculus in the language $\mathcal{L}_0 = \{\&, \to, \wedge, \overline{0}\}$ of type $\langle 2, 2, 2, 0\rangle$. $\mathrm{Fm}_{\mathcal{L}_0}$ will denote the set of all formulae built over a denumerable set of propositional variables using the connectives of $\mathcal{L}_0$. The only inference rule of the calculus is *Modus Ponens* and the axiom schemata are the following (taking $\to$ as the least binding connective):

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi \& \psi \to \varphi$
(A3)	$\varphi \& \psi \to \psi \& \varphi$
(A4)	$\varphi \wedge \psi \to \varphi$
(A5)	$\varphi \wedge \psi \to \psi \wedge \varphi$
(A6)	$\varphi \& (\varphi \to \psi) \to \varphi \wedge \psi$
(A7a)	$(\varphi \to (\psi \to \chi)) \to (\varphi \& \psi \to \chi)$
(A7b)	$(\varphi \& \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
(A8)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A9)	$\overline{0} \to \varphi$

The usual defined connectives are introduced as follows:

$\varphi \vee \psi$	as	$((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$	$\neg\varphi$	as	$\varphi \to \overline{0}$
$\varphi \leftrightarrow \psi$	as	$(\varphi \to \psi) \& (\psi \to \varphi)$	$\overline{1}$	as	$\neg\overline{0}$

Also as usual, φ^n will be used as a shorthand for $\varphi \& \overset{n}{\ldots} \& \varphi$, where $\varphi^0 = \overline{1}$. MTL enjoys the following form of local deduction-detachment theorem and substitution rule.

Proposition 2.1. *For each set of formulae $T \cup \{\varphi, \psi, \chi\}$ it holds:*

$$T, \varphi \vdash_{\mathrm{MTL}} \psi \text{ iff there is } n \in \mathbb{N} \text{ such that } T \vdash_{\mathrm{MTL}} \varphi^n \to \psi \qquad (\mathcal{LDT})$$

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{MTL}} \chi(\varphi) \leftrightarrow \chi(\psi). \qquad \text{(Cong)}$$

The algebraic counterpart[7] of MTL logic is the class of the so-called MTL-*algebras*. They are defined as follows.

Definition 2.2 ([24]). *An* MTL-*algebra is an algebra* $\mathcal{A} = \langle A, \&^{\mathcal{A}}, \to^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ *of type* $\langle 2, 2, 2, 2, 0, 0 \rangle$ *such that:*

1. $\langle A, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ *is a bounded lattice.*
2. $\langle A, \&^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ *is a commutative monoid with unit* $\overline{1}^{\mathcal{A}}$.
3. *The operations* $\&^{\mathcal{A}}$ *and* $\to^{\mathcal{A}}$ *form an adjoint pair:* $a \&^{\mathcal{A}} b \leq c$ *iff* $b \leq a \to^{\mathcal{A}} c$.
4. *It satisfies the prelinearity equation:* $(a \to^{\mathcal{A}} b) \vee^{\mathcal{A}} (b \to^{\mathcal{A}} a) = \overline{1}^{\mathcal{A}}$

If the lattice order is total we will say that $\mathcal{A}$ is a linearly ordered MTL-*algebra (or just an* MTL-*chain).*

An additional (unary) negation operation is defined as $\neg^{\mathcal{A}} a = a \to^{\mathcal{A}} \overline{0}^{\mathcal{A}}$. Similarly, an additional (binary) equivalence operation is defined as $a \leftrightarrow^{\mathcal{A}} b = (a \to^{\mathcal{A}} b) \&^{\mathcal{A}} (b \to^{\mathcal{A}} a)$. For the sake of a simpler notation, superscripts in the operations of the algebras will be omitted when they are clear from the context.

The class of all MTL-algebras is a variety which will be denoted as $\mathbb{MTL}$.[8]

Definition 2.3. *Let $\mathbb{K}$ be a class of* MTL-*algebras. We define the consequence relation $\models_{\mathbb{K}}$ in the following way: $T \models_{\mathbb{K}} \varphi$ iff for each $\mathcal{A} \in \mathbb{K}$ and $\mathcal{A}$-evaluation e: $e(\varphi) = \overline{1}^{\mathcal{A}}$ whenever $e[T] \subseteq \{\overline{1}^{\mathcal{A}}\}$.*

We write $\models_{\mathbb{K}} \varphi$ instead of $\emptyset \models_{\mathbb{K}} \varphi$ and $T \models_{\mathcal{A}} \varphi$ instead of $T \models_{\{\mathcal{A}\}} \varphi$. An $\mathcal{A}$-evaluation e such that $e[T] \subseteq \{\overline{1}^{\mathcal{A}}\}$ is called an $\mathcal{A}$-model of T. That $\mathbb{MTL}$ is the proper algebraic semantics for MTL is witnessed by the following completeness result.

Theorem 2.4 ([24]). *Let $T \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}_0}$. Then $T \vdash_{\mathrm{MTL}} \varphi$ if and only if $T \models_{\mathbb{MTL}} \varphi$.*

[7]We assume some basic knowledge on Universal Algebra. All the undefined notions and the notation we will use can be found in [9].

[8]In fact, it is the variety of bounded commutative integral residuated lattices with prelinearity.

This completeness result can be refined by taking into account the following representation of MTL-algebras, strongly related to the prelinearity property of MTL-algebras.

Proposition 2.5 ([24]). *Every* MTL-*algebra is a subdirect product of* MTL-*chains.*

This leads to the completeness of MTL with respect to the class of MTL-chains.

Corollary 2.6. *Let* $T \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}_0}$*. Then* $T \vdash_{\mathrm{MTL}} \varphi$ *if and only if* $T \models_{\{\mathrm{MTL}\text{-}chains\}} \varphi$*.*

Most of the well-known fuzzy logics (among them Łukasiewicz logic, Gödel logic, product logic and Hájek's BL logic)—as well as the Classical Propositional Calculus—can be presented as axiomatic extensions of MTL. Tables 1 and 2 collect some axiom schemata and the axiomatic extensions of MTL that they define.[9] Notice that in extensions of MTL with the divisibility axiom (Div), the additive conjunction $\wedge$ is in fact definable (as BL proves: $\varphi \wedge \psi \leftrightarrow \varphi \& (\varphi \to \psi)$) and therefore it is not considered as primitive connective in their languages. For the sake of homogeneity we will keep $\mathcal{L}_0 = \{\&, \to, \wedge, \overline{0}\}$ as the common language for all extensions of MTL.

Axiom schema	Name
$\neg\neg\varphi \to \varphi$	Involution (Inv)
$\neg\varphi \vee ((\varphi \to \varphi\&\psi) \to \psi)$	Cancellation (Can)
$\neg(\varphi\&\psi) \vee ((\psi \to \varphi\&\psi) \to \varphi)$	Weak Cancellation (WCan)
$\varphi \to \varphi\&\varphi$	Contraction (C)
$\varphi^{n-1} \to \varphi^n$	n-Contraction (C_n)
$\varphi \wedge \neg\varphi \to \overline{0}$	Pseudocomplementation (PC)
$\varphi \wedge \psi \to \varphi\&(\varphi \to \psi)$	Divisibility (Div)
$(\varphi\&\psi \to \overline{0}) \vee (\varphi \wedge \psi \to \varphi\&\psi)$	Weak Nilpotent Minimum (WNM)
$\varphi \vee \neg\varphi$	Excluded Middle (EM)

Table 1: Some usual axiom schemata in fuzzy logics.

Logic	Additional axiom schemata	References
SMTL	(PC)	[37]
ΠMTL	(Can)	[37]
WCMTL	(WCan)	[55]
IMTL	(Inv)	[24]
WNM	(WNM)	[24]
NM	(Inv) and (WNM)	[24]
C_nMTL	(C_n)	[12]
C_nIMTL	(Inv) and (C_n)	[12]
BL	(Div)	[36]
SBL	(Div) and (PC)	[25]
Ł	(Div) and (Inv)	[36, 49]
Π	(Div) and (Can)	[41]
G	(C)	[36, 21, 33]
CPC	(EM)	

Table 2: Some axiomatic extensions of MTL obtained by adding the corresponing additional axiom schemata.

[9]Of course, some of these logics were known well before MTL was introduced. We only want to point out that they can be seen as axiomatic extensions of MTL. Moreover, we introduce only the most prominent axiomatic extensions of MTL (for some additional ones see e.g. [55, 58, 69, 70]). The pseudocomplementation axiom appearing in the table is equivalent to the weak contraction axiom (see e.g. [63]).

MTL is actually an algebraizable logic in the sense of Blok and Pigozzi (see [7]) and $\mathbb{MTL}$ is its equivalent algebraic semantics. This implies that all axiomatic extensions of MTL are also algebraizable and their equivalent algebraic semantics are the subvarieties of $\mathbb{MTL}$ defined by the translations of the axioms into equations. In particular, there is an order-reversing isomorphism between axiomatic extensions of MTL and subvarieties of $\mathbb{MTL}$:

1. If $\Sigma \subseteq \mathrm{Fm}_{\mathcal{L}_0}$ and L is the extension of MTL obtained by adding the formulae of Σ as axiom schemata, then the equivalent algebraic semantics of L is the subvariety of $\mathbb{MTL}$ axiomatized by the equations $\{\varphi \approx \overline{1} \mid \varphi \in \Sigma\}$. We denote this variety by $\mathbb{L}$ and we call its members L-*algebras*. There are two exceptions to that rule: the algebras associated to Ł are called MV-*algebras* following the terminology of Chang in [10] and the algebras associated to the Classical Propositional Calculus (CPC for short) are called, of course, *Boolean algebras*. Moreover, since L-algebras are representable as subdirect product of L-chains, the completeness of MTL with respect to chains is inherited by L.
2. Let $\mathbb{L} \subseteq \mathbb{MTL}$ be the subvariety axiomatized by a set of equations Λ. Then the logic associated to $\mathbb{L}$ is the axiomatic extension L of MTL given by the axiom schemata $\{\varphi \leftrightarrow \psi \mid \varphi \approx \psi \in \Lambda\}$.

2.2. Expansions of MTL

In the literature of t-norm based logics, one can find not only a number of axiomatic extensions but also expansions of MTL by means of expanding the language with new connectives. Well-known examples are the expansions with Baaz's Delta projection connective Δ [4], expansions with an involutive negation $\sim$ [17, 25, 31], expansions with other conjunction or implication connectives [26, 46, 54], or expansions with intermediate truth-constants [65, 61, 27, 29, 23, 68]. In this subsection we introduce two important classes of logics expanding MTL, which encompass almost all logics mentioned above.

First we introduce one particular expansion of MTL, the logic MTL_Δ. It is obtained by enriching the language with the unary connective Δ and adding to the Hilbert-style system of MTL the deduction rule of necessitation (from φ infer $\Delta\varphi$) and the following axiom schemata:

(Δ1) $\Delta\varphi \vee \neg\Delta\varphi$

(Δ2) $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$

(Δ3) $\Delta\varphi \to \varphi$

(Δ4) $\Delta\varphi \to \Delta\Delta\varphi$

(Δ5) $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$

It is easily provable that MTL_Δ enjoys (Cong) but not ($\mathcal{LDT}$). However it enjoys another form of deduction theorem (in fact it is a global deduction-detachment theorem).

Proposition 2.7. *For each set of formulae $T \cup \{\varphi, \psi\}$ the following condition holds:*

$$T, \varphi \vdash_{\mathrm{MTL}_\Delta} \psi \text{ iff } T \vdash_{\mathrm{MTL}_\Delta} \Delta\varphi \to \psi \qquad (\mathcal{DT}_\Delta)$$

Definition 2.8 ([24]). *An* MTL_Δ-algebra *is a structure* $\mathcal{A} = \langle A, \&, \to, \wedge, \vee, \Delta, \overline{0}, \overline{1}\rangle$ *of type* $\langle 2, 2, 2, 2, 1, 0, 0\rangle$ *such that:*

(0) the reduct $\langle A, \&, \to, \wedge, \vee, \overline{0}, \overline{1}\rangle$ is an MTL-algebra,
(1) $\mathcal{A} \models \alpha \approx \overline{1}$ for each $\alpha \in \{\Delta1, \ldots, \Delta5\}$, and
(2) $\mathcal{A} \models \Delta(\overline{1}) \approx \overline{1}$.

The following two classes of logics were introduced in [39] (with the small difference that Δ-core fuzzy logic were called just Δ-*fuzzy logics*) and used in the current form e.g. in [40].

Definition 2.9. *We say that a finitary logic* L *in a countable language is a* core fuzzy logic *if*

- L *expands* MTL*,*
- L *satisfies* (Cong)*,*
- L *satisfies* $(\mathcal{LDT})$*.*

Definition 2.10. *We say that a finitary logic* L *in a countable language is a* Δ-core fuzzy logic *if*

- L *expands* MTL_Δ*,*
- L *satisfies* (Cong)*,*
- L *satisfies* $(\mathcal{DT}_\Delta)$*.*

The following proposition is a direct consequence of [16, Corollary 8 and Theorem 6].

Proposition 2.11. *Let* L *be an expansion of* MTL *(respectively of* MTL_Δ*) satisfying* (Cong)*. Then* L *is a (*Δ*-)core fuzzy logic if and only if it is an axiomatic expansion of* MTL *(*MTL_Δ*).*

In the following definition by the term *additional connective* (*axiom*) of L we understand a connective (axiom) not present in MTL.

Definition 2.12. *Let* L *be a core fuzzy logic and* I *the set of additional connectives of* L*. An* L-algebra *is a structure* $\mathcal{A} = \langle A, \&, \rightarrow, \wedge, \vee, \overline{0}, \overline{1}, \langle c \rangle_{c \in \mathrm{I}} \rangle$ *such that* $\langle A, \&, \rightarrow, \wedge, \vee, \overline{0}, \overline{1} \rangle$ *is an* MTL-*algebra and for each additional axiom* φ *of* L *the identity* $\varphi \approx \overline{1}$ *holds.*

Analogously we define L-algebras for Δ-core fuzzy logics. As in the previous subsection we will denote the class of L-algebras by $\mathbb{L}$. From the axioms of Δ we easily obtain:

Proposition 2.13. *Let* L *be a* Δ*-core fuzzy logic and* $\mathcal{B}$ *an* L*-chain. Then* $\Delta^{\mathcal{B}} x = \overline{1}^{\mathcal{B}}$ *if* $x = \overline{1}^{\mathcal{B}}$ *and* $\Delta^{\mathcal{B}} x = \overline{0}^{\mathcal{B}}$ *otherwise.*

The following proposition collects the basic properties of (Δ-)core fuzzy logics which are either easy observations or consequences of the corresponding papers.

Proposition 2.14. *Let* L *be a (*Δ*-)core fuzzy logic.*

- L *is an implicative logic in the sense of Rasiowa [67].*
- L *is a weakly implicative* fuzzy *logic in the sense of Cintula [16].*
- L *is algebraizable with the same translations as* MTL*.*
- $\mathbb{L}$ *is an equivalent algebraic semantics of* L*.*
- $\mathbb{L}$ *is a variety.*
- *Every* L*-algebra is representable as a subdirect product of* L*-chains.*
- *For every set of formulae* $T \cup \{\varphi\}$*,* $T \vdash_{\mathrm{L}} \varphi$ *if and only if* $T \models_{\{\mathrm{L}\text{-}chains\}} \varphi$*.*

For each core fuzzy logic L we can define the corresponding Δ-fuzzy logic L_Δ resulting from L in the same way as MTL_Δ from MTL. The following result is straightforward.

Proposition 2.15. *For every core fuzzy logic* L*,* L_Δ *is a conservative expansion of* L*.*

3. General completeness results

As we have already mentioned in the previous section, in (Δ-)core fuzzy logics we have completeness w.r.t. the corresponding variety and also w.r.t. the class of chains in that variety. Occasionally we may restrict ourselves even further to some proper subclass of chains thus obtaining finer completeness results.

Next we deal with different types of algebraic completeness, taking into account the cardinality of the set of premises. We define below the notions of strong $\mathbb{K}$-completeness, finite strong $\mathbb{K}$-completeness and $\mathbb{K}$-completeness.

Definition 3.1. *Let* L *be a (Δ-)core fuzzy logic and* $\mathbb{K}$ *a class of* L*-algebras. We say that* L *has the property of:*

- strong $\mathbb{K}$-completeness*,* S$\mathbb{K}$C *for short, when for every set of formulae* $T \cup \{\varphi\}$*:* $T \vdash_{\mathrm{L}} \varphi$ *iff* $T \models_{\mathbb{K}} \varphi$*.*
- finite strong $\mathbb{K}$-completeness*,* FS$\mathbb{K}$C *for short, when for every* finite *set of formulae* $T \cup \{\varphi\}$*:* $T \vdash_{\mathrm{L}} \varphi$ *iff* $T \models_{\mathbb{K}} \varphi$*.*
- $\mathbb{K}$-completeness*,* $\mathbb{K}$C *for short, when for every formula* φ*:* $\vdash_{\mathrm{L}} \varphi$ *iff* $\models_{\mathbb{K}} \varphi$*.*

Of course, the S$\mathbb{K}$C implies the FS$\mathbb{K}$C, and the FS$\mathbb{K}$C implies the $\mathbb{K}$C. In Subsection 3.2 we prove more results about the mutual relationships of these properties. But first we show several equivalent characterizations.

3.1. Equivalent algebraic characterizations

Let us recall the basic properties of the semantical consequence $\models_{\mathbb{K}}$ (see for instance [9, 18]).

1. $\models_{\mathbb{K}} \varphi$ iff $\models_{\mathbf{V}(\mathbb{K})} \varphi$, for every φ.
2. $T \models_{\mathbb{K}} \varphi$ iff $T \models_{\mathbf{Q}(\mathbb{K})} \varphi$, for every finite $T \cup \{\varphi\}$.
3. $T \models_{\mathbb{K}} \varphi$ iff $T \models_{\mathbf{ISP}_{\sigma-f}(\mathbb{K})} \varphi$, for every $T \cup \{\varphi\}$, where $\mathbf{P}_{\sigma-f}$ denotes the operator of reduced products over countably complete filters.

Thus we can obtain the following equivalent algebraic properties for each type of completeness.

Theorem 3.2. *Let* L *be (Δ-)core fuzzy logic. Then:*

1. L *has the* $\mathbb{K}$C *if and only if* $\mathbb{L} = \mathbf{V}(\mathbb{K})$*.*
2. L *has the* FS$\mathbb{K}$C *if and only if* $\mathbb{L} = \mathbf{Q}(\mathbb{K})$*.*
3. L *has the* S$\mathbb{K}$C *if and only if* $\mathbb{L} = \mathbf{ISP}_{\sigma-f}(\mathbb{K})$*.*

Moreover since we may restrict all classes to chains we can obtain new equivalencies. Before we do so we prepare one definition and one lemma.

Definition 3.3 (Directed set of formulae)**.** *Let* L *be a (Δ-)core fuzzy logic. A set of formulae* Ψ *is* directed *if for each* $\varphi, \psi \in \Psi$ *there is* $\chi \in \Psi$ *such that both* $\varphi \to \chi$ *and* $\psi \to \chi$ *are provable in* L *(we call* χ *an upper bound of* φ *and* ψ*).*

Lemma 3.4. *Let* L *be a (Δ-)core fuzzy logic with* S$\mathbb{K}$C*. Then for every set of formulae* T *and every directed set of formulae* Ψ *the following are equivalent:*

- $T \not\vdash_{\mathrm{L}} \psi$ *for each* $\psi \in \Psi$*.*
- *there is an algebra* $\mathcal{A} \in \mathbb{K}$ *and an* $\mathcal{A}$*-evaluation* e *such that* $e[T] \subseteq \{\bar{1}^{\mathcal{A}}\}$ *and* $\bar{1}^{\mathcal{A}} \notin e[\Psi]$*.*

Proof. One direction is obvious. For the other one let us take an unused propositional variable v, and define the set of formulae $T' = T \cup \{\psi \to v \mid \psi \in \Psi\}$. We show that $T' \not\vdash_{\mathrm{L}} v$ by the way of contradiction. Assume that $T' \vdash_{\mathrm{L}} v$. Thus there are finite sets $\hat{T} \subseteq T$ and $\hat{\Psi} \subseteq \Psi$ such that $\hat{T} \cup \{\psi \to v \mid \psi \in \hat{\Psi}\} \vdash_{\mathrm{L}} v$. Let $\delta \in \Psi$ denote an upper bound of formulae from $\hat{\Psi}$. As $T \not\vdash_{\mathrm{L}} \delta$ we know that there is an L-algebra $\mathcal{A}$ and an $\mathcal{A}$-model e of T such that $e(\delta) < \overline{1}$. We define the evaluation e' as $e'(p) = e(p)$ for each $p \neq v$ and $e'(v) = e(\delta)$. Clearly e' is a $\mathcal{A}$-model of $\hat{T} \cup \{\psi \to v \mid \psi \in \hat{\Psi}\}$ and $e'(v) < \overline{1}$, a contradiction.

Now we can use the S$\mathbb{K}$C to obtain an L-chain $\mathcal{B} \in \mathbb{K}$ and a $\mathcal{B}$-model e of T' such that $e(v) < \overline{1}$. Thus $\overline{1} \notin e[\Psi]$ (if $e(\psi) = \overline{1}$ for some $\psi \in \Psi$ then, since e is a model of T', we obtain $e(v) = \overline{1}$, a contradiction). $\square$

Theorem 3.5. *Let* L *be a (Δ-)core fuzzy logic in a propositional language $\mathcal{L}$ and let $\mathbb{K}$ be a class of* L*-chains. Then the following are equivalent:*

(i) L *has the* S$\mathbb{K}$C.

(ii) Every countable L*-chain belongs to* $\mathbf{IS}(\mathbb{K})$.

(iii) Every countable subdirectly irreducible L*-chain belongs to* $\mathbf{IS}(\mathbb{K})$.

Proof. *(i)* $\Rightarrow$ *(ii)*: Let $\mathcal{A}$ be a countable L-chain. Consider a set of pairwise different variables $\{v_a \mid a \in A\}$ and the following set of formulae:

$$T = \{\lambda(v_{a_1}, \ldots, v_{a_n}) \leftrightarrow v_{\lambda^{\mathcal{A}}(a_1,\ldots,a_n)} \mid \lambda \text{ an } n\text{-ary connective of } \mathcal{L} \text{ and } a_1, \ldots, a_n \in A\}.$$

We define the set $\Psi = \{v_{a_1} \vee \ldots \vee v_{a_n} \mid n \in \mathbb{N} \text{ and } a_1, \ldots a_n \in A \setminus \{\overline{1}^{\mathcal{A}}\}\}$. Clearly Ψ is directed and $T \not\vdash_{\mathrm{L}} \psi$ for each $\psi \in \Psi$ (just take $\mathcal{A}$ with the $\mathcal{A}$-evaluation $e(v_a) = a$ as a countermodel).

Now we use the S$\mathbb{K}$C and Lemma 3.4 to obtain an L-chain $\mathcal{B} \in \mathbb{K}$ and a $\mathcal{B}$-evaluation e such that $e[T] \subseteq \{\overline{1}^{\mathcal{B}}\}$ and $e(\psi) < \overline{1}^{\mathcal{B}}$ for each $\psi \in \Psi$. Consider the mapping $f : A \to B$ defined as $f(a) = e(v_a)$. It is clear that f is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$. By construction, if $a < \overline{1}^{\mathcal{A}}$, then $f(a) = e(v_a) < \overline{1}^{\mathcal{B}}$, as $v_a \in \Psi$. Therefore f is an embedding. Indeed, if $a, b \in A$ are such that $f(a) = f(b)$, then $f(a) \leftrightarrow f(b) = \overline{1}^{\mathcal{B}}$, i.e. $f(a \leftrightarrow b) = \overline{1}^{\mathcal{B}}$, and hence $a \leftrightarrow b = \overline{1}^{\mathcal{A}}$, i.e. $a = b$.

(ii) $\Rightarrow$ *(iii)*: Obvious.

(iii) $\Rightarrow$ *(i)*: Suppose that for some T and φ we have $T \not\vdash_{\mathrm{L}} \varphi$. Then there is an L-chain $\mathcal{A}$ and an $\mathcal{A}$-evaluation e such that $e[T] \subseteq \{\overline{1}^{\mathcal{A}}\}$ and $e(\varphi) < \overline{1}^{\mathcal{A}}$. Let $\mathcal{B}$ be the countable subalgebra of $\mathcal{A}$ whose universe is $e[\mathrm{Fm}_{\mathcal{L}}]$. $\mathcal{B}$ is not necessarily subdirectly irreducible but it is representable as a subdirect product of a family of subdirectly irreducible countable L-chains $\{\mathcal{B}_i \mid i \in I\}$. Since $e(\varphi) < \overline{1}^{\mathcal{B}}$, there is some $i \in I$ such that the i-th component of $e(\varphi)$ is not $\overline{1}^{\mathcal{B}_i}$. Thus, $T \not\models_{\mathcal{B}_i} \varphi$. Finally, since by assumption $\mathcal{B}_i$ is embeddable into some algebra of $\mathbb{K}$, we obtain $T \not\models_{\mathbb{K}} \varphi$. $\square$

The following is an immediate consequence of the above theorem and the fact that each (Δ-)core fuzzy logic enjoys the S$\mathbb{K}$C for $\mathbb{K}$ being the class of countable subdirectly irreducible L-chains.

Corollary 3.6. *Let* L *be a (Δ-)core fuzzy logic. Then each countable* L*-chain is embeddable into a countable subdirectly irreducible* L*-chain.*

Now we move to the FS$\mathbb{K}$C. In some cases (e.g. in Łukasiewicz, product, and Basic fuzzy Logic) this property has been proved by using partial embeddability. We recall its definition.

Definition 3.7. *For two algebras $\mathcal{A}$ and $\mathcal{B}$ of the same language $\mathcal{L}$, $\mathcal{A}$ is* partially embeddable *into $\mathcal{B}$ when each finite subset F of $\mathcal{A}$ can be partially embedded into $\mathcal{B}$. I.e. there is a one-to-one mapping $f : F \to B$ such that for each $c \in \mathcal{L}$ and elements $a_1, \ldots, a_n \in F$ satisfying $c^{\mathcal{A}}(a_1, \ldots, a_n) \in F$: $f(c^{\mathcal{A}}(a_1, \ldots, a_n)) = c^{\mathcal{B}}(f(a_1), \ldots, f(a_n))$. A class $\mathbb{K}$ of algebras is* (partially) embeddable into *a class $\mathbb{M}$ if every member of $\mathbb{K}$ is (partially) embeddable into a member of $\mathbb{M}$.*

Theorem 3.8. *Let* L *be a (Δ-)core fuzzy in a finite language $\mathcal{L}$ and let $\mathbb{K}$ be a class of* L*-chains. Then the following are equivalent:*

(i) L *has the* FS$\mathbb{K}$C*.*

(ii) Every countable L*-chain is partially embeddable into* $\mathbb{K}$*.*

(iii) Every L*-chain is partially embeddable into* $\mathbb{K}$*.*

(iv) Every subdirectly irreducible L*-chain is partially embeddable into* $\mathbb{K}$*.*

(v) Every countable subdirectly irreducible L*-chain is partially embeddable into* $\mathbb{K}$*.*

Proof. *(i)* $\Rightarrow$ *(ii)*: Let us take a countable L-chain $\mathcal{A}$ and a finite set $B \subseteq A$. Define set $B' = B \cup \{a \to^{\mathcal{A}} b \mid a, b \in B\}$. Consider a set of pairwise different variables $\{v_a \mid a \in A\}$ (we can do it because $\mathcal{A}$ is countable) and a set of formulae T (notice a difference between this set and the set T from the proof of Theorem 3.5):

$$T = \{\lambda(v_{a_1}, \dots, v_{a_n}) \leftrightarrow v_{\lambda^{\mathcal{A}}(a_1,\dots,a_n)} \mid \lambda \text{ an } n\text{-ary connective and } a_1, \dots, a_n, \lambda^{\mathcal{A}}(a_1, \dots, a_n) \in B'\}.$$

Let φ be the formula $\bigvee_{a \in B' \setminus \{\bar{1}\}} v_a$. Observe that T is finite and $T \not\vdash_{\mathrm{L}} \varphi$ (take the L-chain $\mathcal{A}$ and the $\mathcal{A}$-evaluation $e(v_a) = a$). Thus by the FS$\mathbb{K}$C there is an L-algebra $\mathcal{D} \in \mathbb{K}$ and a $\mathcal{D}$-evaluation e such that $e[T] = \{\bar{1}^{\mathcal{D}}\}$ and $e(\varphi) < \bar{1}^{\mathcal{D}}$. Define a mapping $f : B \to D$ as $f(a) = e(v_a)$. Obviously f is a partial homomorphism. We show that f is one-to-one: if $a, b \in B$ and $a > b$ then $f(a) \to^{\mathcal{D}} f(b) = e(v_a) \to^{\mathcal{D}} e(v_b) = e(v_{a \to^{\mathcal{A}} b}) < \bar{1}^{\mathcal{D}}$ (the first equality is the definition, the second one is the consequence of the fact that $a \to^{\mathcal{A}} b \in B'$ and e is an $\mathcal{D}$-model of T, and the last inequality follows from the fact that $e(v_{a \to^{\mathcal{A}} b}) \leq e(\varphi) < \bar{1}^{\mathcal{D}}$).

(ii) $\Rightarrow$ *(iii)*: Let $\mathcal{A}$ be an L-chain and $B \subseteq A$ a finite partial subalgebra. Then the subalgebra of $\mathcal{A}$ generated by B is countable, so we can apply *(ii)*.

Implications *(iii)* $\Rightarrow$ *(iv)* and *(iv)* $\Rightarrow$ *(v)* are trivial; *(v)* $\Rightarrow$ *(i)* is proved analogously to the implication *(iii)* $\Rightarrow$ *(i)* of Theorem 3.5. ☐

Remark 3.9. *Notice that the implications from (ii), (iii), (iv) or (v) to (i) hold also for infinite languages, whereas the converse ones do not (as shown by the following example).*

Example 3.10. *Consider the language $\mathcal{L}$ resulting from $\mathcal{L}_0$ by adding an infinite countable set $C = \{c_i \mid i \in \omega\}$ of 0-ary connectives. Let G_C be Gödel logic in this language (and no additional axiom), and let $\mathbb{G}_C$ be the corresponding variety of Gödel algebras with infinitely many constants. Clearly, G_C is a core fuzzy logic. Now consider the class $\mathcal{R}_1$ of algebras on $[0,1]$ in which all constants, except from a finite number, are interpreted as 1.*

Consider a finite set $T \cup \{\varphi\}$, such that $T \not\vdash_{\mathrm{G}_C} \varphi$ then also $T \not\vdash_{\mathrm{G}} \varphi$, where we understand the new constants just as propositional variables. Thus by the strong standard completeness of Gödel logic, there is an $[0,1]_{\mathrm{G}}$-evaluation e such that $e[T] \subseteq \{1\}$ and $e(\varphi) < 1$. We construct a G_C-algebra $\mathcal{A}$ resulting from $[0,1]_{\mathrm{G}}$ by setting $c_i^{\mathcal{A}} = e(c_i)$ for c_i occurring in $T \cup \{\varphi\}$ and 1 otherwise. Notice that e can be viewed as $\mathcal{A}$-evaluation and as $\mathcal{A} \in \mathcal{R}_1$ (because $T \cup \{\varphi\}$ contains only finitely many constants) we obtain, $T \not\models_{\mathcal{R}_1} \varphi$. Thus we have shown $\mathrm{FS}\mathcal{R}_1\mathrm{C}$ of G_C.

On the other hand, let us denote by $[0,1]_0$ the Gödel algebra on $[0,1]$ with all constants interpreted into 0. Clearly, any partial subalgebra of $[0,1]_0$ containing 0 does not partially embed into any algebra in $\mathcal{R}_1$.

Nevertheless, we can give the following characterization for the FS$\mathbb{K}$C that holds even for infinite languages.

Theorem 3.11. *Let* L *be a (Δ-)core fuzzy logic and* $\mathbb{K}$ *a class of* L*-chains. Then the following are equivalent:*

(i) L *satisfies the* FS$\mathbb{K}$C*.*
(ii) Every L*-chain belongs to* $\mathbf{ISP}_U(\mathbb{K})$*.*

Proof. $(i) \Rightarrow (ii)$: if L satisfies the FS$\mathbb{K}$C then, by Theorem 3.2, its equivalent variety semantics $\mathbb{L}$ is such that $\mathbb{L} = \mathbf{Q}(\mathbb{K})$. It follows from [19, Lemma 1.5] that every relative finitely subdirectly irreducible member of $\mathbf{Q}(\mathbb{K})$ belongs to $\mathbf{ISP}_U(\mathbb{K})$. Since $\mathbf{Q}(\mathbb{K})$ is a variety, relative finitely subdirectly irreducible members coincide with finitely subdirectly irreducible algebras in the absolute sense, hence with L-chains.

$(ii) \Rightarrow (i)$: if every L-chain belongs to $\mathbf{ISP}_U(\mathbb{K})$, since every L-algebra is representable as subdirect product of L-chains we have that

$$\mathbb{L} \subseteq \mathbf{IP}_{SD}(\mathbf{ISP}_U(\mathbb{K})) \subseteq \mathbf{Q}(\mathbb{K}) \subseteq \mathbb{L}.$$

Therefore by Theorem 3.2, L has the FS$\mathbb{K}$C. □

3.2. Relations between different notions of completeness

First we show that the notions of the S$\mathbb{K}$C and the FS$\mathbb{K}$C are, under certain conditions, equivalent.

Proposition 3.12. *Let* L *be a* (Δ-)*core fuzzy logic and* $\mathbb{K}$ *a class of* L*-chains. Then* L *has the* FS$\mathbb{K}$C *if and only if* L *is the finitary companion of* $\models_{\mathbb{K}}$.

Proof. Let us denote the finitary companion of $\models_{\mathbb{K}}$ as L′. By definition we have $T \vdash_{\mathrm{L}'} \varphi$ iff there is a finite $T' \subseteq T$ and $T' \models_{\mathbb{K}} \varphi$. Thus from $T \vdash_{\mathrm{L}'} \varphi$ we obtain $T \vdash_{\mathrm{L}} \varphi$ (by the FS$\mathbb{K}$C) and vice versa (by soundness). Reverse direction: just observe that for finite T we have $T \vdash_{\mathrm{L}} \varphi$ iff $T \vdash_{\mathrm{L}'} \varphi$. □

Corollary 3.13. *Let* L *be a* (Δ-)*core fuzzy logic and* $\mathbb{K}$ *a class of* L*-chains such that* $\models_{\mathbb{K}}$ *is finitary. Then* L *has the* S$\mathbb{K}$C *if and only if* L *has the* FS$\mathbb{K}$C.

Corollary 3.14. *Let* L *be a* (Δ-)*core fuzzy logic and* $\mathbb{K}$ *a class of* L*-chains such that* $\mathbf{I}(\mathbb{K}) = \mathbf{IP}_U(\mathbb{K})$. *Then* L *has the* S$\mathbb{K}$C *if and only if* L *has the* FS$\mathbb{K}$C.

Proof. Just recall that the equational consequence relative to a class of algebras closed under ultraproducts is finitary. □

Corollary 3.15. *Let* L *be a* (Δ-)*core fuzzy logic and let* $\mathbb{K}$ *be a class of* L*-chains such that* L *enjoys the* FS$\mathbb{K}$C. *Then* L *has the* $\mathbf{SP}_U(\mathbb{K})$C.

Now we show that if a (Δ-)core fuzzy logic does not enjoy the $\mathbb{K}$C, the FS$\mathbb{K}$C or the S$\mathbb{K}$C, then any of its conservative expansions neither does.

Proposition 3.16. *Let* L *be a* (Δ-)*core fuzzy logic in a language* $\mathcal{L}$ *and* L′ *a conservative expansion of* L. *Let further* $\mathbb{K}'$ *be a class of* L′*-chains and* $\mathbb{K}$ *the class of their* $\mathcal{L}$*-reducts. Then:*

- *If* L′ *enjoys the* $\mathbb{K}'$C, *then* L *enjoys the* $\mathbb{K}$C.
- *If* L′ *enjoys the* FS$\mathbb{K}'$C, *then* L *enjoys the* FS$\mathbb{K}$C.
- *If* L′ *enjoys the* S$\mathbb{K}'$C, *then* L *enjoys the* S$\mathbb{K}$C.

Proof. All the implications are proved in a similar way. Let us prove as an example the first one. We want to show that L has the $\mathbb{K}$C and we do it contrapositively: assume $\not\vdash_{\mathrm{L}} \varphi$. Since L′ is a conservative expansion of L, we also have $\not\vdash_{\mathrm{L}'} \varphi$ and so by the $\mathbb{K}'$C we obtain $\not\models_{\mathcal{A}'} \varphi$ for some $\mathcal{A}' \in \mathbb{K}'$. Thus also $\not\models_{\mathcal{A}} \varphi$ for the reduct $\mathcal{A}$ of $\mathcal{A}'$. As $\mathcal{A} \in \mathbb{K}$ we obtain $\not\models_{\mathbb{K}} \varphi$. □

In the case of expansions with Δ, since there is a one-to-one correspondence between L-chains and L_Δ-chains, the result can be improved in the case of S$\mathbb{K}$C and FS$\mathbb{K}$C.

Proposition 3.17. *Let* L *be a core fuzzy logic. We have:* L *has the* S$\mathbb{K}$C *(resp.* FS$\mathbb{K}$C*) with respect to a class of* L*-chains* $\mathbb{K}$ *if and only if* L_Δ *has the* S$\mathbb{K}_\Delta$C *(resp.* FS$\mathbb{K}_\Delta$C*), where* $\mathbb{K}_\Delta$ *is the class of* Δ*-expansions of chains in* $\mathbb{K}$.

Proof. It is an easy consequence of Theorems 3.5 and 3.11. □

Interestingly enough, in Δ-core fuzzy logics the notions of $\mathbb{K}$C and FS$\mathbb{K}$C coincide.

Proposition 3.18. *Let* L *be a* Δ*-core fuzzy logic and* $\mathbb{K}$ *a class of* L*-chains. Then* L *has the* $\mathbb{K}$C *if and only if* L *has the* FS$\mathbb{K}$C*.*

Proof. One direction is obvious. To prove the converse one, assume that $T \nvdash_{\mathrm{L}} \varphi$ for some finite T. Thus the formula $\chi = \bigwedge_{\psi \in T} \Delta\psi \to \varphi$ is not a theorem of L (by $(\mathcal{DT}_\Delta)$). Thus, by the $\mathbb{K}$C there is an L-chain $\mathcal{B} \in \mathbb{K}$ and an $\mathcal{B}$-evaluation e such that $e(\chi) < \overline{1}^{\mathcal{B}}$. As we know the semantics of Δ in any L-chain (Proposition 2.13) we can conclude that $e[T] \subseteq \{\overline{1}^{\mathcal{B}}\}$ and $e(\varphi) < \overline{1}^{\mathcal{B}}$. □

This proposition is not valid for core fuzzy logics and arbitrary class $\mathbb{K}$ as witnessed by [23, Theorems 28–30]. The question for which classes of logics and chains the equivalence holds is one the most interesting open problems as it usually holds in the known cases. In order to solve this problem a better characterization of $\mathbb{K}$C seems to be needed.

Finally, combining the two propositions above we obtain:

Corollary 3.19. *Let* L *be a core fuzzy logic and* $\mathbb{K}$ *a class of* L*-chains. Then* L *has the* FS$\mathbb{K}$C *if and only if* L_Δ *has the* $\mathbb{K}_\Delta$C*.*

4. Distinguished semantics

Researchers in Fuzzy Logic have been traditionally interested in semantics defined over the real unit interval. Such kind of semantics can be found inside the class of MTL-algebras. Indeed, given a left-continuous t-norm $*$ and its residuum $\to$ (defined as $a \to b = \max\{c \mid a * c \leq b\}$), the algebra $[0,1]_* = \langle [0,1], *, \to, \min, \max, 0, 1\rangle$ is an MTL-chain. Notice that $[0,1]_*$ is completely determined by the t-norm. Moreover, it is obvious that in every MTL-chain $\mathcal{A}$ over $[0,1]$, the operation $\&^{\mathcal{A}}$ is a left-continuous t-norm. These chains are traditionally called *standard algebras*. It is well known that a standard algebra $[0,1]_*$ is a BL-chain if and only if $*$ is continuous. Prominent examples of continuous t-norms are the Łukasiewicz t-norm ($a *_{\mathrm{Ł}} b = \max\{0, a+b-1\}$), the product t-norm ($a *_{\Pi} b = ab$) and the minimum t-norm ($a *_{\mathrm{G}} b = \min\{a,b\}$). We will denote these standard algebras by $[0,1]_{\mathrm{Ł}}$, $[0,1]_{\Pi}$ and $[0,1]_{\mathrm{G}}$, respectively. In [48] and [57] it is independently proved that every standard BL-algebra is decomposable as an ordinal sum of isomorphic copies of these three basic components.

Also very close to the standard semantics for a fuzzy logic is the rational-chain semantics, i.e. instead of standard algebras one can also consider MTL-algebras over the rational unit interval $[0,1]^{\mathbb{Q}} = [0,1] \cap \mathbb{Q}$ as another interesting semantics, easily justified by computational reasons.

Another meaningful semantics close to the standard one is that of hyperreals: we consider algebras over any ultrapower of real unit interval.[10] But in this way one obtains something more than just the class of algebras defined over the non-standard real numbers. Indeed, an ultrapower modulo a principal filter gives again a chain over the reals, so this semantics actually includes not only the proper hyperreal chains but also the standard chains. Therefore, it makes sense to consider the semantics given only by those ultrapowers which are a proper extension of the reals. We will call it *strict hyperreal semantics*. Finally, the semantics based on finite MTL-chains is also interesting for fuzzy logics for its simplicity and its connection to decidability.

In the following we explore the completeness properties with respect to these five kinds of semantics for any (Δ-)core fuzzy logic L.[11]

Notation We will use the term:

[10] This kind of semantics for fuzzy logics has been already considered in a recent paper [30].
[11] For an alternative chain semantics not covered in the present paper see [44].

- *standard* or *real-chain* completeness ($\mathcal{R}$C) to denote $\mathbb{K}$-completeness when $\mathbb{K}$ is the class of L-chains whose lattice reduct is $[0,1]$;[12]

- *rational-chain* completeness ($\mathcal{Q}$C) to denote $\mathbb{K}$-completeness when $\mathbb{K}$ is the class of L-chains whose lattice reduct is $[0,1]^{\mathbb{Q}}$;

- *hyperreal-chain* completeness ($\mathcal{R}^{\star}$C) to denote $\mathbb{K}$-completeness when $\mathbb{K}$ is the class of L-chains whose lattice reduct is any ultrapower of $[0,1]$;

- *strict hyperreal-chain* completeness ($\mathcal{R}^{\star}_{s}$C) to denote $\mathbb{K}$-completeness when $\mathbb{K}$ is the class of L-chains whose lattice reduct is an ultrapower of $[0,1]$ which is a proper extension of the real unit interval;[13]

- *finite-chain* completeness ($\mathcal{F}$C) to denote $\mathbb{K}$-completeness when $\mathbb{K}$ is the class of finite L-chains.

Of course, we also define the two stronger notions of completeness, i.e. FS$\mathbb{K}$C and S$\mathbb{K}$C for $\mathbb{K} \in \{\mathcal{R}, \mathcal{Q}, \mathcal{R}^{\star}, \mathcal{R}^{\star}_{s}, \mathcal{F}\}$.

We can easily show that one of these hyperreal semantics is in fact redundant in propositional logics. Moreover, in Subsection 4.2 we will prove that, when it comes to completeness properties, the rational semantics is equivalent to the hyperreal ones. Analogous equivalence results, with more complex proofs, will be obtained for first-order logics in Subsection 5.3.

Proposition 4.1. *For every (Δ-)core fuzzy logic* L *we have:*

(1) L *enjoys the* S$\mathcal{R}^{\star}$C *if and only if* L *enjoys the* S$\mathcal{R}^{\star}_{s}$C.

(2) L *enjoys the* FS$\mathcal{R}^{\star}$C *if and only if* L *enjoys the* FS$\mathcal{R}^{\star}_{s}$C.

(3) L *enjoys the* $\mathcal{R}^{\star}$C *if and only if* L *enjoys the* $\mathcal{R}^{\star}_{s}$C.

Proof. Since the hyperreal semantics contains the strict hyperreal semantics by definition, the direction from right to left in the three cases is straightforward. Conversely, the proof is also easy if one takes into account the results of Theorem 3.2 and the fact that standard chains are subalgebras of the strict hyperreal ones. □

In the cases of standard and rational-chain semantics, the strong completeness has been always proved in the literature by showing that the logic enjoys an apparently stronger property: the embedding property. However, in the previous section we have proved that they are actually equivalent. These rational-chain and real-chain embedding properties have been already studied in [22] for axiomatic extensions of MTL. We will consider now also the corresponding property for the remaining semantics:

Notation Given a (Δ-)core fuzzy logic L we define:

- L has the *real-chain embedding property* ($\mathcal{R}$-E, for short) iff any countable L-chain[14] can be embedded into a standard L-chain.

[12]Although it would be more homogeneous to use always the term *real-chain* we rather prefer to respect the strong tradition in Fuzzy Logic that has been using *standard* instead.

[13]In order to obtain an ultrapower of $[0,1]$ which does not result into an isomorphic image of $[0,1]$ we must prevent the ultrafilter from being principal, as it is well known, but also from being closed under intersections of countable families (we will justify this requirement in Subsection 5.3).

[14]When dealing with a class that does not contain the trivial chain (such as the classes of standard, rational and hyperreal chains) it is obvious that we cannot embed there the trivial chain. Therefore, strictly speaking, in the embedding properties we should write "any *non-trivial* countable L-chain"; however, as it is always clear from the context, we will often omit this obvious constraint.

- the rational-chain, hyperreal-chain, strict hyperreal-chain, and finite-chain embedding properties are defined accordingly (we use shorthands: $\mathcal{Q}$-E, $\mathcal{R}^\star$-E, $\mathcal{R}_s^\star$-E, and $\mathcal{F}$-E).

Sometimes standard or rational-chain completeness properties can be refined to some subclass of standard or rational algebras; sometimes even it is enough to consider only one algebra. When the standard completeness (resp. rational-chain completeness) can be proved with respect to a particular standard algebra (resp. rational algebra) which is the intended semantics for the logic, we call it *canonical* standard completeness (resp. *canonical* rational-chain completeness). These kinds of completeness are a matter of special interest when one considers the logic of the variety generated by the algebra defined by one particular standard or rational algebra, because sometimes this algebra is the intended semantics for the logic.

Definition 4.2. *Let $*$ be a left-continuous t-norm.* L_* *will denote the axiomatic extension of* MTL *whose equivalent algebraic semantics is* $\mathbf{V}([0,1]_*)$.

It is clear, by definition, that for every left-continuous t-norm $*$, the logic L_* enjoys the $\mathcal{R}$C restricted to $[0,1]_*$, i.e. the canonical $\mathcal{R}$C. For continuous t-norms we can prove even more:

Proposition 4.3 ([23])**.** *For every continuous t-norm $*$, the logic* L_* *has the canonical* FS$\mathcal{R}$C.

Nevertheless, it can be improved to S$\mathcal{R}$C only for Gödel logic:

Proposition 4.4 ([23])**.** *An axiomatic extension* L *of* BL *has the* S$\mathcal{R}$C *iff* L $=$ G.

4.1. Standard completeness

As mentioned, the usual strategy to prove the S$\mathcal{R}$C has consisted in showing in a constructive way that every countable chain, is embeddable into a standard chain of the same variety. This kind of construction was used ad hoc for Gödel logic G in [36], and for the logics NM and WNM in [24]. It was later generalized to MTL in [47] and refined in [56] as we sketch now:

Completion of countable MTL-chains: Let $\mathcal{A}$ be a countable MTL-chain. A standard MTL-chain $[0,1]_*$ and an embedding $h : \mathcal{A} \hookrightarrow [0,1]_*$ are built by following next steps:

- For every $a \in A$, $suc(a)$ is either the successor of a in the order of $\mathcal{A}$ if it exists or $suc(a) = a$ otherwise.
- $B = \{\langle a, 1\rangle \mid a \in A\} \cup \{\langle a, q\rangle \mid \exists a' \in A$ such that $a \neq a'$ and $suc(a') = a,\ q \in \mathbb{Q} \cap (0,1)\}$.
- Consider the lexicographical order $\preceq$ on B.
- Define the following monoidal operation on B:

$$\langle a, q\rangle \circ \langle b, r\rangle = \begin{cases} \min_{\preceq}\{\langle a, q\rangle, \langle b, r\rangle\} & \text{if } a \&^{\mathcal{A}} b = \min\{a, b\} \\ \langle a \&^{\mathcal{A}} b, 1\rangle & \text{otherwise.} \end{cases}$$

- The ordered monoid $\langle A, \&^{\mathcal{A}}, \overline{1}^{\mathcal{A}}, \leq\rangle$ is embeddable into $\langle B, \circ, \langle \overline{1}^{\mathcal{A}}, 1\rangle, \preceq\rangle$ by mapping every $a \in A$ to $\langle a, 1\rangle$.
- $\mathcal{B} = \langle B, \circ, \langle \overline{1}^{\mathcal{A}}, 1\rangle, \preceq\rangle$ is a densely ordered countable monoid with maximum and minimum, so it is isomorphic to a monoid $\mathcal{B}' = \langle [0,1]^{\mathbb{Q}}, \circ', 1, \preceq'\rangle$. Obviously, $\langle A, \&^{\mathcal{A}}, \overline{1}^{\mathcal{A}}, \leq\rangle$ is also embeddable into $\mathcal{B}'$. Let h be such embedding. Moreover, restricted to $h[A]$, the residuum of $\circ'$ exists, call it $\Rightarrow$, and $h(a) \Rightarrow h(b) = h(a \rightarrow^{\mathcal{A}} b)$.
- $\mathcal{B}'$ is completed to $[0,1]$ by defining:

$$\forall \alpha, \beta \in [0,1] \quad \alpha * \beta = \sup\{x \circ' y \mid x \leq \alpha,\ y \leq \beta,\ x, y \in [0,1]^{\mathbb{Q}}\}.$$

- $*$ is a left-continuous t-norm, so it defines a standard MTL-algebra $[0,1]_*$, and h is the desired embedding.

Therefore, we obtain the following sufficient condition for the S$\mathcal{R}$C:

Proposition 4.5. *Let* L *be an axiomatic extension of* MTL*. If for every countable* L*-chain, its completion given by the construction described above is an* L*-chain, then* L *enjoys the* S$\mathcal{R}$C.

The S$\mathcal{R}$C for MTL, SMTL, G, WNM and C$_n$MTL can be proved by applying the previous proposition. Nevertheless, some important equationally definable properties are not preserved under this construction and thus the method does not work to prove the S$\mathcal{R}$C for some axiomatic extensions of MTL. In [22] the authors prove that the completion of Jenei and Montagna does not preserve divisibility, cancellation and involution in general. Actually, it is well known that Ł and Π do not enjoy the S$\mathcal{R}$C (see e.g. [36]). Moreover, in [55] it is proved that the S$\mathcal{R}$C also fails for the logics BL, SBL, ΠMTL, and WCMTL. But the S$\mathcal{R}$C for IMTL can still be proved in [22] by modifying the previous construction. We sketch it again.

Completion of countable involutive MTL**-chains:** Let $\mathcal{A}$ be a countable IMTL-chain. A standard IMTL-chain $[0,1]_*$ and an embedding $h : \mathcal{A} \hookrightarrow [0,1]_*$ are built by following next steps:

- Define the ordered monoid $\langle B, \circ, \langle \overline{1}^{\mathcal{A}}, 1\rangle, \preceq\rangle$ as before.
- Modify the monoidal operation in the following way:

$$\langle a, q\rangle \otimes \langle b, r\rangle = \begin{cases} \langle \overline{0}^{\mathcal{A}}, 1\rangle & \text{if } a = suc(\neg b), q + r \leq 1 \\ \langle a, q\rangle \circ \langle b, r\rangle & \text{otherwise.} \end{cases}$$

- As before, the ordered monoid $\langle A, \&^{\mathcal{A}}, \overline{1}^{\mathcal{A}}, \leq\rangle$ is embeddable into $\langle B, \otimes, \langle \overline{1}^{\mathcal{A}}, 1\rangle, \preceq\rangle$ which is isomorphic to a monoid $\mathcal{B}' = \langle [0,1]^{\mathbb{Q}}, \circ', \preceq'\rangle$. Again, $\mathcal{B}'$ is completed to $[0,1]$ by defining:

$$\forall \alpha, \beta \in [0,1] \quad \alpha * \beta = \sup\{x \circ' y \mid x \leq \alpha,\, y \leq \beta,\, x, y \in [0,1]^{\mathbb{Q}}\}.$$

- $*$ is a left-continuous t-norm with an involutive negation, so it defines a standard IMTL-algebra $[0,1]_*$, and h is the desired embedding.

Proposition 4.6. *Let* L *be an axiomatic extension of* IMTL*. If for every countable* L*-chain, its completion given by the construction above is an* L*-chain, then* L *enjoys the* S$\mathcal{R}$C.

The S$\mathcal{R}$C for IMTL, NM, and C$_n$IMTL can be proved by applying this construction. In [31] the authors use the real-embedding property of some core fuzzy logics (like MTL, SMTL, etc.) and show this property (and hence the strong standard completeness) for the expansions of these logics with Δ and an independent involutive negation $\sim$ (analogously for partial embeddings and finite strong standard completeness). We do not want to go into details here about the logics L$_\sim$ and the method mentioned above (see [31] for the details). We only observe that as we now know the equivalence between $\mathcal{R}$-E and S$\mathcal{R}$C, we could use their method in general and so we can conclude (also using Proposition 3.16) that: the logic L has the S$\mathcal{R}$C (FS$\mathcal{R}$C respectively) iff the logic L$_\sim$ has the S$\mathcal{R}$C (FS$\mathcal{R}$C respectively).

As regards to the FS$\mathcal{R}$C, it is obviously satisfied by the logics that enjoy the S$\mathcal{R}$C. For the remaining logics it has been studied in many papers. Interestingly, rather than using the most straight equivalence given in Theorem 3.2, it has been usually proved by showing that the class of all chains is partially embeddable into the class of standard chains (i.e. using Theorem 3.8). This has been done for Π (in [41]), Ł (see e.g. [36]), BL, and SBL (in [14]) by using essentially Gurevich-Kokorin Theorem. ΠMTL also enjoys the FS$\mathcal{R}$C as it was proved by Horčík in [43] by means of a different construction. Since his method it is not so well known, we sketch it now.

Horčík's method: Take a ΠMTL-chain $\mathcal{A} = \langle A, \&, \rightarrow, \wedge, \vee, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ and a finite subset $G \subseteq A$. Let $\mathcal{S}$ be the submonoid of $\mathcal{A}$ generated by G. By using Dickson's lemma in [43] it is proved that $\mathcal{S}$ is residuated and the residuum is given by: $a \rightarrow b = \max\{z \in S : a\&z \leq b\}$. Thus, the enriched submonoid $\mathcal{S} = \langle S, \&, \rightarrow, \wedge, \vee, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ is a countable MTL-chain. Moreover, since its monoidal operation is just the restriction of the monoidal operation of $\mathcal{A}$, it is clear that it is also cancellative, hence $\mathcal{S}$ is a ΠMTL-chain.

Now define a new chain over the set $S' = \{\langle s, r\rangle \mid s \in S \setminus \{\overline{0}^{\mathcal{A}}\}, r \in (0, 1]\} \cup \{\langle \overline{0}^{\mathcal{A}}, 1\rangle\}$, with the lexicographical order $\leq_{lex}$ and the following operations:

$$\langle a, x\rangle \&' \langle b, y\rangle = \langle a\&b, xy\rangle$$

$$\langle a, x\rangle \rightarrow' \langle b, y\rangle = \begin{cases} \langle a \rightarrow b, 1\rangle & \text{if } a\&(a \rightarrow b) < b, \\ \langle a \rightarrow b, \min\{1, y/x\}\rangle & \text{otherwise.} \end{cases}$$

So $\mathcal{S}' = \langle S', \&', \rightarrow', \leq_{lex}, \langle \overline{0}^{\mathcal{A}}, 1\rangle, \langle \overline{1}^{\mathcal{A}}, 1\rangle\rangle$ is an MTL-chain and there is an embedding $\Psi : \mathcal{S} \rightarrow \mathcal{S}'$ defined by $\Psi(a) = \langle a, 1\rangle$. Moreover $\mathcal{S}'$ is cancellative.

Finally, as proved in [42], the set S' is order isomorphic to the real unit interval $[0, 1]$, so there is a standard ΠMTL-chain $\mathcal{B}$ and an isomorphism $\Phi : \mathcal{S}' \rightarrow \mathcal{B}$. The function $\Phi \circ \Psi$ is a partial embedding of G into $\mathcal{B}$.

Horčík's method has been used in [55] to prove the FS$\mathcal{R}$C for several logics, WCMTL among them. Finally, for some known axiomatic extensions of MTL the FS$\mathcal{R}$C fails (see e.g. [59] and later in Example 4.14), but in all these cases the $\mathcal{R}$C is also false.

Open problem 4.7. *For which core fuzzy logics does the implication* $\mathcal{R}$C $\Rightarrow$ FS$\mathcal{R}$C *hold?*[15]

4.2. Hyperreal-chain and Rational-chain Completeness

As we will see, hyperreal-chain and rational-chain completeness have many things in common so we deal with them together in this section. Let L be a (Δ-)core fuzzy logic, then by $\mathbb{L}_{\mathcal{Q}}$ and $\mathbb{L}_{\mathcal{R}^\star}$ we denote the classes of elements of $\mathbb{L}$ whose lattice reduct is respectively $[0, 1]^{\mathbb{Q}}$ and some ultrapower of $[0, 1]$.

Lemma 4.8. *Let* L *be a (Δ-)core fuzzy logic. Then* $\mathbf{ISP}_U(\mathbb{L}_{\mathcal{Q}}) = \mathbf{IS}(\mathbb{L}_{\mathcal{R}^\star})$.

Proof. Clearly, it suffices to prove that $\mathbb{L}_{\mathcal{Q}} \subseteq \mathbf{IS}(\mathbb{L}_{\mathcal{R}^\star})$ and $\mathbb{L}_{\mathcal{R}^\star} \subseteq \mathbf{ISP}_U(\mathbb{L}_{\mathcal{Q}})$. Now let $\mathcal{A} \in \mathbb{L}_{\mathcal{Q}}$, $\mathcal{B} \in \mathbb{L}_{\mathcal{R}^\star}$ and $\mathcal{A}_0$ and $\mathcal{B}_0$ be their lattice reducts. Then $\mathcal{A}_0$ and $\mathcal{B}_0$ are elementarily equivalent, being totally and densely ordered with maximum and minimum. Therefore, by the Keisler-Shelah theorem they have isomorphic ultrapowers, say $\mathcal{A}_0^\star$ and $\mathcal{B}_0^\star$. The ultraproduct also induces the structure of L-algebra in $\mathcal{A}_0^\star$ and $\mathcal{B}_0^\star$: first define the realization of operations in the direct product, and then consider the quotient modulo the ultrafilter. Thus from $\mathcal{A}_0^\star$ and $\mathcal{B}_0^\star$ we obtain L-algebras $\mathcal{A}^\star$ and $\mathcal{B}^\star$ which are ultraproducts of $\mathcal{A}$ and $\mathcal{B}$ respectively and whose lattice reducts are isomorphic. Now $\mathcal{A}_0^\star$ is an ultraproduct of $\mathcal{B}_0$ which is in turn an ultraproduct of $[0, 1]$. Thus $\mathcal{A}^\star$, having as lattice reduct an ultraproduct of $[0, 1]$, is a member of $\mathbb{L}_{\mathcal{R}^\star}$, and $\mathcal{A}$, being a subalgebra of $\mathcal{A}^\star$, is in $\mathbf{IS}(\mathbb{L}_{\mathcal{R}^\star})$.

We prove the opposite inclusion. By the Löwenheim-Skolem theorem, $\mathcal{B}$ has a countable and dense elementary subalgebra $\mathcal{B}_e$ whose lattice reduct, being countable, dense and with maximum and minimum, is isomorphic to $[0, 1]^{\mathbb{Q}}$. Hence $\mathcal{B}_e \in \mathbb{L}_{\mathcal{Q}}$. Now $\mathcal{B}$ and $\mathcal{B}_e$ are elementarily equivalent, therefore $\mathcal{B}$ and $\mathcal{B}_e$ have isomorphic ultrapowers. Summing-up, $\mathcal{B}$ embeds into an ultrapower of $\mathcal{B}_e$, which is a member of $\mathbb{L}_{\mathcal{Q}}$. This ends the proof. □

Theorem 4.9. *Let* L *be a (Δ-)core fuzzy logic. The following are equivalent:*

1. L *has the* FS$\mathcal{Q}$C.

[15] When the class of standard chains is restricted to some special kind of chain, in order to obtain a canonical completeness result, we already know that this implication fails as several examples of logics with truth-constants show in [23, 29].

2. L *has the* S$\mathcal{R}^\star$C.
3. L *has the* S$\mathcal{Q}$C.
4. L *has the* FS$\mathcal{R}^\star$C.

Furthermore, L *has the* $\mathcal{Q}$C *if and only if* L *has the* $\mathcal{R}^\star$C.

Proof. 1. $\Rightarrow$ 2.: It is a direct consequence of the previous lemma and Theorems 3.11 and 3.5.

2. $\Rightarrow$ 3.: We assume $T \nvdash_{\mathrm{L}} \varphi$ and want to show that $T \not\models_{\mathbb{L}_{\mathcal{Q}}} \varphi$. By the S$\mathcal{R}^\star$C we obtain an L-chain $\mathcal{B} \in \mathbb{L}_{\mathcal{R}^\star}$ and a $\mathcal{B}$-evaluation e such that $e[T] \subseteq \{1\}$ and $e(\varphi) < 1$. Let

$$S = \{e(p) \mid p \text{ a variable in } T \cup \{\varphi\}\}.$$

Then S is countable, and by the Löwenheim-Skolem theorem there is a countable elementary subalgebra $\mathcal{C}$ of $\mathcal{B}$ containing S. Since it is an elementary subalgebra of $\mathcal{B}$, $\mathcal{C}$ is a densely ordered L-chain. Since $\mathcal{C}$ is countable and has maximum and minimum, its lattice reduct is isomorphic to $[0,1]^{\mathbb{Q}}$, therefore $\mathcal{C} \in \mathbb{L}_{\mathcal{Q}}$. Moreover since $S \subseteq C$, e is also a $\mathcal{C}$-evaluation, $e[T] \subseteq \{1\}$, and $e(\varphi) < 1$. Thus $T \not\models_{\mathbb{L}_{\mathcal{Q}}} \varphi$.

As the implications 3. $\Rightarrow$ 1. and 2. $\Rightarrow$ 4. are trivial, all we need to complete the proof is the implication 4. $\Rightarrow$ 2. This is a simple consequence of Corollary 3.14.

The proof of the final claim: assume that L enjoys the $\mathcal{Q}$C and suppose that $\nvdash_{\mathrm{L}} \varphi$ for some formula φ. Then, by the $\mathcal{Q}$C, there is some rational L-chain $\mathcal{A}$ and an evaluation e on $\mathcal{A}$ such that $e(\varphi) \neq 1$. By Lemma 4.8, we have $\mathcal{A} \in \mathbf{ISP}_U(\mathbb{L}_{\mathcal{Q}}) = \mathbf{IS}(\mathbb{L}_{\mathcal{R}^\star})$, so $\mathcal{A}$ can be embedded into a hyperreal L-chain, and hence we have a hyperreal countermodel for φ. The converse direction is proved in the same way as the implication 2. $\Rightarrow$ 3. □

Notice that this theorem extends a previous result from [22] that proved that the FS$\mathcal{Q}$C and the S$\mathcal{Q}$C are equivalent for axiomatic extensions of MTL. Now we turn our attention to the relation between standard completeness and rational-chain and hyperreal-chain completeness. First, we consider the weaker completeness properties.

Theorem 4.10. *Let* L *be a (Δ-)core fuzzy logic with* $\mathcal{R}$C. *Then* L *has the* $\mathcal{Q}$C.

Proof. Assume that L enjoys the $\mathcal{R}$C and suppose that $\nvdash_{\mathrm{L}} \varphi$ for some formula φ. Then, by the $\mathcal{R}$C, there is some standard L-chain $\mathcal{A}$ and an evaluation e on $\mathcal{A}$ such that $e(\varphi) \neq 1$. Let $\mathcal{B}$ be the countable subalgebra of $\mathcal{A}$ generated by the values of the subformulae of φ. Now, again by the Löwenheim-Skolem theorem, we can obtain a countable elementary subalgebra $\mathcal{C}$ of $\mathcal{A}$ extending $\mathcal{B}$. Then $\mathcal{C}$ is isomorphic to a rational L-chain where φ is also refuted. Thus, L enjoys the $\mathcal{Q}$C. □

By using Corollary 3.15 and Theorem 4.9, we obtain:

Proposition 4.11. *Let* L *be a (Δ-)core fuzzy logic with the* FS$\mathcal{R}$C. *Then* L *has the* S$\mathcal{R}^\star$C *and the* S$\mathcal{Q}$C.

This proposition applies for example to the logics Ł, Π, G, BL, SBL, ΠMTL[16] and WCMTL which are only finite strong standard complete, but they are strong rational-chain complete. This result is apparently a bit strange and deserves some explanation. Next we will comment about this fact in the case of Ł and Π.

Remark 4.12. *It is well known that all standard* MV*-chains (resp. standard* Π*-chains) are isomorphic to the canonical one defined over* $[0,1]$ *by the Łukasiewicz (resp. product) t-norm and its residuum* $[0,1]_{\text{Ł}}$ *(resp.* $[0,1]_\Pi$*). It is also well known that the variety and quasivariety generated by* $[0,1]_{\text{Ł}}$ *(resp.* $[0,1]_\Pi$*) coincide with the variety and quasivariety generated by* $[0,1]_{\text{Ł}}^{\mathbb{Q}}$ *(resp.* $[0,1]_\Pi^{\mathbb{Q}}$*). So, it is clear that the infinite-valued Łukasiewicz (resp. product) logic has the finite strong completeness with respect to* $[0,1]_{\text{Ł}}$ *and* $[0,1]_{\text{Ł}}^{\mathbb{Q}}$ *(resp.* $[0,1]_\Pi$ *and* $[0,1]_\Pi^{\mathbb{Q}}$*). Moreover, the example that*

[16]The S$\mathcal{Q}$C for ΠMTL had been already proved in [22].

refutes the $\mathcal{SRC}$ *for Łukasiewicz (resp. product) logic also proves that Łukasiewicz (resp. product) logic is not strong complete with respect to the chain* $[0,1]_{Ł}^{\mathbb{Q}}$ *(resp.* $[0,1]_{\Pi}^{\mathbb{Q}}$*), i.e. the canonical* $\mathcal{SQC}$ *fails as well. However, this failure does not prevent these logics from enjoying the* $\mathcal{SQC}$*. The main fact behind this seemingly strange result is that, unlike the situation in standard chains, there are infinitely many non-isomorphic rational chains. Therefore, the* $\mathcal{SQC}$ *is actually a completeness result with respect to an infinite family of chains.*

We present in Figure 1 a diagram[17] representing explicitly the relations between standard, rational-chain and hyperreal-chain completeness and embedding properties. Taking into account the Jenei and Montagna's method described in the last subsection, in [22] the authors gave the following definition also used in the diagram.

Definition 4.13 ([22])**.** *Let* L *be an axiomatic extension of* MTL*. We say that* L *has the* $\mathcal{Q}$*-E*$^+$ *iff it has the* $\mathcal{Q}$*-E and given a rational* L*-chain* $\mathcal{A}$*, the extension of* $\mathcal{A}$ *to a standard chain defined in the last step of the corresponding embedding method (depending on whether* $\mathcal{A}$ *is or is not involutive) is also an* L*-chain.*

Figure 1 depicts also the implications that we can refute. This is done by means of some counterexamples. First we will present as example one axiomatic extension of BL firstly given in [60] and used in [22] to refute some implications.

Example 4.14. *Let* $\Pi^\star$ *be the logic defined as the axiomatic extension of* BL *obtained by adding the following schema:*

$$(\varphi \wedge \neg\varphi \to \overline{0}) \wedge ((\varphi \to \varphi \& \varphi) \to \neg\varphi \vee \varphi)$$

It is obvious that $\Pi^\star$*-chains are* SBL*-chains that have no idempotents different from the top and the bottom of the chain. Namely, an obvious computation proves that,*

(1) $[0,1]_\Pi$ *is the only standard* $\Pi^\star$*-chain.*

(2) There are $\Pi^\star$*-chains that are not* Π*-chains. In fact, the chains of the variety are the ones obtained by removing the idempotents separating the components (the idempotents different from the top and the bottom of the chain) in any ordinal sum of product chains.*

Therefore, in the logic $\Pi^\star$ *all the standard completeness properties fail, as well as the* $\mathcal{R}$*-E and the* $\mathcal{Q}$*-E*$^+$*, but it still enjoys the* $\mathcal{Q}$*-E (and thus,* $\mathcal{SQC}$*,* $\mathcal{FSQC}$ *and* $\mathcal{QC}$*), as proved in [22].*

Therefore, the example refutes the implications: $\mathcal{Q}$-E $\Rightarrow$ $\mathcal{Q}$-E$^+$, $\mathcal{Q}$-E $\Rightarrow$ $\mathcal{R}$-E, $\mathcal{SQC} \Rightarrow \mathcal{SRC}$, $\mathcal{FSQC} \Rightarrow \mathcal{FSRC}$ and $\mathcal{QC} \Rightarrow \mathcal{RC}$. The implication $\mathcal{FSRC} \Rightarrow \mathcal{SRC}$ is refuted by many counterexamples, for instance: Ł, Π, BL, SBL or ΠMTL.

Finally, some implications are neither proved nor refuted and are listed below as open problems.

Open problem 4.15. *For which core fuzzy logics are the following implications true:* $\mathcal{R}$*-E* $\Rightarrow$ $\mathcal{Q}$*-E*$^+$*,* $\mathcal{RC} \Rightarrow \mathcal{FSRC}$*, and* $\mathcal{QC} \Rightarrow \mathcal{FSQC}$*?*

The standard and rational-chain completeness and embedding properties for the considered examples are collected in Table 3.

4.3. FEP, SFMP, FMP *and finite-chain completeness*

Finite chains provide also an interesting many-valued semantics for some of the considered logics. We will consider again the three kinds of completeness properties. As we will see, these properties are closely related to some well-known algebraic properties, namely the finite embeddability property, the strong finite model property and the finite model property. First, let us recall the involved definitions.

[17] This diagram is an extension of the one given in [22] for axiomatic extensions of MTL.

Logic	$\mathcal{R}$C	FS$\mathcal{R}$C	S$\mathcal{R}$C = $\mathcal{R}$-E	$\mathcal{Q}$C	FS$\mathcal{Q}$C = S$\mathcal{Q}$C = $\mathcal{Q}$-E	$\mathcal{Q}$-E$^+$
MTL, IMTL, SMTL, WCMTL, ΠMTL	Yes	Yes	No	Yes	Yes	No
BL, SBL, Ł, Π	Yes	Yes	No	Yes	Yes	No
$\Pi^\star$	No	No	No	Yes	Yes	No
G, WNM, NM, C$_n$MTL, C$_n$IMTL	Yes	Yes	Yes	Yes	Yes	Yes
CPC	No	No	No	No	No	No

Table 3: Standard and rational-chain completeness and embedding properties for some axiomatic extensions of MTL.

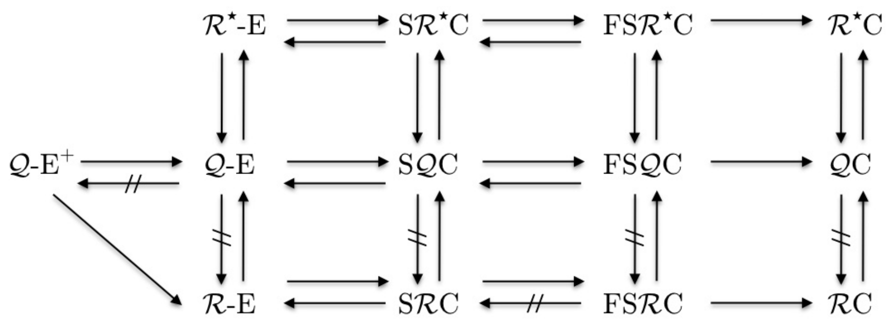

Figure 1: Standard, rational-chain and hyperreal-chain completeness properties.

Definition 4.16. *Given a class $\mathbb{K}$ of algebras, $\mathbb{K}_{fin}$ will denote the class of its finite members. We say that a class $\mathbb{K}$ of algebras has:*

- *the* finite embeddability property *(*FEP*, for short) if and only if it is partially embeddable into $\mathbb{K}_{fin}$.*
- *the* strong finite model property *(*SFMP*, for short) if and only if every quasiequation that fails to hold in $\mathbb{K}$ can be refuted in some member of $\mathbb{K}_{fin}$.*
- *the* finite model property *(*FMP*, for short) if and only if every equation that fails to hold in $\mathbb{K}$ can be refuted in some member of $\mathbb{K}_{fin}$.*

It is clear that a variety has the FMP if and only if it is generated by its finite members and a quasivariety has the SFMP if and only if it is generated (as a quasivariety) by its finite members. Therefore, we obtain the following result:

Theorem 4.17. *Let* L *be a (Δ-)core fuzzy logic. Then:*

(i) L *enjoys the $\mathcal{F}$*C *if and only if* $\mathbb{L}$ *enjoys the* FMP*.*

(ii) L *enjoys the* FS$\mathcal{F}$C *if and only if* $\mathbb{L}$ *enjoys the* SFMP*. Moreover, if the language is finite, these properties are also equivalent to the* FEP *for* $\mathbb{L}$*.*

As regards to the S$\mathcal{F}$C, we know from the general results in Section 3 that it is equivalent to the $\mathcal{F}$-E. In fact, it is equivalent to the fact that all chains are finite and there is maximum length, as the following proposition shows.

Proposition 4.18. *Let* L *be a (Δ-)core fuzzy logic. The following are equivalent:*

(i) L *enjoys the* S$\mathcal{F}$C*,*
(ii) L *enjoys the $\mathcal{F}$-E,*

(iii) *all* L*-chains are finite,*

(iv) *there is a natural number* n *such that the length of each* L*-chain is less or equal than* n*, and*

(v) *there is a natural number* n *such that* $\vdash_{\mathrm{L}} \bigvee_{i<n}(x_i \to x_{i+1})$.

Proof. The equivalence of (i), (ii) and (iii) is trivial.

$(iii) \Rightarrow (iv)$: If all L-chains are finite then there must a bound for their length, because otherwise by means of an ultraproduct we could build an infinite L-chain.

$(iv) \Rightarrow (v)$: Assume (iv) and take an arbitrary L-chain $\mathcal{A}$ and elements $a_0, \ldots a_n \in A$. Since A has at most n elements it is impossible that $a_0 > a_1 > \ldots > a_n$, thus there is some k such that $a_k \leq a_{k+1}$, i.e. $a_k \to^{\mathcal{A}} a_{k+1} = \overline{1}^{\mathcal{A}}$. Therefore, $\mathcal{A} \models \bigvee_{i<n}(x_i \to x_{i+1}) \approx \overline{1}$.

$(v) \Rightarrow (iii)$: Suppose that $\vdash_{\mathrm{L}} \bigvee_{i<n}(x_i \to x_{i+1})$ and take an arbitrary L-chain $\mathcal{A}$. We know that $\mathcal{A} \models \bigvee_{i<n}(x_i \to x_{i+1}) \approx \overline{1}$. If there would be $n+1$ different elements in A then we could choose $a_0, \ldots a_n \in A$ such that $a_0 > a_1 > \ldots > a_n$. Then $a_i \to^{\mathcal{A}} a_{i+1} \neq \overline{1}^{\mathcal{A}}$, for every $i < n$, and $\mathcal{A}$ would falsify the equation, a contradiction. □

Corollary 4.19. *For every* (Δ-)*core fuzzy logic* L *and every natural number* n*, the axiomatic extension* L_n *obtained by adding the schema* $\bigvee_{i<n}(x_i \to x_{i+1})$*, is a* ($\Delta$-)*core fuzzy logic which is strongly complete with respect the* L*-chains of length less or equal than* n*, and hence enjoys the* S$\mathcal{F}$C.

Open problem 4.20. *For which core fuzzy logics is the implication* $\mathcal{F}$C $\Rightarrow$ FS$\mathcal{F}$C *true?*

In [8] the authors introduce a very useful method for proving the FEP for the variety of commutative and integral residuated lattices. Then Ono (private communication) noticed that the method works for MTL, SMTL and IMTL as well. In this section we overview a simplification of the above method. We recall the following result from [8].

Lemma 4.21. *Let* $\mathbb{V}$ *be a variety, and let* $\mathbb{V}_{si}$ *be the class of all subdirectly irreducible members of* $\mathbb{V}$*. Then* $\mathbb{V}$ *has the* FEP *whenever* $\mathbb{V}_{si}$ *does.*

Definition 4.22. *A partial order* $\leq$ *is an* inverse well quasi order *(*iwqo *for short) iff its ascending chains and its antichains are all finite.*

We also recall the following:

Lemma 4.23. *(Dickson Lemma). The product of two iwqo is an iwqo.*

Let us fix from now on a subdirectly irreducible (hence linearly ordered) MTL-algebra $\mathcal{A}$ and its finite partial subalgebra $\mathcal{P}$. Without loss of generality we assume that $\overline{0}, \overline{1} \in P$.

Lemma 4.24. *The submonoid* $\mathcal{M}$ *of* $\mathcal{A}$ *generated by* $\mathcal{P}$ *is an iwqo and residuated. Moreover, if* $a, b, a \to b \in M$*, then the residuum* $a \Rightarrow b$ *of* a *and* b *in* $\mathcal{M}$ *is* $a \to b$.

Proof. Let $P = \{p_1, \ldots, p_n\}$. Then every element $m \in M$ has the form $p_1^{h_1} \cdot \ldots \cdot p_n^{h_n}$. Clearly, the map Φ sending $\langle h_1, \ldots, h_n \rangle$ to $p_1^{h_1} \cdot \ldots \cdot p_n^{h_n}$ is an order preserving monoid homomorphism from $\langle \mathbb{N}^n, +, \geq \rangle$ into $\mathcal{M}$. It is clear that $\mathbb{N}$ is an iwqo with the inverse natural order, therefore $\mathbb{N}^n$ ordered componentwise is also an iwqo by Lemma 4.23. Finally, Φ is order-preserving, therefore $\mathcal{M}$ is an iwqo. Indeed, the presence of an infinite antichain in $\mathcal{M}$ would imply the presence of an infinite antichain in $\mathbb{N}^n$, and the presence of an infinite ascending chain in $\mathcal{M}$ would imply the presence of either an infinite ascending chain or an infinite antichain in $\mathbb{N}^n$.

Since $\mathcal{M}$ is a totally ordered iwqo, it follows that every non-empty subset of M has a maximum. In particular, for all $a, b \in M$ the set $\{m \in \mathcal{M} \mid a \cdot m \leq b\}$ has a maximum, and such maximum is the residuum of a and b in $\mathcal{M}$ (denoted by $a \Rightarrow b$). Now clearly $a \Rightarrow b \leq a \to b$, as $M \subseteq A$. If in addition $a, b, a \to b \in M$, then $a \to b$ is the maximum $z \in \mathcal{M}$ such that $z \cdot a \leq b$, therefore $a \Rightarrow b = a \to b$. □

Lemma 4.25. *For every* $p \in P$*, the set* $M \Rightarrow p = \{m \Rightarrow p \mid m \in M\}$ *is finite.*

Proof. Suppose not. Then since $\mathcal{M}$ is linearly ordered, $M \Rightarrow p$ contains either an infinite ascending chain or an infinite descending chain. The first case is excluded because $\mathcal{M}$ is an iwqo. On the other hand, if $m_1 \Rightarrow p > m_2 \Rightarrow p > \ldots > m_n \Rightarrow p > \ldots$ is a descending chain, then it must be $m_1 < m_2 < \ldots < m_n < \ldots$, which is impossible because $\mathcal{M}$ is an iwqo. $\square$

Corollary 4.26. *The set $M \Rightarrow P = \{m \Rightarrow p \mid m \in M, p \in P\}$ is finite.*

Lemma 4.27. *$M \Rightarrow P$ is closed under $\Rightarrow$.*

Proof. Let $m_1 \Rightarrow p_1$, $m_2 \Rightarrow p_2 \in M \Rightarrow P$. Since $\mathcal{M}$ is a residuated lattice wrt $\Rightarrow$, we have that $(m_1 \Rightarrow p_1) \Rightarrow (m_2 \Rightarrow p_2) \in M$. By residuation we obtain $(m_1 \Rightarrow p_1) \Rightarrow (m_2 \Rightarrow p_2) = (m_2 \cdot (m_1 \Rightarrow p_1)) \Rightarrow p_2$. Since M is closed under $\cdot$ and $\Rightarrow$, $m_2 \cdot (m_1 \Rightarrow p_1) \in M$, and $(m_1 \Rightarrow p_1) \Rightarrow (m_2 \Rightarrow p_2) = (m_2 \cdot (m_1 \Rightarrow p_1)) \Rightarrow p_2 \in M \Rightarrow P$. $\square$

Summing-up, $M \Rightarrow P$ is a finite implicative subreduct of $\mathcal{M}$ (equipped with implication $\Rightarrow$). We now define a monoid operation $\odot$ such that $\Rightarrow$ is the residuum of $\odot$ in $M \Rightarrow P$.

Definition 4.28. *Let for $x, y \in M \Rightarrow P$,*

$$x \odot y = \min\{z \in M \Rightarrow P : x \leq y \Rightarrow z\}. \tag{1}$$

(Note that such a minimum exists because $M \Rightarrow P$ is finite and linearly ordered; also note that definition (1) implies that $x \odot y \geq x \cdot y$). The algebra obtained in this way is denoted by $\mathcal{M} \Rightarrow \mathcal{P}$.

Lemma 4.29. *$\odot$ is a commutative and weakly increasing monoid operation, and $\Rightarrow$ is its residuum in $\mathcal{M} \Rightarrow \mathcal{P}$. Moreover if $a, b, a \cdot b \in M \Rightarrow P$, then $a \cdot b = a \odot b$. Thus $\mathcal{M} \Rightarrow \mathcal{P}$ is an* MTL*-chain and has $\mathcal{P}$ as a partial subalgebra.*

Proof. Since $x \Rightarrow (y \Rightarrow z) = y \Rightarrow (x \Rightarrow z)$, the definition of $\odot$ immediately implies that $\odot$ is commutative. That $\odot$ is weakly increasing also follows from the definition of $\odot$ and from the fact that $\Rightarrow$ is weakly increasing in the second argument and weakly decreasing in the first one. We now prove that

$$(x \odot y) \Rightarrow z = x \Rightarrow (y \Rightarrow z) \tag{2}$$

which immediately implies that $\Rightarrow$ is the residuum of $\odot$. Using the residuation property in $\mathcal{M}$ and the definition of $\odot$, we obtain:

$$\begin{array}{rl} u \leq x \Rightarrow (y \Rightarrow z) & \text{iff} \quad x \leq u \Rightarrow (y \Rightarrow z) \\ & \text{iff} \quad x \leq y \Rightarrow (u \Rightarrow z) \\ & \text{iff} \quad x \odot y \leq u \Rightarrow z \\ & \text{iff} \quad u \leq (x \odot y) \Rightarrow z, \end{array}$$

which immediately gives (2). Finally, associativity follows from the definition of $\odot$ and from (2):

$$\begin{array}{rl} (x \odot y) \odot z \leq u & \text{iff} \quad ((x \odot y) \odot z) \Rightarrow u = 1 \\ & \text{iff} \quad (x \odot y) \Rightarrow (z \Rightarrow u) = 1 \\ & \text{iff} \quad x \Rightarrow (y \Rightarrow (z \Rightarrow u)) = 1 \\ & \text{iff} \quad x \Rightarrow ((y \odot z) \Rightarrow u) = 1 \\ & \text{iff} \quad (x \odot (y \odot z)) \Rightarrow u = 1 \\ & \text{iff} \quad x \odot (y \odot z) \leq u, \end{array}$$

Finally assume that $a, b, a \cdot b \in M \Rightarrow P$. Then we have $a \cdot b \leq z$ iff $a \leq b \Rightarrow z$ iff $a \odot b \leq z$. Thus $a \cdot b = a \odot b$. $\square$

We have thus shown:

Theorem 4.30. *The variety of* MTL*-algebras has the* FEP.

The result may be extended to SMTL and to IMTL:

Theorem 4.31. *Let $\mathbb{V}$ be either the variety of* IMTL*-algebras or the variety of* SMTL*-algebras. Then $\mathbb{V}$ has the* FEP*.*

Proof. For SMTL-algebras, the construction of Theorem 4.30 works without changes. Indeed if $\overline{0}, \overline{1} \in P$, then for any $m \in M$, $m \Rightarrow \overline{0}$ is either $\overline{0}$ or $\overline{1}$, therefore the same is true in $\mathcal{M} \Rightarrow \mathcal{P}$ (cf. Lemmata 4.24 and 4.27). Hence $\mathcal{M} \Rightarrow \mathcal{P}$ is an SMTL-algebra.

For IMTL-algebras, one may assume without loss of generality that $\mathcal{P}$ is closed under $\neg$. (Since $\neg$ is involutive, closing under $\neg$ does not destroy finiteness). Let us construct the algebra $\mathcal{M} \Rightarrow \mathcal{P}$ as in Definition 4.28. To conclude the proof, it is sufficient to show that $\neg$ is involutive in $\mathcal{M} \Rightarrow \mathcal{P}$. Let $x = m \Rightarrow p \in M \Rightarrow P$. We first prove that in $\mathcal{M}$ one has $z \leq m \Rightarrow p$ iff $z \cdot m \cdot \neg p = 0$, where $\neg p$ is the negation of p in $\mathcal{M}$ (by Lemma 4.24, the negations of p in $\mathcal{M}$ and in $\mathcal{A}$ are the same). The left-to-right implication is trivial; for the opposite direction, if $z \cdot m \cdot \neg p = 0$, then $z \cdot m \leq \neg\neg p = p$, and finally $z \leq m \Rightarrow p$. This implies that $m \Rightarrow p = \neg(m \cdot \neg p)$. Thus every element of $\mathcal{M}$ is the negation in $\mathcal{M}$ of some element of $\mathcal{M}$, therefore it coincides with its double negation (the identity $\neg\neg\neg x = \neg x$ holds in any MTL-algebra). By Lemma 4.27, $\mathcal{M} \Rightarrow \mathcal{P}$ is closed under $\Rightarrow$, therefore it is closed under negation. Hence $\neg\neg x = x$ also in $\mathcal{M} \Rightarrow \mathcal{P}$. $\square$

Logic	$\mathcal{F}$C = FMP	FS$\mathcal{F}$C = FEP = SFMP	S$\mathcal{F}$C = $\mathcal{F}$-E
MTL, IMTL, SMTL	Yes	Yes	No
WCMTL, ΠMTL, Π	No	No	No
BL, SBL, Ł, G, NM, WNM, C_nMTL, C_nIMTL	Yes	Yes	No
$Ł_n$, G_n, CPC	Yes	Yes	Yes

Table 4: FEP, FMP and finite-chain completeness properties for some axiomatic extensions of MTL.

5. Completeness in first-order logics

We start by recalling the definitions of the basic concepts of predicate fuzzy logic (for detailed description see a recent survey paper [40]). Let us assume from now on that L is some fixed (Δ-)core fuzzy logic.

In first-order fuzzy logics we restrict the semantics to L-chains only. For each L-chain $\mathcal{A}$, an $\mathcal{A}$-structure for a predicate language Γ is $\mathbf{M} = \langle M, \langle P_{\mathbf{M}} \rangle_{P \in \Gamma}, \langle f_{\mathbf{M}} \rangle_{f \in \Gamma} \rangle$ where $M \neq \emptyset$, for each n-ary predicate symbol P, $P_{\mathbf{M}}$ is an n-ary $\mathcal{A}$-fuzzy relation on M (a mapping $M^n \to A$), and for each n-ary function symbol f, $f_{\mathbf{M}}$ is a mapping $M^n \to M$. Having this, one defines for each formula φ, the *truth value* $\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v}$ of φ in $\mathbf{M}$ determined by the L-chain $\mathcal{A}$ and an evaluation v of free variables of φ in M in the usual (Tarskian) way. In particular, the truth value of a universally quantified formula is the infimum (if it exists) of the truth values of all its instances, similarly for $\exists$ and supremum. A structure $\mathbf{M}$ is *safe* if the truth value is defined for each φ and v.

By $\langle \mathbf{M}, \mathcal{A} \rangle \models \varphi$ we denote that $\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v} = \overline{1}^{\mathcal{A}}$ for each $\mathbf{M}$-evaluation v. When $\mathcal{A}$ is known from the context we write $\mathbf{M} \models \varphi$. We say that $\langle \mathbf{M}, \mathcal{A} \rangle$ is a model of a theory (i.e. a set of *sentences*) T to mean that $\mathcal{A}$ is an L-chain, $\mathbf{M}$ is a safe $\mathcal{A}$-structure and $\langle \mathbf{M}, \mathcal{A} \rangle \models \alpha$ for each $\alpha \in T$. To simplify matters, we use the expression "$\langle \mathbf{M}, \mathcal{A} \rangle$ is a model" meaning that $\langle \mathbf{M}, \mathcal{A} \rangle$ is a model of the empty theory. If we say "for each model $\langle \mathbf{M}, \mathcal{A} \rangle$" we mean "for each L-chain $\mathcal{A}$ and each safe $\mathcal{A}$-structure $\mathbf{M}$". Finally, by $\|\varphi(a_1, \ldots, a_n)\|^{\langle \mathbf{M}, \mathcal{A} \rangle}$ we mean $\|\varphi(x_1, \ldots, x_n)\|^{\mathcal{A}}_{\mathbf{M},v}$ for $v(x_i) = a_i$.

Now we define the corresponding predicate logic for propositional (Δ-)core fuzzy logics. We use the axioms used in the monograph [36] to obtain the predicate variant of Basic fuzzy Logic BL and the axioms for *crisp* equality (see [40]) for more details).

Definition 5.1. *Let* L *be a (Δ-)core fuzzy logic and Γ a predicate language with equality. The logic* L$\forall$ *has the deduction rules of* L *and generalization (from φ infer $(\forall x)\varphi$) and its axioms are:*

(P)	*the axioms resulting from the axioms of* L *by the substitution of propositional variables with formulae of* Γ,
$(\forall 1)$	$(\forall x)\varphi(x) \to \varphi(t)$, *where t is substitutable for x in φ,*
$(\exists 1)$	$\varphi(t) \to (\exists x)\varphi(x)$, *where t is substitutable for x in φ,*
$(\forall 2)$	$(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$, *where x is not free in χ,*
$(\exists 2)$	$(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$, *where x is not free in χ,*
$(\forall 3)$	$(\forall x)(\chi \vee \varphi) \to (\chi \vee (\forall x)\varphi)$, *where x is not free in χ,*
(≈ 1)	$(\forall x)(\forall y)((x \approx y) \vee \neg(x \approx y))$,
(≈ 2)	$(\forall x)x \approx x$,
(≈ 3)	$(\forall x)(\forall y)(\forall \vec{z})(x \approx y \to (\varphi(x, \vec{z}) \leftrightarrow \varphi(y, \vec{z})))$, *where y is substitutable for x in φ.*

The completeness theorem for first-order BL was proven in [36] and the completeness theorems of other predicate fuzzy logics defined in the literature were proven in the corresponding papers. The following general formulation is from the paper [39].

Theorem 5.2. *Let* L *be a (Δ-)core fuzzy logic, Γ a countable predicate language, T a theory, and φ a formula. Then the following are equivalent:*

- $T \vdash_{\mathrm{L}\forall} \varphi$.
- $\langle \mathbf{M}, \mathcal{A}\rangle \models \varphi$ *for each (countable) model* $\langle \mathbf{M}, \mathcal{A}\rangle$ *of the theory* T.

For the results in the next subsection we also need the following definitions of elementary equivalence, elementary embedding, full diagram and σ-embedding.

Definition 5.3. *Two models $\langle \mathbf{M}_1, \mathcal{B}_1\rangle$ and $\langle \mathbf{M}_2, \mathcal{B}_2\rangle$ of a predicate language Γ are* elementarily equivalent *if for each Γ-formula φ it holds: $\langle \mathbf{M}_1, \mathcal{B}_1\rangle \models \varphi$ iff $\langle \mathbf{M}_2, \mathcal{B}_2\rangle \models \varphi$.*

Definition 5.4. *An* elementary embedding *of a model $\langle \mathbf{M}_1, \mathcal{B}_1\rangle$ of a language Γ_1 into a model $\langle \mathbf{M}_2, \mathcal{B}_2\rangle$ of a language $\Gamma_2 \supseteq \Gamma_1$ is a pair $\langle f, g\rangle$ such that:*

1. *f is an injection of the domain of $\mathbf{M}_1$ into the domain of $\mathbf{M}_2$.*
2. *g is an embedding of $\mathcal{B}_1$ into $\mathcal{B}_2$.*
3. *$g(\|\varphi(a_1, \dots, a_n)\|^{\langle \mathbf{M}_1, \mathcal{B}_1\rangle}) = \|\varphi(f(a_1), \dots, f(a_n))\|^{\langle \mathbf{M}_2, \mathcal{B}_2\rangle}$ holds for each Γ_1-formula $\varphi(x_1, \dots, x_n)$ and $a_1, \dots, a_n \in \mathbf{M}_1$.*

Clearly if a model $\langle \mathbf{M}_1, \mathcal{B}_1\rangle$ can be elementarily embedded into a model $\langle \mathbf{M}_2, \mathcal{B}_2\rangle$, then it is elementarily equivalent to the Γ_1-reduct of $\langle \mathbf{M}_2, \mathcal{B}_2\rangle$.

Definition 5.5. *Let* L *be a logic in a propositional language $\mathcal{L}$, $\mathcal{B}$ an* L*-chain, Γ be a predicate language and $\langle \mathbf{M}, \mathcal{B}\rangle$ a model. Then we define:*

- *$\Gamma_{\langle \mathbf{M}, \mathcal{B}\rangle}$ is the predicate language resulting from Γ by adding a constant c_a for each $a \in M$ and a nullary predicate symbol P_b for each $b \in B$.*
- *$\mathbf{M}^*$ is the $\Gamma_{\langle \mathbf{M}, \mathcal{B}\rangle}$-model resulting from $\mathbf{M}$ by setting $(c_a)_{\mathbf{M}^*} = a$ for each $a \in M$ and $(P_b)_{\mathbf{M}^*} = b$ for each $b \in B$.*
- $\mathrm{FDIAG}\langle \mathbf{M}, \mathcal{B}\rangle = Th(\langle \mathbf{M}^*, \mathcal{B}\rangle)$ *(the set of all sentences true in $\langle \mathbf{M}^*, \mathcal{B}\rangle$).*

The set $\mathrm{FDIAG}\langle \mathbf{M}, \mathcal{B}\rangle$ is called the *full diagram* of the model $\langle \mathbf{M}, \mathcal{B}\rangle$ (this is a strengthening of the notion of diagram defined in the context of fuzzy logics in [39]).

Definition 5.6. *Let $\mathcal{A}, \mathcal{B}$ be two algebras of the same type with (defined) lattice operations. We say that an embedding $f : \mathcal{A} \to \mathcal{B}$ is a σ-embedding if $f(\sup C) = \sup f[C]$ (whenever $\sup C$ exists) and $f(\inf D) = \inf f[D]$ (whenever $\inf D$ exists) for each countable $C, D \subseteq A$.*

5.1. General completeness results

Like in the propositional case we introduce several notions of completeness w.r.t. a class of algebras $\mathbb{K}$. We restrict ourselves to at most countable predicate languages (in fact, we could define the notion of κ-$\mathbb{K}$-completeness for each cardinal κ, but this would make the paper unnecessarily complex).

Definition 5.7. *Let* L *be a (Δ-)core fuzzy logic. We say that* L$\forall$ *has the* S$\mathbb{K}$C *if for each countable language Γ, theory T, and formula φ the following are equivalent:*

- $T \vdash_{\mathrm{L}\forall} \varphi$.
- $\langle \mathbf{M}, \mathcal{A}\rangle \models \varphi$ *for each $\mathcal{A} \in \mathbb{K}$ and each countable model $\langle \mathbf{M}, \mathcal{A}\rangle$ of the theory T.*

We say that L$\forall$ *has the* FS$\mathbb{K}$C *if the above condition holds for* finite *theories. Finally, we say that* L$\forall$ *has the* $\mathbb{K}$C *if the above condition holds for the empty theory.*

Lemma 5.8. *Let* L *be a (Δ-)core fuzzy logic enjoying the* S$\mathbb{K}$C*. Then for each language Γ, theory T, and a directed*[18] *set of formulae Ψ the following are equivalent:*

- $T \not\vdash_{\mathrm{L}\forall} \Psi$.
- *there is a chain $\mathcal{A} \in \mathbb{K}$ and a model $\langle \mathbf{M}, \mathcal{A}\rangle$ of T such that $\langle \mathbf{M}, \mathcal{A}\rangle \not\models \Psi$.*

Proof. The proof is the same as in the propositional case (see Lemma 3.4). The only difference is that instead of using an unused propositional variable we extend the predicate language by a new nullary predicate. □

As in the propositional case we provide a characterization for the strong completeness with respect to an arbitray class of chains, but here the equivalent property is not purely algebraic (as it was in Theorem 3.5) but it is written in a model-theoretic fashion.

Theorem 5.9. *Let* L *be a (Δ-)core fuzzy logic. Then the following are equivalent:*

(i) L$\forall$ *has the* S$\mathbb{K}$C.

(ii) For every countable L*-chain $\mathcal{A}$ and every model $\langle \mathbf{M}, \mathcal{A}\rangle$ there is an* L*-chain $\mathcal{B} \in \mathbb{K}$ and a model $\langle \mathbf{M}', \mathcal{B}\rangle$ such that $\langle \mathbf{M}, \mathcal{A}\rangle$ can be elementarily embedded into $\langle \mathbf{M}', \mathcal{B}\rangle$.*

(iii) For every countable L*-chain $\mathcal{A}$ and every model $\langle \mathbf{M}, \mathcal{A}\rangle$ there is an* L*-chain $\mathcal{B} \in \mathbb{K}$ and a model $\langle \mathbf{M}', \mathcal{B}\rangle$ such that $\langle \mathbf{M}, \mathcal{A}\rangle$ is elementarily equivalent to $\langle \mathbf{M}', \mathcal{B}\rangle$.*

Proof. The only non-trivial part to prove is $(i) \Rightarrow (ii)$: Let us define $T = \mathrm{FDIAG}\langle \mathbf{M}, \mathcal{A}\rangle$ and $\Psi = \{\varphi \text{ is a sentence} \mid \varphi \notin \mathrm{FDIAG}\langle \mathbf{M}, \mathcal{A}\rangle\}$. Observe that Ψ is directed and $T \not\vdash_{\mathrm{L}\forall} \Psi$. Thus there is an L-chain $\mathcal{B} \in \mathbb{K}$ and a model $\langle \mathbf{M}', \mathcal{B}\rangle$ of T such that $\langle \mathbf{M}', \mathcal{B}\rangle \not\models \Psi$ (due to the S$\mathbb{K}$C and the previous lemma).

We define $f(a) = (c_a)_{\mathbf{M}'}$ and $g(b) = \|P_b\|^{\langle \mathbf{M}', \mathcal{B}\rangle}$ for every $a \in \mathbf{M}$ and $b \in \mathcal{A}$. Is $\langle f, g\rangle$ an elementary embedding of the model $\langle \mathbf{M}, \mathcal{A}\rangle$ into the model $\langle \mathbf{M}', \mathcal{B}\rangle$? We have to prove the three parts of the definition of an elementary embedding:

Part 1.: Assume that $a \neq b$ and $f(a) = f(b)$, i.e. $(c_a)_{\mathbf{M}'} = (c_b)_{\mathbf{M}'}$. Thus $\langle \mathbf{M}', \mathcal{B}\rangle \models c_a \approx c_b$. As $\langle \mathbf{M}', \mathcal{B}\rangle \models \neg(c_a \approx c_b)$ (since $\langle \mathbf{M}', \mathcal{B}\rangle \models T$) we obtain a contradiction.

Part 3.: We write a chain of equalities: $g(\|\varphi(a_1, \dots a_n)\|^{\langle \mathbf{M}, \mathcal{A}\rangle}) = \|P_{\|\varphi(a_1,\dots a_n)\|^{\langle \mathbf{M}, \mathcal{A}\rangle}}\|^{\langle \mathbf{M}', \mathcal{B}\rangle} = \|P_{\|\varphi(c_{a_1},\dots c_{a_n})\|^{\langle \mathbf{M}^*, \mathcal{A}\rangle}}\|^{\langle \mathbf{M}', \mathcal{B}\rangle} = \|\varphi(c_{a_1}, \dots c_{a_n})\|^{\langle \mathbf{M}', \mathcal{B}\rangle} = \|\varphi(f(a_1), \dots f(a_n))\|^{\langle \mathbf{M}', \mathcal{B}\rangle}$ (the first equality is the definition of g, the second one is straightforward, the third one is due to the fact that $\langle \mathbf{M}', \mathcal{B}\rangle \models T$ and the last one is the definition of f).

[18] The definition is analogous to the propositional case, see Definition 3.3. By $T \not\vdash_{\mathrm{L}\forall} \Psi$ we mean that $T \not\vdash_{\mathrm{L}\forall} \psi$ for each $\psi \in \Psi$ (analogously for $\models$).

Part 2.: We show that g is a homomorphism. Assume for simplicity that λ is a binary connective. We write the chain of simple equalities (observe that we use Part 3.): $g(\lambda^{\mathcal{A}}(x,y)) = g(\|\lambda(P_x,P_y)\|^{\langle \mathbf{M},\mathcal{A}\rangle}) = \|\lambda(P_x,P_y)\|^{\langle \mathbf{M}',\mathcal{B}\rangle} = \lambda^{\mathcal{B}}(\|P_x\|^{\langle \mathbf{M}',\mathcal{B}\rangle}, \|P_y\|^{\langle \mathbf{M}',\mathcal{B}\rangle}) = \lambda^{\mathcal{B}}(g(x),g(y))$.

We show that g is one-to-one: let us assume that $a \neq b$ and $g(a) = g(b)$, i.e. $\|P_a\|^{\langle \mathbf{M}',\mathcal{B}\rangle} = \|P_b\|^{\langle \mathbf{M}',\mathcal{B}\rangle}$. Thus $\langle \mathbf{M}',\mathcal{B}\rangle \models P_a \leftrightarrow P_b$. As $(P_a \leftrightarrow P_b) \in \Psi$ we obtain a contradiction. □

Nevertheless, all existing proofs of standard completeness for predicate fuzzy logics L∀ are not based on this model-theoretic property but on the following algebraic property: every countable L-chain has a σ-embedding on an L-chain over $[0,1]$. Thus one may wonder if for any class $\mathbb{K}$ of L-chains, strong completeness of L∀ with respect to $\mathbb{K}$ implies that every countable L-chain is σ-embeddable into a chain from $\mathbb{K}$. In other words, one may wonder whether this algebraic condition is not only sufficient but also necessary for the strong $\mathbb{K}$-completeness. This question will receive a negative answer in Subsection 5.3. Now we give its model-theoretic counterpart which, by comparing it to the conditions in Theorem 5.9, already suggests that in general it will not be equivalent to the strong $\mathbb{K}$-completeness.

Theorem 5.10. *Let* L *be a* (Δ-)*core fuzzy logic. Then the following are equivalent:*

(i) *For every countable* L*-chain* $\mathcal{A}$ *and every countable model* $\langle \mathbf{M},\mathcal{A}\rangle$ *there is an* L*-chain* $\mathcal{B} \in \mathbb{K}$ *and a model* $\langle \mathbf{M}',\mathcal{B}\rangle$ *such that* $\langle \mathbf{M},\mathcal{A}\rangle$ *can be elementarily embedded into* $\langle \mathbf{M}',\mathcal{B}\rangle$ *via* $\langle f,g\rangle$ *where* f *is an isomorphism.*

(ii) *Every countable* L*-chain* $\mathcal{A}$ *can be* σ*-embedded into some* L*-chain* $\mathcal{B} \in \mathbb{K}$.

Both (i) and (ii) imply that L∀ *has* S$\mathbb{K}$C.

Proof. The implication $(ii) \Rightarrow (i)$ and the final claim are simple. We prove $(i) \Rightarrow (ii)$. First, we define the predicate language Γ_0 which consists of two binary predicates S and I. Let us define the model $\langle \mathbf{M},\mathcal{A}\rangle$ with domain A in the following way:

- $\|\mathrm{S}(a,b)\|^{\langle \mathbf{M},\mathcal{A}\rangle} = a$ iff $a < b$ and $\overline{0}$ otherwise
- $\|\mathrm{I}(a,b)\|^{\langle \mathbf{M},\mathcal{A}\rangle} = a$ iff $a > b$ and $\overline{1}$ otherwise

Due to (i) there is an L-chain $\mathcal{B} \in \mathbb{K}$ and a model $\langle \mathbf{M}',\mathcal{B}\rangle$ such that $\langle \mathbf{M},\mathcal{A}\rangle$ can be elementarily embedded in $\langle \mathbf{M}',\mathcal{B}\rangle$ via $\langle f,g\rangle$ and f is an isomorphism. We will show that g is the σ-embedding we are looking for.

All we have to show is that g preserves infinite suprema and infima. We show that g preserves suprema; the proof for infima is analogous. Let us take a set $D \subseteq A$ such that $\sup D = d$ and $d \notin D$. Define the set $\overleftarrow{d} = \{a \in A \mid a < d\}$ and observe that $\sup \overleftarrow{d} = d$. We obtain the following chain of equalities:

$$
\begin{aligned}
g(d) &= g(\|(\exists x)\mathrm{S}(x,d)\|^{\langle \mathbf{M},\mathcal{A}\rangle}) && (1)\\
&= \|(\exists x)\mathrm{S}(x,f(d))\|^{\langle \mathbf{M}',\mathcal{B}\rangle} && (2)\\
&= \sup\{\|\mathrm{S}(z,f(d))\|^{\langle \mathbf{M}',\mathcal{B}\rangle} \mid z \in M'\} && (3)\\
&= \sup\{\|\mathrm{S}(f(a),f(d))\|^{\langle \mathbf{M}',\mathcal{B}\rangle} \mid a \in M\} && (4)\\
&= \sup\{g(\|\mathrm{S}(a,d)\|^{\langle \mathbf{M},\mathcal{A}\rangle}) \mid a \in M\} && (5)\\
&= \sup(\{g(\|\mathrm{S}(a,d)\|^{\langle \mathbf{M},\mathcal{A}\rangle}) \mid a < d\} \cup \{g(\|\mathrm{S}(a,d)\|^{\langle \mathbf{M},\mathcal{A}\rangle}) \mid a \geq d\}) && (6)\\
&= \sup g[\overleftarrow{d}\,] && (7)
\end{aligned}
$$

The first equality is due to the semantics of S in $\langle \mathbf{M},\mathcal{A}\rangle$, the second one is the third property of elementary embedding, the third one is the definition of Tarskian semantics (we know that the supremum exist as $\langle \mathbf{M}',\mathcal{B}\rangle$ is safe), the fourth one is due to the isomorphism f, the fifth one is the third property of elementary embedding, the sixth one is trivial, and the last one is due to the definition of S in $\langle \mathbf{M},\mathcal{A}\rangle$ and the fact that $g(\overline{0}) = \overline{0}$.

We clearly know that $g(d)$ is an upper bound of $g[D]$. Assume that it is not the least upper bound, i.e. there is $a \in B$ such that for each $x \in D$ we have $g(x) \leq a < g(d)$. As $g(d) = \sup g[\overleftarrow{d}]$ there is an element in $y \in \overleftarrow{d}$ such that $g(x) \leq a < g(y)$. Thus, when $x = y$ we have a contradiction. $\square$

Like in the propositional case (Propositions 3.16 and 3.18) we can prove:

Proposition 5.11. *Let* L *and* L′ *be (*Δ*-)core fuzzy logics such that* L′∀ *is a conservative expansion of* L∀. *Let* $\mathbb{K}'$ *be a class of* L′*-chains and let* $\mathbb{K}$ *be the class of their* L*-reducts. Then:*

- *If* L′∀ *enjoys the* $\mathbb{K}'$C*, then* L∀ *enjoys the* $\mathbb{K}$C.
- *If* L′∀ *enjoys the* FS$\mathbb{K}'$C*, then* L∀ *enjoys the* FS$\mathbb{K}$C.
- *If* L′∀ *enjoys the* S$\mathbb{K}'$C*, then* L∀ *enjoys the* S$\mathbb{K}$C.

Proposition 5.12. *Let* L *a* Δ*-core fuzzy logic. Then* L∀ *has the* $\mathbb{K}$C *if and only if* L∀ *has the* FS$\mathbb{K}$C.

Using Theorem 5.9 we can prove an analog of Proposition 3.17 for the S$\mathbb{K}$C. We cannot prove an analog for the FS$\mathbb{K}$C as, unlike in the propositional case, we have no characterization for this notion in predicate logics.

Proposition 5.13. *Let* L *be a core fuzzy logic and* $\mathbb{K}$ *a class of* L*-chains. Then* L∀ *has the* S$\mathbb{K}$C *if and only if* L_Δ∀ *has the* S$\mathbb{K}_\Delta$C*, where* $\mathbb{K}_\Delta$ *is the class of* Δ*-expansions of chains in* $\mathbb{K}$.

We conclude this subsection by showing the expected relationship between completeness in predicate and in propositional logics.

Theorem 5.14. *Let* L *be a (*Δ*-)core fuzzy logic and* $\mathbb{K}$ *a class of* L*-chains. If* L∀ *has the* S$\mathbb{K}$C *(*FS$\mathbb{K}$C *or* $\mathbb{K}$C *respectively), then* L *has the* S$\mathbb{K}$C *(*FS$\mathbb{K}$C *or* $\mathbb{K}$C *respectively).*

Proof. We give a proof for the case of S$\mathbb{K}$C, the other two are analogous. Take a predicate language Γ containing just a nullary predicate constant v' for each propositional variable v. Let ψ' be the predicate Γ-formula corresponding to the propositional formula ψ. Observe that there is a clear correspondence between models $\langle \mathbf{M}, \mathcal{A} \rangle$ and $\mathcal{A}$-evaluations and between proofs in L and L∀.

Assume that $T \models_{\mathbb{K}} \varphi$. Then clearly $\langle \mathbf{M}, \mathcal{A} \rangle \models \varphi'$ for each $\mathcal{A} \in \mathbb{K}$ and each model $\langle \mathbf{M}, \mathcal{A} \rangle$ of $\{\psi' \mid \psi \in T\}$. Thus by the S$\mathbb{K}$C of L∀ we obtain $\{\psi' \mid \psi \in T\} \vdash_{\mathrm{L}\forall} \varphi'$ and so clearly $T \vdash_{\mathrm{L}} \varphi$. $\square$

5.2. Disproving the FS$\mathbb{K}$C *in a constructive way*

In this subsection we propose a general method to *constructively* disprove the FS$\mathbb{K}$C of the logic L∀ whenever we can constructively disprove the S$\mathbb{K}$C of the logic L and some other technical assumptions are met (details below).

Note that the usual way of disproving the FS$\mathbb{K}$C (or even the $\mathbb{K}$C) is based on results about the position of the classes of $\mathbb{K}$-tautologies in the arithmetical hierarchy, e.g. $\mathcal{R}$C for Łukasiewicz predicate logic was disproved by showing the that set of standard tautologies is Π_2-complete (see [66]). Sometimes the proof of these results can be seen as 'constructive', as the authors reduce the true arithmetics to the set of $\mathbb{K}$-tautologies, thus obtaining non-arithmeticity of the latter and disproving the $\mathbb{K}$C as well. This technique was used for refuting the $\mathcal{R}$C for many logics over BL (see [53] and the survey paper [38] for more details). There is also a paper [55] where the author refutes the FS$\mathcal{R}$C for a wide class of fuzzy logics using similar techniques. As we will see our method is really constructive, simpler, and more general than the ones sketched above.

Let us take a finite propositional language $\mathcal{L}$. We define a predicate language Γ_0 which consists of a functional symbol $\dot{\lambda}$ for each $\lambda \in \mathcal{L}$, a nullary function symbol $\dot{v}$ for each propositional variable and unary predicates $\mathbf{Ev}$ and $\mathbf{T}$. By $\dot{\varphi}$ we denote the closed Γ_0-term corresponding to a formula φ (in the obvious way). Furthermore we define the theory:

$$T_0 = \{\mathbf{Ev}(\dot{\lambda}(x_1, \dots x_n)) \leftrightarrow \lambda(\mathbf{Ev}(x_1), \dots, \mathbf{Ev}(x_n)) \mid \lambda \in \mathcal{L}\} \cup \{\mathbf{T}(x) \to \mathbf{Ev}(x)\}.$$

The proofs of the following two lemmata are almost straightforward.

Lemma 5.15. *Let $\langle \mathbf{M}, \mathcal{A}\rangle \models T_0$. The mapping $e : \mathrm{Fm}_{\mathcal{L}} \to \mathcal{A}$ defined as $e(\varphi) = ||\dot{\varphi}||^{\langle \mathbf{M}, \mathcal{A}\rangle}$ is an $\mathcal{A}$-model of the propositional theory $\{\varphi \mid \mathbf{T}_{\mathbf{M}}(\dot{\varphi}) = \overline{1}^{\mathcal{A}}\}$.*

Given any L-chain $\mathcal{A}$, any propositional theory T and any evaluation e which is an $\mathcal{A}$-model of T, we define a special model $\langle \mathbf{Fm}, \mathcal{A}\rangle^{e,T}$ with domain $\mathrm{Fm}_{\mathcal{L}}$ as:

- $\dot{\lambda}_{\mathbf{Fm}}(\varphi_1, \dots \varphi_n) = \lambda(\varphi_1, \dots, \varphi_n)$
- $\mathbf{Ev}_{\mathbf{Fm}}(\varphi) = e(\varphi)$
- $\mathbf{T}_{\mathbf{Fm}}(\varphi) = \overline{1}^{\mathcal{A}}$ if $\varphi \in T$ and $\overline{0}^{\mathcal{A}}$ otherwise.

Notice that in $\langle \mathbf{Fm}, \mathcal{A}\rangle^{e,T}$ we have $\dot{\varphi}_{\mathbf{Fm}} = \varphi$.

Lemma 5.16. *Let $\mathcal{A}$ be an* L*-chain, T a propositional theory and e an $\mathcal{A}$-model of T. Then $\langle \mathbf{Fm}, \mathcal{A}\rangle^{e,T} \models T_0$.*

Definition 5.17. *Let* L *be a (Δ-)core fuzzy logic. We say that a propositional theory T is* encodable *in* L$\forall$ *if there is a* finite *set of formulae T_1 in a predicate language $\Gamma \supseteq \Gamma_0$ such that the following conditions hold for each* L*-chain $\mathcal{A}$:*

$$\textit{for each } \mathcal{A}\textit{-structure } \langle \mathbf{M}, \mathcal{A}\rangle \textit{ if } \langle \mathbf{M}, \mathcal{A}\rangle \models T_0 \cup T_1, \textit{ then } T \subseteq \{\varphi \mid \mathbf{T}_{\mathbf{M}}(\dot{\varphi}) = \overline{1}^{\mathcal{A}}\} \tag{1}$$

$$\textit{for each } \mathcal{A}\textit{-model } e \textit{ of } T \textit{ there is a model } \langle \mathbf{M}, \mathcal{A}\rangle \textit{ of } T_1 \textit{ extending}^{19} \langle \mathbf{Fm}, \mathcal{A}\rangle^{e,T} \tag{2}$$

Of course any *finite* theory T is encodable by $T_1 = \{\mathbf{T}(\dot{\varphi})\}$, where φ is the conjunction of all formulae in T.

Definition 5.18. *Let* L *be a (Δ-)core fuzzy logic. We say that* L *has a* simple *counterexample to the* S$\mathbb{K}$C *if there is an encodable theory T^c and a formula φ^c such that $T^c \not\models_{\mathrm{L}} \varphi^c$ and $T^c \models_{\mathbb{K}} \varphi^c$.*

We give two examples to illustrate the notion of encodable theory.

Proposition 5.19. *Łukasiewicz logic has a* simple *counterexample to the* S$\mathcal{R}$C.

Proof. It is well known (see e.g. [36]) that the theory $T^c = \{np_1 \to p_2 \mid n \text{ natural}\} \cup \{\neg p_1 \to p_2\}$ and the formula $\varphi^c = p_2$ provide a counterexample to the S$\mathcal{R}$C of Łukasiewicz logic (where $\varphi \oplus \psi = \neg\varphi \to \psi$ and $n\varphi$ is $\varphi \oplus \varphi \cdots \oplus \varphi$ n-times). We show that T^c is an encodable theory. We define a predicate language Γ as the extension of Γ_0 by a new unary predicate $\mathbf{Q}$ and a theory T_1 as:

$$T_1 = \{\mathbf{Q}(\dot{p}_1), \mathbf{Q}(x) \to \mathbf{Q}(x \dot{\oplus} \dot{p}_1), \mathbf{Q}(x) \to \mathbf{T}(x \dot{\to} \dot{p}_2), \mathbf{T}(\dot{\neg}\dot{p}_1 \dot{\to} \dot{p}_2)\}$$

To prove condition (1) observe that $\langle \mathbf{M}, \mathcal{A}\rangle \models T_1$ entails $\{np_1 \mid n \text{ natural}\} \subseteq \{\varphi \mid \mathbf{Q}_{\mathbf{M}}(\dot{\varphi}) = 1\}$. The rest is simple. To prove condition (2) just define $\mathbf{Q}_{\mathbf{M}}(\varphi) = 1$ if $\varphi = np_1$ for some natural n and 0 otherwise. $\square$

Proposition 5.20. *The logic* ΠMTL *has a* simple *counterexample to the* S$\mathcal{R}$C.

Proof. In [45] is shown that $T^c = \{\neg\neg r, p \to q, \neg p \to q\} \cup \{(p^n \to r) \to q \mid n \geq 0\} \cup \{\varphi \wedge \psi \to \varphi \& (\varphi \to \psi) \mid \varphi, \psi \text{ formulae in } p, r\}$ and the formula $\varphi^c = q$ provide a counterexample to the S$\mathcal{R}$C of ΠMTL (where φ^n is $\varphi \& \varphi \dots \& \varphi$ n-times). We show that T^c is an encodable theory. We define a predicate language Γ as the extension of Γ_0 by new unary predicates $\mathbf{F}$ and $\mathbf{R}$ and theories:

$$T_F = \{\mathbf{F}(\dot{p}), \mathbf{F}(\dot{r}), \mathbf{F}(\dot{\overline{0}})\} \cup \{\mathbf{F}(x) \wedge \mathbf{F}(y) \to \mathbf{F}(x \mathbin{\dot{\lambda}} y) \mid \lambda \in \{\&, \to, \wedge\}\}$$

$$T_R = \{\mathbf{R}(\dot{p}), \mathbf{R}(x) \to \mathbf{R}(x \mathbin{\dot{\&}} \dot{p}), \mathbf{R}(x) \to \mathbf{T}((x \mathbin{\dot{\to}} \dot{r}) \mathbin{\dot{\to}} \dot{q})\}$$

$$T_1 = T_F \cup T_R \cup \{\mathbf{T}(\dot{\neg}\dot{\neg}\dot{r}), \mathbf{T}(\dot{p} \mathbin{\dot{\to}} \dot{q}), \mathbf{T}(\dot{\neg}\dot{p} \mathbin{\dot{\to}} \dot{q}), \mathbf{F}(x) \wedge \mathbf{F}(y) \to \mathbf{T}(x \dot{\wedge} y \mathbin{\dot{\to}} x \dot{\&} (x \mathbin{\dot{\to}} y))\}$$

The proof that the set T_1 satisfies both conditions of encodability is analogous to the case of Łukasiewicz logic. $\square$

[19] By extended model we mean a model with the same domain and added realizations of new predicate symbols.

Theorem 5.21. *Let* L *be a* (Δ-)*core fuzzy logic with a* simple *counterexample to the* S$\mathbb{K}$C*. Then* L$\forall$ *has not the* FS$\mathbb{K}$C.

Proof. Let T_1 be the theory providing the encoding of T^c. Let us define the theory $T_2 = T_0 \cup T_1$. We show that $T_2 \not\vdash_{\mathrm{L}\forall} \mathbf{Ev}(\dot{\varphi}^c)$ and $\langle \mathbf{M}, \mathcal{A}\rangle \models \mathbf{Ev}(\dot{\varphi}^c)$ for each $\mathcal{A} \in \mathbb{K}$ and each model $\langle \mathbf{M}, \mathcal{A}\rangle$ of the theory T_2.

From $T^c \not\vdash_{\mathrm{L}} \varphi^c$ we know that there is an L-chain $\mathcal{A}$ and an $\mathcal{A}$-model e of T^c such that $e(\varphi^c) < \overline{1}$. Using condition (2) we know there is a model $\langle \mathbf{M}, \mathcal{A}\rangle$ of T_1 expanding the model $\langle \mathbf{Fm}, \mathcal{A}\rangle^{e,T^c}$ and so $\langle \mathbf{M}, \mathcal{A}\rangle$ is a model of T_2. Because $\|\mathbf{Ev}(\dot{\varphi}^c)\|^{\langle \mathbf{M}, \mathcal{A}\rangle} = \|\mathbf{Ev}(\dot{\varphi}^c)\|^{\langle \mathbf{Fm}, \mathcal{A}\rangle} = \|\mathbf{Ev}(\varphi^c)\|^{\langle \mathbf{Fm}, \mathcal{A}\rangle} = e(\varphi^c) < \overline{1}$ we obtain $T_2 \not\vdash_{\mathrm{L}\forall} \mathbf{Ev}(\dot{\varphi}^c)$.

We prove the second claim by the way of contradiction: assume that there is an algebra $\mathcal{A} \in \mathbb{K}$ and a model $\langle \mathbf{M}, \mathcal{A}\rangle$ of T_2 such that $\langle \mathbf{M}, \mathcal{A}\rangle \not\models \mathbf{Ev}(\dot{\varphi}^c)$. Using Lemma 5.15 we obtain the $\mathcal{A}$-model e of $\{\mathbf{T_M}(\dot{\varphi}) = 1\}$ defined as $e(\varphi) = ||\mathbf{Ev}(\dot{\varphi})||^{\langle \mathbf{M}, \mathcal{A}\rangle}$. Thus $e(\varphi^c) < \overline{1}$. However, using condition (1) we know that $T^c \subseteq \{\varphi \mid \mathbf{T_M}(\dot{\varphi}) = 1\}$, i.e. that e is a model of T^c and so we obtain a contradiction with $T^c \models_{\mathbb{K}} \varphi^c$. $\square$

Corollary 5.22. Ł$\forall$ *and* ΠMTL$\forall$ *do not enjoy the* FS$\mathcal{R}$C.

The rest of this subsection is devoted to the study of some sufficient conditions for a set of formulae to be encodable. Later, in Subsection 5.3, we will see some application of these results for particular logics and semantics.

$\diamond$

In this part we use the machinery of formal languages and grammars (see e.g. [11]). Let us recall that a formal grammar $\mathcal{G}$ consists of:

- a finite set of terminal symbols $\mathcal{T}$;
- a finite set of nonterminal symbols $\mathcal{N}$;
- a finite set of production rules with a left- and a right-hand side consisting of a sequence of terminal and nonterminal symbols $\mathcal{R}$;
- a start symbol $S \in \mathcal{N}$.

For a set of symbols $\mathcal{T}$, a formal language in the alphabet $\mathcal{T}$ is a set of finite strings of symbols from $\mathcal{T}$. The elements of a language are called *words*. Each formal grammar defines (generates) a formal language, whose words are constructed by applying production rules to a sequence of symbols which initially contains just the start symbol. A rule may be applied to a sequence of symbols by replacing an occurrence of the symbols on the left-hand side of the rule with those that appear on the right-hand side. Let us denote the language in the alphabet $\mathcal{T}$ generated by a grammar $\mathcal{G}$ as $L(\mathcal{G})$. A grammar is context-free if all the production rules have on the left sides just one non-terminal symbol. For our needs we define a special auxiliary subclass of context-free grammars.

Definition 5.23. *Let A be a language in the alphabet $\mathcal{T}$. We say that a grammar $\mathcal{G} = \langle \mathcal{T}, \mathcal{N}, \mathcal{R}, S\rangle$ is A-*cautious *iff for each $N \in \mathcal{N}$ we have: $L(\langle \mathcal{T}, \mathcal{N}, \mathcal{R}, N\rangle) \subseteq A$.*

Observe that if $\mathcal{G}$ is A-cautious then obviously $L(\mathcal{G}) \subseteq A$. Notice that the set of propositional formulae in a finite language $\mathcal{L}$ and a finite set of propositional variables VAR (let us denote this set as $\mathrm{Fm}_{\mathcal{L}}^{\mathrm{V}AR}$) is a formal language generated by the context-free grammar $(\mathrm{V}AR \cup \mathcal{L}, \{S\}, \mathcal{R}, S)$ where $\mathcal{R}$ consists of (we are using Polish notation for the sake of simplicity):

$$
\begin{aligned}
&S \mapsto v && \text{for each } v \in \mathrm{V}AR\\
&S \mapsto \lambda && \text{for each 0-ary } \lambda \in \mathcal{L}\\
&S \mapsto \lambda S \overset{n}{\ldots} S && \text{for each } n\text{-ary } \lambda \in \mathcal{L}, n \geq 1
\end{aligned}
$$

Theorem 5.24. *Let* L *be a* (Δ-)*core fuzzy logic in a finite propositional language* $\mathcal{L}$ *and* T *a propositional theory, such that* $T = L(\mathcal{G})$ *for some* $\mathrm{Fm}_{\mathcal{L}}$*-cautious context-free grammar* $\mathcal{G}$*. Then* T *is encodable in* L$\forall$.

Proof. Let us without loss of generality assume that there is no rule such that a non-terminal symbol appears on its right side more than once. First notice that if some propositional theory $T = L(\mathcal{G})$ for some grammar $\mathcal{G}$, then $T \subseteq \mathrm{Fm}_{\mathcal{L}}^{\mathrm{V}AR}$ for some finite set $\mathrm{V}AR$ of propositional variables and that $\mathcal{G}$ is $\mathrm{Fm}_{\mathcal{L}}^{\mathrm{V}AR}$-cautious.

Take $\mathcal{G} = \langle \mathcal{T}, \mathcal{N}, \mathcal{R}, T\rangle$ (the reasons for denoting the initial symbol by T will become clear in the next few lines). Let us define $\Gamma = \Gamma_0 \cup \{\mathbf{N} \mid N \in \mathcal{N}\}$, where $\mathbf{N}$'s are unary predicates (notice that we also added a predicate $\mathbf{T}$ which was already present in Γ_0). As $\mathcal{G}$ is $\mathrm{Fm}_{\mathcal{L}}$-cautious we know that from any non-terminal we can derive just formulae. Thus, if α is a right side of a rule $R \in \mathcal{R}$ then the word $\bar{\alpha}$ resulting from α by replacing:

- each $\lambda \in \mathcal{L}$ by $\dot{\lambda}$
- each $v \in VAR$ by $\dot{v}$
- each non-terminal K by an object variable x_K

is clearly a Γ_0-term. Observe that if α is a formula then $\bar{\alpha} = \dot{\alpha}$.

For each rule R of the form $N \mapsto \alpha$, let $\mathcal{N}_R$ be the set of non-terminals appearing in α. We define a theory:

$$T_1 = \{ \bigwedge_{K \in \mathcal{N}_R} \mathbf{K}(x_K) \to \mathbf{N}(\bar{\alpha}) \mid R \in \mathcal{R}\}.$$

(We understand the empty conjunction as the truth constant $\bar{1}$.) Now we have to prove both conditions on encodability. To prove condition (1) we take any model $\langle \mathbf{M}, \mathcal{A}\rangle \models T_1$ and for each $N \in \mathcal{N}$ and each formula φ derivable in $\langle \mathcal{T}, \mathcal{N}, \mathcal{R}, N\rangle$ we show that $\mathbf{N}_{\mathbf{M}}(\dot{\varphi}) = 1$. Thus as a consequence we obtain $T \subseteq \{\varphi \mid \mathbf{T}_{\mathbf{M}}(\dot{\varphi}) = 1\}$. We do it by induction on the minimal length of derivation of φ in a grammar $\langle \mathcal{T}, \mathcal{N}, \mathcal{R}, N\rangle$: first assume that φ is derivable by one step, i.e. there is the rule $N \mapsto \varphi$. Thus there is a formula $\bar{1} \to \mathbf{N}(\dot{\varphi}) \in T_1$ and so $\mathbf{N}_{\mathbf{M}}(\dot{\varphi}) = 1$. Second, assume that the length of derivation is $n + 1$. Assume that the first rule used was the rule R of the form $N \mapsto \alpha$; for each $K \in \mathcal{N}_R$ there must be a formula φ_K derivable in $\langle \mathcal{T}, \mathcal{N}, \mathcal{R}, K\rangle$ in at most n steps, such that if in the term $\bar{\alpha}$ we replace each occurrence of variable x_K by term $\dot{\varphi}_K$ we obtain term $\dot{\varphi}$. Now we can use the induction assumption to obtain that $\mathbf{K}_{\mathbf{M}}(\dot{\varphi}_K) = 1$. We also know that $\bigwedge_{K \in \mathcal{N}_R} \mathbf{K}(x_K) \to \mathbf{N}(\bar{\alpha}) \in T_1$. Take an instance of this formula for $x_K = \dot{\varphi}_K$. As it holds in $\langle \mathbf{M}, \mathcal{A}\rangle$ we obtain that $\mathbf{N}_{\mathbf{M}}(\dot{\varphi}) = 1$.

To prove condition (2) just define $\mathbf{N}_{\mathbf{M}}(\varphi) = 1$ if φ is derivable in the grammar $\langle \mathcal{T}, \mathcal{N}, \mathcal{R}, N\rangle$ and 0 otherwise. $\square$

Notice that the encodability does not depend on the logic but just on the "form" of the theory in question. Also notice the "constructive" nature of the above proof: given a formal description of the theory T we construct the encoding theory T_1.

Let us show how Theorem 5.24 applies to the example given for Łukasiewicz logic:[20] $T^c = \{np_1 \to p_2 \mid n \text{ natural}\} \cup \{\neg p_1 \to p_2\}$. This theory is clearly encodable by an $\mathrm{Fm}_{\{\to,\neg,\oplus\}}$-cautious context-free grammar

$$\langle \{p_1, p_2, \to, \neg, \oplus\}, \{T, Q\}, \mathcal{R}, T\rangle$$

with the set of rules $\mathcal{R}$ consisting of (again, we are using Polish notation for the sake of simplicity):

$$\begin{array}{ll} S \mapsto \to\neg p_1 p_2 & \quad Q \mapsto p_1 \\ S \mapsto \to Q p_2 & \quad Q \mapsto \oplus Q p_1 \end{array}$$

[20] However the other example mentioned above, for ΠMTL logic, does not fulfill the sufficient condition given in Theorem 5.24.

5.3. Completeness w.r.t. distinguished semantics

As in the propositional case we study completeness w.r.t. five distinguished classes of algebras: the classes of standard (real) chains $\mathcal{R}$, rational chains $\mathcal{Q}$, hyperreal chains $\mathcal{R}^\star$, strict hyperreal chains $\mathcal{R}^\star_s$ and finite chains $\mathcal{F}$. In the first-order case, much less is known (mainly due to lack of some characterization of completeness and finite strong completeness).

We start with the standard semantics. In the paper [56] Montagna and Ono considered the issue of standard completeness for MTL∀. They realized that the embedding construction used in the proof of the S$\mathcal{R}$C for MTL as defined in [47] does not work for the first-order case because in general it does not preserve infima and suprema. They modified it slightly (as we have already presented in Subsection 4.1) in such a way that it gives a σ-embedding so they obtained a proof of the S$\mathcal{R}$C for MTL. The same method works also for other prominent fuzzy logics. Table 5 collects the known results regarding standard completeness for some predicate fuzzy logics, where the negative results follow either from our knowledge of the situation in the propositional case using the constructive method described in Subsection 5.2 or from the known arithmetical hierarchy results surveyed in [38].

Logic	$\mathcal{R}$C	FS$\mathcal{R}$C, S$\mathcal{R}$C	$\mathcal{Q}$C, FS$\mathcal{Q}$C, S$\mathcal{Q}$C	$\mathcal{F}$C, FS$\mathcal{F}$C, S$\mathcal{F}$C
MTL∀, IMTL∀, SMTL∀	Yes	Yes	Yes	No
WCMTL∀, ΠMTL∀	?	No	?	No
BL∀, SBL∀	No	No	No	No
BL∀$^+$, SBL∀$^+$	No	No	Yes	No
Ł∀, Π∀	No	No	Yes	No
Ł$_n$∀, G$_n$∀	No	No	No	Yes
G∀, NM∀, WNM∀, C$_n$MTL∀, C$_n$IMTL∀	Yes	Yes	Yes	No
CPC∀	No	No	No	Yes

Table 5: Standard, rational-chain and finite completeness properties for some axiomatic extensions of MTL∀.

We continue with the finite-chain semantics. First recall that there are many equivalent conditions for a propositional logic L to have the S$\mathcal{F}$C (see Proposition 4.18). Observe that one of them is that there are only finite L-chains. Thus, we can clearly conclude:

Proposition 5.25. *Let* L *be a propositional (*Δ*-)core fuzzy logic. Then the following are equivalent:*

(i) L *enjoys the* S$\mathcal{F}$C*,*
(i) L∀ *enjoys the* S$\mathcal{F}$C*,*
(ii) L *enjoys the* $\mathcal{F}$*-E,*
(iii) all L*-chains are finite,*
(iv) there is a natural number n *such that the length of each* L*-chain is less or equal than* n*, and*
(v) there is a natural number n *such that* $\vdash_{\mathrm{L}} \bigvee_{i<n}(x_i \to x_{i+1})$.

In order to deal with weaker notions of finite-chain completeness, consider the following two formulae[21] (cf. [4]):

$$
\begin{array}{ll}
(C\exists) & (\exists y)((\exists x)\varphi(x) \to \varphi(y)) \\
(C\forall) & (\exists y)(\varphi(y) \to (\forall x)\varphi(x))
\end{array}
$$

We can easily obtain the following proposition.

[21] They are essential for the notion of *witnessed* completeness, see [39].

Proposition 5.26. *Let* L *be a propositional* (Δ-)*core fuzzy logic such that* L$\forall$ *enjoys the* $\mathcal{F}$C. *Then* $\vdash_{\mathrm{L}\forall} (C\exists)$ *and* $\vdash_{\mathrm{L}\forall} (C\forall)$.

The examples in [36, Lemma 5.3.6] can be used to show that $(C\exists)$ is unprovable in G$\forall$ and $(C\forall)$ is unprovable both in G$\forall$ and in $\Pi\forall$. We can also easily show that $\nvdash_{\mathrm{NM}\forall} (C\forall)$. Thus we have just disproved the $\mathcal{F}$C in many fuzzy logics. On the other hand, $\vdash_{\text{Ł}\forall} (C\exists)$ and $\vdash_{\text{Ł}\forall} (C\forall)$. Here we could either observe that an analogy of Corollary 5.22 holds also for finite-chain semantics (as the simple counterexample to the S$\mathcal{R}$C is also a simple counterexample to the S$\mathcal{F}$C of propositional Łukasiewicz logic), or we could notice that [36, Theorem 5.4.30] disproves $\mathcal{F}$C of predicate Łukasiewicz logic. See all the results in Table 5. The table for Δ expansions of prominent fuzzy logics would be the same.

Finally, we prove a slightly stronger variant of the proposition above for Δ-core fuzzy logics. Consider the following formula:

$$(C\Delta) \qquad \Delta(\exists x)\varphi \to (\exists x)\Delta\varphi$$

We can easily obtain the following proposition and its corollary.

Proposition 5.27. *Let* L *be a propositional* Δ*-core fuzzy logic such that* L$\forall$ *enjoys the* $\mathbb{K}$C. *Then* $\vdash_{\mathrm{L}\forall} (C\Delta)$ *iff each* L*-chain from* $\mathbb{K}$ *has a co-atom.*

Corollary 5.28. *Let* L *be a propositional* Δ*-core fuzzy logic such that* L$\forall$ *enjoys the* $\mathcal{F}$C. *Then each* L*-chain has a co-atom.*

◇

We now turn to rational- and (strict) hyperreal-chain completeness. We start by an important lemma from the theory of Abelian ℓ-groups.

Lemma 5.29. *Every totally ordered countable Abelian* ℓ*-group* $\mathcal{G}$ *embeds into a densely ordered countable Abelian* ℓ*-group* $\mathcal{G}'$ *by a* σ*-embedding.*

Proof. Let G^+ denote the set of strictly positive elements of $\mathcal{G}$. We distinguish two cases:

Case (a): G^+ has no minimum. We claim that in this case $\mathcal{G}$ is densely ordered. Indeed, given $a, b \in \mathcal{G}$ with $a < b$, there is a $c \in \mathcal{G}$ such that $0 < c < b - a$, because $b - a \in G^+$ and G^+ has no minimum. Hence $a < a + c < b$, and $\mathcal{G}$ is densely ordered. Thus letting $\mathcal{G}' = \mathcal{G}$ we have that $\mathcal{G}'$ is densely ordered and countable, and $\mathcal{G}$ embeds into $\mathcal{G}'$ by the identity embedding, which is clearly a σ-embedding.

Case (b): G^+ has minimum m. We claim that in this case if a subset X of $\mathcal{G}$ has a supremum s, then $s \in X$, that is, every supremum is a maximum. Suppose $s = \sup(X)$. Then since $s - m < s$, there is $x \in X$ such that $s - m < x \leq s$. It follows that $0 \leq s - x < m$ and since m is the minimum of G^+, it must be $s - x = 0$ and $s = x \in X$, as desired. By the same argument we can show that every infimum is a minimum. Since any embedding of $\mathcal{G}$ into any ℓ-group $\mathcal{G}'$ preserves maxima and minima, and since every supremum is a maximum and every infimum is a minimum, it follows that any embedding of $\mathcal{G}$ into an ℓ-group $\mathcal{G}'$ is a σ-embedding. Thus it suffices to find a densely ordered and countable ℓ-group in which $\mathcal{G}$ embeds. For this, it suffices to take $\mathcal{G} \times_{lex} \mathbb{Q}$, that is, the ℓ-group whose domain is the Cartesian product $G \times \mathbb{Q}$, whose sum and inverse are defined pointwise, that is, $\langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle$ and $-\langle a, b\rangle = \langle -a, -b\rangle$, and whose order $\preceq$ is defined by $\langle a, b\rangle \preceq \langle c, d\rangle$ iff either $a < c$ or $a = c$ and $b \leq d$. □

Corollary 5.30. *Every countable* MV*-chain embeds into a rational* MV*-chain by a* σ*-embedding.*

Proof. Let $\mathcal{A}$ be a countable MV-chain. By using Lemma 5.29 and Mundici's Γ functor (see e.g. [13]) we obtain that $\mathcal{A}$ is σ-embeddable into a countable densely ordered rational MV-chain, which in turn is order isomorphic to $[0, 1]^{\mathbb{Q}}$, hence $\mathcal{A}$ is σ-embeddable into a rational MV-chain. □

Theorem 5.31. Ł$\forall$ *has the* S$\mathbb{Q}$C.

Proof. It is a direct consequence of Corollary 5.30 and Theorem 5.10. □

Remark 5.32. *The claim of Corollary 5.31 was pointed out to the fifth author of the present paper by Petr Hájek on 2006, and more recently by Tommaso Flaminio and Enrico Marchioni. Since we need it in the sequel, and since as far as we know there is no published proof, we have proved it here. However, we do not claim any priority.*

Lemma 5.29 can also be used to prove the strong rational completeness for other predicate logics. Indeed, by using that every Π-chain is isomorphic to the negative cone of a totally ordered Abelian ℓ-group with an added bottom element (see e.g. [15]) one obtains that every countable Π-chain is embeddable into a rational Π-chain by a σ-embedding, and thus $\Pi\forall$ enjoys the S$\mathcal{Q}$C. Nevertheless, the rational completeness properties fail for BL$\forall$ and SBL$\forall$.

Theorem 5.33. BL$\forall$ *and* SBL$\forall$ *do not enjoy the* $\mathcal{Q}$C.[22]

Proof. Consider the formula $(\forall x)(\chi\&\varphi) \to (\chi\&(\forall x)\varphi)$, where x is not free in χ. In [36, page 102] it is proved that this formula is satisfied by every model on a densely ordered BL-chain, so in particular it is a rational tautology. However, thanks to a countermodel found by Félix Bou (see [24]), we know that it is not a tautology for all BL-chains. Indeed, let $\mathcal{C}$ the ordinal sum of Łukasiewicz three-element chain defined over $\{0, \frac{1}{4}, \frac{1}{2}\}$ and the standard Π-chain defined over the real interval $[\frac{1}{2}, 1]$; consider the subalgebra $\mathcal{C}'$ defined over the subuniverse $C' = C \setminus \{\frac{1}{2}\}$ and let $*$ be its monoidal operation. Then $\mathcal{C}'$ does not satisfy the formula as $\frac{1}{4} * \inf(\frac{1}{2}, 1] = 0 \neq \frac{1}{4} = \inf\{\frac{1}{4} * a \mid a \in (\frac{1}{2}, 1]\}$. Now we will turn this counterexample to an SBL-chain showing that the formula is not provable in SBL$\forall$ neither. Just consider the ordinal sum of the two-element G-chain defined over $\{0, \frac{1}{8}\}$, the Łukasiewicz three element chain over $\{\frac{1}{8}, \frac{1}{4}, \frac{1}{2}\}$ and the standard Π-chain over $[\frac{1}{2}, 1]$. The formula fails on the subchain obtained by removing $\frac{1}{2}$, since $\frac{1}{4} * \inf(\frac{1}{2}, 1] = \frac{1}{8} \neq \frac{1}{4} = \inf\{\frac{1}{4} * a \mid a \in (\frac{1}{2}, 1]\}$. □

Let BL$\forall^+$ be the extension of BL$\forall$ with the axiom schema $\forall x(\varphi\&\psi(x)) \to \varphi\&\forall x\psi(x)$. Observe that it is not the first-order version of a core fuzzy logic as defined above, but a pure first-order axiomatic extension of the first-order version of a core fuzzy logic. Nevertheless, we can still consider a slightly modified notion of the S$\mathcal{Q}$C for such a logic, namely it will be strong completeness with respect to models over rational BL-chains.[23]

Theorem 5.34. BL$\forall^+$ *enjoys the* S$\mathcal{Q}$C.

Proof. Let Σ be the set of all instances of the schema $\forall x(\varphi\&\psi(x)) \to \varphi\&\forall x\psi(x)$. Assume that $T \not\vdash_{\mathrm{BL}\forall^+} \varphi$. Then $T \cup \Sigma \not\vdash_{\mathrm{BL}\forall} \varphi$ and so by the completeness of BL$\forall$ there is a structure $\langle \mathbf{M}, \mathcal{A}\rangle$ such that $\mathcal{A}$ is a countable BL-chain, $\langle \mathbf{M}, \mathcal{A}\rangle \models T \cup \Sigma$ and $\langle \mathbf{M}, \mathcal{A}\rangle \not\models \varphi$. We will prove the following fact:

Claim: There is a structure $\langle \mathbf{M}', \mathcal{A}'\rangle$ such that $\mathcal{A}'$ is a densely ordered countable BL-chain, $\mathcal{A} \subseteq \mathcal{A}'$ and for every sentence α, $\|\alpha\|^{\mathcal{A}'}_{\mathbf{M}'} = \|\alpha\|^{\mathcal{A}}_{\mathbf{M}}$.

For every pair $\langle a_1, a_2\rangle \in A^2$ such that $a_1 < a_2$ and the open interval (a_1, a_2) is empty (i.e. a_2 is the successor of a_1) we perform the following construction:

1. Assume that a_1 and a_2 are in the same Wajsberg component[24] $\mathcal{C}$ of $\mathcal{A}$. Recall that $\mathcal{C}$ must be either a negative cone $\mathcal{G}^-$ or an MV-algebra $\mathbf{\Gamma}(\mathcal{G}, u)$ for some countable totally ordered Abelian group $\mathcal{G}$. We replace $\mathcal{G}$ with its divisible extension $\mathcal{G}'$ (which is still countable).

[22]For logics of the form L$_*\forall$ where $*$ is a continuous t-norm given by a finite ordinal sum of basic components it has been described in [28] which of them enjoy the $\mathcal{Q}$C (or, equivalently, which of them prove the schema $(\forall x)(\chi\&\varphi) \to (\chi\&(\forall x)\varphi)$).

[23]Notice that it is not necessary to require that these models satisfy the additional schema: we get it for free because their underlying BL-chains are densely ordered.

[24]For the role of Wajsberg hoops in the structure of BL-chains see [1].

2. If a_1 and a_2 are in different Wajsberg components, then a_2 is an idempotent (otherwise $a_1 < (a_2)^2 < a_2$). If a_1 is idempotent as well, we add an isomorphic copy of $[0,1]_{\mathrm{G}}^{\mathbb{Q}}$ between a_1 and a_2. If a_1 is not idempotent, then it lies in some Wajsberg component $\mathcal{C}$ of the form $\mathcal{G}^-$ or $\mathbf{\Gamma}(\mathcal{G}, u)$, which is discretely ordered. Moreover, a_1 is the coatom of $\mathcal{C}$ and $((a_1)^2, a_1) = \emptyset$. Indeed, if $(a_1)^2 < z < a_1$, then $a_1 \leq a_1 \to z < 1$ and hence, since a_1 is coatom, $a_1 = a_1 \to z$. But then $z = a_1 \wedge z = a_1 \& (a_1 \to z) = (a_1)^2$, a contradiction. $\mathcal{C}$ will be substituted by a densely ordered component when we consider the pair $\langle (a_1)^2, a_1 \rangle$.

Let $\mathcal{A}'$ be the densely ordered countable BL-chain resulting from this construction. Take $\mathbf{M}' = \mathbf{M}$. We prove $\|\alpha\|_{\mathbf{M}}^{\mathcal{A}} = \|\alpha\|_{\mathbf{M}'}^{\mathcal{A}'}$ by induction on the complexity of α.

- If α is atomic, it is clear because $\mathbf{M}' = \mathbf{M}$.
- If α is a combination by propositional connectives of simpler formulae, the result is obvious.
- Assume that $\alpha = \exists x \beta$. It is enough to see that the suprema of $\mathcal{A}$ are preserved in $\mathcal{A}'$. Suppose that $a = \sup_{\mathcal{A}} C$ for some $C \subseteq A$. We must prove $a = \sup_{\mathcal{A}'} C$. If $a \in C$ we are done. Assume $a \notin C$. Then a cannot be in a discretely ordered Wajsberg component of $\mathcal{A}$ and it cannot be an idempotent such that for some $b < a$ $(b, a) = \emptyset$. It follows that in some left neighbourhood of a we have not added new elements and thus $a = \sup_{\mathcal{A}'} C$.
- Assume that $\alpha = \forall x \beta(x)$ (let us assume for simplicity that β has no free variables besides x). The previous argument does not work, as some infima may fail to be preserved, but still we can prove that all *relevant* infima are preserved, i.e. those needed for the interpretation of universal formulae. Suppose by the way of contradiction that $a = \inf_{\mathcal{A}}\{\|\beta(d)\|_{\mathbf{M}}^{\mathcal{A}} \mid d \in M\}$ is not preserved in $\mathcal{A}'$. Then we must have added some new element z such that $\|\forall x \beta(x)\|_{\mathbf{M}}^{\mathcal{A}} = a < z$ and $z \leq \|\beta(d)\|_{\mathbf{M}}^{\mathcal{A}}$ for every $d \in M$. In particular, a must be a proper infimum. It cannot be an idempotent, otherwise either it would be the minimum of a densely ordered Wajsberg component or it would be a limit point of elements from different components, but in both cases we cannot have added z. Therefore, a is a non-idempotent element in the interior of some discretely ordered Wajsberg component $\mathcal{C}$ of $\mathcal{A}$, and hence a is the coatom of $\mathcal{C}$ and the elements of $\{\|\beta(d)\|_{\mathbf{M}}^{\mathcal{A}} \mid d \in M\}$ are not in $\mathcal{C}$. But this leads to the following contradiction: $a^2 = \|\forall x \beta(x)\|_{\mathbf{M}}^{\mathcal{A}} \& \|\forall y \beta(y)\|_{\mathbf{M}}^{\mathcal{A}} < a = \|\forall y \beta(y)\|_{\mathbf{M}}^{\mathcal{A}} = \|\forall x(\forall y \beta(y) \& \beta(x))\|_{\mathbf{M}}^{\mathcal{A}} \leq a^2$, the last equality holds because $\inf_{\mathcal{A}}\{\|\beta(d)\|_{\mathbf{M}}^{\mathcal{A}} \mid d \in M\} = a = \inf\{a \& \|\beta(d)\|_{\mathbf{M}}^{\mathcal{A}} \mid d \in M\}$) and the last inequality hold because $\forall x(\forall y \beta(y) \& \beta(x)) \to \forall y \beta(y) \& \forall x \beta(x) \in \Sigma$ and $\langle \mathbf{M}, \mathcal{A} \rangle \models \Sigma$.

Now, using the claim, we consider the densely ordered countable BL-chain $\mathcal{A}'$. It is isomorphic to a BL-chain over $[0,1]^{\mathbb{Q}}$ and hence we have a structure on the rational unit interval where the formulae in T are true, while φ is false, as desired. $\square$

Observe that if we define SBL$\forall^+$ as the extension of SBL$\forall$ with the axiom schema $\forall x(\varphi \& \psi(x)) \to \varphi \& \forall x \psi(x)$, we can prove that it enjoys the S$\mathcal{Q}$C in a completely analogous way, as the construction of the previous theorem applied to a countable SBL-chain gives a countable densely ordered SBL-chain. In contrast, these logics do not enjoy the real completeness properties. Indeed, since all standard BL-chains are dense and hence satisfy the additional axiom, the set of standard tautologies of BL$\forall^+$ (resp. SBL$\forall^+$) coincides with the set of standard tautologies of BL$\forall$ (resp. SBL$\forall$), which is known to be non recursively enumerable (not even arithmetical).

All the results are collected in Table 5 and they show that, as in the propositional case, the rational-chain completeness properties do not imply the corresponding standard completeness properties. On the contrary, the reverse implications do hold for every predicate (Δ-)core fuzzy logic as we will prove below, and it allows to complete the table.

Before we prove it we need to consider the relations of rational chain-completeness properties with hyperreal-chain and strict-hyperreal-chain completeness properties. To this end, we will make use of the following trick that allows to construct a classical first-order structure from an

$\mathcal{A}$-structure for a (Δ-)core fuzzy logic and vice versa. Let us, for simplicity, restrict to predicate languages Γ without function symbols.

Let L be a (Δ-)core fuzzy logic in propositional language $\mathcal{L}$, $\mathcal{A}$ an L-chain and $\langle \mathbf{M}, \mathcal{A}\rangle$ a model. Without loss of generality we may assume that the domain M of $\mathbf{M}$ and the domain A of $\mathcal{A}$ are disjoint. We associate to $\langle \mathbf{M}, \mathcal{A}\rangle$ a classical first-order structure $\mathcal{A}_{\mathbf{M}}$ as follows:

- The domain of $\mathcal{A}_{\mathbf{M}}$ is the union of A and M.
- For every operation $f \in \mathcal{L}$, the model $\mathcal{A}_{\mathbf{M}}$ has a function f^* of the same arity, defined by $f^*_{\mathcal{A}_{\mathbf{M}}}(x_1, \dots, x_n) = f(x_1, \dots, x_n)$ if $x_1, \dots, x_n \in A$, and $f^*_{\mathcal{A}_{\mathbf{M}}}(x_1, \dots, x_n) = \overline{0}^{\mathcal{A}}$ otherwise.
- For every Γ-formula $\varphi(x_1, \dots, x_n)$ of L$\forall$, in the free variables shown, $\mathcal{A}_{\mathbf{M}}$ has a function $\varphi^*(x_1, \dots, x_n)$ defined by $\varphi^*_{\mathcal{A}_{\mathbf{M}}}(d_1, \dots, d_n) = ||\varphi(d_1, \dots, d_n)||^{\langle \mathbf{M}, \mathcal{A}\rangle}$ if $d_1, \dots, d_n \in M$, and $\varphi^*_{\mathcal{A}_{\mathbf{M}}}(d_1, \dots, d_n) = \overline{0}^{\mathcal{A}}$ otherwise.
- Finally $\mathcal{A}_{\mathbf{M}}$ has two unary predicates, $M_{\mathbf{M}}(x)$, interpreted as $x \in M$, and $A_{\mathbf{M}}(x)$, interpreted as $x \in A$; and one binary predicate $a \leq_{\mathbf{M}} b$ interpreted as $a \to^{\mathcal{A}} b = 1^{\mathcal{A}}$.

Clearly the following the formulae are satisfied in $\mathcal{A}_{\mathbf{M}}$:

(a) $\forall x(M(x) \leftrightarrow \neg A(x))$.

(b) The relativizations[25] to $A(x)$ of all sentences which are true in $\mathcal{A}$.

(c) For every n-ary $f \in \mathcal{L}$, the formula $\forall \overrightarrow{x} A(f^*(\overrightarrow{x}))$, where $\overrightarrow{x}$ stands for $x_1, \dots, x_n$ and $\forall \overrightarrow{x}$ stands for $\forall x_1 \dots \forall x_n$.

(d) For every Γ-formula $\varphi(\overrightarrow{x})$, the axiom $\forall \overrightarrow{x} A(\varphi^*(\overrightarrow{x}))$.

(e) For every n-ary $f \in \mathcal{L}$ and for every Γ-formulae $\varphi_1(\overrightarrow{x}), \dots, \varphi_n(\overrightarrow{x})$, the formula

$$\forall \overrightarrow{x} \left(\overrightarrow{M}(\overrightarrow{x}) \to ((f(\varphi_1, \dots, \varphi_n))^*(\overrightarrow{x}) \approx f^*(\varphi_1^*(\overrightarrow{x}), \dots, \varphi_n^*(\overrightarrow{x}))\right),$$

where $\overrightarrow{M}(\overrightarrow{x})$ stands for $M(x_1) \wedge \ldots \wedge M(x_m)$.

(f) For every Γ-formula $\varphi(x, x_1, ..., x_n)$, the formulae

(f1) $\forall z \forall \overrightarrow{x}((\overrightarrow{M}(\overrightarrow{x}) \wedge A(z)) \to ((\forall x \varphi)^*(\overrightarrow{x}) \approx z \leftrightarrow$
$\leftrightarrow \forall u(A(u) \to (u \leq z \leftrightarrow \forall w(M(w) \to u \leq \varphi^*(w, \overrightarrow{x})))))$, and

(f2) $\forall z \forall \overrightarrow{x}((\overrightarrow{M}(\overrightarrow{x}) \wedge A(z)) \to ((\exists x \varphi)^*(\overrightarrow{x}) \approx z \leftrightarrow$
$\leftrightarrow \forall u(A(u) \to (u \geq z \leftrightarrow \forall w(M(w) \to u \geq \varphi^*(w, \overrightarrow{x})))))$.

Conversely, given a classical model $\mathbf{C}$ of (a)–(f) one can construct $\mathcal{A}_{\mathbf{C}}$ and $\mathbf{M}_{\mathbf{C}}$ as follows: the domain of $\mathcal{A}_{\mathbf{C}}$ is the set $A = \{x \in C \mid \mathbf{C} \models A(x)\}$, the domain of $\mathbf{M}_{\mathbf{C}}$ is the set $M = \{x \in C \mid \mathbf{C} \models M(x)\}$, and the operations f of $\mathcal{A}_{\mathbf{C}}$ (predicates P of $\mathbf{M}_{\mathbf{C}}$) are the restrictions of the functions $f^*_{\mathbf{C}}$ to A (respectively $P^*_{\mathbf{C}}$ to M. Note that axioms (a)–(f) guarantee that $\mathcal{A}_{\mathbf{C}}$ is an L-chain and $\mathbf{M}_{\mathbf{C}}$ is a safe first-order $\mathcal{A}_{\mathbf{C}}$-structure and for every Γ-formula φ and for every $d_1, \dots, d_n \in M$, $\|\varphi(d_1, \dots, d_n)\|^{\langle \mathbf{M}_{\mathbf{C}}, \mathcal{A}_{\mathbf{C}}\rangle} = \varphi^*_{\mathbf{C}}(d_1, \dots, d_n)$.

Theorem 5.35. *Let* L *be a (Δ-)core fuzzy logic. The following are equivalent:*

(1) L$\forall$ *has the* $\mathcal{R}^\star$C.

(2) L$\forall$ *has the* $\mathcal{Q}$C.

[25] The relativization Θ^A of a formula Θ to A is defined inductively as follows: $\Theta^A = \Theta$ if Θ is atomic: ^{A} commutes with (classical) connectives; $(\forall x \Theta)^A = \forall x(A(x) \to \Theta^A)$ and $(\exists x \Theta)^A = \exists x(A(x) \wedge \Theta^A)$.

(3) $\mathrm{L}\forall$ *has the* $\mathcal{R}_s^\star$C.

Moreover, if $\mathrm{L}\forall$ *has the* $\mathcal{R}$C *then these three equivalent conditions hold. Finally, the same claims hold for strong and finite strong completeness notions as well.*

Proof. The proof we are going to present does not depend on the cardinality of the set T, thus we prove all the claims at once. First notice that since the hyperreal chains contain the standard chains it is obvious that the $\mathcal{R}$C implies the $\mathcal{R}^\star$C. Also observe that the implication (3) $\Rightarrow$ (1) is trivial.

Now we prove (1) $\Rightarrow$ (2): assume that $T \not\vdash_{\mathrm{L}\forall} \varphi$. From $\mathcal{R}^\star$C (FS$\mathcal{R}^\star$C or S$\mathcal{R}^\star$C respectively) we obtain an L-chain $\mathcal{A}$ over an ultrapower of $[0,1]$ (possibly $[0,1]$ itself) and a model $\langle \mathbf{M}, \mathcal{A}\rangle$ of T such that $\langle \mathbf{M}, \mathcal{A}\rangle \not\models \varphi$. Construct the classical first-order structure $\mathcal{A}_{\mathbf{M}}$ as shown above, and take (by the Löwenheim-Skolem theorem) a countable elementary substructure $\mathbf{C}$ of $\mathcal{A}_{\mathbf{M}}$. Thus $\langle \mathbf{M}_{\mathbf{C}}, \mathcal{A}_{\mathbf{C}}\rangle$ is a model of T and $\langle \mathbf{M}_{\mathbf{C}}, \mathcal{A}_{\mathbf{C}}\rangle \not\models \varphi$. Because $\leq_{\mathbf{C}}$ is a dense linear order on A with maximum and minimum and thus is isomorphic to $[0,1]^{\mathbb{Q}}$, we can consider that $\mathcal{A}_{\mathbf{C}}$ is a rational L-chain.

Finally, we prove (2) $\Rightarrow$ (3): assume that $T \not\vdash_{\mathrm{L}\forall} \varphi$. From $\mathcal{Q}$C (FS$\mathcal{Q}$C or S$\mathcal{Q}$C respectively) we obtain an L-chain $\mathcal{A}$ with the domain $[0,1]^{\mathbb{Q}}$ and a model $\langle \mathbf{M}, \mathcal{A}\rangle$ of T such that $\langle \mathbf{M}, \mathcal{A}\rangle \not\models \varphi$.

$[0,1]^{\mathbb{Q}}$ and $[0,1]$ (considered as totally ordered sets) are densely ordered and have maximum and minimum. Since the theory of dense linear orders with maximum and minimum is complete, $[0,1]^{\mathbb{Q}}$ and $[0,1]$ have isomorphic ultrapowers, $([0,1]^{\mathbb{Q}})^I/U$ and $[0,1]^J/W$, where I and J are suitable index sets and U and W are ultrafilters, on I and on J respectively, giving proper extensions of $[0,1]^{\mathbb{Q}}$ and $[0,1]$ respectively. Now consider the classical structure $\mathcal{A}_{\mathbf{M}}$, and take its ultrapower $\mathbf{C} = (\mathcal{A}_{\mathbf{M}}^I)/U$. This is an elementary extension of the former structure and thus $\langle \mathbf{M}_{\mathbf{C}}, \mathcal{A}_{\mathbf{C}}\rangle$ is a model of T and $\langle \mathbf{M}_{\mathbf{C}}, \mathcal{A}_{\mathbf{C}}\rangle \not\models \varphi$. To complete the proof just notice that the lattice reduct of $\mathcal{B}$ is $([0,1]^{\mathbb{Q}})^I/U$, which is isomorphic to a non-standard ultrapower of $[0,1]$.

□

We will end by showing that, as stated before, the strong $\mathbb{K}$-completeness does not imply in general the σ-embeddability. First we justify the requirement made in the definition of the strict hyperreal semantics which prevented the ultrafilters from being closed under intersections of countable families.

Lemma 5.36. *Let U be an ultrafilter over a set I closed under intersections of countable families. Then the ultrapower $[0,1]^I/U$ is isomorphic to $[0,1]$.*

Proof. The claim is trivial if U is a principal ultrafilter. Thus, assume that U is not principal. In this case $\mid I \mid$ must be a measurable cardinal. It is known from Set Theory that if U is closed under intersections of countable families, then it is also closed under intersection of families of any cardinal $\kappa <\mid I \mid$. Moreover, since $\mid I \mid$ is a measurable cardinal, it must be strictly bigger than the cardinal of $[0,1]$. Now for any sequence $\langle a_i : i \in I\rangle$ and any $\alpha \in [0,1]$ we define the set $I_\alpha = \{i \in I \mid a_i = \alpha\}$. Of course, $\{I_\alpha \mid \alpha \in [0,1]\}$ is a partition of I whose cardinal is smaller than $\mid I \mid$. Then $\bigcap\{I \setminus I_\alpha \mid \alpha \in [0,1]\} = I \setminus \bigcup\{I_\alpha \mid \alpha \in [0,1]\} = \emptyset \notin U$, hence there is (a unique) α such $I_\alpha \in U$ which means that $\langle a_i : i \in I\rangle/U$ and $\langle \alpha, \dots, \alpha, \dots\rangle/U$ are the same element in the ultrapower. Then by mapping every $\alpha \in [0,1]$ to $\langle \alpha, \dots, \alpha, \dots\rangle/U$ we obtain a surjection on $[0,1]^I/U$, so this ultrapower is isomorphic to $[0,1]$. □

Now, after showing a special property of the non-standard ultrapowers, we will be able to prove the result.

Lemma 5.37. *In every non-standard ultrapower $[0,1]^\star = [0,1]^I/U$ (i.e. where U is a non-principal ultrafilter on I and it is not closed under intersections of countable families) there is no strictly decreasing countable sequence converging to* 0.

Proof. By the assumption, there is a countable sequence $I_0, ..., I_n, ...$ of subsets of I such that for every n, $I_n \in U$ and $\bigcap_{n\in\omega} I_n \notin U$. Let for $n > 0$, $J_n = I \setminus I_{n-1}$, $Y_n = J_n \setminus \bigcup_{i<n} J_i$, and let $Y_0 = I \setminus \bigcup_{i\in\omega, i>0} Y_i$. Then it is easily seen that $\{Y_n : n \in \omega\}$ is a partition of I such that for all n,

$Y_n \notin U$. Now suppose by contradiction that $\alpha_0 = \langle a_{0i} : i \in I\rangle/U, \ldots, \alpha_n = \langle a_{ni} : i \in I\rangle/U, \ldots$ is a countable decreasing sequence with limit 0. Without loss of generality we may assume $a_{ij} > 0$. Indeed since $\alpha_i > 0$, the set $N = \{j : a_{ij} = 0\}$ is not in U, therefore if we change the value of a_{ij} in N, the equivalence class does not change. Thus we may safely replace $\langle a_{ij} : j \in I\rangle$ by $\langle a'_{ij} : j \in I\rangle$ where $a'_{ij} = a_{ij}$ if $a_{ij} > 0$ and $a'_{ij} = 1$ otherwise. Moreover we can assume $a_{1i} \geq a_{2i} \geq \ldots \geq a_{ni} \geq \ldots$ (if not, we can replace a_{ij} by $\min\{a_{kj} : k \leq i\}$ without changing the equivalence classes). Now define $\beta = \langle b_i : i \in I\rangle/U$ where $b_i = \frac{a_{0i}}{2}$ if $i \in Y_0, \ldots$, $b_i = \frac{a_{ki}}{2}$ if $i \in Y_k, \ldots$. Note that for all n, $b_n > 0$, therefore $\beta > 0$. Moreover for every n the set $\{i \in I : b_i < a_{ni}\}$ contains $\bigcup_{m \geq n} Y_m$, therefore it is in U. Thus by the ultraproduct theorem we have $0 < \beta < \alpha_n$ for every n, and a contradiction has been reached. □

Theorem 5.38. *Ł∀ has the* $\mathrm{S}\mathcal{R}^\star\mathrm{C}$ *and* $\mathrm{S}\mathcal{R}^\star_s\mathrm{C}$*, but there is a non-trivial countable* MV*-chain which cannot be σ-embedded into any (strict) hyperreal-chain.*

Proof. Both completeness properties hold due to Theorems 5.35 and 5.31. Let $\mathcal{A}$ be any non-Archimedean MV-chain without an atom (e.g. apply Mundici's Γ functor to the Abelian ℓ-group $\mathbb{Q} \times \mathbb{Q}$ with strong unit $\langle 1, 0\rangle$) Clearly, $\mathcal{A}$ cannot be embedded into the standard MV-algebra (as it is non-Archimedean). Further, there has to be a countable set of positive non-zero elements of $\mathcal{A}$ with infimum 0, thus the existence of a σ-embedding of $\mathcal{A}$ into a non-standard hyperreal MV-chain $\mathcal{A}^*$ would imply the existence of a countable decreasing sequence in $\mathcal{A}^*$ converging to 0, which contradicts Lemma 5.37. □

6. Conclusions

In the first part of this paper we have carried out a general investigation on semantics for propositional core and Δ-core fuzzy logics. We have obtained several useful characterizations for the completeness properties of these logics which, in fact, show that the methods that have been often used in the literature in order to prove completeness results were based on conditions not only sufficient but also necessary. We have also described many relations between different completeness properties with respect to several distinguished semantics. Nevertheless, a significant question arose in the investigation and it is left without answer: In which cases (in the absence of Δ connective) does the $\mathbb{K}$C imply the FS$\mathbb{K}$C? We have seen that, no matter if $\mathbb{K}$ is the semantics of real, rational, hyperreal, strict hyperreal or finite chains, this couple of properties always happen to be equivalent in the prominent logics. We think that in order to solve this problem some better algebraic characterization of the $\mathbb{K}$C would be needed.

In the second part of the paper we have tried to break a new ground by bringing the investigation on completeness properties to the not so deeply developed area of first-order core fuzzy logics. Here we have obtained again a good characterization of a completeness property, the S$\mathbb{K}$C, in model-theoretic terms, and we have shown that the algebraic property commonly used to prove the S$\mathbb{K}$C, the σ-embeddability, is in fact just a sufficient but not necessary condition. The restriction to particular semantics has also revealed several interesting relations between them. This investigation has left several open problems as well:

- Are the three completeness properties ($\mathbb{K}$C, FS$\mathbb{K}$C and S$\mathbb{K}$C) for first-order fuzzy logics equivalent as it happens in the prominent examples with the five distinguished semantics? If not, for which semantics $\mathbb{K}$ and which classes of logics are they equivalent? Again, the lack of good equivalencies for $\mathbb{K}$C and FS$\mathbb{K}$C makes the problem still too hard to be solved.

- We have shown that in some special cases the failure of propositional S$\mathbb{K}$C entails a failure of first-order FS$\mathbb{K}$C. Are there other interesting relations between completeness properties in the propositional and in the first-order level?

- We have shown that the standard completeness properties imply the corresponding rational (and hyperreal and strict hyperreal) completeness properties for first-order logics. Some examples (all of them extensions of BL∀) show that the converse is not true. How can we decide the rational completeness properties for other kinds logics (for instance, ΠMTL∀ and WCMTL∀) where a deep algebraic knowledge of the corresponding variety is not available?

Acknowlegdements: We are grateful to Félix Bou for his reading and useful comments to Section 5 and to Mauro Di Nasso for discussion about ultrapowers of $[0, 1]$ and measurable cardinals.

The first author's work was supported by project 1M0545 of the Ministry of Education, Youth, and Sports of the Czech Republic. The work of the second, third, fourth and sixth author was supported by the Spanish project MULOG2 (TIN2007-68005-C04). The third author's work was supported by the grants MTM2004-03101 of the Spanish Ministry of Education and Science, including FEDER funds of the European Union, and 2005SGR00083 of D.U.R.S.I of the Generalitat de Catalunya. The sixth authors work was supported by grant 2006-BP-A-10043 of the Departament dEducació i Universitats of the Generalitat de Catalunya.

References

[1] Paolo Aglianò and Franco Montagna. Varieties of BL-algebras I: General properties. *Journal of Pure and Applied Algebra*, 181:105–129, 2003.

[2] Claudi Alsina, Ernest Trillas, and Llorenç Valverde. Some logical connectives for fuzzy set theory. *Journal of Mathematical Analysis and Applications*, 93:15–26, 1983.

[3] Arnon Avron. A constructive analysis of RM. *Journal of Symbolic Logic*, 52(4):939–951, 1987.

[4] Matthias Baaz. Infinite-valued Gödel logic with 0-1-projections and relativisations. In Petr Hájek, editor, *Gödel'96: Logical Foundations of Mathematics, Computer Science, and Physics*, volume 6 of *Lecture Notes in Logic*, pages 23–33. Springer-Verlag, Brno, 1996.

[5] Matthias Baaz, Agata Ciabattoni, and Franco Montagna. Analytic calculi for monoidal t-norm based logic. *Fundamenta Informaticae*, 59(4):315–332, 2004.

[6] Libor Běhounek. On the difference between traditional and deductive fuzzy logic. *Fuzzy Sets and Systems*, 159(10):1153–1164, 2008.

[7] Willem J. Blok and Don Pigozzi. *Algebraizable Logics*, volume 396 of *Memoirs of the American Mathematical Society*. American Mathematical Society, Providence, 1989.

[8] Willem J. Blok and Clint J. van Alten. The finite embeddability property for residuated lattices, pocrims and BCK-algebras. *Algebra Universalis*, 48:253–271, 2000.

[9] Stanley Burris and H.P. Sankappanavar. *A Course in Universal Algebra*, volume 78 of *Graduate Texts in Mathematics*. Springer-Verlag, 1981.

[10] Chen Chung Chang. Algebraic analysis of many-valued logics. *Transactions American Mathematical Society*, 88:456–490, 1958.

[11] Noam Chomsky. *Syntactic Structures*. Walter de Gruyter, Berlin, 2nd edition, 2002.

[12] Agata Ciabattoni, Francesc Esteva, and Lluís Godo. T-norm based logics with n-contraction. *Special Issue on SOFSEM2002 of Neural Network World*, 12(5):453–460, 2002.

[13] Roberto Cignoli, Itala M.L. D'Ottaviano, and Daniele Mundici. *Algebraic Foundations of Many-Valued Reasoning*, volume 7 of *Trends in Logic*. Kluwer, Dordercht, 1999.

[14] Roberto Cignoli, Francesc Esteva, Lluís Godo, and Antoni Torrens. Basic fuzzy logic is the logic of continuous t-norms and their residua. *Soft Computing*, 4(2):106–112, 2000.

[15] Roberto Cignoli and Antoni Torrens. Standard completeness of Hájek basic logic and decompositions of BL-chains. *Soft Computing*, 9(12):862–868, 2005.

[16] Petr Cintula. Weakly implicative (fuzzy) logics I: Basic properties. *Archive for Mathematical Logic*, 45(6):673–704, 2006.

[17] Petr Cintula, Erich Peter Klement, Radko Mesiar, and Mirko Navara. Fuzzy logics with an additional involutive negation. To appear in the book Fuzzy Logic and Related Structures: Proceedings of Linz Seminar 2005 (Siegfried Gottwald, Petr Hájek, Erich Peter Klement eds.), 2009.

[18] Janusz Czelakowski. *Protoalgebraic Logics*, volume 10 of *Trends in Logic*. Kluwer, Dordercht, 2001.

[19] Janusz Czelakowski and Wieslaw Dziobiak. Congruence distributive quasivarieties whose finitely subdirectly irreducible members form a universal class. *Algebra Universalis*, 27:128–149, 1990.

[20] Antonio Di Nola, Francesc Esteva, Lluís Godo, and Franco Montagna. Varieties of BL-algebras. *Soft Computing*, 9(12):875–888, 2005.

[21] Michael Dummett. A propositional calculus with denumerable matrix. *Journal of Symbolic Logic*, 27:97–106, 1959.

[22] Francesc Esteva, Joan Gispert, Lluís Godo, and Franco Montagna. On the standard and rational completeness of some axiomatic extensions of the monoidal t-norm logic. *Studia Logica*, 71(2):199–226, 2002.

[23] Francesc Esteva, Joan Gispert, Lluís Godo, and Carles Noguera. Adding truth-constants to continuous t-norm based logics: Axiomatization and completeness results. *Fuzzy Sets and Systems*, 158:597–618, 2007.

[24] Francesc Esteva and Lluís Godo. Monoidal t-norm based logic: Towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124(3):271–288, 2001.

[25] Francesc Esteva, Lluís Godo, Petr Hájek, and Mirko Navara. Residuated fuzzy logics with an involutive negation. *Archive for Mathematical Logic*, 39(2):103–124, 2000.

[26] Francesc Esteva, Lluís Godo, and Franco Montagna. The ŁΠ and ŁΠ$\frac{1}{2}$ logics: Two complete fuzzy systems joining Łukasiewicz and product logics. *Archive for Mathematical Logic*, 40(1): 39–67, 2001.

[27] Francesc Esteva, Lluís Godo, and Carles Noguera. On rational Weak Nilpotent Minimum logics. *Journal of Multiple-Valued Logic and Soft Computing*, 12(1–2):9–32, 2006.

[28] Francesc Esteva, Lluís Godo, and Carles Noguera. First-order t-norm based fuzzy logics with truth-constants: Distinguished semantics and completeness properties. To appear in *Annals of Pure and Applied Logic*, 2009.

[29] Francesc Esteva, Lluís Godo, and Carles Noguera. On expansions of t-norm based logics with truth-constants. To appear in the book Fuzzy Logic and Related Structures: Proceedings of Linz Seminar 2005 (Siegfried Gottwald, Petr Hájek, Erich Peter Klement eds.), 2009.

[30] Tommaso Flaminio. Strong non-standard completeness for fuzzy logics. *Soft Computing*, 12 (4):321–333, 2007.

[31] Tommaso Flaminio and Enrico Marchioni. T-norm based logics with an independent involutive negation. *Fuzzy Sets and Systems*, 157(4):3125–3144, 2006.

[32] Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, Amsterdam, 2007.

[33] Kurt Gödel. Zum intuitionistischen Aussagenkalkül. *Anzieger Akademie der Wissenschaften Wien*, 69:65–66, 1932.

[34] Siegfried Gottwald. *A Treatise on Many-Valued Logics*, volume 9 of *Studies in Logic and Computation*. Research Studies Press, Baldock, 2001.

[35] Petr Hájek. Basic fuzzy logic and BL-algebras. *Soft Computing*, 2(3):124–128, 1998.

[36] Petr Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordercht, 1998.

[37] Petr Hájek. Observations on the monoidal t-norm logic. *Fuzzy Sets and Systems*, 132(1): 107–112, 2002.

[38] Petr Hájek. Arithmetical complexity of fuzzy predicate logics—a survey. *Soft Computing*, 9 (12):935–941, 2005.

[39] Petr Hájek and Petr Cintula. On theories and models in fuzzy predicate logics. *Journal of Symbolic Logic*, 71(3):863–880, 2006.

[40] Petr Hájek and Petr Cintula. Triangular norm predicate fuzzy logics. To appear in the book Fuzzy Logic and Related Structures: Proceedings of Linz Seminar 2005 (Siegfried Gottwald, Petr Hájek, Erich Peter Klement eds.), 2009.

[41] Petr Hájek, Lluís Godo, and Francesc Esteva. A complete many-valued logic with product conjunction. *Archive for Mathematical Logic*, 35(3):191–208, 1996.

[42] Rostislav Horčík. *Algebraic Properties of Fuzzy Logics*. PhD thesis, Czech Technical University, Faculty of Electrical Egineering, Prague, 2005.

[43] Rostislav Horčík. Standard completeness theorem for ΠMTL. *Archive for Mathematical Logic*, 44(4):413–424, 2005.

[44] Rostislav Horčík. Decidability of cancellative extension of monoidal t-norm logic. *Logic Journal of the Interest Group of Pure and Applied Logic*, 14:827–843, 2006.

[45] Rostislav Horčík. On the failure of standard completeness in ΠMTL for infinite theories. *Fuzzy Sets and Systems*, 158:619–624, 2007.

[46] Rostislav Horčík and Petr Cintula. Product Łukasiewicz logic. *Archive for Mathematical Logic*, 43(4):477–503, 2004.

[47] Sándor Jenei and Franco Montagna. A proof of standard completeness for Esteva and Godo's logic MTL. *Studia Logica*, 70(2):183–192, 2002.

[48] C.H. Ling. Representation of associative functions. *Publicationes Mathematicae Debrecen*, 12:189–212, 1965.

[49] Jan Łukasiewicz. O logice trójwartościowej (On three-valued logic). *Ruch filozoficzny*, 5: 170–171, 1920.

[50] Jan Łukasiewicz and Alfred Tarski. Untersuchungen über den Aussagenkalkül. *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie*, 23(iii):30–50, 1930.

[51] George Metcalfe and Franco Montagna. Substructural fuzzy logics. *Journal of Symbolic Logic*, 72(3):834–864, 2007.

[52] Franco Montagna. Generating the variety of BL-algebras. *Soft Computing*, 9(12):869–874, 2005.

[53] Franco Montagna. On the predicate logics of continuous t-norm BL-algebras. *Archive for Mathematical Logic*, 44:97–114, 2005.

[54] Franco Montagna. Subreducts of MV-algebras with product and product residuation. *Algebra Universalis*, 53(1):109–137, 2005.

[55] Franco Montagna, Carles Noguera, and Rostislav Horčík. On weakly cancellative fuzzy logics. *Journal of Logic and Computation*, 16(4):423–450, 2006.

[56] Franco Montagna and Hiroakira Ono. Kripke semantics, undecidability and standard completeness for Esteva and Godo's logic MTL∀. *Studia Logica*, 71(2):227–245, 2002.

[57] Paul S. Mostert and Allen L. Shields. On the structure of semigroups on a compact manifold with boundary. *The Annals of Mathematics, Second Series*, 65:117–143, 1957.

[58] Carles Noguera. *Algebraic Study of Axiomatic Extensions of T-Norm Based Fuzzy Logics*. PhD thesis, University of Barcelona, Barcelona, 2006.

[59] Carles Noguera, Francesc Esteva, and Joan Gispert. On some varieties of MTL-algebras. *Logic Journal of the Interest Group of Pure and Applied Logic*, 13:443–466, 2005.

[60] Antonio Di Nola, Francesc Esteva, Pere Garcia, Lluís Godo, and Salvatore Sessa. Subvarieties of BL-algebras generated by single-component chains. *Archive for Mathematical Logic*, 41: 673–685, 2002.

[61] Vilém Novák, Irina Perfilieva, and Jiří Močkoř. *Mathematical Principles of Fuzzy Logic*. Kluwer, Dordrecht, 2000.

[62] Hiroakira Ono. Proof-theoretic methods in non-classical logic: an introduction. In M. Takahashi, M. Okada, and M. Dezani-Ciancaglini, editors, *Theories of Types and Proofs*, volume 2 of *MSJ Memoirs*, pages 207–254. Mathematical Society of Japan, Tokyo, 1998.

[63] Hiroakira Ono. Logic without contraction rule and residuated lattices. To appear in the book for the Festschrift of Prof. R. K. Meyer, edited by E. Mares, 2007.

[64] Hiroakira Ono and Yuichi Komori. Logic without the contraction rule. *Journal of Symbolic Logic*, 50:169–201, 1985.

[65] Jan Pavelka. On fuzzy logic I, II, III. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 25:45–52, 119–134, 447–464, 1979.

[66] Mathias E. Ragaz. *Arithmetische Klassifikation von Formelmengen der unendlichwertigen Logik*. PhD thesis, Swiss Federal Institute of Technology, Zürich, 1981.

[67] Helena Rasiowa. *An Algebraic Approach to Non-Classical Logics*. North-Holland, Amsterdam, 1974.

[68] Petr Savický, Roberto Cignoli, Francesc Esteva, Lluís Godo, and Carles Noguera. On product logic with truth constants. *Journal of Logic and Computation*, 16(2):205–225, 2006.

[69] San-Min Wang, Bao-Shu Wang, and Dao-Wu Pei. A fuzzy logic for an ordinal sum t-norm. *Fuzzy Sets and Systems*, 149(2):297–307, 2005.

[70] San-Min Wang, Bao-Shu Wang, and Fang Ren. NMŁ, a schematic extension of F. Esteva and L. Godo's logic MTL. *Fuzzy Sets and Systems*, 149(2):285–295, 2005.

[71] Lotfi A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.