ON MODAL EXPANSIONS OF LEFT-CONTINUOUS T-NORM LOGICS

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Modal fuzzy logics is a research topic that has attracted increasing attention in the last years. Several papers have been published treating different aspects, see for instance [6] for a modal expansion of Łukasiewicz logic, [3, 4, 2] for modal expansions of Gödel fuzzy logic, [1] for modal logics over finite residuated lattices, and more recently [7] for a modal expansion of Product fuzzy logic.

In this paper we extend the latter approach (based on infinitary proof systems) by considering the axiomatization problem of the modal expansion of the propositional logic of a left-continuous t-norm (with rational truth constants and Monteiro-Baaz Δ operator) using crisp accessibility relations in the Kripke models over such t-norm. We provide explicit strongly complete axiomatizations solving such problem for a large family of t-norms (including all ordinal sums of Łukasiewicz and Product t-norms). Unfortunately, the proof is not general enough to deal with all such axiomatization problems, e.g., the problem remains open for Gödel t-norm.

The technique used in the strong completeness proof involves building a canonical model, and such approach requires to previously know a strongly complete axiomatization of the propositional logic of a left-continuous t-norm (with rational truth constants and Monteiro-Baaz Δ operator). Therefore, before jumping into the modal discussion we discuss several issues of this non modal logic.

For an arbitrary left-continuous t-norm $*$, in [8] we introduced an axiomatic system L_*^∞ that was proved to be strongly complete with respect to the algebra denoted by $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$ and consisting of expanding the MTL-chain $[\mathbf{0},\mathbf{1}]_*$ (i.e., the one with domain the unit real interval and associated with $*$) with the Δ operator and rational constants. The axiomatization there given is obtained adding to any known axiomatization of MTL_Δ the usual book-keeping axioms for $\&, \rightarrow$ and Δ connectives, and the following infinitary *density rule*

$$\mathsf{D}^\infty : \ \frac{\{(\varphi \rightarrow \overline{c}) \vee (\overline{c} \rightarrow \psi)\}_{c\in[0,1]_{\mathbb{Q}}}}{\varphi \rightarrow \psi}$$

Note that in this axiomatization, the only axioms depending on the particular t-norm $*$ are the book-keeping axioms.

Unfortunately, the inference rule D^∞ turns out to be not well-behaved when the logic is expanded with modalities, in particular it is not clear how to prove that it is closed under $\Box$. Taking inspiration from some of the inference rules presented in [5], we search for alternative axiomatizations using a family of infinitary *conjunctive* inference rules with better behaviour. For each real $x \in [0,1]$ let the infinitary rule R_x^∞ be defined as:

$$\mathsf{R}_x^\infty : \ \frac{\{(\varphi \rightarrow \overline{c}) \wedge (\overline{d} \rightarrow \psi)\}_{d\in[0,x)_{\mathbb{Q}},c\in(x,1]_{\mathbb{Q}}}}{\varphi \rightarrow \psi}$$

It holds that a linearly ordered algebra (of MTL_Δ with rational constants) validates the density rule D^∞ if and only if it validates all the conjunctive inference rules in $\{\mathsf{R}_x^\infty : x \in [0,1]\}$. We will say that a left-continuous t-norm $*$ **is conjunctively axiomatizable** whenever the logic of $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$ can be axiomatized using only conjunctive inference rules. Although not all left-continuous t-norms are conjunctively axiomatizable (e.g., Gödel t-norm is not), we have identified a lot of t-norms that are so: for example, all ordinal sums of Łukasiewicz and Product t-norms.

Given a left-continuous t-norm $*$, let $\Vdash_{\mathbb{SM}_*}$ denote the (local) modal logic arising from the Kripke models with a crisp accessibility relation and evaluations over the algebra $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$, and let $\Vdash_{\mathbb{KM}_*}$ denote the logic induced by those Kripke models with evaluations over algebras of the generalized quasi-variety generated by $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$.

We define K_* to be the axiomatic system obtained expanding L_*^∞ with the $\Box$ and $\Diamond$ operators and the following set of axioms and inference rule (where $c \in [0,1]_{\mathbb{Q}}$):

(K)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	($\Diamond$1)	$\Box(\varphi \to \overline{c}) \leftrightarrow (\Diamond\varphi \to \overline{c})$
($\Box$1)	$\Box(\overline{c} \to \varphi) \leftrightarrow (\overline{c} \to \Box\varphi)$	(FS1)	$(\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$
($\Box$2)	$\Delta\Box\varphi \to \Box\Delta\varphi$	($\mathsf{N}_\Box$)	Necessitation: $\vdash \varphi$ infer $\vdash \Box\varphi$

First, one can check that for any $\Gamma \cup \{\varphi\} \subseteq Fm$,

$$\Gamma \vdash_{K_*} \varphi \;\Leftrightarrow (\Gamma \cup Th(K_*))^\# \vdash_{L_*^\infty} \varphi^\#,$$

where $\#$ translates purely modal formulas (i.e., formulas of the form $\Box\psi$ or $\Diamond\psi$) to fresh propositional variables. On the other hand, assuming $*$ is conjunctively axiomatizable, we can prove that if $\Gamma \vdash_{K_*} \varphi$ then $\Box\Gamma \vdash_{K_*} \Box\varphi$.

Moreover, the completeness proof for K_* can be done resorting to the usual canonical model construction. The Kripke model defined for this task involves the algebra $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$ and is given by $\mathfrak{M}_*^{\mathfrak{c}} = \langle W^c, R^c, e^c \rangle$, where:

- $W^c = \{h \in Hom(\mathbf{Fm}^\#, [\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}) : h(Th(K_*)^\#) = \{1\}\}$
- $R^c vw$ if, for any φ such that $v(\varphi_\Box) = 1$, $w(\varphi^\#) = 1$
- $e^c(v,p) = v(p)$ for any propositional variable p.

Finally, in order to show the *Truth Lemma* for $\mathfrak{M}_*^{\mathfrak{c}}$, a previous important result is required: given any $v \in W$ and a modal formula φ such that $w(\varphi^\#) = 1$ for all $w \in W$ with Rvw, it holds that $v(\varphi_\Box) = 1$. This can be proved using the translation of deductions in K_* into L_*^∞, the strong standard completeness of L_*^∞, and taking advantage of the rational truth constants.

Theorem 1. *Let $*$ be a t-norm conjunctively axiomatizable, and $\Gamma \cup \{\varphi\}$ a set of modal formulas. Then*

$$\Gamma \vdash_{K_*} \varphi \quad \Longleftrightarrow \quad \Gamma \Vdash_{\mathbb{SM}_*} \varphi \quad \Longleftrightarrow \quad \Gamma \Vdash_{\mathbb{KM}_*} \varphi.$$

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