A Logic of Belief Revision in Simplicial Complexes

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Abstract

In this paper, we present a model of epistemic modal logic in simplicial semantics with the aim of motivating a new interpretation of belief revision by way of imaging. Our model modifies and expands on recent papers by Eric Goubault, Doman Kniazev, Sergio Rajsbaum, Jérémy Ledent, and Hans van Ditmarsch, which use simplicial complexes as a semantics for epistemic modal logic, leveraging their special structure not present in the usual Kripke setting.[10][6][1][3] Our motivation is similar: we believe simplicial semantics affords us tools for analyzing belief revision not obviously present in Kripke models.

In simplicial semantics, possible worlds are not treated as primitive. Instead, they can be identified with sets of agent perspectives. Perspectives are given, for technical reasons, as sets of literals. In our model, we construct such "worlds" by taking consistent sets of perspectives, with one perspective uniquely associated with each agent. We will call these "worlds" facets. Given such a facet in a simplicial complex S, call it X, we say that Y is *accessible* for agent a from X if X and Y share their unique a-perspective. As is standard, one says that a formula $K_a\varphi$ is true if and only if φ is true at all facets Y which are accessible for agent a from X. One can show, via a categorical equivalence, that this framework is equivalent to a particular class of Kripke models where the accessibility relations form partitions (i.e., models for **S5**).

This is similar to what has been done in the previous literature. There, the simplicial complexes are used to model knowledge. [6][10][1][7][5][3][2][4] If we are to use simplicial complexes as a model for belief, the fact that the axiom T is sound is undesirable. It is easy to see why it is sound in the usual setup, as X is always a-accessible from X.

To get around this, we will introduce distinct simplicial complexes for each agent. More specifically, given a set of perspectives, each agent will have their own simplicial complex over this set, call it S_a. All of the S_a will be subcomplexes of a background complex, call it S, which is not specific to any agent. We then modify the definition of a-accessible facets as follows. Given facets X and Y in S, we say that Y is *accessible** for agent a from X if and only if Y is a facet in S_a and X and Y share their unique a-perspective. If we say that $B_a\varphi$ is true at X if and only if φ is true at all Y that are *a-accessible** from X, It's easy to show that this makes sound **K45**. Additionally, it's easy to see why the axiom T is not sound, as if X is not a facet in S_a, then X is not *a-accessible** from X.

Motivated by the idea of charity towards other agents, we argue that a good notion of "nearness" between two worlds, in the sense of Lewis' work on imaging, in the simplicial setting is given by the size of their intersection.[9] That is, worlds which share more perspectives are closer. More specifically, suppose the formula φ is publicly announced. Then we can define an imaging function R which replaces every facet X in S_a with a set of facets $R(X)$, which consists of those facets which 1: satisfy φ, 2: share their unique a-perspective, and 3: are such that for any facet $Y \in R(X)$, and any facet Z satisfying conditions 1 and 2, $|X \cap Y| \geq |X \cap Z|$. We explore variations of this imaging function and some of their consequences. For instance, we could restrict $R(X)$ to always be a set of facets from the background complex S, and furthermore have S eliminate facets which contradict announced information with each announcement. In this way, S acts as a kind of "memory" for the agents. Another option would be to say that if there are facets in S_a which share the a perspective with X and satisfy φ, then $R(X)$ should be this set, regardless of the size of the intersection. We show, specifically, that a variant of this second

update mechanism is a "nested sphere" model in the sense of Grove, 1988, and therefore satisfies the AGM axioms. [8]

Furthermore, we give soundness proofs for the knowledge and belief modalities in our models. We interpret the belief modality using *accessibility**, and the knowledge modality using *accessibility* for the background complex S. Specifically, for the following language:

$$\varphi ::= P|\bot|\varphi \to \varphi|B_a\varphi|K_a\varphi$$

our simplicial semantics is sound with respect to the axioms of propositional logic, **S5** for the K_a modality, **K45** for the B_a modality, the axiom $K_a\varphi \to B_a\varphi$, and the following axiom for each modality, and any atomic formula P, which we call **NU** for "No Uncertainties":

$$\mathbf{NU} : P \to \bigvee_{a \in A} K_a P$$

Axioms similar to **NU** appear throughout the simplicial literature. [1].

One easy way to see that these soundness results hold is to proceed as much of the existing literature does, by demonstrating a logic preserving categorical equivalence between a category whose objects are our simplicial models, and a category whose objects are a class of Kripke models, where these Kripke models make sound these same axioms. We conclude by discussing how one can extend this equivalence to a proof of completeness.

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