

# Modal Weak Kleene Logics

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The basic weak Kleene logics  $B_3$  and PWK (standing for Bochvar logic and Paraconsistent weak Kleene, respectively) can be introduced as three-valued logics characterized by an infectious non-classical value. While these logics have interesting syntactic properties as examples of variable inclusion logics (see [2]), from the perspective of (abstract) algebraic logic they are very weakly connected to their algebraic counterparts.

Once the language of these logics - that is in the type of classical logic - is enriched with an unary operator  $J_2$ , whose intuitive reading is "it is (classically) true that", the resulting expansions - also known as *external* Kleene logics - gain strong algebraic properties. The operator  $J_2$  allows to define two other operators, and their semantics is explained by the following tables:

$\varphi$	$J_2\varphi$	$\varphi$	$J_0\varphi$	$\varphi$	$J_1\varphi$
1	1	1	0	1	0
1/2	0	1/2	0	1/2	1
0	0	0	1	0	0

The three mentioned connectives form the so-called external operators, in the sense that each formula in which every propositional variable falls under the scope of those operators behaves entirely classically. Therefore the expansions of  $B_3$  and PWK with  $J_2$  are called, respectively, Bochvar external logic  $B_e$  and external paraconsistent weak Kleene logic  $PWK_e$ .

The addition of  $J_2$  is enough to restore the algebraic connection that basic, non-external Kleene logics lacked: both  $B_e$  and  $PWK_e$  are algebraizable, as proved, respectively, in [1] and [4]. These logics share the quasi-variety of Bochvar algebras BCA as their equivalent algebraic semantics. BCA was introduced in [5] and has been recently studied in [3], which provided a representation theorem of the  $J_2$ -free reduct of Bochvar algebras in terms of Płonka sums of Boolean algebras (plus additional operations). Płonka sums [6, 7] are an algebraic construction which allows to construct a new algebra starting from a semilattice direct system +of similar algebras. This tool has revealed its efficacy in the algebraic study of algebras connected with weak Kleene logics and, more in general, for the logics of variable inclusion [2].

In this work we present a study started in [4] on modal weak Kleene external logics. The language of weak Kleene external logics can be expanded with a modal operator  $\Box$ , whose intended meaning is that of standard alethic modal logic, a task first undertaken by Segerberg in [8] with a less general scope. Our work is divided into two parts: the first focuses on Kripke-style semantics, the other on algebraic semantics for modal weak Kleene external systems. We introduce the logics  $B_e^\Box$  and  $PWK_e^\Box$ , respectively modal Bochvar external logic and modal external PWK. The reading of the  $\Box$  modality differs between the two systems, according to the underlying propositional logic. Using a possible world interpretation, the intuitive reading of  $\Box\varphi$  is " $\varphi$  is true at every accessible world" in  $B_e^\Box$ , and " $\varphi$  is non-false at

every accessible world" in  $\text{PWK}_e^\square$ . The logics has been axiomatized and a complete Kripke-style semantics is provided for both. The systems are also decidable and easy to extend axiomatically, obtaining completeness results w.r.t. classes of frames characterized by well-known properties.

In the algebraic part we introduce the global versions of the local modal logics  $\text{B}_e^\square$  and  $\text{PWK}_e^\square$ , respectively  $g\text{B}_e^\square$  and  $g\text{PWK}_e^\square$ . We present a study of the algebraic counterparts of these logics, first introducing the quasi-variety of modal Bochvar algebras  $\text{MBCA}$ , and then identifying the two subclasses  $\text{MBCA}_B$  and  $\text{MBCA}_H$ , which are the equivalent algebraic semantics of  $g\text{B}_e^\square$  and  $g\text{PWK}_e^\square$ , respectively. The choice to move from local to global logics is motivated by the failure of algebraizability for local modal systems, algebraizability that is recovered once we consider their global versions. Building upon the results obtained in [3], we prove a representation theorem for  $\text{MBCA}_B$  and  $\text{MBCA}_H$ , which states that the  $J_2$ -free reduct of a modal Bochvar algebra belonging to these classes is a particular Płonka sum of Boolean algebras with operators. We show how certain relative sub-varieties of these classes correspond to standard extensions of the basic modal logics  $g\text{B}_e^\square$  and  $g\text{PWK}_e^\square$  which are characterized by well-known frame properties from the side of their Kripke semantics.

## References

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