Vietoris endofunctor for closed relations and its de Vries dual*

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Abstract

We generalize the Vietoris endofunctor to the category of compact Hausdorff spaces and closed relations and describe the dual endofunctor on the category of de Vries algebras and subordinations.

Taking the Vietoris hyperspace $\mathbb{V}(X)$ of a compact Hausdorff space X defines an endofunctor $\mathbb{V}$ on the category KHaus of compact Hausdorff spaces and continuous functions. On morphisms, a continuous function $f\colon X \to Y$ is mapped to the function $\mathbb{V}(f)\colon \mathbb{V}(X) \to \mathbb{V}(Y)$ which maps a closed subset F of X to the image $f[F]$ of F under f.

The larger category $\mathsf{KHaus}^{\mathsf{R}}$ of compact Hausdorff spaces and closed relations has been investigated in various works [11, 8, 10, 5, 1]. One appealing feature of $\mathsf{KHaus}^{\mathsf{R}}$ is that it is self-dual. We generalize the Vietoris endofunctor to an endofunctor $\mathbb{V}^{\mathsf{R}}\colon \mathsf{KHaus}^{\mathsf{R}} \to \mathsf{KHaus}^{\mathsf{R}}$. For a closed relation $R \subseteq X \times Y$, we define $\mathbb{V}^{\mathsf{R}}(R)$ by generalizing the well-known Egli-Milner order: for all closed subsets $F \subseteq X$ and $G \subseteq Y$, we set

$$F \ \mathbb{V}^{\mathsf{R}}(R)\ G \iff G \subseteq R[F] \text{ and } F \subseteq R^{-1}[G],$$

where $R[F]$ is the R-image of F in Y and $R^{-1}[G]$ is the R-preimage of G in X. We show that this defines an endofunctor $\mathbb{V}^{\mathsf{R}}\colon \mathsf{KHaus}^{\mathsf{R}} \to \mathsf{KHaus}^{\mathsf{R}}$ that restricts to the Vietoris endofunctor $\mathbb{V}\colon \mathsf{KHaus} \to \mathsf{KHaus}$ and commutes with the self-duality of $\mathsf{KHaus}^{\mathsf{R}}$.

De Vries duality [7] is a duality for KHaus which associates with each compact Hausdorff space X the boolean algebra $\mathcal{RO}(X)$ of regular opens of X equipped with the proximity relation given by $U \prec V$ iff $\mathsf{cl}(U) \subseteq V$. This yields a duality between KHaus and the category DeV of *de Vries algebras*, i.e. pairs $(B, \prec)$ where B is a complete boolean algebra and $\prec$ is a proximity relation on B. A direct pointfree construction of the endofunctor $\mathsf{DeV} \to \mathsf{DeV}$ dual to $\mathbb{V}\colon \mathsf{KHaus} \to \mathsf{KHaus}$ remained an open problem [4, p. 375]. We resolve this problem as follows.

In [1] we extended de Vries duality to $\mathsf{KHaus}^{\mathsf{R}}$. Let $\mathsf{Stone}^{\mathsf{R}}$ be the full subcategory of $\mathsf{KHaus}^{\mathsf{R}}$ consisting of Stone spaces. Stone duality extends to an equivalence between $\mathsf{Stone}^{\mathsf{R}}$ and the category BA^{S} with boolean algebras as objects and subordination relations as morphisms [6, 9, 1]. This yields an equivalence between $\mathsf{KHaus}^{\mathsf{R}}$ and a category whose objects are pairs (B, S) where B is a boolean algebra and S is a subordination relation on B satisfying axioms generalizing the axioms of an $\mathsf{S5}$-modality. Because of this connection, we termed the pairs (B, S) $\mathsf{S5}$-*subordination algebras* and denoted the resulting category by $\mathsf{SubS5}^{\mathsf{S}}$ [1]. The inclusion $\mathsf{DeV}^{\mathsf{S}} \hookrightarrow \mathsf{SubS5}^{\mathsf{S}}$ of the full subcategory $\mathsf{DeV}^{\mathsf{S}}$ consisting of de Vries algebras

*This presentation is based on [3].

†Presenter

is an equivalence, with quasi-inverse obtained by generalizing the MacNeille completion to $\mathsf{S5}$-subordination algebras [2].

In [12], the endofunctor $\mathbb{K}$ on boolean algebras dual to the Vietoris endofunctor $\mathbb{V}$ on Stone spaces was defined. We lift $\mathbb{K}$ to an endofunctor $\mathbb{K}^{\mathsf{S}}$ on BA^{S} equivalent to $\mathbb{V}^{\mathsf{R}}$ on $\mathsf{Stone}^{\mathsf{R}}$. Finally, we lift $\mathbb{K}^{\mathsf{S}}$ to an endofunctor on $\mathsf{SubS5}^{\mathsf{S}}$ equivalent to $\mathbb{V}^{\mathsf{R}}$ on $\mathsf{KHaus}^{\mathsf{R}}$. Composing it with the MacNeille completion yields an endofunctor on $\mathsf{DeV}^{\mathsf{S}}$ equivalent to $\mathbb{V}^{\mathsf{R}}$. This solves the problem mentioned above in the category $\mathsf{SubS5}^{\mathsf{S}}$, in its full subcategory $\mathsf{DeV}^{\mathsf{S}}$, and finally in DeV via a duality between DeV and a wide subcategory of $\mathsf{DeV}^{\mathsf{S}}$.

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