Structurally complete finitary extensions of positive Łukasiewicz logic

Paolo Aglianò[1] and Francesco Manfucci[1]

DIISM, Università di Siena, Siena, Italy

A large class of substructural logics is given by the finitary extensions of the *Full Lambek Calculus* $\mathcal{FL}$ (see [4]). It is well known that all these finitary extensions are algebraizable with an *equivalent algebraic semantics* (in the sense of Blok-Pigozzi [3]) that is at least a quasivariety of algebras. An important subfamily of substructural logics over $\mathcal{FL}$ consists of logics that satisfy both exchange and weakening but lack contraction. In this context, one of the most studied examples is Łukasiewicz's infinite-valued logic, which we denote by $\mathcal{MV}$ [6], and its algebraic counterpart given by the variety of MV-algebras. Since all extensions of $\mathcal{FL}$ have a primitive connective $\mathbf{0}$ that denotes the *falsum*, if $\mathcal{L}$ is an extension of $\mathcal{FL}$, it makes sense to study its *positive* fragment $\mathcal{L}^+$ (i.e. the logic obtained from $\mathcal{L}$ by removing $\mathbf{0}$ from the signature), which is still algebraizable.

In this framework, relevant algebraic structures are given by *hoops*, a particular variety of residuated monoids which were defined in an unpublished manuscript by Büchi and Owens. As a most relevant subvariety, the variety WH of *Wajsberg hoops*, is the equivalent algebraic semantics of $\mathcal{MV}^+$, i.e. the positive fragment of Łukasiewicz logic. Its relevance extends also to the purely algebraic framework, since Wajsberg hoops can be used to describe subdirectly irreducible hoops and the whole variety of hoops can be obtained as the join of iterated powers of the variety of WH, in the sense defined in [2]. In this contribution, we study structural completeness in sub(quasi)varieties of Wajsberg hoops.

A rule is *admissible* in a logic if, when added to its calculus, it does not produce new theorems. A logic $\mathcal{L}$ is *structurally complete* if every admissible rule of $\mathcal{L}$ is derivable in $\mathcal{L}$; it is *hereditarily structurally complete* if every finitary extension of $\mathcal{L}$ is structurally complete. In an algebraizable logic, these notions correspond to the associated quasivariety being *structural* and *primitive* respectively. A quasivariety Q is *structural* if for every subquasivariety $\mathsf{Q}' \subseteq \mathsf{Q}$, $\mathbf{H}(\mathsf{Q}') = \mathbf{H}(\mathsf{Q})$ implies $\mathsf{Q}' = \mathsf{Q}$ (where $\mathbf{H}(\mathsf{Q})$ is the class of all homomorphic images of algebras in Q); moreover we define the *structural core of* Q as the smallest $\mathsf{Q}' \subseteq \mathsf{Q}$ such that $\mathbf{H}(\mathsf{Q}) = \mathbf{H}(\mathsf{Q}')$. A quasivariety Q is *primitive* if every subquasivariety of Q is structural.

Notice that a quasivariety Q is structural if and only if it coincides with its structural core. As a consequence the structural subquasivarieties of a quasivariety Q are exactly those that coincide with the structural cores of Q' for some $\mathsf{Q}' \subseteq \mathsf{Q}$; even more, since $\mathbf{H}(\mathsf{Q})$ is a variety, the structural subquasivarieties of a variety V are exactly the structural cores of V' for some subvariety V' of V.

In this work, we characterize all the structurally complete finitary extensions of $\mathcal{MV}^+$. Moreover, we provide some examples of finitary extensions of $\mathcal{MV}^+$ that are hereditarily structurally complete and others that are not. Algebraically, this corresponds to classifying all the structural quasivarieties and studying some of the primitive quasivarieties of Wajsberg hoops.

Starting from the work of Gispert about MV-algebras ([5]), we begin our analysis from varieties, where the results are almost straightforward. It is known ([1]) that every proper variety of Wajsberg hoops is generated by a finite number of chains of the type $\mathbf{Ł}_n, \mathbf{Ł}_n^\infty$ or $\mathbf{C}_\omega$, where $\mathbf{Ł}_n = \Gamma(\mathbb{Z}, n)$, $\mathbf{Ł}_n^\infty = \Gamma(\mathbb{Z} \times_l \mathbb{Z}, (n, 0))$ and $\mathbf{C}_\omega$ is the negative cone of $\mathbb{Z}$ with the operations defined in the obvious way (the notation for these constructions is the one of Mundici [7]). To be more precise, every proper subvariety of Wajsberg hoops can be associated with a particular kind of triple (I, J, K), called *reduced triple*, where I, J are finite subsets of $\mathbb{N} \setminus \{0\}$, $K \subseteq \{\omega\}$ and if $J \neq \emptyset$ then $K = \emptyset$. The connection is the following: if $P = (I, J, \emptyset)$ then $\mathsf{V}(P) = \mathbf{V}(\{\mathbf{Ł}_i : i \in I\} \cup \{\mathbf{Ł}_j^\infty : j \in J\})$ (the variety generated by all $\mathbf{Ł}_i$ and $\mathbf{Ł}_j^\infty$), if $P = (I, \emptyset, \{\omega\})$ then $\mathsf{V}(P) = \mathbf{V}(\{\mathbf{Ł}_i : i \in I\} \cup \{\mathbf{C}_\omega\})$.

Theorem. *Let $\mathsf{V} = \mathsf{V}(I, J, K)$ be a proper subvariety of Wajsberg hoops. Then V is structural if and only if either $J = \emptyset$, or $J = \{1\}$.*

Notice that, if $\mathsf{V} = \mathsf{V}(I, J, K)$ with either $J = \emptyset$ or $J = \{1\}$, every subquasivariety of V is still a variety of that form, so as an immediate consequence we get that a variety of Wajsberg hoops is structural if and only if it is primitive.

Moving on to quasivarieties, the characterization becomes more difficult to prove, but the results are still quite simple to present. Given a reduced triple, we define $\mathsf{Q}[I, J, \emptyset] = \mathbf{Q}(\{\mathbf{Ł}_i : i \in I\} \cup \{\mathbf{Ł}_{j,1} : j \in J\})$ (the quasivariety generated by all $\mathbf{Ł}_i$ and $\mathbf{Ł}_{j,1} = \Gamma(\mathbb{Z} \times_l \mathbb{Z}, (j, 1))$ [7]) and $\mathsf{Q}[I, \emptyset, \{\omega\}] = \mathbf{Q}(\{\mathbf{Ł}_i : i \in I\} \cup \{\mathbf{C}_\omega\})$. Using this notation, we can characterize all the structural quasivarieties of Wajsberg hoops.

Theorem. *Let Q be a quasivariety of Wajsberg hoops. Then Q is structural if and only if either Q is the structural core of WH or $\mathsf{Q} = \mathsf{Q}[I, J, K]$ for some reduced triple (I, J, K).*

Unlike varieties, we fall short of characterizing all primitive quasivarieties, due to the lack of understanding of the lattice of all the subquasivarieties of Wajsberg hoops. Despite that, we managed to find some examples of primitive and non-primitive quasivarieties.

Proposition. *Let $\mathsf{Q} = \mathsf{Q}[I, J, \emptyset]$ with $(I, J, \emptyset)$ reduced triple; if there exists $m \neq 1$ and there exist $i \in I, j \in J$ such that m divides i and j, then Q is not primitive. On the other hand, if $\mathsf{Q} = \mathsf{Q}[\emptyset, \{p\}, \emptyset]$ where p is a prime number, then Q is primitive.*

In particular, these examples show that in Wajsberg hoops there exist nontrivial primitive proper quasivarieties, but not all structural quasivarieties are primitive.

References

[1] P. Aglianò and G. Panti, *Geometrical methods in Wajsberg hoops*, J. Algebra **256** (2002), 352–374.

[2] W.J. Blok and I.M.A. Ferreirim, *On the structure of hoops*, Algebra Universalis **43** (2000), 233–257.

[3] W.J. Blok and D. Pigozzi, *Algebraizable Logics*, Mem. Amer. Math. Soc., no. 396, American Mathematical Society, Providence, Rhode Island, 1989.

[4] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono, *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, Studies in Logics and the Foundations of Mathematics, vol. 151, Elsevier, Amsterdam, The Netherlands, 2007.

[5] J. Gispert, *Least V-quasivarieties of MV-algebras*, Fuzzy Sets and Systems **292** (2016), 274–284.

[6] Y. Komori, *Super-Łukasiewicz propositional logics*, Nagoya Math. J. **84** (1981), 119–133.

[7] D. Mundici, *Interpretation of AFC^*-algebras in Łukasiewicz sentential calculus*, J.Functional Analysis **65** (1986), 15–63.