A bi-equivalence between topoi with enough points and a localisation of topological groupoids

J.L. Wrigley

Queen Mary, University of London

Our contribution. In this presentation, we will demonstrate a bi-equivalence

$$\mathbf{Topos}^{\text{iso}}_{w.e.p.} \simeq [\mathfrak{W}^{-1}]\mathbf{LogGrpd},$$

where:

- $\mathbf{Topos}^{\text{iso}}_{w.e.p.}$ is the bi-category of topoi with enough points, geometric morphisms, and natural isomorphisms,
- $\mathbf{LogGrpd} \subseteq \mathbf{TopGrpd}$ is a bi-subcategory of the bi-category of topological groupoids – the bi-category of *logical groupoids*,
- and $\mathfrak{W}$ is a *left* bi-calculus of fractions on $\mathbf{LogGrpd}$.

Background on localic representations of topoi. It is often remarked that Grothendieck topoi are a generalisation of topological spaces, in their point-free incarnation, where 'points can have non-trivial isomorphisms'. As proved by Joyal and Tierney [4], every topos $\mathcal{E}$ is *represented* by some *localic* groupoid $\mathbb{X}$, in the sense that $\mathcal{E}$ is equivalent to the *topos of sheaves* $\mathbf{Sh}(\mathbb{X})$.

In [5], Moerdijk demonstrates that a geometric morphism $\mathbf{Sh}(\mathbb{X}) \xrightarrow{f} \mathbf{Sh}(\mathbb{Y})$ is induced by a cospan

$$\begin{tikzcd} \mathbb{W} \arrow[r] \arrow[d] & \mathbb{Y} \\ \mathbb{X} & \end{tikzcd}$$

of homomorphism of localic groupoids, and moreover a bi-equivalence

$$\mathbf{Topos}^{\text{iso}} \simeq \mathbf{ECG}[\Sigma^{-1}] \tag{1}$$

between the bi-category of topoi (with only invertible 2-cells) and a localisation on the *right* of a bi-subcategory $\mathbf{ECG} \subseteq \mathbf{LocGrpd}$ of localic groupoids (where the details of the bi-category fractions are handled in a paper by Pronk [6]).

Topological representation of topoi. Since any topos with enough points can be represented by a *topological* groupoid (see [3]), it is natural to wonder whether a version of the bi-equivalence (1) exists where localic groupoids are replaced by topological groupoids. However, we can demonstrate that:

Proposition 1. *For any bi-subcategory $\mathcal{C} \subseteq \mathbf{TopGrpd}$, and any* right *bi-calculus of fractions Σ on $\mathcal{C}$,*

$$\mathbf{Topos}^{\text{iso}}_{w.e.p.} \not\simeq \mathcal{C}[\Sigma^{-1}].$$

This motivates our adoption of a *left* bi-calculus of fractions in the result:

Theorem 2. *There is a bi-equivalence* $\mathbf{Topos}^{\text{iso}}_{w.e.p.} \simeq [\mathfrak{W}^{-1}]\mathbf{LogGrpd}$.

An application to model theory. A classical result of model theory asserts that an *atomic/ω-categorical* theory $\mathbb{T}$ is characterised, up to bi-interpretability, by the topological automorphism group $\mathrm{Aut}(M)$ of its unique countable model (see [1]), i.e. given atomic theories $\mathbb{T}_1$ and $\mathbb{T}_2$ with countable models M and N,

$$\mathbb{T}_1, \mathbb{T}_2 \text{ are bi-interpretable} \iff \mathrm{Aut}(M) \cong \mathrm{Aut}(N).$$

Recently, Ben Yaacov has shown that *any* theory is characterised up to bi-interpretability by a topological groupoid [2]; however, his groupoid is *not* a groupoid of models for the theory.

From our bi-equivalence, we will deduce a groupoidal extension of the classical Ahlbrandt-Ziegler result: given theories $\mathbb{T}_1, \mathbb{T}_2$,

$$\mathbb{T}_1, \mathbb{T}_2 \text{ are Morita equivalent} \iff \mathbb{X}, \mathbb{Y} \text{ are weakly equivalent},$$

where $\mathbb{X}$ and $\mathbb{Y}$ are representing topological groupoids of models for the classifying topoi of $\mathbb{T}_1$ and $\mathbb{T}_2$ respectively.

References

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[2] I. Ben Yaacov. Reconstruction of non-$\aleph_0$-categorical theories. *Journal of Symbolic Logic*, 87(1):159–187, 2022.

[3] C. Butz and I. Moerdijk. Representing topoi by topological groupoids. *Journal of Pure and Applied Algebra*, 130(3):223–235, 1998.

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