Categorical Continuous Logic

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Continuous logic is obtained by replacing the binary truth values $\{0, 1\}$ by the unit interval $[0, 1]$. It was introduced for the model theory of complete metric structures, see [5] for a recent introduction. I will explain how continuous logic arises naturally when combining categorical logic and duality theory.

A coherent hyperdoctrine is a functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{DL}$ satisfying some axioms (see, e.g., [3, Ch. 5]), where $\mathbf{C}$ is a left exact category of contexts and where $\mathbf{DL}$ is the category of distributive lattices. These hyperdoctrines algebraize theories in coherent logic, more precisely the ones extending the theory of flat functors on $\mathbf{C}$. Composing with Priestley duality, one obtains a functor $\mathbf{C} \to \mathbf{Priestley}$, giving, in model theoretic terms, the spaces of types of the theory. The functors obtained in this way can be axiomatized as the *open polyadic Priestley spaces* [9]. We can replace the Priestley spaces by the more general compact ordered spaces to obtain *open polyadic compact ordered spaces* and it is possible to develop an elementary model theory from this order-topological perspective (Beth definability, omitting types, Makkai conceptual completeness).

In order to come back to the algebraic side, two dualities for compact ordered spaces behave well:

1. The duality between compact ordered spaces and stably continuous frames.
2. The duality obtained in [1, 2] by taking the unit interval $[0, 1]$ as a dualizing object.

Applying either of these dualities yields a different kind of hyperdoctrine. We will call them respectively stably continuous hyperdoctrines and fuzzy hyperdoctrines. Each possibility has its own advantage.

The duality with stably continuous frames allows to draw a connection to topos theory. The classifying toposes of stably continuous hyperdoctrines are the *stably continuous toposes*, specializing the continuous toposes of [6].

On the other hand, the duality of [1, 2] allows for a very straightforward generalization of intuitionistic logic. For instance, Pitts' uniform interpolation theorem [7] still holds by generalizing the proof of [4, 8].

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