The Logic with Unsharp Implication and Negation - Algebraic Approach

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Abstract

It is well-known that intuitionistic logics can be formalized by means of Brouwerian semilattices, i.e. relatively pseudocomplemented semilattices. Then the logical connective implication is considered to be the relative pseudocomplement and conjunction is the semilattice operation meet. If the Brouwerian semilattice has a bottom element 0 then the relative pseudocomplement with respect to 0 is called the pseudocomplement and it is considered as the connective negation in this logic. Our idea is to consider an arbitrary meet-semilattice with 0 satisfying only the Ascending Chain Condition, which is trivially satisfied in finite semilattices, and introduce the connective negation x^0 as the set of all maximal elements z satisfying $x \wedge z = 0$ and the connective implication $x \to y$ as the set of all maximal elements z satisfying $x \wedge z \leq y$. The Ascending Chain Condition means that every chain has a maxima element and it ensures that every non-void subset has maximal elements. Such a negation and implication are "unsharp" since they assign respectively, to one entry x or to two entries x and y belonging to the semilattice, a subset instead of an element of the semilattice. Surprisingly, these kind of negation and implication, respectively, still share a number of properties of the corresponding connectives in intuitionistic logic, in particular the derivation rule Modus Ponens. Moreover, unsharp negation and unsharp implication can be characterized by means of five, respectively seven simple axioms. Several examples are presented. The concepts of a deductive system and of a filter are introduced as well as the congruence determined by such a filter. We finally describe certain relationships between these concepts.

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