Semigroups in Classical Planning

Pere Pardo

University of Luxembourg, Esch-sur-Alzette, Luxembourg
pere.pardo@uni.lu

Classical planning [3] has become a major paradigm in areas of applied computer science, such as robotics, logistics and manufacturing. Although the class of programs under its scope (classical plans) are strickingly simple, current research still depends on traditional formalisms that are largely disconnected from abstract mathematics. (An exception being recent attempts to subsume planning, combined with computer vision or reasoning, into category theory [1].)

To this end, we suggest an algebraic appropach to classical planning that: (1) brings this area closer to mathematical practice; and (2) permits an abstract approach to plans that benefits research in the area of classical planning itself.

In a nutshell, classical planning is the search for deterministic plans that lead to a goal from the initial state. Classical plans $\pi \in A^*$ are built from a set A of primitive actions available to an agent. Given a finite set of logical atoms $At = \{p, q, \dots\}$ and literals $Lit = \{p, -p, q, \dots\}$, a state $s \in S$ is a maximally consistent set of literals. The goal is just a consistent set of literals, and so are the preconditions and effects that define an action a as a pair $a = (pre(a), \mathit{eff}(a))$.

For action updates, one defines first a consistency-preserving update function over sets of literals X, Y

$$X \diamond Y = (X \setminus -Y) \cup Y$$

where $-Y := \{-y : y \in Y\}$. Action or plan executions are then defined by a function $\gamma : S \times A^* \to S$ (technically, a semigroup action) where:

$$\gamma(s,a) = \begin{cases} s \diamond \mathit{eff}(a) & \text{if } s \models pre(a) \\ \mathit{undefined} & \text{otherwise} \end{cases} \qquad \begin{array}{rcl} \gamma(s, \langle\rangle) & = & s \\ \gamma(s, a.\pi) & = & \gamma(\gamma(s,a), \pi) \end{array}$$

(Here, $\langle\rangle$ is the empty plan, and the plan $a.\pi$ is the concatenation of a and π.)

Let us now turn into algebra, by abstracting from the goal and initial state that define a planning problem. Henceforth, a plan is just a finite action sequence.

A semigroup $(G, \cdot)$ consists of an associative operation $\cdot : G \times G \to G$ on a set G. Two immediate semigroups capture the syntax and semantics of plans:

(1) the free (word) semigroup $(A^*, .)$ of plans π built under concatentation '.'

(2) the semigroup $(\|A^*\|, \circ)$ of plan executions $\|\pi\|$ under map composition $\circ$.

Each plan π does correspond to a partial transformation $\|\pi\| : S \to S$ given by $\|\pi\|(s) = \gamma(s, \pi)$. Indeed, (2) is a subsemigroup of $\mathcal{PT}(S)$, the semigroup of partial transformations of S, thoroughly studied in [2]. Semantically, there is thus no difference between actions and plans: they are just partial maps. To replicate this uniformity at the level of syntax, we define a product $\bullet : A \times A \to A$ that reduces plans to actions (for any set A closed under $\bullet$) so as to obtain:

(3) the semigroup $(A, \bullet)$ of actions $a = (pre(a), \mathit{eff}(a))$.

Let $\mathbf{A}_0$ contain all planning actions plus a zero 0, that we introduce with the *false* constant as the action $0 = (\{\bot\}, \{\bot\})$. (Note that $\gamma(s, 0) = \textit{undefined}$ for any $s \in S$.) We define the product in (3) by:

$$a \bullet b = \begin{cases} (\mathit{pre}(a \bullet b), \mathit{eff}(a \bullet b)) & \text{if } \mathit{pre}(b) \cap -(\mathit{pre}(a) \diamond \mathit{eff}(a)) = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

where $\mathit{pre}(a \bullet b) = \mathit{pre}(a) \cup (\mathit{pre}(b) \setminus \mathit{eff}(a))$ and $\mathit{eff}(a \bullet b) = \mathit{eff}(a) \diamond \mathit{eff}(b)$.

After proving that (3) is a semigroup, we verify that its product $\bullet$ is correct:

$$\|a \bullet b\| = \|a.b\| = \|a\| \circ \|b\|.$$

Next we fully characterize in each semigroup (1)–(3): the identity 1, the zero 0 and invertible elements $a = a^{-1}$; the zero divisors $ax = 0 = xa$ (for some x), nilpotents $aa = 0$, idempotents $aa = a$ and their natural partial ordering ($a \leq b$ iff $ab = a = ba$); and also commutativity $ab = ba$.

A function $\mathit{can}(a) = (\mathit{pre}(a), \mathit{eff}(a) \setminus \mathit{pre}(a))$ further identifies non-redundant actions as canonical representatives of behaviourally equivalent actions, in the sense that $\|\mathit{can}(a)\| = \|a\|$. Such non-redundant actions arrange into:

(3') the semigroup $(\mathbf{A}'_0, \bullet')$ of canonical actions, where $a \bullet' b = \mathit{can}(a \bullet b)$

Then we prove an isomorphism ($\leftrightarrow$) between (3') and the partial transformations induced by constructible plans $(\|\mathbf{A}^*\|, \circ)$ (see left figure):

$$\begin{tikzcd} & \mathbf{A} \arrow[dl, dotted, "\bullet"'] \arrow[dr, dotted, "."] & \\ (\mathbf{A}_0, \bullet) \arrow[d, "can(\cdot)"'] & & (\mathbf{A}^*, .) \arrow[ll, "\bullet"] \arrow[d, "\|\cdot\|"] \\ (\mathbf{A}'_0, \bullet') \arrow[rr, leftrightarrow, "\|\cdot\|"] & & (\|\mathbf{A}^*\|, \circ) \end{tikzcd}
\qquad
\begin{tikzcd} & \mathcal{L} \arrow[dr] & \\ id = \mathcal{H} \arrow[ur] \arrow[dr] & & \mathcal{D} = \mathcal{J} \\ & \mathcal{R} \arrow[ur] & \end{tikzcd}$$

Finally, (see right figure) we identify the Green relations and principal ideals in (3'):

$$\begin{array}{rcll} a\mathcal{L}b & \text{iff} & \mathbf{A}'_0\, a = \mathbf{A}'_0\, b & \text{iff } -(\mathit{pre}(a) \,\Delta\, \mathit{pre}(b)) \subseteq \mathit{eff}(a) = \mathit{eff}(b) \\ a\mathcal{R}b & \text{iff} & a\, \mathbf{A}'_0 = b\, \mathbf{A}'_0 & \text{iff } \mathit{pre}(a) = \mathit{pre}(b) \text{ and } \mathit{eff}(a) \,\Delta\, \mathit{eff}(b) = -(\mathit{eff}(a) \Delta \mathit{eff}(b)) \\ a\mathcal{H}b & \text{iff} & a(\mathcal{L} \cap \mathcal{R})b & \text{iff } a = b \\ \mathcal{D} & = & \text{min. equiv.} \supseteq \mathcal{L}, \mathcal{R} & \text{iff } -(\mathit{pre}(a) \cup \mathit{pre}(b)) \subseteq \mathit{eff}(a) \cap \mathit{eff}(b) \text{ and } \mathit{pre}(a) \cap \mathit{eff}(b) = \emptyset \\ a\mathcal{J}b & \text{iff} & \mathbf{A}'_0 a\, \mathbf{A}'_0 = \mathbf{A}'_0 b\, \mathbf{A}'_0 & \qquad \ldots \text{and } \mathit{eff}(a) \,\Delta\, \mathit{eff}(b) \subseteq -(\mathit{eff}(a) \,\Delta\, \mathit{eff}(b)). \end{array}$$

Our results offer a solid and elegant foundation to classical planning, with potential applications in the study of heuristic search functions, plan-space planning and partial-order planning, among other research lines in the area.

References

[1] Angeline Aguinaldo et al. Robocat: A category theoretic framework for robotic interoperability using goal-oriented programming. *IEEE Trans Autom. Sci. Eng.*, 19(3):2637–2645, 2022.

[2] Olexandr Ganyushkin and Volodymyr Mazorchuk. *Classical Finite Transformation Semigroups*. Springer London, 2008.

[3] Malik Ghallab, Dana S. Nau, and Paolo Traverso. *Automated planning - theory and practice*. Elsevier, 2004.