Conditional Esakia Duality

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Classical conditional logic strives to resolve issues that mathematicians and philosophers alike have had with traditional implication. We often have an intuition that $A \to B$ should speak to a connection between A and B — they should not be irrelevant or coincidentally connected. Weiss developed **ICK** from Chellas classical conditional in [1] by adding conditional implication, $\Box\!\!\to$, to basic intuitionistic logic. The conditional intuitively represents a stricter interpretation of implication — $\varphi \Box\!\!\to \psi$ is only the case if φ is relevant to ψ. This intuition is reflected in the frame semantics of **ICK**.

Definition 1. An *ICK-frame* is a tuple $(X, \leq, f)$ where $(X, \leq)$ is a preorder and

$$f : X \times Up(X, \leq) \to Up(X, \leq)$$

is a *selection function* such that $x \leq y$ implies $f(y, a) \subseteq f(x, a)$ for all $a \in Up(X, \leq)$. Proposition letters are interpreted via a valuation which assigns an upset of $(X, \leq)$ to each proposition letter, and $\wedge, \vee, \neg$ and $\to$ are interpreted as usual. For $x \in X$ we let

$$x \Vdash \varphi \Box\!\!\to \psi \quad \text{iff} \quad f(w, [\![\varphi]\!]) \subseteq [\![\psi]\!],$$

where $[\![\varphi]\!] = \{w \in X \mid w \Vdash \varphi\}$.

We can think of f as picking out the worlds relevant to φ at w. While a selection function best approximates our conditional motivations, it is sometimes easier to view it as an upset-indexed a family of relations $\{R_\alpha\}$, where $(\leq \circ R_\alpha \circ \leq) \subseteq R_\alpha$ for each relation. This allows us to view each relation as a modal relation in the sense of intuitionistic normal modal logic [4].

The new connective $\Box\!\!\to$ can be axiomatised by adding to intuitionistic logic the axioms

$$(\varphi \Box\!\!\to (\psi \wedge \theta)) \leftrightarrow ((\varphi \Box\!\!\to \psi) \wedge (\varphi \Box\!\!\to \theta)) \qquad \text{and} \qquad (\varphi \Box\!\!\to \top) \leftrightarrow \top$$

and congruence rules, resulting in the logic **ICK**. We find the following algebraic semantics:

Definition 2. A *conditional Heyting algebra* is a tuple $(\mathcal{A}, \Box\!\!\to)$ consisting of a Heyting algebra $\mathcal{A}$ and a binary operator $\Box\!\!\to$ satisfying $a \Box\!\!\to (b \wedge c) = (a \Box\!\!\to b) \wedge (a \Box\!\!\to c)$ and $a \Box\!\!\to 1 = 1$.

Inspired by the duality for intuitionistic normal modal logic [4, 2] and Weiss' work on **ICK** [3], we define topologised frame semantics as follows.

Definition 3. A *conditional Esakia space* is an Esakia space $\mathbb{X} = (X, \leq, \tau)$ equipped with a family of point-closed relations $\{R_A \mid A \in ClpUp(\mathbb{X})\}$ such that for each $A, B \in ClpUp(\mathbb{X})$:

$$\Box_{R_A}(B) := \{x \in X \mid R_A[x] \subseteq B\} \in ClpUp(\mathbb{X}) \qquad \text{and} \qquad (\leq \circ R_A \circ \leq) = R_A.$$

It is well known that collection of clopen upsets of an Esakia space $\mathbb{X}$ forms a Heyting algebra, denoted by $\mathbb{X}^+$. We can obtain a conditional Heyting algebra $(\mathbb{X}^+, \Box\!\!\to)$ by defining

$$A \Box\!\!\to B = \Box_{R_A}(B).$$

Conversely, every Heyting algebra $\mathcal{A}$ corresponds to an Esakia space $\mathcal{A}_+$ based on the prime filters of $\mathcal{A}$. In particular, we know that every clopen upset of $\mathcal{A}_+$ is of the form $\tilde{a} := \{x \in \mathcal{A}_+ \mid a \in x\}$ for some $a \in \mathcal{A}$. Ergo, the following definition

$$xR_{\tilde{a}}y \qquad \text{iff} \qquad \{b \in \mathcal{A} \mid a \,\Box\!\!\rightarrow b \in x\} \subseteq y,$$

results in a conditional Esakia space $(\mathcal{A}_+, \{R_{\tilde{a}}\})$. In fact, the assignments above give rise to a dual equivalence between categories:

Theorem 4. *The category of conditional Esakia spaces (with suitable morphisms) is dually equivalent to the category of conditional Heyting algebras and homomorphisms.*

This duality allows us to prove several frame completeness results. Beginning with **ICK** we note that if **ICK** $\nvdash \varphi$ then there exists a conditional Heyting algebra such that $\mathcal{A} \not\models \varphi$. Hence there exists a conditional Esakia space $\mathbb{X} = (X, \leq, \tau, \{R_\alpha\})$ and a valuation V such that $(\mathbb{X}, V) \nVdash \varphi$. Forgetting the topology *almost* gives an ICK-frame, except it lacks relations R_α when α is a non-clopen upset. We can fill in these missing relations by setting:

$$S_\alpha = \begin{cases} R_\alpha & \text{if } \alpha \text{ is a clopen upset} \\ \emptyset & \text{otherwise} \end{cases}$$

Since the clopen-indexed relations are unchanged we find that $((X, \leq, \{S_\alpha\}), V) \nVdash \varphi$, so that:

Theorem 5. *The logic* **ICK** *is sound and complete with respect to ICK-frames.*

This example highlights the two prongs of a duality completeness proof. When we extend **ICK** with a collection of axioms we induce both a *frame correspondence condition* and a *counter-example space*. We then need to extend the space to a frame which satisfies this correspondence condition for all upsets while also leaving the clopen relations (which guarantee the counter-example) unchanged. We call these "nice extensions" *fill-ins*, since we are in a sense filling in the missing upset relations on the underlying frame of a conditional Esakia space. Below we list several extensions of **ICK** together with their frame correspondent and (one possible) fill-in that can be used to prove completeness.

Axiom	Frame Condition	Fill-in
$\varphi \,\Box\!\!\rightarrow \varphi$	$R_A[x] \subseteq A$	$R_p[x] \subseteq p$
$p \wedge (p \,\Box\!\!\rightarrow q) \to q$	$p \cap \Box_{R_p}(q) \subseteq q$	$R_p[x] \supseteq p$
$(p \,\Box\!\!\rightarrow q) \to (p \wedge r \,\Box\!\!\rightarrow q)$	$\Box_{R_p}(q) \subseteq \Box_{R_{p\cap q}}(q)$	$R_p[x] := \bigcup_{p \supseteq U} R_U[x]$
$(p \,\Box\!\!\rightarrow q) \wedge (q \,\Box\!\!\rightarrow r) \to p \,\Box\!\!\rightarrow r$	$\Box_{R_p}(q) \cap \Box_{R_q}(r) \subseteq \Box_{R_p}(r)$	$R_p[x] := \bigcup\{R_U[x] \mid R_U[x] \subseteq p\}$
$(p \,\Box\!\!\rightarrow q) \vee (p \,\Box\!\!\rightarrow \neg q)$	$\Box_{R_p}(q) \cup \Box_{R_p}(X \setminus \downarrow q) = X$	The empty relation

References

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