Maximal sublattices of convex geometries

Adam Mata*

Warsaw University of Technology
adam.mata.dokt@pw.edu.pl

The origin of convex geometries lies in combinatorics, and the goal of the study of finite convex geometries was to develop the combinatorial abstraction of convexity. A convex geometry $G = (X, \alpha)$ is a finite closure system which satisfies the *anti-exchange property*, namely for all $x \neq y$ and all closed sets $A \in \mathcal{F} \subseteq 2^X$:

$$x \in \alpha(A \cup \{y\}) \textit{ and } x \notin A \textit{ imply that } y \notin \alpha(A \cup \{x\}).$$

The dual of a convex-geometry is an *antimatroid*: the family of complements of closed sets in G.

The closure lattices of a convex geometry have also been studied from the lattice theoretic point of view [3, 5]. A lattice L is (isomorphic to) the closure lattice of some convex geometry (shortly, we say: L is a convex geometry) iff, L is both:

- *join-semidistributive*: for every $x, y, z \in L$, $x \vee y = x \vee z$ implies that $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, and
- *lower-semimodular*: for every $x, y \in L$, the covering relation $x \prec x \vee y$ implies $x \wedge y \prec y$.

In the nineties, a series of papers studied maximal sublattices and Frattini sublattices (intersection of all maximal sublattices), and considerable progress was done for lattices in classes $\mathcal{D}$ of distributive lattices and $\mathcal{B}$ of McKenzie's bounded lattices as shown in [1, 2]. But not much was known about classes that extends $\mathcal{D}$ and $\mathcal{B}$, as e.g. $\mathcal{CG}$ of (finite) convex geometries and $\mathcal{SD}_\vee$ of join semi-distributive lattices.

In this talk we show some results about maximal sublattices and Frattini sublattices in these two classes. In particular, there is a full description of maximal sublattices in convex geometries of convex dimension 2. The complements of maximal sublattices are precisely order-convex sets of one of the three forms (and satisfying in each case additional technical conditions):

- A singleton, namely a doubly-irreducible element.
- A chain.
- A union of two chains with the common least element.

Further, we present the conditions which have to be kept in order to obtain particular features of the Frattini sublattices. It is worth mentioning that convex geometries of convex dimension 2 are structures dual to SPS lattices, see e. g. [4].

This is a joint work with **K. Adaricheva** and **S. Silberger** from Hofstra University, as well as with **A. Zamojska-Dzienio** from Warsaw University of Technology.

*Speaker.

References

[1] M. Abad and M. E. Adams. The Frattini sublattice of a finite distributive lattice. *Algebra Universalis*, 32:314–329, 1994.

[2] M. E. Adams, R. Freese, J. B. Nation, and J. Schmid. Maximal sublattices and Frattini sublattices of bounded lattices. *J. Austral. Math. Soc. Ser.*, A 63:110–127, 1997.

[3] K. Adaricheva and J.B. Nation. *Convex geometries, in Lattice Theory: Special Topics and Applications, v.2,*. G. Grätzer and F. Wehrung, eds., Springer, Basel, 2016.

[4] G. Czedli and E.T. Schmidt. Slim semimodular lattices I. A visual approach. *ORDER*, 29:481–497, 2012.

[5] P. H. Edelman and R. E. Jamison. The theory of convex geometries. *Geometriae Dedicata*, 19(3):247–270, 1985.