Weak distributive laws between powerspaces over stably compact spaces

Quentin Aristote

Université Paris Cité, CNRS, Inria, IRIF, F-75013, Paris, France
quentin.aristote@irif.fr

In the study of programming languages, effects may be modeled using monads. For instance, the powerset monad $\mathcal{P}$ on the category **Set** models non-determinism: a non-deterministic function from a set X to a set Y can be seen as a (deterministic) function $X \to \mathcal{P}Y$. Similarly, the distribution monad $\mathcal{D}$ and the corresponding functions $X \to \mathcal{D}Y$ model probabilistic non-determinism: the monad $\mathcal{D}$ sends a set X to the set of finitely supported probability distributions on X. Ordinary and probabilistic non-determinism may be combined in **Set** using *weak distributive laws*, and in this work we study how to combine these effects in a topological setting: we construct weak distributive laws between powerspace monads over the category of stably compact spaces, with the hope that weak distributive laws involving probabilistic powerspace monads may also be constructed in future work.

Given two monads $\mathcal{S}$ and $\mathcal{T}$ on a category $\mathcal{C}$ corresponding to two effects, the combination of these effects may be modeled by composing the monads $\mathcal{S}$ and $\mathcal{T}$. This can be done by finding a *distributive law* between the two monads, i.e. a natural transformation $\mathcal{TS} \Rightarrow \mathcal{ST}$ satisfying certain axioms. Such a distributive law yields a new monad $\mathcal{S} \circ \mathcal{T}$ on $\mathcal{C}$ whose underlying functor is the composite of the functors underlying $\mathcal{S}$ and $\mathcal{T}$.

But such distributive laws may not necessarily be unique and may not exist. In particular, there is no distributive law $\mathcal{PP} \Rightarrow \mathcal{PP}$ nor $\mathcal{DP} \Rightarrow \mathcal{PD}$ for combining two layers of non-determinism or non-determinism with probabilistic non-determinisim [9, 1]. Still, there are natural transformations $\lambda^{\mathcal{P}/\mathcal{P}} : \mathcal{PP} \Rightarrow \mathcal{PP}$ and $\lambda^{\mathcal{D}/\mathcal{P}} : \mathcal{DP} \Rightarrow \mathcal{PD}$ that satisfy all but one of the axioms required for them to be distributive laws [2, 6]: these natural transformations are called *weak distributive laws*. Again, weak distributive laws $\mathcal{TS} \Rightarrow \mathcal{ST}$ yield a composite monad, but its underlying functor need not be the composite of the functors underlying $\mathcal{S}$ and $\mathcal{T}$ anymore.

The powerset monad has a topological analogue, the Vietoris monad $\mathcal{V}$ on the category of compact Hausdorff spaces (it sends a compact Hausdorff space to the space of its closed subsets equipped with the Vietoris topology). It was shown recently that the weak distributive law $\lambda^{\mathcal{P}/\mathcal{P}}$ also has a topological analogue $\lambda^{\mathcal{V}/\mathcal{V}} : \mathcal{VV} \Rightarrow \mathcal{VV}$ [7].

What about a topological analogue of $\lambda^{\mathcal{D}/\mathcal{P}}$? Goy conjectures in [5] that the strategy for constructing $\lambda^{\mathcal{V}/\mathcal{V}}$ can be adapted to construct a weak distributive law $\mathcal{RV} \Rightarrow \mathcal{VR}$, where $\mathcal{R}$ is the Radon monad – a topological analogue of the distribution monad. But we show that this conjecture does not hold, the main problem being that the category of free algebras of the Vietoris monad is a strict subcategory of that of closed relations between compact Hausdorff spaces. To get a topological analogue of $\lambda^{\mathcal{D}/\mathcal{P}}$, we thus choose to study weak distributive laws between powerspace monads in the category of stably compact spaces instead [10], as in this category the closed relations match exactly the free algebras of a powerspace monad [8].

An important property of a stably compact space (X, τ) is that it comes with its de Groot dual: X can also be equipped with the topology τ^d whose open sets are the compact saturated subsets of (X, τ). A continuous map $f : X \to Y$ that is also continuous for the dual topologies is then said to be *proper*. There are then two notions of powerspaces, dual to one another: the Smyth powerspace of a stably compact space X is the space $\mathcal{Q}X$ of its compact saturated subsets equipped with the upper Vietoris topology, while its Hoare powerspace is the space $\mathcal{H}X$ of its closed subsets equipped with the lower Vietoris topology. It is well-known that $\mathcal{Q}X^d = (\mathcal{H}X)^d$: we also show that the two monads themselves are de Groot duals of one another, so that their unit and multiplicative laws are proper maps.

The main contribution of this work is to construct a weak distributive law $\mathcal{Q}\mathcal{Q} \Rightarrow \mathcal{Q}\mathcal{Q}$, using a known general construction starting from the identity monad morphism $\mathcal{Q} \Rightarrow \mathcal{Q}$, and a (strong) distributive law $\mathcal{Q}\mathcal{H} \Rightarrow \mathcal{H}\mathcal{Q}$, derived by hand, and which happens to be an isomorphism of monads $\mathcal{Q}\mathcal{H} \cong \mathcal{H}\mathcal{Q}$ with inverse the dual distributive law $\mathcal{H}\mathcal{Q} \Rightarrow \mathcal{Q}\mathcal{H}$, so that $(\mathcal{Q}\mathcal{H}X)^d = \mathcal{H}\mathcal{Q}X^d \cong \mathcal{Q}\mathcal{H}X^d$ and $\mathcal{Q}\mathcal{H}$ is self-dual.

A third kind of powerspace, the Plotkin powerspace, arises naturally as a combination of the Hoare and Smyth powerspaces: there is hope that making this combination categorical would also allow for combining the two weak laws above into a weak law for the Plotkin powerspace monad. Another next step would also be to extend the combination of probabilistic powerspaces over stably compact spaces with the Hoare and Smyth powerspaces [3, 4] to the monadic setting, again by constructing weak distributive laws.

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