Substructural Logics weaker than Commutative Lambek Calculus*

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Lambek Calculus [1] is a substructural logic that omits all the typical structural rules. The system is derived from its origins and can be understood as an algebraic structure, typically known as residuated monoid, and occaionally referred to as a semigroup [2]. The standard Lambek calculus is a system where only the associative law holds, as shown in Definition 1.

From the perspective of substructural logics, research has been conducted on the proof-theoretic and algebraic properties of this system by adding/ removing structural rules such as weakening, contraction, and exchange. Building on the standard Lambek calculus, researchers have identified a weaker system that incorporates only one structure rule. This paper focuses on a system that employs solely the exchange rule $\Gamma, \alpha, \beta, \Sigma \vdash \gamma \Longrightarrow \Gamma, \beta, \alpha, \Sigma$ and demonstrates the existence of a countably infinite series of logics $L_e^1, L_e^2, L_e^3, \ldots, L_e^n, \ldots$, shown in Definition 2, and their closure L_e^*, shown in Definition 3, between the standard system and the system with only exchange.

We discuss the logical systems L_e^n and L_e^*, and their relationship with L and L_e, focusing on modifications to the introduction rules for / and \ without explicitly adding the exchange rule. The analysis argues that L_e^n adn L_e^* aer fundamentally different logical system, both from each other and from L and L_e (Theorem 1). It highlights that L_e^n is stronger than L but equal to or weaker than L_e, as evidenced by the number of provable sequents; namely, the rule $/\mathrm{L}^{\mathrm{k}}$ and $\backslash\ \mathrm{L}^{\mathrm{k}}$ (and consequently $/^*$ and $\backslash\ \mathrm{L}^*$) are provable in L_e but not in L. Additionally, it is natural to consider the commutative Lambek calculus as possessing the algebraic structure of a commutative residuated monoid.

The standard Lambek calculus, which is inherently non-commutative, is characterized by the algebraic structure of a residuated monoid. Furthermore, the commutative Lambek calculus can also naturally be considered to have the algebraic structure of a commutative residuated monoid. However, the above-mentioned L_e^n and L_e^* do not fit into either algebra. We are exploring the translation of these systems into the algebraic structure of operads and plan to detail these effors in future work.

Definition 1 (Lambek Calculus L)**.** *The Lambek calculus L is a system of sequent calculus defined solely by the following inference rules. In particular, / and \ correspond to implications.*

$$\frac{}{\alpha \vdash \alpha}\,\mathrm{Ax} \qquad \frac{\Gamma \vdash \alpha \qquad \Sigma, \alpha, \Delta \vdash \beta}{\Sigma, \Gamma, \Delta \vdash \beta}\,\mathrm{Cut} \qquad \frac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \beta/\alpha}\,/\mathrm{R} \qquad \frac{\alpha, \Gamma \vdash \beta}{\Gamma \vdash \alpha\backslash\beta}\,\backslash\mathrm{R}$$

$$\frac{\Gamma \vdash \alpha \qquad \Sigma, \beta, \Delta \vdash \gamma}{\Sigma, \beta/\alpha, \Gamma, \Delta \vdash \gamma}\,/\mathrm{L} \qquad \frac{\Gamma \vdash \alpha \qquad \Sigma, \beta, \Delta \vdash \gamma}{\Sigma, \Gamma, \alpha\backslash\beta, \Delta \vdash \gamma}\,\backslash\mathrm{L}$$

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Definition 2 (Mildly Commutative Lambek Calculus L_e^n)**.** *Mildly Commutative Lambek Calculus L_e^n is defined by the left introduction rules:* $/\mathrm{L}^0$,$\backslash\mathrm{L}^0$,$/\mathrm{L}^1$,$\backslash\mathrm{L}^1$, $/\mathrm{L}^2$,$\backslash\mathrm{L}^2$, *…,* $/\mathrm{L}^\mathrm{n}$,$\backslash\mathrm{L}^\mathrm{n}$*, and L's standard rules:* Ax,Cut,/R,\R*. All introduction rules* $/\mathrm{L}^\mathrm{k}$ *and* $\backslash\mathrm{L}^\mathrm{k}$ *are defined as follows.*

$$\frac{\Gamma \vdash \alpha \qquad \Sigma, \beta, \delta_1, \delta_2, \ldots, \delta_k, \Delta \vdash \gamma}{\Sigma, \beta/\alpha, \underbrace{\delta_1, \delta_2, \ldots, \delta_k}_{\text{k skip}}, \Gamma, \Delta \vdash \gamma} /\mathrm{L}^\mathrm{k} \qquad \frac{\Gamma \vdash \alpha \qquad \Sigma, \sigma_1, \sigma_2, \ldots, \sigma_k, \beta, \Delta \vdash \gamma}{\Sigma, \Gamma, \underbrace{\sigma_1, \sigma_2, \ldots, \sigma_k}_{\text{k skip}}, \alpha\backslash\beta, \Delta \vdash \gamma} \backslash\mathrm{L}^\mathrm{k}$$

Definition 3 (Mildly Commutative Lambek Calculus L_e^*)**.** *Mildly Commutative Lambek Calculus L_e^* is defined by the rules:* $/\mathrm{L}^*$,$\backslash\mathrm{L}^*$*, and L's standard rules:* Ax,Cut,/R,\R.

$$\frac{\Gamma \vdash \alpha \qquad \Sigma, \beta, \Theta, \Delta \vdash \gamma}{\Sigma, \beta/\alpha, \Theta, \Gamma, \Delta \vdash \gamma} /\mathrm{L}^* \qquad \frac{\Gamma \vdash \alpha \qquad \Sigma, \Theta, \beta, \Delta \vdash \gamma}{\Sigma, \Gamma, \Theta, \alpha\backslash\beta, \Delta \vdash \gamma} \backslash\mathrm{L}^*$$

Theorem 1. *L_e^{n-1} is weaker than L_e^n and L_e^* because the following sequent is not provable in L_e^{n-1}, but is provable in L_e^n and L_e^*. Let $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}, \beta$ be atomic formulas. Then,*

$$\alpha_1, \underbrace{\alpha_2, \ldots, \alpha_{n+1}}_{n\ skip}, \alpha_1\backslash(\alpha_2\backslash(\ldots\backslash(\alpha_n\backslash(\alpha_{n+1}\backslash\beta))\ldots)) \vdash \beta.$$

Proof. First, we prove the case when $n = 1$; *i.e.*, $\alpha_1, \alpha_2, \alpha_1\backslash(\alpha_2\backslash\beta) \vdash \beta$ is provable in L_e^1 and L_e^* but not in L. The sequent is provable in L_e^1 and L_e^* by the rule $\backslash\mathrm{L}^1$. Furthermore, the sequent is not provable in L because the exhaustive proof search is halted by the cut elimination theorem. Next, we prove the case when $n = 2$; *i.e.*, $\alpha_1, \alpha_2, \alpha_3, \alpha_1\backslash(\alpha_2\backslash(\alpha_3\backslash\beta)) \vdash \beta$ is provable in L_e^2 and L_e^* but not in L_e^1. Similarly, we can prove that the sequent is provable in L_e^2 and L_e^* by the rules $\backslash\mathrm{L}^2$ and $\backslash\mathrm{L}^1$. Unlike the standard Lambek calculus, there is no cut elimination theorem in the systems L_e^n. However, we can prove that the sequent is not provable in L_e^1 by analyzing the proof search. Accordingly, we can prove the remaining cases in the same manner. Thus, we conclude that L_e^{n-1} is weaker than L_e^n and L_e^*. □

References

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