Topoi with enough points

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General landscape

This paper is concerned with the general problem of assessing whether a topos has enough points, with motivations coming both from geometry and logic. This problem has been influential in both fields up to the present day (see for example the work of Lurie [Lur04, VII, 4.1] and Barwick et al. [BGH18, II.3], for the case of geometry, and Espíndola and Kanalas [EK23] for the case of logic). We begin with an account of the development of this problem to properly frame our contribution and the significance of our work.

In 1964 Pierre Deligne proved a very celebrated theorem in topos theory.

Theorem (Deligne, [BDSD06, Exposé VI, 9.0])**.** Every locally coherent topos has enough points.

The theorem's original motivation came from algebraic geometry, but after Joyal and Reyes developed the theory of classifying topoi, it was observed by Lawvere that Deligne's theorem was essentially the statement of Gödel's completeness theorem for first order logic in disguise. This realisation crowned Deligne's theorem as a major result in categorical logic, and a source of inspiration for finding other completeness-like results using techniques from topos theory. To the present day Deligne's theorem remains the main argument to show that a wide class of topoi have enough points, and to some extent this paper investigates the limits (and the possibly unexploited potential) of this result.

Following Deligne's theorem, new results eventually emerged showing that further classes of topoi have enough points. Makkai and Reyes proved that *separable* topoi have enough points.

Theorem (Makkai and Reyes, [MR06, Theorem 6.2.4, page 180])**.** Let $\mathcal{C}$ be a countable category with pullbacks and J a Grothendieck topology generated by a countable family of sieves. Then $\mathrm{Sh}(\mathcal{C}, J)$ has enough points.

This result was inspired by the Fourman-Grayson completeness theorem for the logic L_{ω_1,ω_0} (see [FG82]), and indeed it is almost the translation of it into topos-theoretic language through the bridge of classifying topoi. The proof in [MR06] is a bit sketchy, and of model theoretic inspiration, thus it is hard to compare this result to Deligne's.

Recent developments

Lurie has imported Deligne's original argument to the world of ∞-topoi [Lur04, VII, 4.1]. The proof carries with minor adjustments, and under the mild additional assumption that the ∞-topos is hypercomplete. On the logical side, the main advances are due to Espíndola [Esp19, Esp20, EK23]; the most recent categorical analysis with Kanalas delivers a vast generalization of the original results achieved in Espíndola's PhD thesis.

Simultaneously to these developments, the topos theory community has been trying to understand the limits of Deligne's original argument and its possible generalization. Quite independently the authors of this paper and Tim Campion conjectured that every *locally finitely presentable topos could have enough points.* This conjecture finds its motivations in a number of examples, including the fact that coherent topoi are locally finitely presentable ([Joh02, D3.3.12]); presheaf topoi are often not coherent and yet they are always locally finitely presentable, and they are the easiest example of topos with enough points, and finally Hoffmann-Lawson duality

[HL78] from the theory of locales seems to support the conjecture that exponentiable topoi (which include locally finitely presentable topoi) have enough points. Before discussing our contribution, we should say that we did not manage to prove (or disprove) the conjecture. Yet our analysis seems to suggest that these methods cannot be sufficient to deliver a proof of the conjecture, if any such exists.

Our contribution

We employ the notion of collage in our presentation and generalization of Deligne's proof. Diagrams in the collage offer a convenient framework where points (on the left) can interact with objects (on the right) of a topos as if they were in the same category.

$$\begin{tikzcd} p \arrow[r, "z"] & Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \end{tikzcd}$$

After introducing the notion of *improvement*, designed to isolate the central idea of Deligne's proof, we prove our main theorem (which we express in a simplified, easy-to-read form).

Theorem. Let $j : \mathcal{F} \rightarrowtail \mathcal{E}$ be an inclusion of toposes. Suppose that for every point p admits an improvement. If $\mathcal{E}$ has enough points, then $\mathcal{F}$ has enough points.

We show that our refinement of Deligne's argument can be used to recover every existing result of this kind (for 1-topoi), including the most recent ones about κ-coherent κ-topos. Our strategy allows us to relax the assumptions on the site so that one is no longer required to control the cardinality of the set of morphisms.

References

[BDSD06] N Bourbaki, Pierre Deligne, and Bernard Saint-Donat. *Théorie des Topos et Cohomologie Etale des Schémas. Séminaire de Géométrie Algébrique du Bois-Marie 1963-1964 (SGA 4): Tome 2*, volume 270. Springer, 2006.

[BGH18] Clark Barwick, Saul Glasman, and Peter Haine. Exodromy. *arXiv:1807.03281*, 2018.

[EK23] Christian Espíndola and Kristóf Kanalas. Every theory is eventually of presheaf type. *arXiv preprint arXiv:2312.12356*, 2023.

[Esp19] Christian Espíndola. Infinitary first-order categorical logic. *Annals of Pure and Applied Logic*, 170(2):137 – 162, 2019.

[Esp20] Christian Espíndola. Infinitary generalizations of deligne's completeness theorem. *The Journal of Symbolic Logic*, 85(3):1147–1162, 2020.

[FG82] Michael P Fourman and Robin J Grayson. Formal spaces. In *Studies in Logic and the Foundations of Mathematics*, volume 110, pages 107–122. Elsevier, 1982.

[HL78] Karl H Hofmann and Jimmie D Lawson. The spectral theory of distributive continuous lattices. *Transactions of the American Mathematical Society*, 246:285–310, 1978.

[Joh02] Peter T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium: 2 Volume Set*. Oxford University Press UK, 2002.

[Lur04] Jacob Lurie. *Derived algebraic geometry*. PhD thesis, MIT, 2004.

[MR06] Michael Makkai and Gonzalo E Reyes. *First order categorical logic: model-theoretical methods in the theory of topoi and related categories*, volume 611. Springer, 2006.