A comonadic account of Feferman–Vaught–Mostowski theorems

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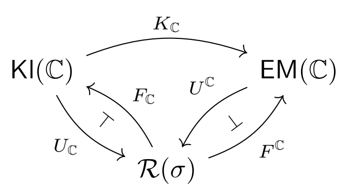
Mostowski [5] showed that, for two pairs of logically equivalent relational structures $A \equiv_{FO} B$ and $A' \equiv_{FO} B'$ in first-order logic, their cartesian products are also logically equivalent $A \times A' \equiv_{FO} B \times B'$. Then, Feferman and Vaught [2] showed that a similar statement holds for arbitrary (potentially infinite) products and coproducts, instead of just binary products. In our work we give a categorical account of these and other Feferman–Vaught–Mostowski type theorems.

Aiming for applications in finite model theory, we reformulate these theorems in the recently introduced setting of *game comonads*. Typically, for a well-behaved fragment $\mathcal{L}$ of first-order logic, there is a comonad $\mathbb{C}$ on the category $\mathcal{R}(\sigma)$ of relational structures in signature σ. It is a standard fact about comonads that we have a pair of adjunctions and a comparison functor



where $\mathsf{Kl}(\mathbb{C})$ is the Kleisli category and $\mathsf{EM}(\mathbb{C})$ is the Eilenberg–Moore category of coalgebras for $\mathbb{C}$. For our typical comonads, the objects of $\mathsf{EM}(\mathbb{C})$ can be viewed as tree-ordered relational structures and, moreover,

$$A \equiv_{\mathcal{L}} B \quad \iff \quad F^{\mathbb{C}}(A) \sim F^{\mathbb{C}}(B)$$

where $\sim$ denotes that the two structures in $\mathsf{EM}(\mathbb{C})$ are bisimilar. In fact, this bisimulation relation encodes that Duplicator/Player II has a winning strategy in the corresponding model comparison game for $\equiv_{\mathcal{L}}$. The structure of $F^{\mathbb{C}}(A)$ encodes all possible positions in this game. See [1] for a recent survey on game comonads.

Coming back to the theorem of Mostowski, we have a functor $\times\colon \mathcal{R}(\sigma) \times \mathcal{R}(\sigma) \to \mathcal{R}(\sigma)$ and we want to show that if

$$F^{\mathbb{C}}(A) \sim F^{\mathbb{C}}(B) \text{ and } F^{\mathbb{C}}(A') \sim F^{\mathbb{C}}(B') \quad \text{then also} \quad F^{\mathbb{C}}(A \times A') \sim F^{\mathbb{C}}(B \times B').$$

This indicates that we need to find a functor $\widetilde{\times} : \mathsf{EM}(\mathbb{C}) \times \mathsf{EM}(\mathbb{C}) \to \mathsf{EM}(\mathbb{C})$ which commutes with the free functors $F^{\mathbb{C}}$ and preserves the bisimulation relation $\sim$.

A suitable candidate $\widetilde{\times}$ can be found by making use of the universal property of products. However, the task becomes more interesting when we abstract away from products and allow operations of arbitrary arity. It turns out that it is more natural to consider only unary operations between possibly different categories. This subsumes the n-ary case since the pointwise

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product of n comonads on the product category is a comonad as well. To this end, assume that we have a functor $H\colon \mathcal{A} \to \mathcal{B}$ and comonads $\mathbb{C}$ and $\mathbb{D}$ on $\mathcal{A}$ and $\mathcal{B}$, respectively. As before, in order to have that

$$F^{\mathbb{C}}(A) \sim F^{\mathbb{C}}(B) \quad \text{implies} \quad F^{\mathbb{D}}(H(A)) \sim F^{\mathbb{D}}(H(B))$$

we need to find a *lifting* of H, that is, a functor $\widetilde{H}\colon \mathsf{EM}(\mathbb{C}) \to \mathsf{EM}(\mathbb{D})$ which preserves bisimulation and commutes with free functors: $F^{\mathbb{D}}(H(A)) \cong \widetilde{H}(F^{\mathbb{C}}(A))$. Observe that comonad morphisms $H\mathbb{C} \Rightarrow \mathbb{D}H$ are not suitable because the lift of H that these induce only commutes with the forgetful functors.

We take inspiration from the theory of monoidal monads (cf. [3, 6]), where the monoidal structure on the base category is lifted to the category of algebras for the monad. Perhaps surprisingly, the monoidal structure plays no role for the lift to exist. By dualising and generalising these results to our situation, we only require a Kleisli law $\mathbb{D}H \Rightarrow H\mathbb{C}$ (also known as an oplax comonad morphism) and $\mathsf{EM}(\mathbb{D})$ with equalisers of coreflexive pairs (ECP). Then the usual Kleisli lift $\widehat{H}$ of H further lifts to the categories of coalgebras:

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{F_{\mathbb{C}}} & \mathsf{Kl}(\mathbb{C}) & \xrightarrow{K_{\mathbb{C}}} & \mathsf{EM}(\mathbb{C}) \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle \widehat{H}} & & \downarrow{\scriptstyle \widetilde{H}} \\ \mathcal{B} & \xrightarrow{F_{\mathbb{D}}} & \mathsf{Kl}(\mathbb{D}) & \xrightarrow{K_{\mathbb{D}}} & \mathsf{EM}(\mathbb{D}) \end{array}$$

Another surprising feature is that the theorems of [6] about bimorphisms generalise to this setting as well. These become crucial when proving that the lifted functor $\widetilde{H}$ preserves the bisimulation relation. In fact, our conditions ensure that $\widetilde{H}$ is a parametric relative right adjoint. To summarise, we prove the following.

Theorem 1. *Let $\mathbb{C}$ and $\mathbb{D}$ be comonads on $\mathcal{R}(\sigma)$ and $\mathcal{R}(\tau)$, capturing logic fragments $\mathcal{L}$ and $\mathcal{K}$, respectively. Assume $\mathsf{EM}(\mathbb{D})$ has ECP and $H\colon \mathcal{R}(\sigma) \to \mathcal{R}(\tau)$ admits a Kleisli law $\mathbb{D}H \Rightarrow H\mathbb{C}$ which lifts H to a parametric relative right adjoint between the categories of coalgebras then*

$$A \equiv_{\mathcal{L}} B \quad \textit{implies} \quad H(A) \equiv_{\mathcal{K}} H(B).$$

Not only many Feferman–Vaught–Mostowski type theorems from the literature are a special case of this theorem but, also, this theorem becomes essential in the theory of game comonads. It allows us to compare logics, show preservation of type-equivalence by transformations, prove locality theorems, etc.

References

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