Category-theoretic Fraïssé theory: an overview

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Abstract

In the talk I will give an overview of countable abstract Fraïssé theory formulated in the language of category theory. We start with a core setup (free completion, characterization of the Fraïssé limit, existence of a Fraïssé sequence), demonstrate it on several examples, and then we sketch further directions in which the core setup can be extended.

Recall that a first-order structure U is *ultrahomogeneous* if every isomorphism $f\colon A \to B$ between finitely generated substructures $A, B \subseteq U$ can be extended to an automorphism $\tilde{f}\colon U \to U$. Most classical countable ultrahomogeneous structures include the *linear order of rationals*, the *random graph*, and the *rational Urysohn metric space*. Study of countable ultrahomogeneous structures goes back to Fraïssé [5] and so is sometimes called *Fraïssé theory*. Model-theoretic treatment is now classical, see e.g. [6].

In 2006, Irwin and Solecki [7] introduced *projective Fraïssé theory*, where instead of embeddings of first-order structures, quotients of topological structures are considered. The (projectively) homogeneous structure is obtained as a limit of an inverse sequence of quotient maps, instead of taking the union of an increasing chain. The particular limit obtained by Irwin and Solecki was the Cantor space endowed with a special closed equivalence relation with the quotient space being the *pseudo-arc*, a well-known continuum. Since then, many continua were realized as quotients of projective Fraïssé limits, see e.g. [11], [2].

It is natural to formulate Fraïssé theory using the language of category theory. This allows for clear and general proofs capturing the essence of the constructions involved. Extra structure like the induced topology of the automorphism group of the Fraïssé limit also arises naturally. Such treatment of Fraïssé theory provides a unified framework: there is essentially no difference between classical Fraïssé theory of first-order structures and projective Fraïssé theory of topological structures. It also provides flexibility: we can easily consider other morphisms than embeddings like left-invertible embeddings, embedding-projection pairs, relational morphisms, or abstract elements of a monoid.

Category-theoretical Fraïssé theory was pioneered by Droste and Göbel [4] who started with a *semi-algebroidal* category of "large" objects $\mathcal{L}$, proved the uniqueness of an $\mathcal{L}$-object homogeneous over the full subcategory $\mathcal{L}_{\text{fin}}$ of *finite objects*, and characterized its existence in the case when $\mathcal{L}_{\text{fin}}$ is essentially countable. On the other hand, Kubiś [9] started with a category of "small" objects $\mathcal{K}$ and introduced the notion of *Fraïssé sequence* in $\mathcal{K}$ (also of uncountable length), which serves as the Fraïssé limit in the category of sequences $\sigma\mathcal{K}$. In applications, $\sigma\mathcal{K}$ is identified with a particular category of large structures we are interested in. Existence of a Fraïssé sequence is closely connected to the notion of *dominating subcategory*, also introduced in [9]. The two views can be combined by working with a pair of categories $\mathcal{K} \subseteq \mathcal{L}$, as done by Caramello [3].

In the talk I shall give an overview of a polished framework. An $\mathcal{L}$-object U is *homogeneous* in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every pair of $\mathcal{L}$-maps from a $\mathcal{K}$-object $f, g\colon x \to U$ there is an automorphism

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$h\colon U \to U$ such that $h \circ g = f$. We say that $\langle \mathcal{K}, \mathcal{L} \rangle$ is a *free completion* (or more precisely, free sequential cocompletion) if $\mathcal{L}$ essentially arises from $\mathcal{K}$ by freely adding colimits of $\mathcal{K}$-sequences. The core of countable abstract Fraïssé theory can be summarized by the following two theorems.

Theorem (Characterization of the Fraïssé limit)**.** Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion and let U be an $\mathcal{L}$-object. Then the following are equivalent.

(1) U is cofinal and homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$,

(2) U is cofinal and injective in $\langle \mathcal{K}, \mathcal{L} \rangle$,

(3) U is the $\mathcal{L}$-colimit of a Fraïssé sequence in $\mathcal{K}$.

Moreover, such U is unique and cofinal in $\mathcal{L}$, and every $\mathcal{K}$-sequence with $\mathcal{L}$-colimit U is Fraïssé in $\mathcal{K}$. Such U is called the *Fraïssé limit* of $\mathcal{K}$ in $\mathcal{L}$.

Theorem (Existence of a Fraïssé sequence)**.** Let $\mathcal{K}$ be a category. Then $\mathcal{K}$ has a Fraïssé sequence if and only if $\mathcal{K}$ is a *Fraïssé category*, i.e. $\mathcal{K} \neq \emptyset$ and

(1) $\mathcal{K}$ is directed,

(2) $\mathcal{K}$ has the amalgamation property,

(3) $\mathcal{K}$ has a countable dominating subcategory.

After explaining the core setup, we demonstrate it on several examples and see how it encompasses classical and projective situations, namely, how $\langle \mathcal{K}, \mathcal{L} \rangle$ being a free completion is verified in applications. Then we sketch several directions in which the core setup can be extended: *weak Fraïssé theory* [10], *metric-enriched categories* [8] and *MU-categories* [1], and *self-generic Fraïssé limits* in situations beyond free completion (joint work in progress with Matheus Duzi Ferreira Costa).

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