Constructing conical and perfect residuated lattices

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Residuated structures play an important role in the field of algebraic logic since they constitute the equivalent algebraic semantics, in the sense of Blok and Pigozzi, of substructural logics (see [2, 3]). These encompass many of the interesting nonclassical logics: intuitionistic logic, fuzzy logics, relevance logics, linear logics and also classical logic as a limit case. Thus, the algebraic investigation of residuated lattices is a powerful tool in the systematic and comparative study of such logics. While many deep results have been obtained in the last decades, the multitude of different kinds of residuated lattices and their rich theory makes their study fairly complicated, and at the present moment large classes of residuated lattices lack a structural description. Because of this, the development of constructions that allow one to obtain new structures from known ones is of utter importance for the understanding of both residuated lattices and substructural logics as a whole.

In this contribution we introduce two constructions that, starting from two algebras, obtain respectively a conical commutative residuated lattice and a perfect (bounded) commutative residuated lattice; the underlying idea is on one side to generalize the constructions in [4] (used by the authors to provide a description of conical idempotent residuated lattices), and on the other side to generalize the generalized disconnected rotation construction (see e.g. [1]).

To be more clear let us first recall some important definitions. A *commutative residuated lattice* is an algebra $\mathbf{A} = (A, \cdot, \to, \wedge, \vee, 1)$ of type $(2, 2, 2, 2, 0)$ where: $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a commutative monoid, and the residuation law holds, i.e. $x \cdot y \leq z$ if and only if $y \leq x \to z$ for any $x, y, z \in A$. A residuated lattice $\mathbf{A}$ is: *integral* if it has a maximum element which coincides with the monoidal unit 1; *conical* if the unit 1 is a conical element, i.e. for each $a \in A$, either $a \leq 1$ or $1 \leq a$. Moreover, we call *bounded* an integral commutative residuated lattice with an extra constant 0 that is the smallest element; we call a bounded commutative residuated lattice *perfect* if it can be seen as the disjoint union of a congruence filter F and the set $\{a \in F : x \to 0 \in F\}$.

Let us now discuss the wanted constructions. We start from a commutative integral residuated lattice $\mathbf{A}$ and an algebra $\mathbf{B}$ with some properties. To be more precise, let us endow both $\mathbf{A}$ and $\mathbf{B}$ with a closure operator γ such that $\gamma(x)$ is a conical and idempotent element for any x. Let then $\gamma[\mathbf{A}]$ and $\gamma[\mathbf{B}]$ be the set of its fix points; one can observe that the set of elements having the same γ-image can be seen as a bubble with the γ-fixed point as top (see the image below for a pictorial intuition). Hence let one start form a commutative integral residuated lattice $\mathbf{A}$ which is the ordinal sum of its bubbles and from an algebra $\mathbf{B}$ such that any $x \in B$ is above 1 and $\mathbf{B} \cup \{\perp, \top\}$ is a residuated lattice where the product between elements of different bubbles is the join. Let then $\gamma_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}$ and $\gamma_{\mathbf{B}} : \mathbf{B} \to \mathbf{B}$ be such maps.

Observe that $\gamma_{\mathbf{A}}[\mathbf{A}]$ and $\gamma_{\mathbf{B}}[\mathbf{B}]$ are isomorphic; let us denote with $'$ such isomorphism and its inverse.

We will now define new operations on the domain $\mathbf{A} \cup \mathbf{B}$; in order to do so, we extend the map $'$ to be a map from $\mathbf{A} \cup \mathbf{B}$ to $\mathbf{A} \cup \mathbf{B}$ is a way such that: for any $x \in A$, $x' = (\gamma_{\mathbf{A}}(x))'$ and for any $x \in B$, $x' = (\gamma_{\mathbf{B}}(x))'$. We call such a map $'$ a *complementation* of $\mathbf{A}$ and $\mathbf{B}$.

Now, for the first construction the idea is to copy $\mathbf{B}$ above $\mathbf{A}$, and consider a new product that extends the one of $\mathbf{A}$ and $\mathbf{B}$ in the following way: if $x \in A$ and $y \in B$ $x \cdot y = x$ if $x \leq y'$, and $x \cdot y = y$ otherwise. We show that such product is residuated and yields a conical residuated lattice, $\mathbf{C}_{\mathbf{A},\mathbf{B}}$. For the second construction, we instead copy $\mathbf{B}$ below $\mathbf{A}$; the product remains the old one on $\mathbf{A}$, $x \cdot y = x$ if $x \in B, y \in A$ and $x' > y$, and it is 0 in all other cases. We demonstrate that such product is residuated, and obtain the corresponding perfect residuated lattice, $\mathbf{P}_{\mathbf{A},\mathbf{B}}$. The following figure sketches the two constructions.

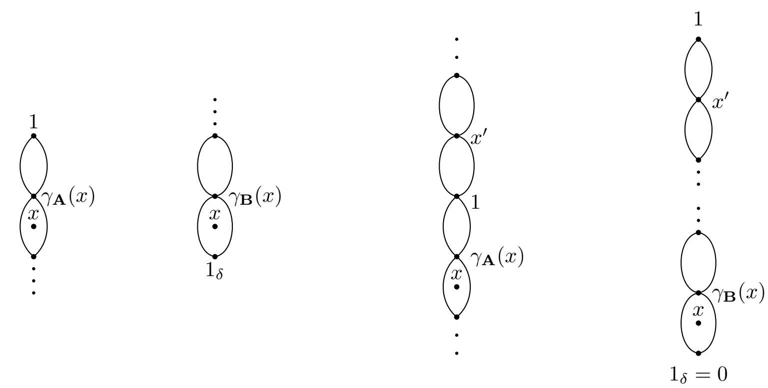

Figure 1: From the left: $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}_{\mathbf{A},\mathbf{B}}$, $\mathbf{P}_{\mathbf{A},\mathbf{B}}$.

Starting from these constructions, we develop and study a connection between subclasses of conical and perfect commutative residuated lattices.

References

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