Domains arising in operator algebras

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Abstract

Continuous dcpos naturally arise in the study of operator algebras as families of certain modules, equipped with the partial order given by inclusion. These domains play a crucial role in noncommutative topology.

I will briefly explain the construction of these domains, give some important examples, and then present a recent structure result showing that in many cases the domains are semilattices.

Background

In 2008, a surprising connection between domain theory and operator algebras was discovered [CEI08], see also [Kei17]: The isomorphism classes of countably generated Hilbert modules over a C^*-algebra A, equipped with the partial order induced by the inclusion of closed submodules, is a domain (a continuous dcpo). Moreover, direct sum of modules defines an abelian addition on this domain, turning it into a domain semigroup. The construction goes back to Cuntz [Cun78], and the said domain semigroup is therefore called the *Cuntz semigroup* $\mathrm{Cu}(A)$.

A C^**-algebra* is a norm-closed $*$-algebra of operators on a Hilbert space. These are often thought of as noncommutative topological spaces, since a C^*-algebra is commutative if and only if it is isomorphic to $C(X) = \{f \colon X \to \mathbb{C} | f \text{ continuous}\}$ for some compact, Hausdorff space X.

For many C^*-algebras, the Cuntz semigroup can be computed explicitly. For example, if $A = \mathbb{C}$, then countably generated Hilbert modules are nothing but separable Hilbert spaces, and these are characterized by their dimension, which gives

$$\mathrm{Cu}(\mathbb{C}) \cong \overline{\mathbb{N}} := \{0, 1, 2, 3, \ldots, \infty\}.$$

For the C^*-algebra $A = C([0, 1])$, every countably generated Hilbert modules is a bundle of separable Hilbert spaces over the base space $[0, 1]$, with the dimension of the fibers varying lower-semicontinuously, which gives

$$\mathrm{Cu}(C([0, 1])) \cong \mathrm{Lsc}([0, 1], \overline{\mathbb{N}}).$$

Other examples of domain semigroups arising as the Cuntz semigroup of a C^*-algebra are $[0, \infty]$ (with the usual addition and order), and the semigroup $\mathrm{LAff}(K)_{++} \cup \{0\}$ of lower-semicontinuous, affine functions $K \to (0, \infty]$ for a Choquet simplex K.

Since 2008, domain semigroups have been studied extensively in the context of C^*-algebras. In particular, the author and collaborators have shown that domain semigroups form a closed, symmetric monoidal category [APT18, APT20].

A C^*-algebra is said to have *stable rank one* if its invertible operators are norm-dense, a property that is known to be equivalent to the ring-theoretic notion of Bass stable range one.

Stable rank one is a finiteness assumption that is automatically satisfied in many situations of interest. Recently [APRT22], a deep structure result was shown for the domain semigroups arising from such C^*-algebras:

Theorem 1. *Given a C^*-algebra A with stable rank one, the Cuntz semigroup $\mathrm{Cu}(A)$ has the Riesz interpolation property, that is, whenever countably generated Hilbert modules E_1, E_2 and F_1, F_2 satisfy*

$$E_j \hookrightarrow F_k$$

for $j = 1, 2$ and $k = 1, 2$, then there exists a countably generated Hilbert module G such that

$$E_j \hookrightarrow G \hookrightarrow F_k$$

for $j = 1, 2$ and $k = 1, 2$.

This shows that for two elements x and y in $\mathrm{Cu}(A)$, the set of lower bounds for $\{x, y\}$ is upward directed and therefore has a supremum, which means that the infimum $x \wedge y$ exists. Thus, the Cuntz semigroup not only has the structure of a domain semigroup, but it is also a continuous inf-semilattice.

References

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