Inception Display Calculi

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In previous years, a formal connection between correspondence phenomena [7] and the theory of display calculi [1] was established, applying results and insights from unified correspondence theory [3]. One of the consequences of the aforementioned connection was the development of proper display calculi for the class of LE-logics [4], together with a method to convert a broad class of axioms (the class of all the analytic inductive inequalities) into rules that can be modularly added to the base calculus without disrupting the admissibility of the cut rule [5].

In this work we extend the framework of proper display calculi for LE-logics to include axiomatic extensions with axioms that are inductive [4] but not necessarily analytic inductive, greatly extending the class of axioms that can be converted into analytic rules. This class covers and properly extends all Sahlqvist axioms. A semantical analysis of the first-order correspondent of inductive inequalities suggests an approach that is similar in nature to that of Schroeder-Heister's Calculus of Higher-level Rules [8], and captures the whole acyclic portion of the substructural hierarchy [2], meaning that we can cope with arbitrary alternations of box-like and diamond-like connectives, as long as certain acyclicity conditions are satisfied.

Our approach is somewhat reminiscent of Negri's systems of rules [6], with the difference that no labelled G3c-like calculus is available for LE-logics, with the consequence that previously existing results cannot be applied to the case at hand. We make use of unified correspondence theory and the algorithm ALBA [4] to uniformly generate analytic rules for the previously mentioned inductive axiomatic extensions, and we call our new framework *Inception Display Calculus*.

Definition of the Inception Display Calculus framework

Inception Display calculi introduce special *side conditions* to the rules of proper display calculi.

Definition 0.1. Let $\mathcal{R}$ and $\mathcal{X}$ be a set of analytic structural rules (see [5]) and a set of structure variables, respectively. If $\Pi \vdash \Sigma$ is derivable using the rules of the base calculus together with $\mathcal{R}$, where $\Pi, \Sigma, \mathcal{R}$ may contain structure variables from $\mathcal{X}$, we write $[\Pi \vdash \Sigma]^{\mathcal{R}}_{\mathcal{X}}$ and we call it a *shallow contract*. A *shallow inception rule* is an analytic structural rule augmented with one or more shallow contracts as side conditions, namely a rule of the following form:

$$\frac{X_1 \vdash Y_1 \quad \cdots \quad X_n \vdash Y_n \qquad [\Pi_1 \vdash \Sigma_1]^{\mathcal{R}_1}_{\mathcal{X}_1} \quad \cdots \quad [\Pi_m \vdash \Sigma_m]^{\mathcal{R}_m}_{\mathcal{X}_m}}{X \vdash Y}$$

Sometimes we write $[\pi]^{\mathcal{R}}_{\mathcal{X}}$ in place of $[\Pi \vdash \Sigma]^{\mathcal{R}}_{\mathcal{X}}$, where π is a derivation of $\Pi \vdash \Sigma$, omitting subscripts and superscripts when they are clear from the context.

Definition 0.2. Let us define inductively *depth-n inception rules* ($n \geq 0$) and *depth-n contracts* ($n \geq 1$).

- Depth-0 inception rules are the analytic structural rules; depth-1 contracts are the shallow contracts and depth-1 inception rules are the shallow inception rules.
- Suppose we defined depth-k inception rules and contracts for every $k < n$, for a certain $n > 1$. A depth-n contract is a side condition of the form $[\Pi \vdash \Sigma]^{\mathcal{R}}_{\mathcal{X}}$, where $\mathcal{R}$ is a set of inception rules of depth smaller than n. A depth-n inception rule is an analytic structural rule augmented with contracts of depth not greater than n.

An *inception rule* (resp. a *contract*) is a depth-n inception rule (resp. depth-n contract) for some $n \geq 1$. We also say that its depth is n. A derivation has *finite dreams* if it contains a finite number of instances of contracts.

In this work, we consider only derivations with finite dreams. As an example, consider the inductive but not analytic inductive axiom $\Box(\Diamond p \circ p) \circ \Diamond p \vdash p$.

	ALBA run computing the inception rule for $\Box(\Diamond p \circ p) \circ \Diamond p \vdash p$:
	$\Box(\Diamond p \circ p) \circ \Diamond p \leq p$
iff	$\forall p \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m}[\mathbf{i} \leq \Box(\Diamond p \circ p) \ \&\ \mathbf{j} \leq p \ \&\ p \leq \mathbf{m} \Rightarrow \mathbf{i} \circ \Diamond \mathbf{j} \leq \mathbf{m}]$
iff	$\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m}[\mathbf{i} \leq \Box(\Diamond \mathbf{m} \circ \mathbf{m}) \ \&\ \mathbf{j} \leq \mathbf{m} \Rightarrow \mathbf{i} \circ \Diamond \mathbf{j} \leq \mathbf{m}]$
iff	$\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m}[\forall \mathbf{n}(\Diamond \mathbf{m} \circ \mathbf{m} \leq \mathbf{n} \Rightarrow \mathbf{i} \leq \Box \mathbf{n}) \ \&\ \mathbf{j} \leq \mathbf{m} \Rightarrow \mathbf{i} \circ \Diamond \mathbf{j} \leq \mathbf{m}]$
iff	$\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m}[\forall \mathbf{n}(\forall \mathbf{k} \forall \mathbf{h}(\mathbf{k} \leq \mathbf{m} \ \&\ \mathbf{h} \leq \mathbf{m} \Rightarrow \Diamond \mathbf{k} \circ \mathbf{h} \leq \mathbf{n}) \Rightarrow \mathbf{i} \leq \Box \mathbf{n}) \ \&\ \mathbf{j} \leq \mathbf{m} \Rightarrow \mathbf{i} \circ \Diamond \mathbf{j} \leq \mathbf{m}]$

The last line of the derivation above gives us the first-order correspondent of $\Box(\Diamond p \circ p) \circ \Diamond p \vdash p$, from which we can obtain the depth-1 inception rule

$$R_0 \ \frac{Y \vdash Z \qquad [X \vdash \check{\Box} N]^{\mathcal{R}}_{\{N\}}}{X \mathbin{\hat{\circ}} \hat{\Diamond} Y \vdash Z}$$

where $\mathcal{R}$ is the singleton containing

$$R_1 \ \frac{K \vdash Z \qquad H \vdash Z}{\hat{\Diamond} K \mathbin{\hat{\circ}} H \vdash N} \ .$$

We show how to derive the axiom $\Box(\Diamond p \circ p) \circ \Diamond p \vdash p$ from the rule just obtained, where adjunction rules are omitted for brevity:

$$R_0 \ \cfrac{\cfrac{\cfrac{p \vdash p \qquad [\pi]^{\mathcal{R}}_{\{N\}}}{\Box(\Diamond p \circ p) \mathbin{\hat{\circ}} \hat{\Diamond} p \vdash p}}{\Box(\Diamond p \circ p) \mathbin{\hat{\circ}} \Diamond p \vdash p}}{\Box(\Diamond p \circ p) \circ \Diamond p \vdash p} \text{, where } \pi \text{ is:} \qquad R_1 \ \cfrac{\cfrac{\cfrac{\cfrac{p \vdash p \qquad p \vdash p}{\hat{\Diamond} p \mathbin{\hat{\circ}} p \vdash W}}{\Diamond p \mathbin{\hat{\circ}} p \vdash W}}{\Diamond p \circ p \vdash W}}{\Box(\Diamond p \circ p) \vdash \check{\Box} W}$$

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