The logic of vague categories

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Categories are cognitive tools humans use to make sense of the world, and interact with it and with each other. They are key to the development and use of language, the construction of knowledge and identity, the formation of evaluation, and decision-making. The literature on categorization is expanding rapidly in fields ranging from cognitive linguistics to social science, management science and AI.

A key issue to the development of the foundations of categorization theory concerns the formalization of the *vague nature* of categories. While mathematical concepts such as 'prime number' or 'circle' have a precise extension, this is not so for concepts such as 'red', 'tall', 'heap' or 'house'. *Vague categories* and concepts admit borderline cases, namely cases for which it is not clear whether the concept should apply or not. For instance, where is the limit between dark blue and light blue? Is a certain object blue, or is it grey or green? The absence of clear-cut boundaries between categories is the main reason why, in most real-life categorization processes, objects are assigned to more than one category, giving rise to the phenomenon of *category-spanning*, which has important consequences on decision-making.

Rough set theory [16] provides the starting point of the formal approach to vagueness proposed in the present contribution, since it accounts for the absence of clear-cut categorical boundaries via the interval induced by the *upper* and *lower approximations* of sets and predicates, arising from an indiscernibility relation on a domain of discourse. In [10], these insights have been extended to the formal environment of *conceptual approximation spaces*, a common generalization of Pawlak's approximation spaces and Wille's formal contexts (aka polarities) [14], on which the present contribution directly builds.

Specifically, the present contribution continues a line of research aimed at introducing and studying logical frameworks specifically designed to reason about categories and categorization, and at using these logics to formalize notions and analyze problems involving categorization arising across disciplines. In [7], building on the general mathematical framework for non-distributive logics developed in [12, 11], the basic normal non-distributive modal logic and some of its axiomatic extensions are interpreted as *epistemic logics of categories and concepts*, and in [8], the corresponding 'common knowledge'-type construction is used to give an epistemic-logical formalization of the notion of *prototype* of a category; in [10, 15], conceptual approximation spaces are proposed as a relational semantics for non-distributive modal logic, which, being interpreted in this context as the logic of *rough concepts*, serves as an encompassing framework for the integration of Rough Set Theory [16] and Formal Concept Analysis (FCA) [14]. Other different but closely related semantics for non-distributive modal logic have been introduced and explored in [4, 6], and generalized to the many-valued semantic setting [5, 13].

In this contribution, building on Bělohlávek's framework of fuzzy formal concepts [1, 2], we present the mathematical and conceptual investigation of the *many-valued polarity-based* relational semantics for non-distributive modal logic. This framework has been initially investigated in [10, Section 7.2]. Further developments in the direction of correspondence theory have been developed in [9], and in [3] it has been applied in the development of unsupervised learning algorithms for outlier detection that also provide explanations of their results.

In our presentation we will discuss the many-valued non-distributive modal logics described above. We will introduce many-valued enriched formal contexts; introduce the semantics and proof theory for the logics; expand on the completeness of this logic; and present results in correspondence and duality in this context. Finally, we will present a generalization of this framework to a framework where the algebra of values is a non-commutative quantale. We will discuss how this shift affects the aforementioned notions and present some further results on correspondence and completeness.

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