Finitely Weighted Kleene Algebra With Tests

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Kleene algebras, going back to [2], are algebraic structures central to automata theory, semantics of programs, and theoretical computer science in general. Kozen [4] has shown that the equational theory of Kleene algebras is complete with respect to the model based on regular languages. Kozen [5] introduces Kleene algebras with tests, a combination of Kleene algebras (programs) and Boolean algebras (tests), and shows that they have non-trivial applications in verification of imperative programs. Ésik and Kuich [3] generalize Kozen's completeness result for Kleene algebras to the case of weighted regular languages, or formal power series. In particular, their result applies to a weighted generalization of Kleene algebras where the semiring of weights is finite, commutative, zero-bounded (or positive) and partially ordered.

We establish two completeness results for a weighted generalization of Kleene algebras with tests. First, we establish completeness with respect to the algebra of weighted guarded languages using a reduction to weighted regular languages similar to the one used by Kozen and Smith [6] in their completeness proof for (non-weighted) Kleene algebras with tests. Second, we establish completeness with respect to weighted transition systems by using a Cayley-like construction going back to Pratt's work [7] on (non-weighted) dynamic algebras. In addition to the assumptions of Ésik and Kuich, however, we need to assume that the semiring of weights is also integral. These results are interesting also because of the connection between weighted Kleene algebras with tests and weighted programs [1], noted in our earlier work [8]. We also argue that finitely weighted Kleene algebras with tests are a natural framework for equational reasoning about weighted programs in cases where an upper bound on admissible weights is assumed.

A *Kleene algebra* [4] is an idempotent semiring X with a unary operation * satisfying, for all $x, y, z \in X$ the following *unrolling* (left column) and *fixpoint* laws (right column):

$$1 + (x \cdot x^*) = x^* \qquad\qquad y + (x \cdot z) \leq z \implies x^* \cdot y \leq z \tag{1}$$

$$1 + (x^* \cdot x) = x^* \qquad\qquad y + (z \cdot x) \leq z \implies y \cdot x^* \leq z\,. \tag{2}$$

a Kleene algebra with tests [5] is a Kleene algebra X with a distinguished $B \subseteq X$ such that $\langle B, +, \cdot, 0, 1\rangle$ is a subalgebra of X and a bounded distributive lattice, and $^-$ is an unary operation on B such that $x \cdot \bar{x} = 0$ and $x + \bar{x} = 1$ for all $x \in B$. Hence, B forms a Boolean algebra. Intuitively, elements of B represent *Boolean tests.* ("If b then x else y" can be expressed as $bx + \bar{b}y$ and "While b do x" as $(bx)^*\bar{b}$; partial correctness is expressed by $bx\bar{c} = 0$.)

Definition 1. *Let S be a finite semiring. A* Kleene S-algebra with tests *is a Kleene algebra with tests X together with a binary operation $\odot : X \times S \to X$ such that (the additive monoid reduct of) X forms a right S-semimodule and*

$$(xy) \odot s = x(y \odot s) = (x \odot s)y \qquad\qquad 1 \odot s^* \leq (1 \odot s)^*$$

Similar to Kleene algebras with tests, the algebraic language for Kleene S-algebras with tests is two-sorted, consisting of *tests* and *expressions*:

$$b, c := \mathtt{p} \mid \bar{b} \mid b + c \mid b \cdot c \mid \mathtt{0} \mid \mathtt{1} \qquad\qquad e, f := \mathtt{a} \mid b \mid e \odot s \mid e + f \mid e \cdot f \mid e^*$$

where $\mathtt{p} \in \Phi$ (a finite set of proposition letters), $\mathtt{a} \in \Sigma$ (a finite set of program letters) and $s \in S$. Expression $e \odot s$ means "execute e and add s to the weight of the current computation".

An atom over Φ is a finite sequence of literals over Φ containing exactly one of $\mathtt{p}$ and $\bar{\mathtt{p}}$ for each $\mathtt{p} \in \Phi$. A guarded string is a string of the form $G_1 \mathtt{a}_1 G_2 \ldots \mathtt{a}_{n-1} G_n$ where the G's are atoms and the $\mathtt{a}$'s are program letters. Fusion product $wG \diamond Hu$ of guarded strings is undefined if $G \neq H$ and wGu otherwise. The set of *guarded formal power series* over a finite semiring S is the set of mappings from the set of guarded strings to S. The *rational operations* on guarded f.p.s. are defined point-wise as follows:

$$(r_1 + r_2)(w) = r_1(w) + r_2(w) \qquad (r_1 \cdot r_2)(w) = \sum \{r_1(v_1) \cdot r_2(w_2) \mid w = v_1 \diamond v_2\}$$
$$(r \odot s)(w) = r(w) \cdot s \qquad r^*(w) = \sum_{n \in \omega} r^n(w)$$

where $r^0 = 1$ and $r^{n+1} = r^n \cdot r$. (Note that the sum is defined since S is assumed to be finite.) A *polynomial* is any guarded f.p.s. r such that the set of guarded strings w where $r(w) \neq 0$ is finite. The set of *rational guarded f.p.s.* is the least set of guarded f.p.s. that contains all polynomials and is closed under the rational operations. The set of rational guarded f.p.s. over S forms a Kleene S-algebra with tests.

Theorem 1. *The equational theory of Kleene S-algebras with tests coincides with the equational theory of the algebra of rational guarded f.p.s. over S.*

An S-transition system is a set with a collection of S-weighted binary relations $M(\mathtt{a})$ on the set for $\mathtt{a} \in \Sigma$ and $\{0, 1\}$-weighted diagonal relations $M(\mathtt{p})$ for $\mathtt{p} \in \Phi$. Binary relations $M(e)$ for arbitrary expressions are defined as expected using familiar matrix operations.

Theorem 2. *An equation $e \approx f$ is valid in all Kleene S-algebras with tests iff $M(e) = M(f)$ in all S-transition systems.*

References

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