On Boolean Topos Constructions by Freyd and Pataraia and their generalizations

Evgeny Kuznetsov*

TSU Andrea Razmadze Mathematical Institute
e.kuznetsov@freeuni.edu.ge

In 1992, A. Pitts proved a somewhat surprising result about Heyting's intuitionistic propositional calculus, IPC [3]. He showed that for each variable p and formula ϕ in IPC there exists a formula $A_p\phi$ (effectively computable from ϕ), containing only variables distinct from p which occur in ϕ, and such that for all formulas ψ not involving p, $\vdash \psi \Rightarrow A_p\phi$ if and only if $\vdash \psi \Rightarrow \phi$. Here, $\vdash$ denotes validity in IPC.

This in particular means that one can model quantification over propositional variables in IPC, which provides an interpretation of the second-order intuitionistic propositional calculus IPC^2 in IPC.

As a corollary, A. Pitts showed that a model of IPC^2 can be constructed with an algebra of truth values isomorphic to any given Heyting algebra. In [3] he also asked whether his result can be generalized further to higher order calculi.

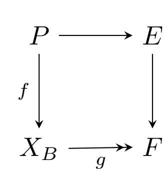
This question can also be reformulated in topos-theoretic terms, asking whether every Heyting algebra occurs as the algebra of all subobjects of an object in a topos.

For the case when the Heyting algebra in question is in fact Boolean, the affirmative answer is the contents of Exercise 9.11 in [1]. The explicit construction of the corresponding topos $\mathcal{F}_B$ is sketched there; Johnstone attributes it to Peter Freyd.

Specifically, [1, Exercise 9.11] suggests expressing a Boolean algebra B as the (directed) union of its finite subalgebras, utilizing the fact that Boolean algebras are locally finite (finitely generated Boolean subalgebras are finite). Then one can describe the topos corresponding to B as a colimit of a directed diagram of toposes and logical functors between them, corresponding to finite subalgebras $B_0 \subseteq B$. Each B_0 is isomorphic to the powerset of the set $\mathrm{at}(B_0)$ of its atoms, and the corresponding topos is $\mathbf{Fin}^{\mathrm{at}(B_0)}$, the product of $\mathrm{at}(B_0)$ many copies of the topos **Fin** of finite sets.

We learned from the late D. Pataraia an alternative construction of what turns out to be an equivalent topos $\mathcal{L}_B$. Namely, using the Stone duality for Boolean algebras, he considered certain explicitly described subcategory of local homeomorphisms over the Stone space $X = X_B$ dual to the Boolean algebra B. Domains of his local homeomorphisms have form $(m_1 \times U_1) \sqcup \ldots \sqcup (m_n \times U_n)$, where $U_1, \ldots, U_n$ are disjoint clopen subsets of X forming a partition of X, and $m_1, \ldots, m_n$ are finite discrete spaces (can be assumed to be of pairwise distinct cardinalities). We have not heard about this kind of construction from anybody else.

To demonstrate that for a given Boolean algebra B the topos $\mathcal{F}_B$ by Freyd and the topos $\mathcal{L}_B$ by Pataraia are isomorphic, we consider a third, intermediate category $\mathcal{M}_B$. The objects of this category are pullbacks of the form shown on the right, where $E \to F$ is any map between finite discrete topological spaces, and g is any surjective continuous map from the Stone space X_B of B to F.



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Note that later, D. Pataraia invented an entirely different construction of a topos, proving that for every Heyting algebra H, there exists a topos with the algebra of subterminals isomorphic to H [2]. However, this work was never published.

In the present work, we generalize the constructions by Freyd and Pataraia and apply the resulting generalization to some classes of Heyting algebras beyond the classes of Boolean algebras and complete Heyting algebras. Important rôle in our investigations plays the notion of *coherent object*, which we recall here.

Let A be an object of a category $\mathbf{C}$ with finite limits.

- A is *compact* if every jointly epimorphic family of subobjects of A admits a finite jointly epimorphic subfamily.
- A is *stable* if, for every pair of morphisms $U \to A \leftarrow V$ with U and V compact, the pullback $U \underset{A}{\times} V$ is compact as well.
- A is *coherent* if it is both compact and stable.

Here, a family of morphisms $(e_i\colon U_i \to A)$ is called *jointly epimorphic* if, given any two morphisms $g, h\colon A \to B$ such that $g \circ e_i = h \circ e_i$ for all i, it follows that $g = h$.

We will use coherent objects to characterize the above three categories by proving the following theorem.

Theorem. *For a given Boolean algebra B, the toposes $\mathcal{F}_B$, $\mathcal{L}_B$ and $\mathcal{M}_B$ described above are equivalent to the subcategory $\mathcal{C}$ of coherent objects in the category of sheaves $\mathbf{Sh}(X_B)$ over the Stone space X_B associated with the Boolean algebra B.*

Note in particular that coherent objects of $\mathbf{Sh}(X_B)$ form a Boolean topos.

In the talk we will discuss possible generalizations to some other classes of Heyting algebras.

One can describe the category of sheaves $\mathbf{Sh}(\mathrm{Spec}(H))$ over the spectral space $\mathrm{Spec}(H)$ corresponding to H in order-topological terms, as certain Esakia spaces over X_H. We will use this description to study analogs of the above three categories, and relate them to the subcategory of coherent objects in $\mathbf{Sh}(\mathrm{Spec}(H))$.

Finally, we consider the case of locally finite algebras and employ the latter construction of taking coherent objects in the corresponding categories of sheaves. Instead of pullbacks of maps between finite sets we will need pullbacks of local homeomorphisms between finite topological spaces. In this case the corresponding inclusion functors are no longer logical.

In the talk we will address several related questions, namely, when do coherent objects of a topos form a topos, and which spectral spaces can be obtained as inverse limits of directed diagrams of local homeomorphisms between finite spaces.

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References

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