The finite model property for lattice based $\mathsf{S4}$

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Distributive modal logics based on classical, positive and intuitionistic logics have been thoroughly investigated (see e.g. [3, 4, 10]). Non-distributive modal logics have received less attention, even though they contain important logics such as quantum logic [5, 8] and substructural logics [7]. Recently the duality and Sahlqvist theory of non-distributive modal logics were studied in [2, 9, 6]. In this abstract we investigate the finite model property of non-distributive modal logics, including a non-distributive version of $\mathsf{S4}$, from an algebraic perspective.

Let $\ell\mathsf{S4}_\Box$ be the logic whose algebraic semantics is given by lattices with a $\Box$ satisfying:

$$\Box 1 \approx 1, \qquad \Box(a \wedge b) \approx \Box a \wedge \Box b, \qquad \Box a \leq a, \qquad \Box a \leq \Box\Box a.$$

Theorem 1. *The logic $\ell\mathsf{S4}_\Box$ has the finite model property.*

Proof. Let A be an algebra with valuation $\sigma\colon Fm \to A$ such that $\sigma(\phi) \neq \sigma(\psi)$ (i.e. $A \not\models \phi \approx \psi$). We construct a finite algebra B such that $B \not\models \phi \approx \psi$. Let Σ be the set of subformulas of ϕ and ψ. Define B to be the smallest $0, 1, \Box, \wedge$-subreduct of A containing $\sigma[\Sigma]$. Then B is finite because $\Box$ is normal and $\mathsf{S4}$. Hence it is complete, so we can define a join in B as

$$a \vee_B b = \bigwedge\{c \in B \mid c \geq a, b\}.$$

One easily checks that if $a, b, a \vee b \in B$, then $a \vee_B b = a \vee b$. Therefore, we can define a valuation $\tau\colon Fm \to B$ by setting $\tau(\chi) = \sigma(\chi)$ if $\chi \in \Sigma$, and extending it to Fm in the natural way. This is well-defined. Indeed, if $\alpha \vee \beta \in \Sigma$, then

$$\tau(\alpha \vee \sigma) = \sigma(\alpha \vee \sigma) = \sigma(\alpha) \vee \sigma(\beta) = \tau(\alpha) \vee \tau(\beta) = \tau(\alpha) \vee_B \tau(\beta),$$

since $\tau(\alpha), \tau(\beta), \tau(\alpha) \vee \tau(\beta) \in B$. The valuation τ is such that $\tau(\phi) \neq \tau(\psi)$. Therefore $\phi \approx \psi$ can be refuted in a finite algebra. $\Box$

We highlight the difference with the classical cases. When proving the finite model property for classical modal logic, one would take B to be the Boolean algebra generated by Σ, and define a suitable box on it. In our case, we cannot consider the lattice generated by Σ, as it could be infinite. Instead, we generate B as a meet-semilattice. Dropping joins from the generating set allows us to add box instead (provided that it is $\mathsf{S4}$), which simplifies the construction.

Next we add a monotone diamond, in line with [2, Section 4], which satisfies the following:

$$\Diamond 0 \approx 0, \qquad \Diamond(a \vee b) \geq \Diamond a \vee \Diamond b.$$

The resulting logic is denoted by $\mathcal{L}_{\Diamond^m}\mathsf{S4}_\Box$. From this we obtain the logic $\mathcal{L}\mathsf{S4}_{\Box\Diamond^m}$ by adding:

$$a \leq \Diamond a \qquad \Diamond\Diamond a \leq \Diamond a.$$

Theorem 2. *The logics $\mathcal{L}_{\Diamond^m}\mathsf{S4}_\Box$ and $\mathcal{L}\mathsf{S4}_{\Box\Diamond^m}$ have the finite model property.*

Proof. We proceed as in the previous proof. The only difference is that we need to define a diamond on B. We define

$$\Diamond_B a = \bigwedge\{b \in B \mid b \geq \Diamond a\} \qquad \text{and} \qquad \Diamond_B a = \bigwedge\{\Diamond b \mid b \in B, \Diamond b \geq \Diamond a, \Diamond b \in B\}$$

in the first and second cases, respectively. One easily checks that if $a, \Diamond a \in B$, then $\Diamond_B a = \Diamond a$. One can also check that $\Diamond_B$ is monotone, and that it is $\mathsf{S4}$ provided that $\Diamond$ is. $\Box$

One might wonder if the diamond can be made normal. The main difficulty lies in non distributivity. In [1, Lemmas 4.5 & 6.2], the proofs rely on distributivity. Another difficulty arises from the fact that one does not need to prove $\Diamond_B(a \vee b) \leq \Diamond_B a \vee \Diamond_B b$, but $\Diamond_B(a \vee_B b) \leq \Diamond_B a \vee_B \Diamond_B b$.

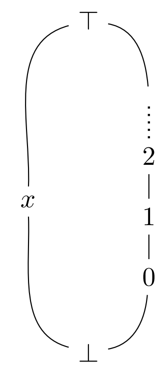

Figure 1: The lattice N_∞

So far, we have treated $\Box$ and $\Diamond$ as two unrelated operators. Guided by [2, Section 4], we may wish to add interaction axioms, such as

$$\Box a \wedge \Diamond b \leq \Diamond(a \wedge b).$$

However, the method for obtaining finite models used above does not readily work in presence of this interaction axiom. We will illustrate where it fails; resolving this is ongoing work. Let A be the lattice N_∞ equipped with an identity box and a diamond sending n to $n+1$ (and sending $\top, x, \bot$ to themselves). The axiom $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$ is satisfied in A. However, it cannot be satisfied in any $B \subseteq A$. Indeed, let n be the maximum of $B \cap \mathbb{N}$. Then $\Diamond n = n + 1$. In line with [1, Theorem 4.2], we wish to have $\Diamond_B m \geq \Diamond m$, which forces $\Diamond_B m = \top$. Then $\Box x \wedge \Diamond_B m = x$, although $\Diamond_B(x \wedge m) = \Diamond_B \bot = \bot$. Therefore, $\Box a \wedge \Diamond_B b \leq \Diamond_B(a \wedge b)$ is refuted in B.

This leaves the finite model property of this logic as an open question. We intend to resolve it by exploring the Kripke-like semantics of non-distributive modal logic developed in [2].

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