# Conjunctive Table Algebras

Jens Kötters and Stefan E. Schmidt

Technische Universität Dresden, Dresden, Germany

**Tables.** The first infinite ordinal is denoted by $\omega$. We formalize a *table* as a set $T \subseteq G^X$, where $X \subseteq \omega$ is a finite set of *column names* (not column numbers), an element $t \in T$ is a *row*, $t(x)$ is the *entry* in row $t$ and column $x$, and $G$ is an arbitrary set. Hence,

$$\mathrm{Tab}(G) = \bigcup \{\mathfrak{P}(G^X) \mid X \subseteq \omega \text{ finite}\} \tag{1}$$

contains all tables with entries in $G$. Note that while $X$ must be finite, a table can have an infinite number of rows if $G$ is infinite.

**Primitive Positive Formulas.** Let $M$ denote a relational signature. A first-order formula over $M$ is *primitive positive* if it is built from atoms using $\{\wedge, \exists\}$. An atom is either a *relational atom* $Rx_1 \ldots x_n$, an *equality atom* $x{=}y$, or one of the special atoms $\mathsf{true}$ (the *tautology*) and $\mathsf{false}$ (the *contradiction*). The set of primitive positive formulas over $M$ is denoted by $\mathrm{PP}(M)$.

**Variables.** We assume that $\omega$ is the countably infinite set of variables. The function free : $\mathrm{PP}(M) \to \mathfrak{P}(\omega)$ maps each formula $\varphi$ to the set of free variables occurring in $\varphi$; for the special atoms, we define $\mathrm{free}(\mathsf{true}) = \emptyset$ and $\mathrm{free}(\mathsf{false}) = \omega$.

**Conjunctive Table Algebras.** Every relational structure $\mathfrak{G}$, with universe $G$ and signature $M$, induces a *solution operation* $(\cdot)^{\mathfrak{G}} : \mathrm{PP}(M) \to \mathrm{Tab}(G)$ that maps each formula $\varphi$ to its *solution set*

$$\varphi^{\mathfrak{G}} := \{t \in G^{\mathrm{free}(\varphi)} \mid \mathfrak{G} \models \varphi[t]\} \subseteq \mathrm{Tab}(G) \quad , \tag{2}$$

where $\mathfrak{G} \models \varphi[t]$ means that $\varphi$ holds in $\mathfrak{G}$ under the variable assignment $t : \mathrm{free}(\varphi) \to G$.

The algebra $\mathbf{PP}(M) := (\mathrm{PP}(M), \wedge, \mathsf{false}, \mathsf{true}, \exists_x, x{=}y, \mathrm{free})_{x,y\in\omega}$ extends $\mathrm{PP}(M)$ with a binary operation $\wedge$ (interpreted as syntactic conjunction), a unary operation $\exists_x$ for each $x \in \omega$ (interpreted as syntactic existential quantification over $x$), the function free : $\mathrm{PP}(M) \to \mathfrak{P}(\omega)$, and it contains all non-relational atoms as distinguished elements. The solution operation homomorphically maps the logical operations to corresponding table operations; we have

$$(\varphi \wedge \psi)^{\mathfrak{G}} = \varphi^{\mathfrak{G}} \bowtie \psi^{\mathfrak{G}}, \quad \mathsf{false}^{\mathfrak{G}} = \emptyset, \quad \mathsf{true}^{\mathfrak{G}} = \{\emptyset\}, \quad (\exists_x \varphi)^{\mathfrak{G}} = \mathrm{del}_x(\varphi^{\mathfrak{G}}), \quad (x{=}y)^{\mathfrak{G}} = E_{xy},$$

where $\bowtie$ is the *natural join*, $\emptyset$ is the *empty table*, $\{\emptyset\}$ is the table with a single empty row, $\mathrm{del}_x$ is a *deletion operation* (deletes column $x$ if it exists), and $E_{xy} := \{t \in G^{\{x,y\}} \mid t(x) = t(y)\}$ is a *diagonal*. Moreover, the *schema* of a table $T \in \mathrm{Tab}(G)$ is uniquely defined by

$$\mathrm{schema}(T) := \begin{cases} X & \text{if } T \in G^X \text{ and } T \neq \emptyset \\ \omega & \text{if } T = \emptyset \end{cases} \quad , \tag{3}$$

and if $\varphi^{\mathfrak{G}} \neq \emptyset$, then also $\mathrm{free}(\varphi) = \mathrm{schema}(\varphi^{\mathfrak{G}})$. This motivates the definition of the table algebra $\mathbf{Tab}(G) := (\mathrm{Tab}(G), \bowtie, \emptyset, \{\emptyset\}, \mathrm{del}_x, E_{xy}, \mathrm{schema})_{x,y\in\omega}$. A *conjunctive table algebra* with *base* $G$ is a subalgebra of $\mathbf{Tab}(G)$.

**Comparison with cylindric set algebras.** Conjunctive table algebras are a database-theoretic variant of cylindric set algebras (of dimension $\omega$). In his survey paper [2, Sect. 7(4)], Németi briefly discusses the charm of such a variant. Németi's universe $\mathrm{Gfs}(G)$ is our $\mathrm{Tab}(G)$. He credits Howard [1] with the approach (although Howard refers to the universe $\mathfrak{P}(\bigcup_{X\subseteq\omega} G^X)$). Howard uses complements, so in that sense, conjunctive table algebras are more generic.

**Main Result.** We present an axiomatization of conjunctive table algebras. The conjunctive table algebras with nonempty base are, up to isomorphism, precisely the projectional semilattices; a *projectional semilattice* is an algebraic structure $(V, \wedge, 0, 1, c_x, d_{xy}, \mathrm{dom})_{x,y\in\omega}$ consisting of an infimum operation $\wedge$, a *bottom element* $0$, a *top element* $1$, a *cylindrification* $c_x : V \to V$ for each $x \in \omega$, a *diagonal* $d_{xy} \in V$ for each $(x, y) \in \omega\times\omega$, and a *domain function* $\mathrm{dom} : V \to \mathfrak{P}(\omega)$, such that the axioms

**(PS0)** $(V, \wedge, 0, 1)$ is a bounded semilattice

**(PS1)** $c_x(0) = 0$

**(PS2)** $u \leq c_x(u)$

**(PS3)** $c_x(u \wedge c_x(v)) = c_x(u) \wedge c_x(v)$

**(PS4)** $c_x(c_y(u)) = c_y(c_x(u))$

**(PS5)** $u \neq 0 \Rightarrow (u \neq c_x(u) \Leftrightarrow u \leq d_{xx})$

**(PS6)** $x \neq y, z \Rightarrow d_{yz} = c_x(d_{yx} \wedge d_{xz})$

**(PS7)** $x \neq y \Rightarrow d_{xy} \wedge c_x(d_{xy} \wedge u) \leq u$

**(PS8)** $u \neq 0 \Rightarrow \mathrm{dom}(u)$ finite

**(PS9)** $\mathrm{dom}(u) = \{x \in \omega \mid u \leq d_{xx}\}$

**(PS10)** $\mathrm{dom}(u) = \emptyset \Rightarrow u = 1$

**(PS11)** $d_{xx} \neq 0$

**(PS12)** $d_{xy} = d_{yx}$

hold for all $u, v \in V$ and $x, y, z \in \omega$.

**Comparison with cylindric algebras.** The axioms **(PS0)**, . . . , **(PS7)** correspond to cylindric algebra axioms (**CA0**), . . . , (**CA7**). Axiom (**CA0**) asserts a Boolean algebra; since we do not consider disjunction and negation, axiom **(PS0)** only asserts a bounded semilattice. The Axioms (**CA1**), (**CA2**), (**CA3**), (**CA4**) and (**CA6**) are identical to **(PS1)**, **(PS2)**, **(PS3)**, **(PS4)** and **(PS6)**, respectively. Cylindric algebra axiom (**CA5**) states $d_{xx} = 1$, reflecting that $x{=}x$ is a tautology; however, the *table semantics* in eq. (2) corresponds to a logic with undefined variables, where $x{=}x$ is not a tautology! We consider **(PS5)** to be a suitable replacement: Under the definition axiom **(PS9)**, axiom (**CA5**) asserts $\mathrm{dom}(u) = \omega$ for all $u \neq 0$; whereas axiom **(PS5)** asserts $\mathrm{dom}(u) = \{x \in \omega \mid c_x(u) \neq u\}$ for all $u \neq 0$; the latter set is known as the *dimension set* $\Delta(u)$ in the terminology of cylindric algebras. Axiom **(PS7)** is the historical axiom (**CA7**); the contemporary axiom (**CA7**) is equivalent but involves negation! Historically, there was also an axiom (**CA8**), stating that $\Delta(u)$ is finite for all $u \in V$. Since $\mathrm{dom}(u) = \Delta(u)$ for $u \neq 0$, we can identify (**CA8**) with **(PS8)**, disregarding the case $u = 0$.

**Variant: Empty Universe.** If axiom **(PS11)** is weakened to $1 \neq 0$, we obtain a characterization of conjunctive table algebras (including base $G = \emptyset$) up to isomorphism.

## References


[1] Charles Malone Howard. *An Approach to Algebraic Logic.* PhD thesis, University of California, Berkeley, 1965.

[2] István Németi. Algebraizations of quantifier logics, an introductory overview. `https://old.renyi.hu/pub/algebraic-logic/survey.html`, January 1997. 12.1th Version.