Decompositions of locally integral involutive residuated structures

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Extended abstract

An *involutive partially-ordered semigroup* (*ipo-semigroup*) is a structure of the form $\mathbf{A} = (A, \leq, \cdot, \sim, -)$ such that $(A, \leq)$ is a partially ordered set and $(A, \cdot)$ is a semigroup with two order-reversing operations $\sim$ and $-$ satisfying *involution* $\sim{-x} = x = -{\sim x}$ and *rotation* $x \cdot y \leq z \iff y \cdot {\sim z} \leq {\sim x} \iff -z \cdot x \leq -y$. In the case that the semigroup has an identity, we call it an *ipo-monoid*. An ipo-semigroup in which the partial order is a lattice order is called an *iℓ-semigroup*.

In the presence of order-reversal and involution, rotation is equivalent to *residuation*:

$$xy \leq z \iff x \leq -(y \cdot {\sim z}) \iff y \leq {\sim}(-z \cdot x).$$

Thus, the multiplication of every ipo-semigroup is residuated in both arguments, with *left* and *right residuals* given by $z/y = -(y \cdot {\sim z})$ and $x \backslash z = {\sim}(-z \cdot x)$, respectively.

We say that an ipo-monoid is *integral* if the global identity 1 is also the top element. In this case, $x \backslash x = 1 = x/x$. More generally, an ipo-semigroup $\mathbf{A}$ has local identities if $x \backslash x = x/x$ for all x, in which case we denote this element by 1_x, and $1_x \cdot x = x$. If, moreover, elements are bounded by their local identities ($x \leq 1_x$), the local identities are positive ($y \leq 1_x \cdot y$), and $x \backslash 1_x = 1_x$, then we say that $\mathbf{A}$ is *locally integral*.

We show that every locally integral ipo-semigroup $\mathbf{A}$ decomposes uniquely into a Płonka sum over a semilattice directed system of integral ipo-monoids. We also solve the reverse problem, that is, we provide necessary and sufficient conditions so that the glueing of a system of integral ipo-monoids becomes an ipo-semigroup. This is a generalization of the results in [1], in which the decomposition and glueing results are proven for locally integral ipo-monoids.

Commutative idempotent locally integral ipo-semigroups are called *locally integral ipo-semilattices* and decompose into a system of Boolean algebras. A structural description of finite commutative idempotent involutive residuated lattices (unital iℓ-semilattices) is given in [2]. We also describe a dual representation for a class containing all finite locally integral ipo-semilattices via semilattice directed systems of partial functions between sets.

This is joint work with José Gil-Férez and Peter Jipsen.

References

[1] J. Gil-Férez, P. Jipsen, and S. Lodhia. The structure of locally integral involutive po-monoids and semirings. In *Relational and algebraic methods in computer science*, volume 13896 of *Lecture Notes in Comput. Sci.*, pages 69–86. Springer, Cham, 2023.

[2] P. Jipsen, O. Tuyt, and D. Valota. The structure of finite commutative idempotent involutive residuated lattices. *Algebra Universalis*, 82(4):Paper No. 57, 23, 2021.