Prenex normal form theorems in intuitionistic arithmetic and the effective topos

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Abstract

The fact that every first-order formula has a prenex normal form in classical logic is known as *prenex normal form theorem*, recognized as one of the most widely used theorems in mathematical logic. However, this theorem does not generally hold for intuitionistic theories. There has been work on this matter for more than half a century.

As studied in [1] and [2], it has been revealed that certain variants of prenex normal form theorem in Heyting arithmetic **HA** are strongly related to *semi-classical axioms* such as Γ-**DNE** (double negation elimination restricted to a class Γ of formulas). Akama et al. [1] initially introduced two syntactically defined classes of formulas, written as E_n and U_n. They are equivalent to Σ_n and Π_n, respectively, in the standard sense over classical logic. Akama et al. then show that $\mathbf{HA} + \Pi_n \vee \Pi_n$-**DNE** proves the prenex normal form theorem $\mathbf{PNFT}(\mathrm{U}_n, \Pi_n)$ from U_n to Π_n. Similarly, $\mathbf{HA} + \Sigma_n$-**DNE** $+ \Pi_n \vee \Pi_n$-**DNE** ensures both $\mathbf{PNFT}(\mathrm{E}_n, \Sigma_n)$ and $\mathbf{PNFT}(\mathrm{U}_n, \Pi_n)$.

In contrast, the situation concerning $\mathbf{PNFT}(\mathrm{E}_n, \Sigma_n)$ alone is more subtle. Fujiwara and Kurahashi [2] gave negative evidence: they show that there is an E_1-formula φ_0 that is not equivalent to any Σ_1-formula over $\mathbf{HA} + \Sigma_1$-**DNE**. This indicates that $\mathbf{PNFT}(\mathrm{E}_1, \Sigma_1)$ does not hold over $\mathbf{HA} + \Sigma_1$-**DNE**. The proof in [2] relies on a syntactic argument using a non-classical axiom, Church's thesis.

The purpose of this talk is to provide a topos-theoretic account on the last negative result concerning prenex normal form theorems in Heyting arithmetic. If an elementary topos $\mathcal{E}$ has a natural number object (NNO), $\mathcal{E}$ can be regarded as a model of Heyting arithmetic according to the standard interpretation of first-order logic. For instance, the *effective topos* $\mathcal{E}ff$, which is a significant example in categorical realizability, satisfies $\mathbf{HA} + \Sigma_1$-**DNE** but does not satisfy Σ_2-**DNE**. As seen from this example, a topos is not a model of classical arithmetic in general. However, by using the concept of *local operator*, it can be always "classicalized".

Local operator (a.k.a. Lawvere-Tierney topology) is the most important tool for creating a new topos from a given one. As a matter of fact, each local operator j in a topos $\mathcal{E}$ corresponds precisely to a subtopos $\mathcal{E}_j$ of $\mathcal{E}$. The logic of $\mathcal{E}_j$ may be different from the logic of $\mathcal{E}$. A typical example is the *double negation operator* $\neg\neg$, which exists in every topos. It is important that the corresponding subtopos always models classical logic even in the case the original topos does not. For example, the associated subtopos $\mathcal{E}ff_{\neg\neg}$ of the effective topos $\mathcal{E}ff$ is categorically equivalent to the category **Set** of sets.

As an illustration of the relationship between prenex normal form theorem and a topos-theoretic structure, we show the following theorem.

Theorem 1. *Let φ be an arithmetical formula and $\mathcal{E}$ an elementary topos with NNO satisfying Σ_n-**DNE**. In addition, suppose that φ is true in $\mathcal{E}$, while not in the subtopos $\mathcal{E}_{\neg\neg}$ associated with the double negation operator. Then there is no Σ_{n+2}-formula equivalent to φ over $\mathbf{HA} + \Sigma_n$-**DNE**.*

The proof is based on a topos-theoretic notion, *transparency*, introduced in [3]. Furthermore, we can find concrete examples within subtoposes of the effective topos $\mathcal{E}\!f\!f$ for this theorem. As is well known in categorical realizability, for every Turing degree d, there is a corresponding local operator j_d in $\mathcal{E}\!f\!f$. In particular, we can consider the local operator $j_{\emptyset^{(n)}}$ for the n-th Turing jump $\emptyset^{(n)}$ of the empty set. The associated subtopos $\mathcal{E}\!f\!f_{j_{\emptyset^{(n)}}}$ satisfies $\mathbf{HA}+\Sigma_{n+1}$-$\mathbf{DNE}$.

Theorem 2. *Let n be an arbitrary natural number. Then there exists an E_{n+1}-formula φ_n such that it is true in $\mathcal{E}\!f\!f_{j_{\emptyset^{(n)}}}$ but not true in $(\mathcal{E}\!f\!f_{j_{\emptyset^{(n)}}})_{\neg\neg} \simeq \mathbf{Set}$.*

The above theorems provide an alternative proof for the theorem in [2] and generalize it.

Corollary 3. *Let n be an arbitrary natural number. For the E_{n+1}-formula φ_n in Theorem 2, there is no Σ_{n+3}-formula equivalent to φ_n over $\mathbf{HA}+\Sigma_{n+1}$-$\mathbf{DNE}$.*

This implies that $\mathbf{PNFT}(\mathrm{E}_{n+1}, \Sigma_{n+1})$ does not hold over $\mathbf{HA}+\Sigma_{n+1}$-$\mathbf{DNE}$.

References

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