# Embeddings of metric Boolean algebras in $\mathbb{R}^N$

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A metric Boolean algebra (see e.g. [1, 2, 3]) consists of a Boolean algebra $\mathbf{A}$, equipped with a strictly positive (finitely-additive) probability measure[1] $m\colon \mathbf{A} \to [0,1]$, which makes $(\mathbf{A}, d_m)$ a metric space, where the distance between any two points $a, b \in A$ is defined as:

$$d_m(a,b) := m(a \bigtriangleup b) = m((a \wedge b') \vee (a' \wedge b)).$$

From a geometrical point of view, it is natural to wonder under which conditions a metric Boolean algebra $(\mathbf{A}, d_m)$, or some of its relevant subspaces, can be isometrically embedded in $\mathbb{R}^N$ (equipped with the Euclidean distance), for a given positive integer $N$. Actually, for $|A| > 2$, there is no such embedding. However, under the assumption that $\mathbf{A}$ is finite (or, more generally, atomic), it makes sense to restrict the question to the subspace $\mathsf{At}(\mathbf{A})$ of its atoms.

A classical result by Morgan [5] states that a metric space $(X, d)$ embeds in $\mathbb{R}^N$ if and only if it is flat and has dimension less or equal to $N$, where $(X, d)$ is flat if the determinant of the matrix $M(\vec{x}_n)$, whose generic entry is $M_{ij} = \frac{1}{2}(d(x_0, x_i)^2 + d(x_0, x_j)^2 - d(x_i, x_j)^2)$, is non-negative for every $n$-simplex (namely every choice of $n+1$ points $\vec{x}_n = \{x_0, \dots, x_n\}$ in $X$) and the dimension of $(X, d)$ is the greatest $N$ (if exists) such that there exists a $N$-simplex with positive determinant.

Given a finite metric Boolean algebra $\mathbf{A}$ with $\mathsf{At}(\mathbf{A}) = \{a_0, a_1, \dots, a_k\}$, it is easily checked that the matrix $M(\vec{x}_n) = \{M_{ij}\}$, $2 \leq n \leq k$ (introduced in Morgan's theorem) has generic entry

$$M_{ij} = (x_0 + x_i)^2 \delta_{ij} + (x_0^2 + x_0 x_1 + x_0 x_j - x_i x_j)(1 - \delta_{ij}),$$

where $x_\alpha = m(a_\alpha)$ (thus $x_\alpha > 0$, for every $\alpha \in \{0, 1, \dots, k\}$). Therefore the form of the determinant can be simplified according to the following.

**Lemma 1.** *Let $M(\vec{x}_n)$, $2 \leq n \leq k$ be the matrix associated to a finite metric atomic Boolean algebra $\mathbf{A}$ with $k+1$ atoms. Then*

$$\det(M(\vec{x}_n)) = 2^{n-1} \left[ \left( \sum_{\alpha=0}^{n} x_0 \cdot \dots \cdot \hat{x}_\alpha \cdot \dots \cdot x_n \right)^2 - (n-1) \left( \sum_{\alpha=0}^{n} x_0^2 \cdot \dots \cdot \hat{x}_\alpha^2 \cdot \dots \cdot x_n^2 \right) \right],$$

*where $\hat{x}_i$ means that $x_i$ has to be omitted.*

It follows, for instance, that the space $(\mathsf{At}(\mathbf{A}), d_m)$ of the $k+1$ atoms of a finite metric Boolean algebra such that $m(a_i) = \frac{1}{k+1}$ (for every $a_i \in \mathsf{At}(\mathbf{A})$) embeds in $\mathbb{R}^k$ with the Euclidean metric and that $\det(M(\vec{x}_2)) > 0$.

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[1] Recall that a *strictly positive* (finitely additive) probability measure over a Boolean algebra $\mathbf{A}$ is a map $m\colon \mathbf{A} \to [0,1]$ such that:

1. $m(\bot) = 1$,
2. $m(a \vee b) = m(a) + m(b)$, for every $a, b \in A$ such that $a \wedge b = \bot$,
3. $m(a) > 0$, for every $a \in A$, $a \neq \bot$.

Upon indicating by $\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ the space of the (finitely additive) probability measures $m$ such that $(\mathrm{At}(\mathbf{A}), d_m)$ admits an isometric embedding into some Euclidean space $\mathbf{R}^N$, in virtue of Morgan's theorem one has

$$\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A})) = \bigcap_{n=3}^{k} C_n \cap \Pi_k,$$

where $C_n = \{\vec{x} \in \mathbb{R}_+^{k+1} \mid \det M(\vec{x}_n) \geq 0\}$, with $3 \leq n \leq k$ and $\Pi_k$ is the interior of the standard $k$-simplex (or probability simplex) of $\mathbb{R}^{k+1}$, namely

$$\Pi_k = \{\vec{x} \in (0,1)^{k+1} \mid \sum_{\alpha=0}^{k} x_\alpha = 1\}.$$

We are interesting in solving the following.
**Problem.** Study the topology of $\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ with the topology induced by $(0,1)^{k+1} \subset \mathbb{R}_+^{k+1}$.

In order to get a solution, we first analyze the topology of $C_n$.

**Lemma 2.** *For each $3 \leq n \leq k$, the space $C_n \cong H_n \times \mathbb{R}_+^{k-n}$ where $H_n$ is a solid half-hypercone in $\mathbb{R}_+^{n+1}$.*

The solution to the above presented problem is given by the following.

**Theorem 3.** *Let $k \geq 3$. Then:*

1. *$\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ is contractible.*
2. *$\mathcal{M}(\mathrm{At}(\mathbf{A})) \setminus \mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ is simply-connected (not contractible).*

In the final part of the talk, we will draw some considerations on the significance of our results for probability theory and on their possibile extensions to the case of infinite (non-atomic) Boolean algebras.

# References


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