When are bounded arity polynomials enough?

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This work is within the context of the classical Galois connection between sets of functions and sets of relations determined by the notion of *preservation*. For a set A, a function $f : A^n \to A$, and a relation $R \subseteq A^k$, we say that f *preserves* R (and write $f \triangleright R$) whenever f applied coordinatewise to n-many tuples belonging to R produces another tuple belonging to R. For a finite set A, the Galois closed sets of functions are the *function clones*, while the Galois closed sets of relations are *relational clones*, which are sets of relations that are closed under positive primitive definitions. A classical and easy result in Universal Algebra states, for an equivalence relation $\theta \subseteq A^2$ and a function $f : A^n \to A$, that

$$f \triangleright \theta \iff \mathsf{trl}_1(f) \triangleright \theta,$$

where $\mathsf{trl}_1(f)$ is the set of all *basic translations* of f, i.e. those unary polynomials that can be produced by evaluating all arguments except possibly one at a constant. If a relation R satisfies the above property, we will write $\Xi_1(R)$.

Before this collaboration, each author had found a different kind of generalization of this result. In [1], the second author and his collaborators were interested in describing all relations R such that $\Xi_1(R)$ holds. Since $\Xi_1(\theta)$ holds for any quasiorder θ (reflexive and transitive binary relation), their work is focused on establishing properties of what they call *generalized quasiorders*, which are relations $R \subseteq A^k$ that are reflexive (contain the constant tuple $(c, \ldots, c)$ for all $c \in A$) and transitive, which we now define. For $a \in A^{k^2}$, we will write $R \models a$ to indicate that every row and column of a when considered as a $k \times k$ matrix is a tuple belonging to R. With this notation, we say that R is transitive if whenever $R \models a$, then the diagonal of a is also an element of R. It is easy to see that this definition of transitivity coincides with the usual definition for binary relations. The authors establish that $\Xi_1(R)$ holds for any generalized quasiorder R.

On the other hand, in [2] the first author shows that properties of a higher arity commutator operation are closely connected to the properties of certain invariant relations called *higher dimensional equivalence relations.* Such relations are naturally coordinatized by higher dimensional cubes and satisfy natural generalizations of the transitive, reflexive, and symmetric properties ordinarily associated with binary relations. Higher dimensional congruences enjoy some nice properties, one of which is a generalization of the above equivalence to the following:

$$f \triangleright \theta \iff \mathsf{trl}_d(f) \triangleright \theta,$$

for a higher dimensional congruence $\theta \subseteq A^{2^d}$ and a function $f : A^k \to A$, where now we take $\mathsf{trl}_d(f)$ to be the set of polynomial functions obtainable from f by evaluating all but up to d-many variables at a constant. If a relation R satisfies this generalization of $\Xi_1(R)$, we will write $\Xi_d(R)$.

These two lines of inquiry prompt the search for a characterization of the those relations R for which $\Xi_d(R)$ holds. This question is closely related to a question about clones: for $M \subseteq A^{A^d}$

a set of d-ary operations on a finite set A, when is it true that

$$M^* = \{f \in \mathrm{Op}(A) : \mathsf{trl}_d(f) \subseteq M\}$$

is a clone? Among the results of this inquiry are definitions of reflexivity and transitivity which, on the one hand are suitable for a very broad class of relations, and on the other hand are each a natural generalization of the older concept. We are able to show that a relation R which is reflexive and transitive in the more general sense satisfies $\Xi_d(R)$ (for d a dimension parameter which we will not define here). Furthermore, we characterize those clones that are the polymorphisms of a set of such relations, for a particular dimension d. One of our conclusions is that each relation R with the property $\Xi_d(R)$ has a positive primitive definition in a particular relation Γ_M that is both reflexive and transitive in our general sense.

References

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