An algebraic semantics for possibilistic finite-valued Łukasiewicz logic

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In the present work, based on the ideas of [1], we analyse an algebraic semantics for a many-valued modal logical system based on the n-valued Łukasiewicz logic $\Lambda(\mathbf{Ł}_n)$. We extend $\Lambda(\mathbf{Ł}_n)$ (for each fixed natural n) to a modal system, by adding a unary operator $\Box$ to the original many-valued propositional language. Our approach strongly relies on the fact that the system $\Lambda(\mathbf{Ł}_n)$ has as an algebraic semantics the subvariety of MV-algebras generated by the n-elements MV-chain $\mathbf{Ł}_n$, that is the algebra with universe $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$, and main operations $x \to y = \min\{1, 1-x+y\}$, $\neg x = 1-x$.

To achieve our aim, we work with a propositional modal language formed by a numerable set of variables Var and the connectives $\langle \to, \neg, \Box, 0\rangle$, thus the set $Form_\Box$ of formulas in this language is built as usual. An $\mathbf{Ł}_n$-valued *possibilistic frame* $\langle W, \pi\rangle$ is given by a non-empty set of worlds W and a function $\pi : W \to \mathbf{Ł}_n$ (called a normalized possibility distribution over W) such that $\bigvee_{w\in W} \pi(w) = 1$. An *$\mathbf{Ł}_n$-valued possibilistic model* is a 3-tuple $M = \langle W, \pi, e\rangle$ where $\langle W, \pi\rangle$ is an $\mathbf{Ł}_n$-valued possibilistic frame and e is a map, called *valuation*, assigning to each propositional variable in Var and each possible world in W an element of $\mathbf{Ł}_n$ (i.e., $e : Var \times W \longrightarrow \mathbf{Ł}_n$).

If $M = \langle W, \pi, e\rangle$ is a $\mathbf{Ł}_n$-valued possibilistic model, the map e can be uniquely extended to a map, assigning to each formula in $Form_\Box$ and each world in W an element of $\mathbf{Ł}_n$ (i.e., $e : Form_\Box \times W \longrightarrow \mathbf{Ł}_n$) satisfying that:

- e is an algebraic homomorphism in its first component, i.e., for the connectives $\to, \neg, 0$,
- $e(\varphi, w) = \bigwedge\{\pi(w') \to e(\varphi, w') : w' \in W\}$

The logical system that we are trying to characterize generalizes the classical possibilistic logic, and it is the many-valued modal system semantically defined by possibilistic models over $\mathbf{Ł}_n$

Our algebraic approach deals with complex algebras that arise from $Ł_n$-valued frames. That is: given an $\mathbf{Ł}_n$-valued possibilistic frame $\langle W, \pi\rangle$ we consider the MV-algebra of functions $Ł_n^W$ and the unary operator $\Box : Ł_n^W \to Ł_n^W$ given by $\Box x(i) = \bigwedge\{\pi(j) \to x(j) : j \in W\}$. We study the quasivariety of algebras generated by these complex algebras, and this quasivariety, together with the abstract theory of algebraizable logics immediately provide an axiomatization for the possibilistic many-valued system over $\mathbf{Ł}_n$. From the way that the system is defined, it turns out to be complete with respect to the logic semantically defined by the $\mathbf{Ł}_n$-valued possibilistic frames. So the logical system determined by frames over $Ł_n$ has an algebraic semantics based on MV-algebras.

The present investigation provides a negative answer to a conjecture of P. Hájek posed in his book [2] which intends to generalize the classical setting, where the possibilistic logic coincides with the modal logic $KD45$. We prove that the logic semantically defined by $\mathbf{Ł}_n$-valued possibilistic frames can not be axiomatized by simply requiring the fuzzy analogues of the classical axioms K,D,4 and 5.

References

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[2] Hájek, P., *Metamathematics of Fuzzy Logic*, Trends in Logic, Kluwer. 1998.